Quantum resonances in the semiclassical regime: a brief survey

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Introduction

- In physics, the introduction of the notion of quantum resonance was motivated by the behavior of various quantities related to scattering experiments, as scattering cross sections, or time delays, ...
At certain energies, these quantities present peaks, which were modelized by a Lorentzian shaped function

\[ w_{a,b} : \lambda \mapsto \frac{1}{\pi} \frac{b/2}{(\lambda - a)^2 + (b/2)^2}. \]

Note that for \( \rho = a - ib/2 \in \mathbb{C} \), one has

\[ w_{a,b}(\lambda) = \frac{1}{\pi} \frac{\text{Im} \rho}{|\lambda - \rho|^2}, \]

and the complex number \( \rho \) was called a resonance.

Such complex values for energies had also appeared for example in the work by Gamow, to explain \( \alpha \)-radioactivity, and the notion of resonance was therefore also associated to some energy decay.
The Schrödinger operator

A quantum particle of mass $m$, moving in a potential $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$, is described by a solution of the time-dependent Schrödinger equation

$$\begin{cases} i\hbar \partial_t \phi(t, x) = P(x, \hbar D_x) \phi(t, x), \\ P(x, \hbar D_x) = -\frac{\hbar^2}{2m} \Delta + V(x). \end{cases}$$

When $E \in \sigma_{disc}(P)$, and $P\phi_0 = E\phi_0$, $\|\phi_0\|_{L^2} = 1$, then

$$\phi(t, x) = e^{-iEt/\hbar} \phi_0(x, \hbar)$$

is a solution, which satisfies the normalization condition

$$\|\phi(t, .)\|_{L^2} = \left( \int |\phi(t, x)|^2 dx \right)^{1/2} = 1.$$ 

$\phi(t, x)$ is called the wave function of the particle, with energy $E$, and $|\phi(t, x)|^2$ is its density of probability of presence at time $t$. 
Now suppose $\psi_0$ is a resonant state (not in $L^2$) corresponding to the resonance $\rho = a - ib/2$. Its time evolution should be written

$$\psi(t) = e^{-ita-tb/2}\psi_0,$$

so that its density of probability of presence at time $t$ is

$$\frac{|\psi(t,x)|^2}{|\psi_0(x)|^2} = e^{-bt},$$

and $b$ is the decay rate of that probability.

In other words: a resonance $\rho \in \mathbb{C}$ corresponds to a quasi-particle with lifetime $1/\Im \rho$. The more a resonance is close to the real axis, the more its presence leads to physically observable effects (can also be seen on Breit-Wigner peaks).
Poles of the resolvent

- We suppose that $V \in C^\infty(\mathbb{R}^n)$ is real valued, and that
  $$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\sigma-|\alpha|}),$$
  for some $\sigma > 0$.

- Then $P = -\hbar^2 \Delta + V$ is self-adjoint with domain $H^2(\mathbb{R}^n)$. In particular for $\Im z > 0$, $R(z) = (P - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is well defined.

**Theorem and Definition 1:**

The resolvent

$$\{\Re z > 0, \Im z > 0\} \ni z \mapsto R(z) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n)$$

can be meromorphically continued to the lower half-plane $\{\Re z > 0, \Im z < 0\}$. Its poles are called resonances of $P$. 
Classical scattering

▶ A classical particle with mass 1, when placed in the conservative force field \( F(x) = -\nabla V(x) \), follows the trajectory \((x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))\) given by

\[
\begin{align*}
\dot{x}(t) &= \xi(t), \\
\dot{\xi}(t) &= -\nabla V(x(t)).
\end{align*}
\]

▶ This is Newton’s Equation written in Hamiltonian form: In the phase space \( T^* \mathbb{R}^d \), the trajectory is an integral curve of the vector field

\[
H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi
\]

associated to the total energy of the particle \( p(x, \xi) = \frac{1}{2}\xi^2 + V(x) \)

▶ Note that:

\[
Pu(x) = \text{Op}_h^w(p(x, \xi))u(x) = \int e^{i(x-y) \cdot \xi / h} p\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \quad \frac{d\xi}{(2\pi h)^n}
\]

▶ Scattering: \( V \to 0 \) at \( \infty \), and we consider trajectories going to infinity. The aim could be to recover \( V \) knowing what goes out of the interaction region when one throw particles towards that region.
Let $E > 0$ be fixed. For any $\alpha \in S^{n-1}$, $z \in \alpha^\perp \sim \mathbb{R}^{n-1}$, there is one and only one integral curve for $H_p$

$$\gamma_\pm(t, z, \alpha, E) = (x_\pm(t, z, \alpha, E), \xi_\pm(t, z, \alpha, E)) \in p^{-1}(E)$$

such that

$$\lim_{t \to \pm \infty} |x_\pm(t, z, \alpha) - \sqrt{2E}\alpha t - z| = 0,$$

$$\lim_{t \to \pm \infty} |\xi_\pm(t, z, \alpha) - \sqrt{2E}\alpha| = 0.$$

The classical scattering map at energy $E$ is the map $(z, \omega) \mapsto (z', \theta)$. 
The scattering amplitude

We suppose for simplicity that $V$ is a very short range potential:

$$\partial^\alpha V(x) = O(\langle x \rangle^{-\sigma-|\alpha|}), \text{ for some } \sigma > \frac{n+1}{2}.$$

Let $E > 0$ be fixed. For any $\omega, \theta \neq \omega \in S^{n-1}$, there exists one and only one $u \in L^2_{loc}(\mathbb{R}^n)$ such that

$$\begin{cases}
Pu = Eu, \\
u(x, h) = e^{i \sqrt{E} x \cdot \omega / h} + \mathcal{A}(\omega, \theta, E, h) e^{i \sqrt{E} x \cdot \theta / h} |x|^{(1-n)/2} + o(|x|^{(1-n)/2}),
\end{cases}$$

as $|x| \to +\infty$ with $\frac{x}{|x|} = \theta$.

The scattering amplitude at energy $E$ is the operator

$$\mathcal{A}(E) : L^2(S^{n-1}) \to L^2(S^{n-1})$$

with kernel $\mathcal{A}(\omega, \theta, E, h)$. The scattering matrix at energy $E$ is

$$S(E) = \mathcal{I} - 2i\pi c(E, h) \mathcal{A}(E),$$

for some normalization constant $c(E, h)$. 
More general potentials

- We suppose now that $V$ is a short range potential:

$$\partial^\alpha V(x) = O(\langle x \rangle^{-\sigma-|\alpha|}), \text{ for some } \sigma > 1,$$

and we set $P = P_0 + V$, $P_0 = -\hbar^2 \Delta$.

- For any given $u_-$, one can find $u$, and then $u_+$, such that

$$\| e^{-itP_0/\hbar} u_- - e^{-itP_0/\hbar} u \| \to 0 \text{ as } t \to -\infty,$$

$$\| e^{-itP_0/\hbar} u_+ - e^{-itP_0/\hbar} u \| \to 0 \text{ as } t \to +\infty.$$

- The Wave operators $W_\pm$ are defined by $u = W_- u_- = W_+ u_+$.

- Notice the intertwining property: $PW_\pm = W_\pm P_0$. 

\[ e^{-itP/\hbar} u \]
\[ e^{-itP_0/\hbar} u_+ \]
\[ u = W_+ u_+ = W_- u_- \]
\[ u_+ \]
\[ u_- \]
\[ e^{-itP_0/\hbar} u_- \]
The scattering amplitude

- The scattering operator for $(P, P_0)$ is defined by
  
  \[ S = (W_+)^{-1}W_+ : u_+ \mapsto u_. \]

- Since $P_0 S - S P_0 = 0$, one has
  
  \[ S = \int_0^{+\infty} S(E) d\mu_P(E), \quad S(E) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \]

The operator $S(E)$ is the scattering matrix at energy $E$.

- $T(E) = \frac{1}{2i\pi}(I d - S(E))$ is an integral operator, with a smooth kernel out of $\theta = \omega$. Up to a multiplicative constant, $T(E)$ is the scattering amplitude at energy $E$.  

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Meromorphic extension of the scattering amplitude

Theorem and Definition 2:
Suppose $V$ is a long range potential. The scattering amplitude $\mathbb{R}^*_+ \ni E \mapsto A(E)$ can be meromorphically continued to the lower half-plane $\{ \text{Re } z > 0, \text{Im } z < 0 \}$. Its poles are called resonances of $P$.

When is $V$ compactly supported, one can prove that for $\chi_1 \prec \chi_2 \in C_0^\infty(\mathbb{R}^n)$,

$$A(\omega, \theta, E, h) = i\pi E^{\frac{n-2}{2}} \langle (P - (E + i0))^{-1} [h^2 \Delta, \chi_2] e^{i \sqrt{E} x \cdot \omega/h}, [h^2 \Delta, \chi_1] e^{i \sqrt{E} x \cdot \theta/h} \rangle$$

Then one can extend $A(\omega, \theta, E, h)$ to complex values of $E$ by

$$A(\omega, \theta, z, h) = i\pi z^{\frac{n-2}{2}} \langle R(z)[h^2 \Delta, \chi_2] e^{i \sqrt{z} x \cdot \omega/h}, [h^2 \Delta, \chi_1] e^{i \sqrt{z} x \cdot \theta/h} \rangle$$
Analytic dilation

▶ For \( \mu \in \mathbb{R} \), we set

\[
U_\mu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad U_\mu \phi(x) = e^{d\mu/2} \phi(xe^\mu), \quad \text{for} \; \phi \in C_0^\infty(\mathbb{R}^d)
\]

and we see

\[
U_\mu P U_\mu^* = e^{-2\mu}(-h^2 \Delta) + V(xe^\mu).
\]

Definition

The function \( V : \mathbb{R}^d \to \mathbb{R} \) is dilation-analytic when

\[
\mu \mapsto V(xe^\mu)(-h^2 \Delta + 1)^{-1}
\]

extends as an analytic (w.r.t. \( \mu \)) family of compact operators on \( L^2(\mathbb{R}^d) \).

▶ For example, if \( V \in C^\infty(\mathbb{R}^d) \) extends as holomorphic function in

\[
\Omega_{\theta_0} = \{ x \in \mathbb{C}^d, \; |\text{Im} \; x| \leq \tan \theta_0 \langle \text{Re} \; x \rangle \},
\]

where it satisfies

\[
V(x) \to 0 \; \text{as} \; |x| \to +\infty,
\]

then \( V \) is dilation-analytic.
Resonances

⚠️ For such a \( V \),

\[
U_\mu P U^*_\mu = e^{-2\mu(-h^2\Delta)} + [V(xe^{i\mu})(-h^2\Delta + 1)^{-1}]( -h^2\Delta + 1),
\]

so that one can define \( P_\theta = U_{i\theta} P U^*_{i\theta} \) for \( \theta \in \mathbb{R}^+ \) small enough.

⚠️ By Weyl’s Theorem

\[
\sigma_{\text{ess}}(P_\theta) = e^{-2i\theta} \sigma_{\text{ess}}(-h^2\Delta) = e^{-2i\theta} \mathbb{R}^+.
\]

Definition 3

Let \( V \) be analytic-dilation in \(|\mu| < \delta\). The set \( \text{Res}(P) \) of resonances of \( P \) is

\[
\text{Res}(P, h) = \bigcup_{0 < \theta < \delta} \sigma_{\text{disc}}(P_\theta) \cap (0, \mathbb{R}^+, e^{-2i\theta} \mathbb{R}^+).
\]
Complex distortion

Now we assume only that \( V(x) \) has an analytic extension to

\[
\Sigma = \{ x \in \mathbb{C}^n; \ |\text{Im} \ x| < \delta \ |\text{Re} \ x| \text{ and } \ |\text{Re} \ x| > C \}
\]

for some \( \delta, C > 0 \), and that \( V(x) \to 0 \) as \( |x| \to \infty \), \( x \in \Sigma \).

Let \( F: \mathbb{R}^n \to \mathbb{R}^n \) be s. t. \( F(x) = 0 \) for \( |x| < C \) and \( F(x) = 1 \) for \( |x| \gg 1 \). For \( \mu \in \mathbb{R} \) small enough, we define \( U_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) as the unitary operator

\[
U_\mu \varphi(x) = \left| \det \left( dxe^{\mu F(x)} \right) \right|^{1/2} \varphi(xe^{\mu F(x)})
\]

For \( \theta \in \mathbb{R} \) small enough, we denote

\[
P_\theta = U_{i\theta} PU_{i\theta}^{-1}
\]

the distorted operator. As before, the set of resonances of \( P \) is

\[
\text{Res}(P, h) = \bigcup_{0<\theta<\delta} \sigma_{\text{disc}}(P_\theta) \cap (0, \mathbb{R}^+, e^{-2i\theta} \mathbb{R}^+).
\]
All these definitions coincide!

**Theorem : (Helffer-Martinez)**

When their domain of validity overlap, these different definitions of resonance (as well as more sophisticated ones) coincide.
Scattering in 1d

► The (stationary) Schrödinger equation in 1d is

\[-h^2 u'' + Vu = Eu.\]

► The scattering matrix \( S(E, h) \) gives outgoing data from incoming data: for \( V \in C_0^\infty(\mathbb{R}, \mathbb{R}) \), and with \( k = \sqrt{E} \), for any solution \( u \) (not in \( L^2 \) !), one has

\[
\begin{align*}
  u_{1x < -R} &= p_- e^{ikx/h} + q_- e^{-ikx/h}, \quad u_{1x > R} = p_+ e^{-ikx/h} + q_+ e^{ikx/h},
\end{align*}
\]

Then \( S(E) \) is defined by

\[
\begin{pmatrix}
q_+ \\
q_-
\end{pmatrix} = S(E, h) \begin{pmatrix}
p_+ \\
p_-
\end{pmatrix}.
\]

► On can show that

\[
S(E, h) = \frac{1}{a^*(E, h)} \begin{pmatrix}
1 & -b^*(E, h) \\
b(E, h) & 1
\end{pmatrix}
\]

where \( a(E) \) and \( b(E) \) are analytic. The resonances are the zeros of \( a^*(E) \).
Now we suppose $P = \hbar^2 D^2 + V(x)$ with $V$ holomorphic in
\{ $x \in \mathbb{C}, \mid \text{Im} \, x \mid < \tan \theta_0 \mid \text{Re} \, x \mid$ \}, and that $V$ is short range in that set. We have

$$P_\theta u(x) = U_\theta PU_{-\theta} u(x) = (-\hbar^2 u''(x) + V(xe^{i\theta})u(x).$$

We denote by $J_{d,g}^{d,g}$ the solutions of $(P - E)u = 0$ characterized by

$$\begin{cases}
J_{\pm}^r(x, E, h) \sim e^{\pm i \sqrt{Ex}/h} \text{ as } \text{Re} \, x \to +\infty, \\
J_{\pm}^l(x, E, h) \sim e^{\pm i \sqrt{Ex}/h} \text{ as } \text{Re} \, x \to -\infty.
\end{cases}$$

Then, for real $E$’s, $J_{\pm, \theta}^{r,l} : x \mapsto J_{\pm}^{r,l}(xe^{i\theta})$ form a basis of solutions to $P_\theta u = Eu$ in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$ respectively.

For $E \in \mathbb{C}$ with $-2\theta_0 < \arg E < 0, \theta < \theta_0$ the only solution in $L^2(\mathbb{R}^+)$ is $J_{+, \theta}^r(x, E, h)$, and $J_{-, \theta}^l(x, E, h)$ is the only one in $L^2(\mathbb{R}^-)$.

$E \in \text{Res}(P)$ iff there is an outgoing solution, i.e. $a^*(E, h) = 0$. 

$\hbar^2 D^2$ + $V(x)$ with $V$ holomorphic in
\{ $x \in \mathbb{C}, \mid \text{Im} \, x \mid < \tan \theta_0 \mid \text{Re} \, x \mid$ \}, and that $V$ is short range in that set. We have

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For $E \in \mathbb{C}$ with $-2\theta_0 < \arg E < 0, \theta < \theta_0$ the only solution in $L^2(\mathbb{R}^+)$ is $J_{+, \theta}^r(x, E, h)$, and $J_{-, \theta}^l(x, E, h)$ is the only one in $L^2(\mathbb{R}^-)$.

$E \in \text{Res}(P)$ iff there is an outgoing solution, i.e. $a^*(E, h) = 0$. 

Le puits dans l’île

- We suppose now that \( V \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) has an analytic extension to
  \[ \Sigma = \{ x \in \mathbb{C}^n; \ |\text{Im} \, x| < \delta \langle \text{Re} \, x \rangle \} \]
  for some \( \delta > 0 \), and that \( V(x) = O(\langle x \rangle^{-\sigma}), \ \sigma > 0 \), as \( |x| \to \infty \), \( x \in \Sigma \). We also assume that \( V \) has a non-degenerate local minimum at \( x_0 \), and ”looks like”

- Let \( K \) be a compact set containing a vicinity of \( x_0 \). We denote \( P^D \) the Dirichlet realization of \( P \) in \( K \). The spectrum of \( P^D \) is discrete and we denote its eigenvalues by \( E_1(h), E_2(h), \ldots \).
Shape resonances

Theorem (Helffer-Sjöstrand)

There exists a bijection $b(h)$ between the spectrum of $P^D$ in $D(E_0, Ch)$ and the set of resonances of $P$ in $D(E_0, Ch)$. Moreover

$$|b(z) - z| = O(e^{-(2S_0/h)})$$

where $S_0$ is the Agmon distance between $x_0$ and the sea.
Barrier-top resonances

We suppose that $V$ has an analytic extension to

$$\Sigma = \{ x \in \mathbb{C}^n; \ |\text{Im} \ x| < \delta \langle \text{Re} \ x \rangle \}$$

for some $\delta > 0$, and that $V(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$. Moreover, we assume

$$V(x) = E_0 - \sum_{j=1}^{n} \lambda_j^2 x_j^2 + O(|x|^3).$$

and that the trapped set at energy $E_0$ is $\{0\}$.

Theorem (Sjöstrand, Briet-Combes-Duclos)

Let $C > 0$ be different from $\sum_{j=1}^{n} (\alpha_j + \frac{1}{2}) \lambda_j$ for all $\alpha \in \mathbb{N}^n$. Then, for $h > 0$ small enough, there exists a bijection $b_h$ between the sets $\text{Res}_0(P) \cap D(E_0, Ch)$ and $\text{Res}(P) \cap D(E_0, Ch)$ counted with their multiplicity, where

$$\text{Res}_0(P) = \left\{ z_\alpha^0 = E_0 - ih \sum_{j=1}^{n} (\alpha_j + \frac{1}{2}) \lambda_j; \ \alpha \in \mathbb{N}^n \right\},$$

such that $b_h(z) - z = o(h)$. 
A homoclinic orbit in 1d

We suppose here that $d = 1$ and that

$$V(x)$$

Theorem (Fujiié–R.)

The resonances $(z_k)_{k \in \mathbb{N}}$ of $P$ in $D(E_0, Ch)$ satisfies

$$z_k = E_0 + \lambda \frac{S_0 - (2k + 1)\pi h}{2|\ln h|} - i \frac{h}{|\ln h|} \frac{\lambda \ln 2}{2} + O(h/(\ln h)^2),$$

where $V(x) = E_0 - \frac{\lambda^2}{2}x^2 + O(x^3)$. 
The Time delay:

For $E \in ]0, +\infty[$, $S(E, h)$ is unitary, and the scattering phase is defined as $\theta(E, h) = \frac{1}{2i} \ln \det S(E, h)$. 
We consider the semiclassical Schrödinger operator on $\mathbb{R}^n$, $n \geq 1$,

$$P = -\hbar^2 \Delta + V(x)$$

where $V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

We define the resonances by analytic distortion, assuming that $V(x)$ has an analytic extension to

$$\Sigma = \{ x \in \mathbb{C}^n; \ | \text{Im} \ x| < \delta | \text{Re} \ x| \text{ and } | \text{Re} \ x| > C \}$$

for some $\delta, C > 0$, and $V(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in \Sigma$. 

![Diagram](attachment:image.png)
The trapped set

- We have seen three situations where one has been able to show existence and give precise location, in the semiclassical regime, of resonances close to some $E_0 \in \mathbb{R}$. In these three cases, the trapped set at energy $E_0$ is not empty:

**Definition:**

The trapped set at energy $E > 0$ is

$$K(E_0) = \{(x, \xi) \in p^{-1}(E_0); \ t \mapsto \exp(tH_p)(x, \xi) \text{ is bounded}\}$$

We recall that $p(x, \xi) = \xi^2 + V(x)$ is the semiclassical symbol of $P = Op_h^w(p)$, and that $H_p$ is the vector field on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ given by $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi = \begin{pmatrix} 2\xi \\ -\nabla V(x) \end{pmatrix}$.

- Examples:
  - Le puits dans l’isle (shape resonances).
  - A barrier top
  - The homoclinic case
What if the trapped set is empty?

**Theorem (Martinez)**

Suppose \( K(E_0) = \emptyset \). Then \( P \) has no resonance in

\[
Z_h(E_0) = [E_0 - \varepsilon, E_0 + \varepsilon] + i[-C h |\ln h|, 0]
\]

for some \( \varepsilon > 0 \) and all \( C > 0 \). Moreover, for all \( \chi \in C_0^\infty(\mathbb{R}^n) \), there exists \( N = N(C) \) such that for \( z \in Z_h(E_0) \),

\[
\| \chi(P - z)^{-1} \chi \| \lesssim h^{-N}.
\]

- If \( V \) is analytic in a whole neighborhood of \( \mathbb{R}^n \), Briet–Combes–Duclos and Helffer–Sjöstrand have proved that \( P \) has no resonance in a fixed neighborhood of \( E_0 \).
- Notice that if \( \chi \in C_0^\infty(\mathbb{R}^n) \) is supported where the distortion operator does nothing, then \( \chi(P - z)^{-1} \chi = \chi(P_\theta - z)^{-1} \chi \), so that the estimate on the resolvent implies that there is no resonance in \( Z_h(E_0) \).
Resolvent estimate for real energies

**Theorem**

For $E_0 > 0$, the following properties are equivalent.

i) $K(E_0) = \emptyset$.

ii) There exists $\varepsilon > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$,

\[
\sup_{z \in [E_0-\varepsilon, E_0+\varepsilon]} \|\chi(P - z)^{-1}\chi\| \lesssim h^{-1}.
\]

**Theorem (Bony-Burq-R)**

For $E_0 > 0$, the following properties are equivalent.

i) $K(E_0) \neq \emptyset$.

ii) There exists $\chi \in C_0^\infty(\mathbb{R}^n)$ such that, for all $\varepsilon > 0$,

\[
\sup_{z \in [E_0-\varepsilon, E_0+\varepsilon]} \|\chi(P - z)^{-1}\chi\| \gtrsim h^{-1}|\ln h|.
\]

Moreover, when $K(E_0) \neq \emptyset$, the above estimate is true for any $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ near $\pi_x K(E_0)$. 
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Theorem

For $E_0 > 0$, the following properties are equivalent.

i) $K(E_0) = \emptyset$.

ii) There exists $\varepsilon > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \| \chi (P - z)^{-1} \chi \| \lesssim h^{-1}.$$

Theorem (Bony-Burq-R)

For $E_0 > 0$, the following properties are equivalent.

i) $K(E_0) \neq \emptyset$.

ii) There exists $\chi \in C_0^\infty(\mathbb{R}^n)$ such that, for all $\varepsilon > 0$,

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \| \chi (P - z)^{-1} \chi \| \gtrsim h^{-1} |\ln h|.$$

Moreover, when $K(E_0) \neq \emptyset$, the above estimate is true for any $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ near $\pi_x K(E_0)$. 
Proof (1)

\(\Rightarrow\) (i) by the first theorem. We only have to prove that (i) \(\Rightarrow\) (ii). We set

\[ h^{-1} M(h) = \sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \| \chi(\hat{P} - z)^{-1} \chi \|. \]

Using Kato smoothness, we have

\[ \int_{\mathbb{R}} \| \chi 1_{[E_0 - \varepsilon, E_0 + \varepsilon]}(P)e^{isP}u \|_{L^2}^2 \, ds \lesssim h^{-1} M(h) \| u \|^2. \]

Therefore, if \( \varphi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon]; [0, 1]) \) with \( \varphi(E_0) = 1 \),

\[ \int_{\mathbb{R}} \| \chi \varphi(P)e^{itP/h}u \|_{L^2}^2 \, dt \lesssim M(h) \| u \|^2. \]

\(\Rightarrow\) Now \( \varphi(P)\chi^2 \varphi(P) = \text{Op}_h^w(a) \), where

\[ a(x, \xi; h) = a_0(x, \xi) + hS_h(1), \quad a_0(x, \xi) = \chi^2(x)\varphi(p(x, \xi))^2. \]

Thus we have

\[ \int_{\mathbb{R}} (\text{Op}_h^w(a)e^{itP/h}u, e^{itP/h}u) \, dt \lesssim M(h) \| u \|^2. \]
Proof (2)

Now let $u = u_\rho$ be a normalized gaussian coherent state centered at $\rho \in K(E_0) (u = T^h_\rho(e^{-x^2/2h}))$. We have seen in Clotilde’s talk that

$$\lim_{h \to 0} \left( \text{Op}_h^w(a)e^{itP/h}u, e^{itP/h}u \right) = a_0(\exp(tH_p)(\rho)),$$

for $|t| \leq C|\ln h|$, where $C > 0$.

Since $\exp(tH_p) \in K(E_0)$ for all $t$, we have $a_0(\exp(tH_p)(\rho)) = 1$, and

$$\left( \text{Op}_h^w(a)e^{itP/h}u, e^{itP/h}u \right) \geq \frac{1}{2},$$

provided $h$ is small enough. We obtain

$$C|\ln h| \leq \int_{|t| \leq C|\ln h|} \left( \text{Op}_h^w(a)e^{itP/h}u, e^{itP/h}u \right) dt \lesssim M(h)\|u\|^2.$$

so that, as stated,

$$|\ln h| \lesssim M(h).$$
A universal upper bound

**Theorem (Burq)**

*For a broad class of operators $P$, whatever the trapped set is, there is no resonance in*

$$[E_0 - \varepsilon, E_0 + \varepsilon] + i\left[e^{-C/h}, 0\right]$$

*for some $\varepsilon, C > 0$. Moreover*

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \|\chi(P - z)^{-1}\chi\| \lesssim e^{C/h}.$$  

▶ This theorem is optimal, as shown by the ”Puits dans l’isle” case.
The barrier top case

- We suppose that $V$ has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n; \ |\text{Im} \ x| < \delta \langle \text{Re} \ x \rangle\}$$

for some $\delta > 0$, and that $V(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$. Moreover, we assume

$$V(x) = E_0 - \sum_{j=1}^{n} \lambda_j^2 x_j^2 + O(|x|^3).$$

and that $K(E_0) = \{0\}$.

- We already know that there exists $\delta > 0$ such that $P$ has no resonance in $D(E_0, \delta h)$. 
Resolvent estimates in the barrier top case

**Theorem (Alexandrova-Bony-R)**

There exists $\varepsilon > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$, and for $|z - E_0| \leq \varepsilon$,

$$
\| \chi (P - z)^{-1} \chi \| \lesssim h^{-1} |\ln h|,
$$

**Theorem (Bony-Fujiié-R-Zerzeri)**

There exists $\varepsilon > 0$ such that, for all $C > 0$ and $h$ small enough,

i) The operator $P$ has no resonances in

$$
[E_0 - \varepsilon, E_0 + \varepsilon] + i[\neg Ch, 0] \setminus D(E_0, 2Ch).
$$

ii) There exists $K > 0$ such that

$$
\| \chi (P - z)^{-1} \chi \| \lesssim h^{-K} \prod_{z_\alpha \in \text{Res}(P) \cap D(E_0, 2Ch)} |z - z_\alpha|^{-1},
$$

for all $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[\neg Ch, Ch]$.
Resonant estimates in the barrier top case

**Theorem (Alexandrova-Bony-R)**

There exists $\varepsilon > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$, and for $|z - E_0| \leq \varepsilon$.

$$\|\chi(P - z)^{-1}\chi\| \lesssim h^{-1}|\ln h|,$$

**Theorem (Bony-Fujiié-R-Zerzeri)**

There exists $\varepsilon > 0$ such that, for all $C > 0$ and $h$ small enough,

1) The operator $P$ has no resonances in

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$$\|\chi(P - z)^{-1}\chi\| \lesssim h^{-K} \prod_{z_\alpha \in \text{Res}(P) \cap D(E_0, 2Ch)} |z - z_\alpha|^{-1},$$

for all $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, Ch]$. 
The Schrödinger group

- We denote by $e^{-itP/h}u_0$ the solution $u$ to the time-dependent Schrödinger equation

$$\left\{ \begin{align*}
    i\hbar \partial_t \phi(t, x) &= P(x, \hbar D_x)\phi(t, x), \\
    u|_{t=0} &= u_0.
\end{align*} \right.$$ 

- If $\lambda_0 \in \mathbb{R}$ is an isolated eigenvalue of $P$, and $\psi \in \mathcal{C}_0^\infty([\lambda_0 - \delta, \lambda_0 + \delta])$ for some $\delta > 0$ small enough, one has

$$e^{-itP/h}\psi(P) = e^{-it\lambda_0/h}\Pi_{\lambda_0}\psi(\lambda_0),$$

where $\Pi_{\lambda_0}$ is the spectral (orthogonal) projection onto the eigenspace associated to $\lambda_0$.

- Is (how) the behavior of the quantum evolution related to the resonances?
Definition

Let $z$ be a resonance for the operator $P$. The associated (generalized) spectral projection is

$$\Pi_z = -\frac{1}{2i\pi} \oint_{\gamma_z} (P - \zeta)^{-1} d\zeta,$$

as an operator from $L^2_{\text{comp}}$ to $L^2_{\text{loc}}$. The multiplicity of a resonance is the rank of $\Pi_z$.

Notice that $\Pi_z$ is not an orthogonal projection.
The shape resonances case

When the resonances are close to the real axis, one can expect that they truly influence the quantum evolution. In the case of shape resonances we have the

**Theorem : (Nakamura-Stefanov-Zworski)**

Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) and write \( \chi = \chi_1 + \chi_2 \) with \( \chi_2 \) not supported near the well, and \( \chi_1 \) not supported near the sea. Let \( \psi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon]) \) for some \( \varepsilon > 0 \) small enough. Then for any \( \mu > 0 \), there exists \( T > 0 \) such that

\[
\chi e^{-itP/h} \chi \psi(P) = \sum_{z_\alpha \in \text{Res}(P) \cap D(E_0, \mu h)} e^{-itz_\alpha/h} \chi_1 \prod_{z_\alpha} \chi_1 \psi(P) + O(h^\infty) + \chi_2 O((t - T)^+ - \infty) \chi_2,
\]

for all \( t \geq 0 \).

Notice that this makes sense only for \( t > T \).
The barrier top case

Suppose that we are in the barrier-top situation. The resonances \( z_\alpha \) are close to the
\[
z_\alpha^0 = E_0 - i\hbar \sum_{j=1}^n (\alpha_j + \frac{1}{2}) \lambda_j.
\]

**Theorem : (Bony-Fujiie-R-Zerzeri)**

Let \( \mu > 0 \) be different from \( \sum_{j=1}^n (\alpha_j + \frac{1}{2}) \lambda_j \) for all \( \alpha \in \mathbb{N}^n \).
Suppose that all the \( z_\alpha^0 \) in \( D(E_0, \mu \hbar) \) are simple.
Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) and \( \psi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon]) \) for some \( \varepsilon > 0 \) small enough.
Then, there exists \( K = K(\mu) > 0 \) such that

\[
\chi e^{-itP/\hbar} \chi \psi(P) = \sum_{z_\alpha \in \text{Res}(P) \cap D(E_0, \mu \hbar)} e^{-itz_\alpha/\hbar} \chi \prod_{z_\alpha} \chi \psi(P)
\]

\[
+ O(\hbar^\infty) + O(e^{-\mu t \hbar^{-K}}),
\]

for all \( t \geq 0 \).

This makes sense only for \( t > K|\ln \hbar| \). When \( t/|\ln \hbar| \to +\infty \) as \( \hbar \to 0 \), the sum over the resonances is negligible and this result says only
\[
\chi e^{-itP/\hbar} \chi \psi(P) = O(\hbar^\infty).
\]