Integral binary Hamiltonian forms and their waterworlds

Jouni Parkkonen Frédéric Paulin

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Abstract

We give a graphical theory of integral indefinite binary Hamiltonian forms $f$ analogous to the one by Conway for binary quadratic forms and the one of Bestvina-Savin for binary Hermitian forms. Given a maximal order $\mathcal{O}$ in a definite quaternion algebra over $\mathbb{Q}$, we define the waterworld of $f$, analogous to Conway’s river and Bestvina-Savin’s ocean, and use it to give a combinatorial description of the values of $f$ on $\mathcal{O} \times \mathcal{O}$. We use an appropriate normalisation of Busemann distances to the cusps (with an algebraic description given in an independent appendix), and the $\text{SL}_2(\mathcal{O})$-equivariant Ford-Voronoi cellulation of the real hyperbolic 5-space.

1 Introduction

In the beautiful little book [Con], Conway uses Serre’s tree $X_\mathbb{Z}$ of the modular lattice $\text{SL}_2(\mathbb{Z})$ in $\text{SL}_2(\mathbb{R})$ (see [Ser2]), considered as an equivariant deformation retract of the upper halfplane model of the hyperbolic plane $\mathbb{H}^2$, in order to give a graphical theory of binary quadratic forms $f$. The components $C$ of $\mathbb{H}^2 - X_\mathbb{Z}$ consist of points closer to a given cusp $p/q \in \mathbb{P}^1(\mathbb{Q})$ of $\text{SL}_2(\mathbb{Z})$ than to all the other ones. When $f$ is indefinite, anisotropic and integral over $\mathbb{Z}$, Conway constructs a line $R(f)$ in $X_\mathbb{Z}$, called the river of $f$, separating the components $C$ of $\mathbb{H}^2 - X_\mathbb{Z}$ such that $f(p,q) > 0$ from the ones with $f(p,q) < 0$. This allows a combinatorial description of the values taken by $f$ on integral points.

Bestvina and Savin in [BeS] have given an analogous construction when $\mathbb{R}$ is replaced by $\mathbb{C}$, $\mathbb{Z}$ by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary extension $K$ of $\mathbb{Q}$, $\mathbb{H}_\mathbb{R}^2$ by $\mathbb{H}_K^2$ and $X_\mathbb{Z}$ by Mendoza’s spine $X_{\mathcal{O}_K}$ in $\mathbb{H}_K^2$ for the Bianchi lattice $\text{SL}_2(\mathcal{O}_K)$ in $\text{SL}_2(\mathbb{C})$ (see [Men]). They construct a subcomplex $O(f)$ of $X_{\mathcal{O}_K}$, called the ocean of $f$, for any indefinite anisotropic integral binary Hermitian form $f$ over $\mathcal{O}_K$, separating the components of $\mathbb{H}_K^2 - X_{\mathcal{O}_K}$ on whose point at infinity $f$ is positive from the negative ones, and prove that it is homeomorphic to a 2-plane.

In this paper, we give analogs of these constructions and results for Hamilton’s quaternions and maximal orders in definite quaternion algebras over $\mathbb{Q}$.

Let $\mathbb{H}$ be the standard Hamilton quaternion algebra over $\mathbb{R}$, with conjugation $x \mapsto \overline{x}$, reduced norm $n$ and reduced trace $\text{tr}$. Let $\mathcal{O}$ be a maximal order in a definite quaternion algebra $A$ over $\mathbb{Q}$ with class number $h_A$ and discriminant $D_A$, for instance the Hurwitz

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2 See also [Wei, Hat]

3 that is, $A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$
order $\mathcal{O} = \mathbb{Z} + \mathbb{Z} i + \mathbb{Z} j + \mathbb{Z} \frac{1+i+j+k}{2}$, in which case $h_A = 1$ and $D_A = 2$.\textsuperscript{4} The Hamilton-Bianchi group $\mathrm{SL}_2(\mathcal{O})$, which is defined using Dieudonné’s determinant, is a lattice in $\mathrm{SL}_2(\mathbb{H})$. It acts discretely on the real hyperbolic 5-space $\mathbb{H}^5_\mathbb{R}$ with finite volume quotient, and the number of cusps of the hyperbolic orbifold $\mathrm{SL}_2(\mathcal{O}) \backslash \mathbb{H}^5_\mathbb{R}$ is $h_A^2$.

Analogously to [Men] in the complex case, we give in Section 3 an appropriate normalisation of the Busemann distance to the cusps, and we construct the Ford-Voronoi cell decomposition of $\mathbb{H}^5_\mathbb{R}$ for $\mathrm{SL}_2(\mathcal{O})$, so that the interior of the Ford-Voronoi cell $\mathcal{H}_A$ consists of the points in $\mathbb{H}^5_\mathbb{R}$ closer to a given cusp $\alpha \in \mathbb{P}^1(\mathcal{O})$ of $\mathrm{SL}_2(\mathcal{O})$ than to all the others. If $X_\mathcal{O}$ is the codimension 1 skeleton of the Ford-Voronoi cellulation, called the spine of $\mathrm{SL}_2(\mathcal{O})$, then the hyperbolic 5-orbifold $\mathrm{SL}_2(\mathcal{O}) \backslash \mathbb{H}^5_\mathbb{R}$ retracts by strong deformations onto the finite 4-dimensional orbihedron $\mathrm{SL}_2(\mathcal{O}) \backslash X_\mathcal{O}$. Using uniform 3-, 4- and 5-polytopes, we give in Example 4.5 when $D_A = 2$ and in Example 4.6 when $D_A = 2$, a complete description of the quotient $\mathrm{SL}_2(\mathcal{O}) \backslash X_\mathcal{O}$ and of the link of its vertex. For instance, if $\mathcal{O}$ is the Hurwitz order, then $\mathrm{SL}_2(\mathcal{O}) \backslash X_\mathcal{O}$ is obtained by identifying opposite faces and taking the quotient by its symmetry group of the 24-cell (the self-dual convex regular Euclidean 4-polytope with Schläfli symbol $\{3, 4, 3\}$).

Following H. Weyl [Wey], we will call Hamiltonian form a Hermitian form over $\mathbb{H}$ with anti-involution the conjugation. We refer to Subsection 2.3 and for instance to [PP2] for background.\textsuperscript{5} Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a binary Hamiltonian form, with

$$f(u, v) = a \mathbf{n}(u) + \mathbf{tr}(\mathbf{b} v) + c \mathbf{n}(v),$$

which is integral\textsuperscript{6} over $\mathcal{O}$ and indefinite.\textsuperscript{7} Its group of automorphs is the arithmetic lattice $\mathrm{SU}_f(\mathcal{O}) = \{g \in \mathrm{SL}_2(\mathcal{O}) : f \circ g = f\}$.

If $C$ is a Ford-Voronoi cell for $\mathrm{SL}_2(\mathcal{O})$, let $F(C) = \frac{f(a, b)}{\mathbf{n}(a + b)}$ where $ab^{-1} \in \mathbb{P}^1(\mathcal{O})$ is the cusp of $C$. We will say that $C$ is respectively positive, negative or flooded if $F(C) > 0$, $F(C) < 0$ or $F(C) = 0$. Contrarily to the real and complex cases, there are always flooded Ford-Voronoi cells, since by taking a $\mathbb{Z}$-basis of $\mathcal{O}$, the Hamiltonian form $f$ becomes an integral binary quadratic form over $\mathbb{Z}$ with $8 \geq 5$ variables, hence always represents 0. Our countably many flooded Ford-Voronoi cells are thus the analogues of Conway’s two lakes for an indefinite isotropic integral binary quadratic form over $\mathbb{Z}$. On the components of $\mathbb{H}^5_\mathbb{R} - X_\mathcal{O}$ along the lakes, Conway proved that the values of such a form consist in an infinite arithmetic progression. An analogous result holds in our case, that we only state when the class number is one in this introduction in order to simplify the statement (see Proposition 5.2.)

**Proposition 1.1.** If $h_A = 1$, given a flooded Ford-Voronoi cell $C$, there exists a finite set of nonconstant affine maps $\{\varphi : \mathbb{H} \to \mathbb{R} : i \in F\}$ defined over $\mathbb{Q}$ such that the set of values of $F$ on the Ford-Voronoi cells meeting $C$ is $\bigcup_{i \in F} \varphi_i(\mathcal{O})$.

In order to simplify the next statement, assume from now on in this introduction that the flooded Ford-Voronoi cells are pairwise disjoint. We define the waterworld $\mathcal{W}(f)$ of $f$ as the subcomplex of the spine separating positive Ford-Voronoi cells from negative ones, that is, $\mathcal{W}(f)$ is the union of the cells of $X_\mathcal{O}$ contained in (the boundary of) both a positive

\textsuperscript{4}See for instance [Vig] and Subsection 2.1.
\textsuperscript{5}See [PP2] also for a sharp asymptotic result on the average number of their integral representations.
\textsuperscript{6}Its coefficients satisfy $a, c \in \mathbb{Z}$ and $b \in \mathcal{O}$
\textsuperscript{7}its discriminant $\Delta(f) = n(b) - ac$ is positive
and a negative Ford-Voronoi cell. The coned-off waterworld $CW(f)$ is the union of $W(f)$ and, for all cells $\sigma$ of $W(f)$ contained in a flooded Ford-Voronoi cell $W_\alpha$, of the cone with base $\sigma$ and vertex at infinity $\alpha$. The following result (see Section 5) in particular says that $CW(f)$ is a piecewise hyperbolic polyhedral 4-plane contained in the spine of $SL_2(\mathcal{O})$ except for its ideal cells.

**Theorem 1.2.** The closest point mapping from the coned-off waterworld $CW(f)$ to the hyperbolic hyperplane of $H^5_R$ whose boundary is the projective set of zeros $\{[u:v] \in P^1(H) : f(u,v) = 0\}$ of $f$, is an $SU_f(\mathcal{O})$-equivariant homeomorphism.

Section 2 recalls the necessary information on the definite quaternion algebras over $Q$, the Hamilton-Bianchi groups, and the binary Hamiltonian forms. Section 3 gives the construction of the normalized Busemann distance to the cusp, and uses it to give a quantitative reduction theory à la Hermite (see for instance [Bor2]) for the arithmetic group $SL_2(\mathcal{O})$. We describe the Ford-Voronoi cellulation for $SL_2(\mathcal{O})$ and its spine $X_\mathcal{O}$ in Section 4. We define the waterworlds and prove their main properties in Section 5. The noncommutativity of $H$ and the isotropic property of $f$ require at various point of this text a different approach than the one in [BeS].

Recall (see for instance [PP2, §7] and Section 3) that there is a correspondence between positive definite binary Hamiltonian forms with discriminant $−1$ and the upper halfspace model of the real hyperbolic 5-space. In the independent Appendix A, we give an algebraic formula for the Busemann distance of a point $x \in H^5_R$ to a cusp $\alpha \in P^1_r(A)$ in terms of the covolume of the $\mathcal{O}$-flag associated with $\alpha$, with respect to the volume of the positive definite binary Hamiltonian form associated with $x$, analogous to the one of Mendoza in the complex case. Furthermore, in the proof of Theorem 3.5, we use the upper bound on the minima of positive definite binary Hamiltonian forms given in [ChP]: If $\gamma_2(\mathcal{O})$ is the upper bound, on all such forms $f$ with discriminant $−1$, of the lower bound of $f(u,v)$ on all nonzero $(u,v) \in \mathcal{O} \times \mathcal{O}$, then

$$\gamma_2(\mathcal{O}) \leq \sqrt{DA}.$$  

(1)

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## 2 Backgrounds

We refer to [PP2] for more informations on the objects considered in this paper, and we only recall what is strictly needed.

### 2.1 Background on definite quaternion algebras over $Q$

A quaternion algebra over a field $F$ is a four-dimensional central simple algebra over $F$. We refer to [Vig] for generalities on quaternion algebras. A real quaternion algebra is isomorphic either to $M_2(\mathbb{R})$ or to Hamilton’s quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, with basis elements $1, \ i, \ j, \ k$ as a $\mathbb{R}$-vector space, with unit element $1$ and $i^2 = j^2 = k^2 = −1, \ ij = −ji = k$. We define the conjugate of $x = x_0 + x_1i + x_2j + x_3k$ in $\mathbb{H}$ by $\overline{x} = x_0 − x_1i − x_2j − x_3k$, its reduced trace by $\text{tr}(x) = x + \overline{x}$, and its reduced norm by $n(x) = x\overline{x} = \overline{x}x$. Note that $n(xy) = n(x)n(y)$, $\text{tr}(\overline{x}) = \text{tr}(x)$ and $\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in \mathbb{H}$. For every matrix
Let $A$ be a quaternion algebra over $\mathbb{Q}$. We say that $A$ is definite (or ramified over $\mathbb{R}$) if the real quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathbb{H}$, and we then fix an identification between $A$ and a $\mathbb{Q}$-subalgebra of $\mathbb{H}$. The reduced discriminant $D_A$ of $A$ is the product of the primes $p \in \mathbb{N}$ such that the quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ over $\mathbb{Q}_p$ is a division algebra. Two definite quaternion algebras over $\mathbb{Q}$ are isomorphic if and only if they have the same reduced discriminant, which can be any product of an odd number of primes (see [Vig, page 74]).

A Z-lattice $I$ in $A$ is a finitely generated $\mathbb{Z}$-module generating $A$ as a $\mathbb{Q}$-vector space. An order in $A$ is a unitary subring $\mathcal{O}$ of $A$ which is a Z-lattice. In particular, $A = \mathbb{Q}\mathcal{O} = \mathcal{O}\mathbb{Q}$. Each order of $A$ is contained in a maximal order. For instance $\mathcal{O} = \mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z} k \frac{1+i+j+k}{2}$ is a maximal order, called the Hurwitz order, in $A = \mathbb{Q}+\mathbb{Q}i+\mathbb{Q}j+\mathbb{Q}k$ with $D_A = 2$. Let $\mathcal{O}$ be an order in $A$. The reduced norm $n$ and the reduced trace $tr$ take integral values on $\mathcal{O}$. The invertible elements of $\mathcal{O}$ are its elements of reduced norm 1. Since $\mathcal{O} = tr(x) - x$, any order is invariant under conjugation.

The left order $\mathcal{O}_l(I)$ of a Z-lattice $I$ is $\{x \in A : xI \subseteq I\}$. A left fractional ideal of $\mathcal{O}$ is a $\mathbb{Z}$-lattice of $A$ whose left order is $\mathcal{O}$. A left ideal of $\mathcal{O}$ is a left fractional ideal of $\mathcal{O}$ contained in $\mathcal{O}$. A (left) ideal class of $\mathcal{O}$ is an equivalence class of nonzero left fractional ideals of $\mathcal{O}$ for the equivalence relation $m \sim m'$ if $m' = cm$ for some $c \in A^\times$. The class number $h_A$ of $A$ is the number of ideal classes of a maximal order $\mathcal{O}$ of $A$. It is finite and independent of the maximal order $\mathcal{O}$, and we have $h_A = 1$ if and only if $D_A = 2, 3, 5, 7, 13$ (see for instance [Vig]).

The reduced norm $n(m)$ of a nonzero left ideal $m$ of $\mathcal{O}$ is the greatest common divisor of the norms of the nonzero elements of $m$. In particular, $n(\mathcal{O}) = 1$. By [Rei, p. 59], we have

$$n(m) = [\mathcal{O} : m]^\frac{1}{2}. \quad (2)$$

The reduced norm of a nonzero left fractional ideal $m$ of $\mathcal{O}$ is $\frac{n(cm)}{n(c)}$ for any $c \in \mathbb{N} - \{0\}$ such that $cm \subseteq \mathcal{O}$. By Equation (2), if $m, m'$ are nonzero left fractional ideals of $\mathcal{O}$ with $m' \subseteq m$, we have

$$\frac{n(m')}{n(m)} = [m : m']^\frac{1}{2}. \quad (3)$$

For $K = \mathbb{H}$ or $K = A$, we consider $K \times K$ as a right module over $K$ and we denote by $P_1^+(K) = (K \times K - \{0\})/K^\times$ the right projective line of $K$, identified as usual with the Alexandrov compactification $K \cup \{\infty\}$ where $[1 : 0] = \infty$ and $[x : y] = xy^{-1}$ if $y \neq 0$.

### 2.2 Background on Hamilton-Bianchi groups

The Dieudonné determinant\(^8\) Det is the group morphism from the group $GL_2(\mathbb{H})$ of invertible $2 \times 2$ matrices with coefficients in $\mathbb{H}$ to $\mathbb{R}_+^*$, defined by

$$(\text{Det} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right))^2 = n(ad) + n(bc) - tr(a \overline{\sigma} bd) = n(ac^{-1}dc - bc) \text{ if } c \neq 0. \quad (4)$$

\(^8\)See [Fue, Die, Asl].
It is invariant under the adjoint map $g \mapsto g^*$. Let $\text{SL}_2(\mathbb{H})$ be the group of $2 \times 2$ matrices with coefficients in $\mathbb{H}$ and Dieudonné determinant $1$. We refer for instance to [Kel] for more information on $\text{SL}_2(\mathbb{H})$.

The group $\text{SL}_2(\mathbb{H})$ acts linearly on the left on the right $\mathbb{H}$-module $\mathbb{H} \times \mathbb{H}$, and its homogeneous action on $\mathbb{P}^1(\mathbb{H})$, induced by its linear action on $\mathbb{H} \times \mathbb{H}$, is the action by homographies on $\mathbb{H} \cup \{\infty\}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} (az + b)(cz + d)^{-1} & \text{if } z \neq \infty, -c^{-1}d \\ ac^{-1} & \text{if } z = \infty, c \neq 0 \\ \infty & \text{otherwise} \end{cases}.$$ 

We use the upper halfspace model $\{(z, r) : z \in \mathbb{H}, r > 0\}$ with Riemannian metric $ds^2(z, r) = \frac{ds^2_{(z)} + dr^2}{r^2}$ for the real hyperbolic space $\mathbb{H}^5_R$ with dimension $5$. Its space at infinity $\partial_{\infty}\mathbb{H}^5_R$ is hence $\mathbb{H} \cup \{\infty\}$. The action of $\text{SL}_2(\mathbb{H})$ by homographies on $\partial_{\infty}\mathbb{H}^5_R$ extends to a left action on $\mathbb{H}^5_R$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \left( \frac{(az + b)(cz + d) + a\bar{c}r^2}{n(cz + d) + r^2n(c)}, \frac{r}{n(cz + d) + r^2n(c)} \right).$$

(5)

In this way, the group $\text{PSL}_2(\mathbb{H})$ is identified with the group of orientation preserving isometries of $\mathbb{H}^5_R$.

For any order $\mathcal{O}$ in a definite quaternion algebra $A$ over $\mathbb{Q}$, the Hamilton-Bianchi group

$$\Gamma_{\mathcal{O}} = \text{SL}_2(\mathcal{O}) = \text{SL}_2(\mathbb{H}) \cap \mathcal{M}_2(\mathcal{O})$$

is a nonuniform arithmetic lattice in the connected real Lie group $\text{SL}_2(\mathbb{H})$ (see for instance [PP1, page 1104] for details). In particular, the quotient real hyperbolic orbifold $\Gamma_{\mathcal{O}} \backslash \mathbb{H}^5_R$ has finite volume.

The action by homographies of $\Gamma_{\mathcal{O}}$ preserves the right projective space $\mathbb{P}^1(\mathbb{H}) = A \cup \{\infty\}$, which is the set of fixed points of the parabolic elements of $\Gamma_{\mathcal{O}}$ acting on $\mathbb{H}^5_R \cup \partial_{\infty}\mathbb{H}^5_R$. In particular, the topological quotient space $\Gamma_{\mathcal{O}} \backslash (\mathbb{H}^5_R \cup \mathbb{P}^1(\mathbb{A}))$ is the compactification of the finite volume hyperbolic orbifold $\Gamma_{\mathcal{O}} \backslash \mathbb{H}^5_R$ by its (finite) space of ends.

### 2.3 Background on binary Hamiltonian forms

A binary Hamiltonian form $f$ is a map $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ with

$$f(u, v) = a \, n(u) + tr(\overline{u} b v) + c \, n(v)$$

whose coefficients $a = a(f)$, $b = b(f)$ and $c = c(f)$ satisfy $a, c \in \mathbb{R}$, $b \in \mathbb{H}$. Note that $f((u, v)\lambda) = n(\lambda) f(u, v)$ for all $u, v, \lambda \in \mathbb{H}$.

The matrix $M(f)$ of $f$ is the Hermitian matrix $\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$, so that

$$f(u, v) = \left( \begin{pmatrix} u \\ v \end{pmatrix} \right)^* \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$ 

The discriminant of $f$ is

$$\Delta(f) = n(b) - ac.$$
An easy computation shows that the Dieudonné determinant of $M(f)$ is equal to $|\Delta(f)|$.

The linear action on the left on $\mathbb{H} \times \mathbb{H}$ of the group $SL_2(\mathbb{H})$ induces an action on the right on the set of binary Hamiltonian forms $f$ by precomposition. The matrix of $f \circ g$ is $M(f \circ g) = g^* M(f) g$. For every $g \in SL_2(\mathbb{H})$, we have

$$\Delta(f \circ g) = \Delta(f).$$

Given an order $\mathcal{O}$ in a definite quaternion algebra over $\mathbb{Q}$, a binary Hamiltonian form $f$ is integral over $\mathcal{O}$ if its coefficients belong to $\mathcal{O}$. Note that such a form $f$ takes integral values on $\mathcal{O} \times \mathcal{O}$. The lattice $\Gamma_\mathcal{O} = SL_2(\mathcal{O})$ of $SL_2(\mathbb{H})$ preserves the set of indefinite binary Hamiltonian forms $f$ that are integral over $\mathcal{O}$. The stabilizer in $\Gamma_\mathcal{O}$ of such a form $f$ is its group of automorphs

$$SU_f(\mathcal{O}) = \{ g \in \Gamma_\mathcal{O} : f \circ g = f \}.$$

For every indefinite binary Hamiltonian form $f$, with $a = a(f)$, $b = b(f)$ and $\Delta = \Delta(f)$, let

$$\mathcal{C}_\infty(f) = \{ [u : v] \in \mathbb{P}_1^\perp(\mathbb{H}) : f(u, v) = 0 \}$$

and

$$\mathcal{C}(f) = \{(z, r) \in \mathbb{H} \times [0, +\infty[ : f(z, 1) + ar^2 = 0 \}.$$

In $\mathbb{P}_1^\perp(\mathbb{H}) = \mathbb{H} \cup \{ \infty \}$, the set $\mathcal{C}_\infty(f)$ is the 3-sphere of center $-b/a$ and radius $\sqrt{\Delta/|a|}$ if $a \neq 0$, and it is the union of $\{ \infty \}$ with the real affine hyperplane $\{ z \in \mathbb{H} : \text{tr}(\overline{z}b) + c = 0 \}$ of $\mathbb{H}$ otherwise. For every $g \in SL_2(\mathbb{H})$,

$$\mathcal{C}_\infty(f \circ g) = g^{-1} \mathcal{C}_\infty(f) \quad \text{and} \quad \mathcal{C}(f \circ g) = g^{-1} \mathcal{C}(f).$$

The values of $f$ are positive on (the representatives in $\mathbb{H} \times \mathbb{H}$ in) one of the two components of $\mathbb{P}_1^\perp(\mathbb{H}) - \mathcal{C}_\infty(f)$ and negative on the other one.

The set $\mathcal{C}(f)$ is the (4-dimensional) hyperbolic hyperplane in $\mathbb{H}^5_\mathbb{R}$ with boundary at infinity $\mathcal{C}_\infty(f)$. If $f$ is integral over $\mathcal{O}$, $SU_f(\mathcal{O})/\mathcal{C}(f)$ is a finite volume hyperbolic 4-orbifold, since $SU_f(\mathcal{O})$ is arithmetic and by Borel-Harish-Chandra’s theorem.

### 3 On the reduction theory of binary Hamiltonian forms and Hamilton-Bianchi lattices

In this section, we study the geometric reduction theory of positive definite binary Hamiltonian forms, as in Mendoza [Men] for the Hermitian case. The results will be useful in Section 5. We start by recalling the correspondence between $\mathbb{H}^5_\mathbb{R}$ and positive definite binary Hamiltonian forms with discriminant $-1$.

Let $\mathcal{D}$ be the 6-dimensional real vector space of binary Hamiltonian forms, and $\mathcal{D}^+$ its open cone of positive definite ones. The multiplicative group $\mathbb{R}^+_\mathbb{R}$ of positive real numbers acts on $\mathcal{D}^+$ by multiplication. We will denote by $\mathbb{R}^+_\mathbb{R} f$ the orbit of $f$ and by $\mathbb{P}_+ \mathcal{D}^+$ the quotient space $\mathcal{D}^+/\mathbb{R}^+_\mathbb{R}$. It identifies with the image of $\mathcal{D}^+$ in the projective space $\mathbb{P}(\mathcal{D})$ of $\mathcal{D}$.

Let $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ be the symmetric $\mathbb{R}$-bilinear form (with signature $(4, 2)$) on $\mathcal{D}$ such that for every $f \in \mathcal{D}$,

$$\langle f, f \rangle_{\mathcal{D}} = -2\Delta(f).$$
That is, for all \( f, f' \in \mathcal{L} \), we have
\[
\langle f, f' \rangle_{\mathcal{L}} = a(f) c(f') + c(f) a(f') - \text{tr} \left( \overline{b(f)} b(f') \right).
\] (8)

By Equation (6), we have, for all \( f, f' \in \mathcal{L} \) and \( g \in \text{SL}_2(\mathbb{H}) \)
\[
\langle f \circ g, f' \circ g \rangle_{\mathcal{L}} = \langle f, f' \rangle_{\mathcal{L}}.
\] (9)

Let \( \mathcal{L}_1^+ \) the submanifold of \( \mathcal{L}^+ \) consisting of the forms with discriminant \(-1\), and let \( \Theta : \mathbb{H}_R^5 \rightarrow \mathcal{L}_1^+ \) be the homeomorphism such that, for every \((z, r) \in \mathbb{H}_R^5\),
\[
M(\Theta(z, r)) = \frac{1}{r} \begin{pmatrix} 1 & -z \\ -z & n(z) + r^2 \end{pmatrix}.
\]
The fact that this map is well defined and is a homeomorphism follows by checking that its composition by the canonical projection \( \mathcal{L}_1^+ \rightarrow \mathbb{P}_{+} \mathcal{L}_1 \) is the inverse of the homeomorphism denoted by
\[
\Phi : \mathbb{R}^+_+ f \mapsto \left( -\frac{b(f)}{a(f)}, \frac{\sqrt{-\Delta(f)}}{a(f)} \right)
\]
in [PP2, Prop. 22]. By loc. cit., the map \( \Theta \) is hence (anti-)equivariant under the actions of \( \text{SL}_2(\mathbb{H}) : \) For all \( x \in \mathbb{H}_R^5 \) and \( g \in \text{SL}_2(\mathbb{H}) \), we have
\[
\Theta(gx) = \Theta(x) \circ g^{-1}.
\] (10)

Let \( \mathcal{O} \) be a maximal order in a definite quaternion algebra \( A \) over \( \mathbb{Q} \). For every \( \alpha \in A \), let
\[
I_\alpha = \mathcal{O} \alpha + \mathcal{O},
\]
which is a left fractional ideal of \( \mathcal{O} \). Let \( f_\alpha \) be the binary Hamiltonian form with matrix
\[
M(f_\alpha) = \frac{1}{n(I_\alpha)} \begin{pmatrix} 1 & -\alpha \\ -\alpha & n(\alpha) \end{pmatrix}.
\]
Note that \( f_\alpha \) is a positive scalar multiple of the norm form associated with \( \alpha \); for all \( z \in \mathbb{H} \),
\[
f_\alpha(u, v) = (\overline{u} \overline{v}) M(f_\alpha) \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{n(I_\alpha)} n(u - \alpha v).
\]
The form \( f_\alpha \) depends on the choice of the maximal order \( \mathcal{O} \), though its homothety class \( \mathbb{R}^+ f_\alpha \) does not.

Let \( f_\infty \) be the binary Hamiltonian form whose matrix is \( M(f_\infty) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), that is, \( f_\infty : (u, v) \mapsto n(v) \). Note that for every \( \alpha \in \mathbb{P}_1^1(A) = A \cup \{\infty\} \), the form \( f_\alpha \) is nonzero and degenerate (its discriminant is equal to 0), and \( \mathbb{R}^+ f_\alpha \) belongs to the boundary of \( \mathbb{P}_+ \mathcal{L}_1 \) in \( \mathbb{P}(\mathcal{L}) \). The map \( \Phi^{-1} : \mathbb{H}_R^5 \rightarrow \mathbb{P}(\mathcal{L}) \) given by \( x \mapsto \mathbb{R}^+ \Theta(x) \) extends continuously to a \( \text{SL}_2(A) \)-(anti-)equivariant homeomorphism between \( \mathbb{H}_R^5 \cup \mathbb{P}_1^1(A) \) and its image in \( \mathbb{P}(\mathcal{L}) \) by sending \( \alpha \) to \( \mathbb{R}^+ f_\alpha \) for every \( \alpha \in \mathbb{P}_1^1(A) \). The next result makes precise the scaling factor for the action of \( \text{SL}_2(A) \) on the forms \( f_\alpha \) for \( \alpha \in \mathbb{P}_1^1(A) \).

**Proposition 3.1.** For all \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(A) \) and \( \alpha = [x : y] \in \mathbb{P}_1^1(A) \), we have
\[
f_{g \alpha} \circ g = \frac{n(\mathcal{O}x + \mathcal{O}y)}{n(\mathcal{O}(ax + by) + \mathcal{O}(cx + dy))} f_\alpha.
\]
Note that this implies that $f_{g\cdot\alpha} \circ g = f_\alpha$ if $g \in \text{SL}_2(\mathcal{O})$.

**Proof.** The result is left to the reader when $\alpha = \infty$ or $g \cdot \alpha = \infty$, hence we assume that $\alpha, g \cdot \alpha \neq \infty$. We have the following beautiful (and probably well-known) formula.

**Lemma 3.2.** For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{H})$ and $z, w \in \mathbb{H}$ such that $g \cdot z, g \cdot w \neq \infty$, we have

$$n(g \cdot z - g \cdot w) = \frac{1}{n(cz+d)n(cw+d)} n(z-w).$$

**Proof.** Since

$$\begin{pmatrix} az + b & aw + b \\ cz + d & cw + d \end{pmatrix} = g \begin{pmatrix} z & w \\ 1 & 1 \end{pmatrix}$$

and by taking the square of the Dieudonné determinant (see Equation (4)), we have

$$n(g \cdot z - g \cdot w) = n((az + b)(cz + d)^{-1} - (aw + b)(cw + d)^{-1})$$

$$= \frac{1}{n(cw + d)} n((az + b)(cz + d)^{-1}(cw + d) - (aw + b))$$

$$= \frac{1}{n(cz + d)n(cw + d)} n((az + b)(cz + d)^{-1}(cw + d)(cz + d) - (aw + b)(cz + d))$$

$$= \frac{1}{n(cz + d)n(cw + d)} n(z-w). \quad \Box$$

Hence for all $z \in \mathbb{H}$ such that $g \cdot z \neq \infty$,

$$f_{g\cdot\alpha} \circ g(z,1) = n(cz+d) f_{g\cdot\alpha}(g \cdot z,1) = \frac{n(cz+d)}{n(I_{g\cdot\alpha})} n(g \cdot z - g \cdot \alpha)$$

$$= \frac{1}{n(I_{g\cdot\alpha})n(\alpha + d)} n(z - \alpha) = \frac{n(I_{\alpha})}{n(I_{g\cdot\alpha})n(\alpha + d)} f_\alpha(z,1).$$

The result easily follows. \qed

For all $\alpha \in \mathbb{P}^1(A) = A \cup \{\infty\}$ and $x \in \mathbb{H}_\mathbb{R}^5$, let us define the **distance from $x$ to the point at infinity $\alpha$** by

$$d_\alpha(x) = \langle f_\alpha, \Theta(x) \rangle_{\mathcal{O}}.$$

See Appendix A for an alternate description of the map $d_\alpha : \mathbb{H}_\mathbb{R}^5 \to \mathbb{R}$.

The next result gives a few computations and properties of these maps $d_\alpha$ (which depend on the choice of maximal order $\mathcal{O}$). We will see afterwards that $\ln d_\alpha$ is an appropriately normalised Busemann function for the point at infinity $\alpha$.

**Proposition 3.3.** (1) For all $(z,r) \in \mathbb{H}_\mathbb{R}^5$ and $\alpha \in A$, we have

$$d_\alpha(z,r) = \frac{1}{r n(I_{\alpha})} (n(z - \alpha) + r^2),$$

and $d_\infty(z,r) = \frac{1}{r}$.

(2) For all $x \in \mathbb{H}_\mathbb{R}^5$ and $\alpha = [u : v] \in \mathbb{P}^1(A)$, we have

$$d_\alpha(x) = \frac{\Theta(x)(u,v)}{n(\mathcal{O}u + \mathcal{O}v)}.$$
(3) For all \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(A) \) and \( \alpha = [x : y] \in \mathbb{P}^1(A) \), we have

\[
d_{g^\alpha} \circ g = \frac{n(\partial x + \partial y)}{n(\partial(ax + by) + \partial(cx + dy))} \, d_{\alpha}.
\]

In particular, if \( g \in \text{SL}_2(\mathcal{O}) \) and \( \alpha \in \mathbb{P}^1(A) \), then \( d_{g^\alpha} \circ g = d_{\alpha} \).

**Proof.** (1) Since \( M(f_\alpha) = \frac{1}{n(I_\alpha)} \left( \begin{array}{cc} 1 & -\alpha \\ -1 & n(\alpha) \end{array} \right) \) and \( M(\Theta(z, r)) = \frac{1}{r} \left( \begin{array}{cc} 1 & -z \\ -1 & n(z) + r^2 \end{array} \right) \), we have, by Equation (8),

\[
d_{\alpha}(z, r) = \langle f_\alpha, \Theta(z, r) \rangle_{\mathcal{O}} = \frac{1}{r \, n(I_\alpha)} \left( n(z) + r^2 + n(\alpha) - \text{tr}(\alpha) \right) = \frac{n(z - \alpha) + r^2}{r \, n(I_\alpha)}.
\]

The computation of \( d_\infty \) is similar and easier.

(2) Let \( x = (z, r) \in \mathbb{H}^5_R \) and \( f = \Theta(x) \). If \( v \neq 0 \), then \( \alpha = uw^{-1} \), and by the definition of \( \Theta \) and Assertion (1),

\[
\frac{f(u, v)}{n(\partial u + \partial v)} = \frac{f(\alpha, 1)}{n(I_\alpha)} \frac{1}{n(I_\alpha)} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) M(f) \left( \begin{array}{c} \alpha \\ 1 \end{array} \right)
= \frac{n(\alpha) - \bar{\alpha} z - \bar{\alpha} \alpha + n(z) + r^2}{r \, n(I_\alpha)} = \frac{n(z - \alpha) + r^2}{r \, n(I_\alpha)} = d_{\alpha}(x).
\]

Similarly, if \( v = 0 \), then \( \frac{f(u, v)}{n(\partial u + \partial v)} = f(1, 0) = \frac{1}{r} = d_{\alpha}(x) \).

(3) For every \( w \in \mathbb{H}^5_R \), using the (anti-)equivariance property of \( \Theta \), Equation (9) and Proposition 3.1, we have

\[
d_{g^\alpha} \circ g(w) = \langle f_{g^\alpha}, \Theta(gw) \rangle_{\mathcal{O}} = \langle f_{g^\alpha} \circ g^{-1}, \Theta(w) \rangle_{\mathcal{O}}
= \frac{n(\partial x + \partial y)}{n(\partial(ax + by) + \partial(cx + dy))} \langle f_{\alpha}, \Theta(w) \rangle_{\mathcal{O}}
= \frac{n(\partial x + \partial y)}{n(\partial(ax + by) + \partial(cx + dy))} \, d_{\alpha}(w) \quad \square
\]

Since \( \text{SL}_2(\mathcal{O}) \) is a noncocompact lattice with cofinite volume in \( \text{SL}_2(\mathbb{H}) \) and set of parabolic fixed points at infinity \( \mathbb{P}^1_r(A) \), there exists (see for instance [Bow]) a \( \Gamma \)-equivariant family of horoballs in \( \mathbb{H}^5_R \) centered at the points of \( \mathbb{P}^1_r(A) \), with pairwise disjoint interiors. Since \( \text{SL}_2(\mathcal{O}) \backslash \mathbb{H}^5_R \) may have several cusps, there are various choices for such a family, and we now use the normalized distance to the points of \( \mathbb{P}^1_r(A) \) in order to define a canonical such family, and we give consequences on the structure of the orbifold \( \text{SL}_2(\mathcal{O}) \backslash \mathbb{H}^5_R \).

For all \( \alpha \in \mathbb{P}^1_r(A) \) and \( s > 0 \), we define the normalized horoball centered at \( \alpha \) with radius \( s \) as

\[
B_{\alpha}(s) = \{ x \in \mathbb{H}^5_R : d_{\alpha}(x) \leq s \}.
\]

The terminology is justified by the following result, which proves in particular that \( B_{\alpha}(s) \) is indeed a (closed) horoball. Recall that the Busemann function \( \beta : \partial \infty \mathbb{H}^5_R \times \mathbb{H}^5_R \times \mathbb{H}^5_R \to \mathbb{R} \) is defined, with \( t \mapsto \xi_t \) any geodesic ray with point at infinity \( \xi \in \partial \infty \mathbb{H}^5_R \), by

\[
(\xi, x, y) \mapsto \beta_{\xi}(x, y) = \lim_{t \to +\infty} d(x, \xi_t) - d(y, \xi_t).
\]
Proposition 3.4. Let $\alpha \in \mathbb{P}_r^1(A)$ and $s > 0$.

1. There exists $c_\alpha \in \mathbb{R}$ such that $\ln d_\alpha(x) = \beta_\alpha((0, 1)) + c_\alpha$ for every $x \in \mathbb{H}_\mathbb{R}^5$.

2. If $\alpha \in A$, then $B_\alpha(s)$ is the Euclidean ball of center $\left(\alpha, \frac{sn(I_\alpha)}{2}\right)$ and radius $\frac{sn(I_\alpha)}{2}$. If $\alpha = \infty$, then $B_\alpha(s)$ is the Euclidean halfspace consisting of all $(z, r)$ with $r \geq \frac{s}{2}$.

3. For all $g \in SL_2(\mathcal{O})$, we have $g(B_\alpha(s)) = B_{g\alpha}(s)$.

Proof. (1) If $\alpha = \infty$, then for every $(z, r) \in \mathbb{H}_\mathbb{R}^5$, we have $d_\alpha(z, r) = \frac{1}{r}$ and $\beta_\infty((z, r), (0, 1)) = \beta_\infty((0, r), (0, 1)) = -\ln r$, hence the result holds with $c_\infty = 0$.

If $\alpha \in A$, since $SL_2(A)$ acts transitively on $\mathbb{P}_r^1(A)$, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(A)$ be such that $\alpha = g \cdot \infty$. Recall that the Busemann function is invariant under the diagonal action of $SL_2(\mathbb{H})$ on $\partial_\infty \mathbb{H}_\mathbb{R}^5 \times \mathbb{H}_\mathbb{R}^5 \times \mathbb{H}_\mathbb{R}^5$ and is an additive cocycle in its two variables in $\mathbb{H}_\mathbb{R}^5$. By Proposition 3.3 (3) since $\infty = [1 : 0]$, we hence have for every $x \in \mathbb{H}_\mathbb{R}^5$

$$\ln d_\alpha(x) = \ln d_{g\infty}(g^{-1}x) = \ln \frac{d_{\infty}(g^{-1}x)}{\text{n}(\delta a + \delta c)} = \beta_{\infty}(g^{-1}x, (0, 1)) - \ln \text{n}(\delta a + \delta c) = \beta_{\alpha}(x, (0, 1)) + \beta_{\alpha}((0, 1), g(0, 1)) - \ln \text{n}(\delta a + \delta c).$$

Hence the result holds, and taking $x = (0, 1)$, we have by Proposition 3.3 (1)

$$c_\alpha = \ln \frac{\text{n}(\alpha) + 1}{\text{n}(I_\alpha)}.$$

(2) If $\alpha \in A$, for every $(z, r) \in \mathbb{H}_\mathbb{R}^5$, by Proposition 3.3 (1), we have $d_\alpha(z, r) \leq s$ if and only if $\text{n}(z - \alpha) + r^2 \leq s \text{n}(I_\alpha)$, that is, if and only if $\text{n}(z - \alpha) + (r - \frac{sn(I_\alpha)}{2})^2 \leq \left(\frac{sn(I_\alpha)}{2}\right)^2$. The second claim of Assertion (2) is immediate.

(3) This follows from Proposition 3.3 (3). \[\square\]

The following result extends and generalizes a result for $D_A = 2$ of [Spe, §5].

Theorem 3.5. (1) For all distinct $\alpha, \beta \in \mathbb{P}_r^1(A)$, the normalized horoballs $B_\alpha(1)$ and $B_\beta(1)$ have disjoint interior. Furthermore, their intersection is nonempty if and only if $\alpha = \infty$ and $\beta \in \mathcal{O}'$, or $\beta = \infty$ and $\alpha \in \mathcal{O}'$, or $\alpha, \beta \neq \infty$ and $I_\alpha I_\beta = \mathcal{O}'(\alpha - \beta)$, in which case they meet in one and only one point.

(2) We have

$$\mathbb{H}_\mathbb{R}^5 = \bigcup_{\alpha \in \mathbb{P}_r^1(A)} B_\alpha(\sqrt{D_A}).$$

Before proving this result, let us make two remarks.

(i) Note that $B_0(1)$ and $B_\infty(1)$ intersect (exactly at their common boundary point $(0, 1)$) whatever the definite quaternion algebra $A$ over $\mathbb{Q}$ is. Thus the constant $s = 1$ in Assertion (1) is optimal. The family $(B_\alpha(1))_{\alpha \in \mathbb{P}_r^1(A)}$ is a (canonical) family of maximal (closed) horoballs centered at the parabolic fixed points of $SL_2(\mathcal{O})$ with pairwise disjoint interiors. Since $SL_2(\mathcal{O})$ is a lattice (hence is geometrically finite with convex hull of its
limit set equal to the whole $\mathbb{H}^5_\mathbb{R}$), the quotient $\text{SL}_2(\mathcal{O}) \setminus \left( \mathbb{H}^5_\mathbb{R} - \bigcup_{\alpha \in \mathbb{P}^1(A)} B_{\alpha}(1) \right)$ is compact (see for instance [Bow]).

(ii) Assertion (2) is a quantitative version of the standard geometric reduction theory (see for instance [GR, Bor1, Leu]) for the structure of the arithmetic orbifold $\text{SL}_2(\mathcal{O}) \setminus \mathbb{H}^5_\mathbb{R}$. It indeed implies that if $\mathcal{R}$ is a finite subset of $\text{SL}_2(A)$ such that $\mathcal{R} \cdot \infty$ is a set of representatives of $\text{SL}_2(\mathcal{O}) \setminus \mathbb{P}^1_r(A)$, and if $\mathcal{P}_\gamma$ is a fundamental domain for the action on $\mathbb{H}$ of the stabilizer of $\infty$ in $\gamma^{-1} \text{SL}_2(\mathcal{O}) \gamma$ for every $\gamma \in \mathcal{R}$, then a weak fundamental domain for the action of $\text{SL}_2(\mathcal{O})$ on $\mathbb{H}^5_\mathbb{R}$ is the finite union $\bigcup_{\gamma \in \mathcal{R}} \gamma \mathcal{J}_\gamma$ where $\mathcal{J}_\gamma$ is the Siegel set

$$\mathcal{J}_\gamma = (\mathcal{P}_\gamma \times [0, +\infty[) \cap \gamma^{-1} B_{\gamma \infty}(\sqrt{D_A}) \, .$$

**Proof.** (1) Note that two horoballs centered at distinct points at infinity, which are not disjoint but have disjoint interior, meet at one and only one common boundary point. Hence the last claim of Assertion (1) follows from the first two ones.

First assume that $\alpha = \infty$, so that $\beta \in A$. By Proposition 3.4 (2), we have $B_{\infty}(1) = \{ (z, r) \in \mathbb{H}^5_\mathbb{R} : r \geq 1 \}$ and $B_{\beta}(1)$ is the horoball centered at $\beta$ with Euclidean diameter $n(I_\beta)$ (see the picture below). They hence meet if and only if $n(I_\beta) \geq 1$, and their interiors meet if and only if $n(I_\beta) > 1$. But since $\mathcal{O} \subset I_\beta$, by Equation (3), we have $n(I_\beta) \leq n(\mathcal{O}) = 1$ with equality if and only if $I_\beta = \mathcal{O}$, that is, $\beta \in \mathcal{O}$. The result follows.

![Diagram](https://via.placeholder.com/150)

Up to permuting $\alpha$ and $\beta$ and applying the above argument, we may now assume that $\alpha, \beta \neq \infty$. The Euclidean balls $B_{\alpha}(1)$ and $B_{\beta}(1)$ meet if and only if the distance $d_{\alpha\beta}$ between their Euclidean center is less than or equal to the sum of their radii $r_\alpha$ and $r_\beta$, and their interior meet if and only if $d_{\alpha\beta} < r_\alpha + r_\beta$. By Proposition 3.4 (2) and by the multiplicativity of the reduced norms (see [Rei, Thm. 24.11 and p. 181]), we have (see the above picture)

$$d_{\alpha\beta}^2 - (r_\alpha + r_\beta)^2 = \left( n(\alpha - \beta) + \left( \frac{n(I_\alpha)}{2} - \frac{n(I_\beta)}{2} \right)^2 \right) - \left( \frac{n(I_\alpha)}{2} + \frac{n(I_\beta)}{2} \right)^2 \nonumber$$

$$= n(\alpha - \beta) - n(I_\alpha) n(I_\beta) = n(\alpha - \beta) - n(I_\alpha I_\beta) \, .$$

Since $\alpha - \beta \in I_\alpha I_\beta$ and again by Equation (3), we have $n(\alpha - \beta) \geq n(I_\alpha I_\beta)$, with equality if and only if $I_\alpha I_\beta = \mathcal{O}(\alpha - \beta)$. The result follows.

(2) For every $x \in \mathbb{H}^5_\mathbb{R}$, let $(u, v)$ in $\mathcal{O} \times \mathcal{O} - \{0\}$ realizing the minimum on $\mathcal{O} \times \mathcal{O} - \{0\}$ of the positive definite binary Hamiltonian form $\Theta(x)$, whose discriminant is $-1$. Let $\alpha = [u : v]$. Then by Proposition 3.3 (2) and by Equation (1), we have, since the norm of an integral left ideal is at least 1,

$$d_\alpha(x) = \frac{\Theta(x)(u, v)}{n(\mathcal{O} u + \mathcal{O} v)} \leq \sqrt{D_A} \, .$$
This proves the result. □

The following observation, which is closely related with the proof of Assertion (1) of Theorem 3.5, will be useful later on.

**Remark 3.6.** For all $\alpha \neq \beta \in A$, the hyperbolic distance between $B_\alpha(1)$ and $B_\beta(1)$ is

$$d(B_\alpha(1), B_\beta(1)) = \ln \frac{n(\alpha - \beta)}{n(I_\alpha I_\beta)}.$$ 

**Proof.** This follows from the easy exercise in real hyperbolic geometry saying that the distance in the upper halfspace model of the real hyperbolic $n$-space between two horospheres $H, H'$ with Euclidean radius $r, r'$, and with Euclidean distance between their points at infinity equal to $\lambda$, is $d(H, H') = \ln \frac{\lambda^2}{4rr'}$.

This exercise uses the facts that the common perpendicular between two disjoint horoballs is the geodesic line through their points at infinity and that the (signed) hyperbolic length of an arc of Euclidean circle centered at a point at infinity with angles with the horizontal hyperplane between $\alpha$ and $\pi/2$ is $-\ln \tan \frac{\alpha}{2}$.

**4 The spine of SL$_2(\mathcal{O})$**

Let $A$ be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathcal{O}$ be a maximal order in $A$. In this section, following [Men, BeS] when the field $\mathbb{H}$ is replaced by $\mathbb{C}$, the order $\mathcal{O}$ by the ring of integers of a quadratic imaginary extension of $\mathbb{Q}$, and $\mathbb{H}^5_\mathbb{R}$ by $\mathbb{H}^3_\mathbb{R}$, we describe a canonical $\text{SL}_2(\mathcal{O})$-invariant cell decomposition of the 5-dimensional real hyperbolic space $\mathbb{H}^5_\mathbb{R}$.

For every $\alpha \in \mathbb{P}_r^1(A)$, the **Ford-Voronoi cell** of $\alpha$ for the action of $\text{SL}_2(\mathcal{O})$ on $\mathbb{H}^5_\mathbb{R}$ is the set $\mathcal{H}_\alpha$ of points not farther from $\alpha$ than from any other element of $\mathbb{P}_r^1(A)$:

$$\mathcal{H}_\alpha = \{x \in \mathbb{H}^5_\mathbb{R} : \forall \beta \in \mathbb{P}_r^1(A), \ d_\alpha(x) \leq d_\beta(x)\}.$$ 

**Proposition 4.1.** Let $\alpha \in \mathbb{P}_r^1(A)$.

1. For all $g \in \text{SL}_2(\mathcal{O})$, we have $g(\mathcal{H}_\alpha) = \mathcal{H}_{g\alpha}$.

2. We have $B_\alpha(1) \subset \mathcal{H}_\alpha \subset B_\alpha(\sqrt{DA})$.

3. The Ford-Voronoi cell $\mathcal{H}_\alpha$ is a noncompact 5-dimensional convex hyperbolic polytope, whose proper cells are compact, and the stabilizer of $\alpha$ in $\text{SL}_2(\mathcal{O})$ acts cocompactly on its boundary $\partial \mathcal{H}_\alpha$.

4. If $\beta \in \mathbb{P}_r^1(A) - \{\alpha\}$, then $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ have disjoint interior and their (possibly empty) intersection is contained in the hyperbolic hyperplane that intersects perpendicularly the geodesic line with points at infinity $\alpha$ and $\beta$ at the point where the distances to both points at infinity coincide.

$^9\text{called minimal set in [Men]}$ in the complex case
Thus

$$\mathbb{H}_R^3 = \bigcup_{\alpha \in \mathbb{P}_1^1(A)} \mathcal{H}_\alpha$$

is a $SL_2(\mathcal{O})$-invariant cell decomposition of $\mathbb{H}_R^3$, whose codimension 1 skeleton will be studied in the remainder of this section. We will see in Examples 4.5 and 4.6 that the inclusions in Assertion (2) of this proposition, as well as the one of Theorem 3.5, are sharp when $D_A = 2, 3$.

**Proof.** (1) This follows from Proposition 3.3 (3).

(2) The inclusion on the left hand side follows from Theorem 3.5 (1): If $x \in B_\alpha(1)$ and $x \notin \mathcal{H}_\alpha$, then there exists $\beta \in \mathbb{P}_1^1(A) - \{\alpha\}$ such that $d_\beta(x) < d_\alpha(x) \leq 1$, thus the interiors of $B_\alpha(1)$ and $B_\beta(1)$ have nonempty intersection, a contradiction. If $x \notin B_\alpha(\sqrt{D_A})$, then by Theorem 3.5 (2), there exists $\beta \in \mathbb{P}_1^1(A) - \{\alpha\}$ such that $x \in B_\beta(\sqrt{D_A})$. Hence $d_\beta(x) \leq \sqrt{D_A} < d_\alpha(x)$, so that $x \notin \mathcal{H}_\alpha$.

(3) and (4) Since $\ln d_\alpha$ is a Busemann function with respect to the point at infinity $\alpha$ by Proposition 3.4 (1), for every $\beta \in \mathbb{P}_1^1(A) - \{\alpha\}$, the set $\mathcal{H}_{\alpha,\beta} = \{x \in \mathbb{H}_R^3 : d_\alpha(x) \leq d_\beta(x)\}$ is a (closed) hyperbolic halfspace. Its boundary $\{x \in \mathbb{H}_R^3 : d_\alpha(x) = d_\beta(x)\}$ is the hyperbolic hyperplane that intersects perpendicularly the geodesic line with points at infinity $\alpha$ and $\beta$ at the point where the distances to both points at infinity coincide. Being the intersection of the locally finite family of hyperbolic halfspaces $(\mathcal{H}_{\alpha,\beta})_{\beta \in \mathbb{P}_1^1(A) - \{\alpha\}}$, and containing the horoball $B_\alpha(1)$, the Ford-Voronoi cell $\mathcal{H}_\alpha$ is a noncompact 5-dimensional convex hyperbolic polytope. Since $\alpha$ is a bounded parabolic fixed point of the lattice $SL_2(\mathcal{O})$ and by Assertion (2), the stabilizer of $\alpha$ in $SL_2(\mathcal{O})$ acts cocompactly on $\partial \mathcal{H}_\alpha$, and hence the boundary cells of $\mathcal{H}_\alpha$ are compact.

The horoballs $B_0(1)$ and $B_\infty(1)$ with disjoint interiors meet at $(0, 1) \in \mathbb{H}_R^2$, and at most two horoballs with disjoint interior can meet at a given point of $\mathbb{H}_R^2$. Thus, the Ford-Voronoi cells at 0 and at $\infty$ have nonempty intersection, which is a compact 4-dimensional hyperbolic polytope. This intersection $\Sigma_\mathcal{O} = \mathcal{H}_0 \cap \mathcal{H}_\infty$ is called the fundamental cell of the spine of $SL_2(\mathcal{O})$. We will describe it in Example 4.5 when $D_A = 2$ and in Example 4.6 when $D_A = 3$.

**Lemma 4.2.** Let $\alpha \in \mathbb{P}_1^1(A)$ be such that $e = \mathcal{H}_\infty \cap \mathcal{H}_0 \cap \mathcal{H}_\alpha$ is a 3-dimensional cell in the boundary of $\Sigma_\mathcal{O}$. Then

$$\min\{n(I_\alpha), n(I_{\alpha^{-1}})\} \geq \frac{1}{D_A},$$

and the horizontal projection of $e$ to $\mathbb{H}$ is contained in the Euclidean hyperplane

$$\{z \in \mathbb{H} : \text{tr}(\overline{\alpha} z) = 1 + n(\alpha) - n(I_\alpha)\}.$$

**Proof.** Note that $\alpha \neq 0, \infty$. By Proposition 4.1 (2), the intersection $B_\infty(\sqrt{D_A}) \cap B_0(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ contains $e$, hence the intersections $B_\infty(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ and $B_0(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ are nonempty. Since $B_\infty(\sqrt{D_A})$ is the Euclidean halfspace of points $(z, r)$ with $r \geq \sqrt{D_A}$ and $B_\alpha(\sqrt{D_A})$ is a Euclidean ball tangent to the horizontal plane with diameter $\sqrt{D_A} n(I_\alpha)$ by Proposition 3.4 (2), this implies that $\sqrt{D_A} n(I_\alpha) \geq \frac{1}{\sqrt{D_A}}$, so that $D_A n(I_\alpha) \geq 1$. Since $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ belongs to $SL_2(\mathcal{O})$ and maps 0 to $\infty$ and $\alpha$ to $\alpha^{-1}$,
and by Proposition 3.4 (3), the intersection $B_\infty(\sqrt{D_A}) \cap B_{\alpha-1}(\sqrt{D_A})$ is nonempty, hence similarly $D_A \mathbf{n}(I_{\alpha-1}) \geq 1$.

The set of points equidistant to 0 and $\infty$ is the open Euclidean upper hemisphere of radius 1 centered at 0, and the set of points equidistant to $\alpha$ and $\infty$ is the open Euclidean upper hemisphere of radius $\sqrt{\mathbf{n}(I_{\alpha})}$ centered at $\alpha$. The projection to $\mathbb{H}$ of the intersection of these hemispheres is contained in the affine Euclidean hyperplane of $\mathbb{H}$ perpendicular to the real vector line containing $\alpha$ that passes through the projection $\lambda \alpha$ with $\lambda > 0$ to that line of any point at Euclidean distance 1 from 0 and at Euclidean distance $\sqrt{\mathbf{n}(I_{\alpha})}$ from $\alpha$. An easy computation (considering the two cases when $\mathbf{n}(\alpha) > 1$ as in the above picture or when $\mathbf{n}(\alpha) \leq 1$) using right angled triangles gives that $\lambda = \frac{1+\mathbf{n}(\alpha)-\mathbf{n}(I_{\alpha})}{2 \mathbf{n}(\alpha)}$. Since $(u, v) \mapsto \frac{1}{2} \text{tr}(\overline{u} v)$ is the standard Euclidean scalar product on $\mathbb{H}$, this gives the result. \[\square\]

The spine\textsuperscript{10} of $\text{SL}_2(\mathcal{O})$ is the codimension 1 skeleton of the cell decomposition into Ford-Voronoï cells of $\mathbb{H}_{\mathbb{R}}^5$, that is

$$X_{\mathcal{O}} = \bigcup_{\alpha \neq \beta \in \mathbb{P}_1^1(A)} \mathcal{H}_\alpha \cap \mathcal{H}_\beta = \bigcup_{\alpha \in \mathbb{P}_1^1(A)} \partial \mathcal{H}_\alpha.$$  

It is an $\text{SL}_2(\mathcal{O})$-invariant piecewise hyperbolic polyhedral complex of dimension 4.\textsuperscript{11} Note that the stabilizers in $\text{SL}_2(\mathcal{O})$ of the cells of $X_{\mathcal{O}}$ may be nontrivial.

For every hyperbolic cell $C$ of $X_{\mathcal{O}}$ and every $\alpha \in \mathbb{P}_1^1(A)$ such that $C \subset \partial \mathcal{H}_\alpha$, the radial projection along geodesic rays with point at infinity $\alpha$ from $C$ to the horosphere $\partial B_{\alpha}(1)$ is a homeomorphism onto its image, and the pull-back of the flat induced length metric on this horosphere endows $C$ with a structure of a compact Euclidean polytope. This Euclidean structure does not depend on the choice of $\alpha$, since the (possibly empty) intersection $\mathcal{H}_\alpha \cap \mathcal{H}_\beta$ is equidistant to $B_\alpha(1)$ and $B_\beta(1)$ for all distinct $\alpha, \beta$ in $\mathbb{P}_1^1(A)$. It is well known (see for instance [Ait]) that these Euclidean structures on the cells of $X_{\mathcal{O}}$ endow $X_{\mathcal{O}}$ with the structure of a $\text{CAT}(0)$ piecewise Euclidean polyhedral complex.

Furthermore, $X_{\mathcal{O}}$ is a $\text{SL}_2(\mathcal{O})$-invariant deformation retract of $\mathbb{H}_{\mathbb{R}}^5$ along the geodesic rays with points at infinity the points in $\mathbb{P}_1^1(A)$ and since the quotient orbifold with boundary $\text{SL}_2(\mathcal{O}) \backslash (\mathbb{H}_{\mathbb{R}}^5 - \bigcup_{\alpha \in \mathbb{P}_1^1(A)} B_\alpha(1))$ is compact, the quotient space $\text{SL}_2(\mathcal{O}) \backslash X_{\mathcal{O}}$ is a finite locally $\text{CAT}(0)$ piecewise Euclidean orbihedral complex.

We end this section by a description of the cell structure of $\text{SL}_2(\mathcal{O}) \backslash X_{\mathcal{O}}$ in some particular cases. Recall that the order $\mathcal{O}$ is left-Euclidean if for all $a, b \in \mathcal{O}$ with $b \neq 0$, there

\textsuperscript{10}called in the complex case minimal incidence set in [Men], and cut locus of the cusp in [HP, §5] when the class number is one

\textsuperscript{11}We refer for instance to [BrH] for the definitions related to polyhedral complexes, $\text{CAT}(0)$ spaces and orbihedra.
exists $c, d \in \mathcal{O}$ with $a = cb + d$ and $n(d) < n(b)$, or, equivalently, if for every $\alpha \in A$, there exists $c \in \mathcal{O}$ such that $n(\alpha - c) < 1$. By for instance [Vig, p. 156], $\mathcal{O}$ is left-Euclidean if and only if $D_A \in \{2, 3, 5\}$.

**Proposition 4.3.** The Hamilton-Bianchi group $SL_2(\mathcal{O})$ acts transitively on the set of 4-dimensional cells of its spine $X_\mathcal{O}$ if and only if $D_A \in \{2, 3, 5\}$. In these cases, the horizontal projection of the fundamental cell $\Sigma_\mathcal{O}$ to $\mathbb{H}$ is the Euclidean Voronoi cell of 0 for the $\mathbb{Z}$-lattice $\mathcal{O}$ in the Euclidean space $\mathbb{H}$.

**Proof.** If $SL_2(\mathcal{O})$ acts transitively on the 4-dimensional cells of $X_\mathcal{O}$, then $X_\mathcal{O} = SL_2(\mathcal{O}) \Sigma_\mathcal{O}$, and the stabilizer of $\infty$ in $SL_2(\mathcal{O})$ acts transitively on the 4-dimensional cells in $\partial \mathcal{H}_\infty$, since \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathcal{O}) \) preserves $\Sigma_\mathcal{O} = \mathcal{H}_\infty \cap \mathcal{H}_0$ and exchanges $\mathcal{H}_\infty$ and $\mathcal{H}_0$. This stabilizer consists of the upper triangular matrices with coefficients in $\mathcal{O}$, hence with diagonal coefficients invertible in $\mathcal{O}$. The orbit of 0 under this stabilizer is exactly $\mathcal{O}$. Since $\Sigma_\mathcal{O}$ is compact and contained in the open Euclidean upper hemisphere centered at 0 with radius 1, by horizontal projection on $\mathbb{H}$, this proves that $\mathbb{H}$ is covered by the open balls of radius 1 centered at the points of $\mathcal{O}$. Hence $\mathcal{O}$ is left-Euclidean.

Conversely, if $\mathcal{O}$ is left-Euclidean, then the class number of $\mathcal{O}$ is 1, and $SL_2(\mathcal{O})$ acts transitively on the Ford-Voronoi cells. In order to prove that $SL_2(\mathcal{O})$ acts transitively on the 4-dimensional cells of $X_\mathcal{O}$, we hence only have to prove that the stabilizer of $\infty$ in $SL_2(\mathcal{O})$ acts transitively on the 4-dimensional cells of $\partial \mathcal{H}_\infty$. For this, let $\alpha \in A$ be such that $\mathcal{H}_\infty \cap \mathcal{H}_\alpha$ is a 4-dimensional cell in $\partial \mathcal{H}_\infty$. Let us prove that $\alpha \in \mathcal{O}$, which gives the result. Due to problems caused by the noncommutativity of $\mathbb{H}$, the proof of [BeS, Prop. 4.3] does not seem to extend exactly. We will use instead the following elementary lemma of independent interest, see also [Spe, §5] for the first claim.

**Lemma 4.4.** If $\mathcal{O}$ is left-Euclidean, then the group $SL_2(\mathcal{O})$ is generated by $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B_w = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for $w \in \mathcal{O}$ and $C_{u,v} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ for $u, v \in \mathcal{O}^\times$. In particular, the anti-homography $z \mapsto \overline{z}$ normalizes the action by homographies of $SL_2(\mathcal{O})$ on $\mathbb{H}$.

With $\{w_1, w_2, w_3, w_4\}$ a $\mathbb{Z}$-basis of $\mathcal{O}$, the set

$$\{J, B_{w_1}, B_{w_2}, B_{w_3}, B_{w_4}\} \cup \{C_{u,v} : u, v \in \mathcal{O}^\times\}$$

is a nice finite generating set for $SL_2(\mathcal{O})$, but we shall not need this.

**Proof.** The last claim follows from the first one, since $J^{-1} = J$, $B_{w}^{-1} = B_{-w}$, $C_{u,v}^{-1} = C_{u^{-1},v^{-1}}$ and for all $z \in \mathbb{H}$, we have

$$J \cdot \overline{z} = J \cdot z, \quad B_w \cdot \overline{z} = B_{-w} \cdot z, \quad C_{u,v} \cdot \overline{z} = C_{u^{-1},v^{-1}} \cdot z.$$ 

Let $G$ be the subgroup of $SL_2(\mathcal{O})$ generated by the matrices $J, B_w, C_{u,v}$ for $w \in \mathcal{O}$ and $u, v \in \mathcal{O}^\times$ (their Dieudonné determinant is indeed 1). Let us prove that any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O})$ belongs to $G$, by induction on the integer $n(c)$. If $c = 0$, then $M = C_{a,d} B_{a^{-1}b}$.
belongs to $G$. Otherwise, since $\mathcal{O}$ is left-Euclidean, there exists $w, c' \in \mathcal{O}$ such that $a = wc + c'$ and $\mathbf{n}(c') < \mathbf{n}(c)$. Hence

$$M = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ c' & b - w d \end{pmatrix}$$

belongs to $G$ by induction. \hfill \Box

Now, assume for a contradiction that $\alpha \notin \mathcal{O}$. Since $\mathcal{O}$ is left-Euclidean, there exists $c \in \mathcal{O}$ such that $\mathbf{n}(\alpha - c) < 1$. Up to replacing $\alpha$ by $\alpha - c$, since translations by $\mathcal{O}$ preserve $\mathcal{H}_\infty$, we may assume that $0 < \mathbf{n}(\alpha) < 1$. For every $\beta \in A$ and $\beta' \in \mathbb{P}^1(A) - \{\beta\}$, let us denote by $S_{\beta, \beta'}$ the Euclidean upper hemisphere centered at $\beta$ equidistant from the points at infinity $\beta$ and $\beta'$. In particular, $S_{0, \infty}$ has radius 1. The inversion with respect to the sphere containing $S_{0, \infty}$ acts by an orientation-reversing isometry on $\mathbb{H}_0^2$, and acts on the boundary at infinity $\mathbb{P}^1(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$ by $z \mapsto \frac{z}{\mathbf{n}(z)} = \frac{1}{\mathbf{n}(\alpha)}$. By Lemma 4.4, it hence normalizes $\text{SL}_2(\mathcal{O})$ and, in particular, sends $S_{\alpha, \infty}$ to $S_{\frac{\alpha}{\mathbf{n}(\alpha)}, 0}$, and fixes $S_{0, \infty}$ (see the above picture). Since $\mathbf{n}(\alpha) < 1$, the hemisphere $S_{\alpha, \infty}$ is therefore below the union of $S_{0, \infty}$ and $S_{\frac{\alpha}{\mathbf{n}(\alpha)}, 0}$, which contradicts the fact that $\mathcal{H}_\infty \cap \mathcal{H}_{\alpha}$, which is contained in $S_{\alpha, \infty}$, is a 4-dimensional cell in $\partial \mathcal{H}_\infty$.

In order to prove the last claim of Proposition 4.3, note that $\mathbf{n}(I_\alpha) = 1$ if $\alpha \in \mathcal{O}$, and that the above proof shows that the 4-dimensional cells contained in $\partial \mathcal{H}_\infty$ and meeting the fundamental cell along a 3-dimensional cell are contained in spheres centered at points in $\mathcal{O}$. Therefore, by Lemma 4.2, the horizontal projection of $\Sigma_\mathcal{O}$ is the intersection of the halfspaces containing 0 and bounded by the Euclidean hyperplanes with equation $\text{tr}(\pi z) = \mathbf{n}(\alpha)$ for all $\alpha \in \mathcal{O}$. Since this hyperplane is the set of points $z$ in the Euclidean space $\mathbb{H}$ equidistant to 0 and $\alpha$, this proves that the horizontal projection of $\Sigma_\mathcal{O}$ is indeed the Voronoi cell at 0 of the $\mathbb{Z}$-lattice $\mathcal{O}$. \hfill \Box

**Example 4.5.** Let $A = \mathbb{Q} + \mathbb{Q} i + \mathbb{Q} j + \mathbb{Q} k \subset \mathbb{H}$ be the definite quaternion algebra over $\mathbb{Q}$ with $D_A = 2$, and let $\mathcal{O} = \mathbb{Z} + \mathbb{Z} i + \mathbb{Z} j + \mathbb{Z} \frac{1 + i + j + k}{2}$ be the (maximal) Hurwitz order in $A$. The Hurwitz order $\mathcal{O}$ is the lattice of type $F_4 = D_4^*$, whose Voronoi cell $C_{24}$ is (up to homothety) the 24-cell, which is the (unique) self-dual, regular, convex Euclidean 4-polytope, whose Schläfli symbol is $\{3, 4, 3\}$. See for instance [CS, p. 119] for more details and references.

The vertices of $C_{24}$ are the 24 quaternions $\frac{1}{2}(1 + i)u$, where $u$ is one of the 24 unit Hurwitz quaternions, that is, an element of $\mathcal{O}^\times = \{ \pm 1, \pm i, \pm j, \pm k, \pm \frac{1+i+j+k}{2} \}$. The group of Euclidean symmetries of the 24-cell consists of the 1152 elements $z \mapsto uzv^{-1}$,
$z \mapsto u z v^{-1}$ of $O(4)$, where either $u$ and $v$ are unit Hurwitz integers or $u/\sqrt{2}$ and $v/\sqrt{2}$ are vertices of $C_{24}$.\textsuperscript{12} By Proposition 4.3, the fundamental cell of $\text{SL}_2(\mathcal{O})$ is

$$\Sigma_\mathcal{O} = \{(z,t) \in \mathbb{H}^5_{\mathbb{R}} : z \in C_{24}, \mathbf{n}(z) + t^2 = 1\}.$$  

The quotient $\text{SL}_2(\mathcal{O}) \backslash X_\mathcal{O}$ is obtained by identifying the opposite 3-dimensional cells of $\Sigma_\mathcal{O}$ (which are 24 regular octahedra) by translations by elements of $\mathcal{O}$, and by taking the quotient by the stabilizer of $(0,1)$ in $\text{SL}_2(\mathcal{O})$, which is the group of order 288 generated by $J$ and the $C_{u,v}$ with $u,v \in \mathcal{O}^\times$. In particular, all the vertices of $X_\mathcal{O}$ are in the same orbit under $\text{SL}_2(\mathcal{O})$.

Speiser [Spe, §5] observed that the estimate of Proposition 4.1 (2) is sharp in this example: $\mathbb{H}^5_{\mathbb{R}}$ is indeed completely contained in $\bigcup_{\alpha \in \mathcal{P}^*_i(A)} B_\alpha(\sqrt{2})$, and the orbit that contains all the vertices of $\Sigma_\mathcal{O}$ is not contained in the union of the interiors of the horoballs $B_\alpha(\sqrt{2})$. Furthermore, Speiser proved that the point

$$v_0 = \left(\frac{1+i}{2}, \frac{1}{\sqrt{2}}\right)$$

belongs to the boundary of exactly 10 horoballs $B_\alpha(\sqrt{2})$, the ones with $\alpha$ in

$$E = \left\{\infty, 0, 1, i, 1+i, \frac{1+i \pm j \pm k}{2}, \frac{1}{1-i} = \frac{1+i}{2}\right\}.$$  

In particular, $v_0$ is a vertex of the spine $X_\mathcal{O}$, contained in the boundary of exactly 10 Ford-Voronoi cells $\mathcal{H}_\alpha$ for $\alpha$ in this set.

The set $E$ contains exactly 5 pairs $\{\alpha, \beta\}$ of distinct elements such that the interiors of the horoballs $B_\alpha(\sqrt{2})$ and $B_\beta(\sqrt{2})$ are disjoint, these pairs being $\{\infty, \frac{1}{1-i}\}$, $\{0, 1+i\}$, $\{1, i\}$, $\{\frac{1+i+j+k}{2}, \frac{1+i-j-k}{2}\}$ and $\{\frac{1+i+j-k}{2}, \frac{1+i-j+k}{2}\}$. If $\{\alpha, \beta\}$ is one of these pairs, the Ford-Voronoi cells $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ intersect only at $v_0$. For all other pairs in $E$, the intersection is a higher-dimensional cell.

\textsuperscript{12}The mappings $z \mapsto u z v^{-1}$ and $z \mapsto -(u)z(-v)^{-1}$ are obviously equal, as well as $z \mapsto u \mp v^{-1}$ and $z \mapsto -(u) \mp (-v)^{-1}$, but there are no further identities in this set of isometries. Note also that only 576 of the isometries arise from Hurwitz integers.
As $0, 1, i, 1 + i, \frac{1+i+j+k}{2}$ are in $\mathcal{O}$ and $\frac{1}{1+i}$ is not in $\mathcal{O}$, there are 8 Ford-Voronoi cells incident to $v_0$ that intersect $\mathcal{H}_\infty$ in a 4-dimensional 24-cell (see the above picture on the left, which represents the intersection with the plane in $\mathbb{H}$ containing $0, 1, i$ of the closures of the equidistant spheres and planes between some pairs of elements in $\{\infty, 0, 1, i, 1 + i, \frac{1+i+j+k}{2}\}$, so that the horizontal projection of $v_0$ is the common intersection points of the straight lines). A similar property holds for all the other Ford-Voronoi cells incident to $v_0$: For example, $\mathcal{H}_0$ intersects in a 4-dimensional cell the Ford-Voronoi cells $\mathcal{H}_\infty, \mathcal{H}_1, \mathcal{H}_1 \mathcal{H}_1 \mathcal{H}_1$, but not $\mathcal{H}_1$ by Theorem 3.5 (1), since $I_0 I_{i+1} = \mathcal{O} \neq \mathcal{O}(1 + i)$. Thus the pattern of pairwise intersections into 4-dimensional cells of these 10 Ford-Voronoi cells is given by the above picture on the right, and the number of 24-cells containing $v_0$ is exactly $40 = (10 \times 8)/2$, one for each edge of this intersection pattern.

The boundary of each $\mathcal{H}_\alpha$ is tiled by 24-cells, combinatorially forming the 24-cell honeycomb. The dual of this honeycomb is the 16-cell honeycomb. Therefore, the link$^{13}$ of the vertex $v_0$ in the tessellation of $\partial \mathcal{H}_\alpha$ for all $\alpha \in E$ is the dual of the 16-cell, which is the boundary of the 4-cube, such that the intersection of the link with each of the 8 24-cells is a 3-cube.

Gluing together the ten boundaries of 4-cubes (that have been subdivided in eight 3-cubes each) according to the above intersection pattern proves that the link of $v_0$ in the spine $X_\mathcal{O}$ is the 3-skeleton of the 5-cube (which is the 5-dimensional regular polytope with Schl"afli symbol $\{4, 3, 3, 3\}$).

**Example 4.6.** The maximal order of the definite quaternion algebra $\left(\frac{-1 - 3}{Q}\right)$ of discriminant $D_A = 3$ is $\mathbb{Z}[1, i, \frac{i+j}{2}, \frac{1+j}{2}]$, see [Vig, p. 98]. By the unique $\mathbb{Q}$-linear map from $\left(\frac{-1 - 3}{Q}\right)$ to $\mathbb{H}$ sending $1$ to $1$, $i$ to $j$, $j$ to $k\sqrt{3}$ and $k$ to $-i\sqrt{3}$, we identify $\left(\frac{-1 - 3}{Q}\right)$ with the $\mathbb{Q}$-subalgebra $A$ of $\mathbb{H}$ generated by $1$, $i\sqrt{3}$, $j$ and $k\sqrt{3}$, and the maximal order is then identified with $\mathcal{O} = \mathbb{Z}[1, \rho, j, \rho j]$, where

$$\rho = \frac{1 + i\sqrt{3}}{2}.$$

The group of units of $\mathcal{O}$ as order 12:

$$\mathcal{O}^\times = \{\pm 1, \pm j, \pm \rho, \pm \rho^2, \pm \rho j, \pm \rho^2 j\}.$$

The elements of the maximal order $\mathcal{O} = \mathbb{Z}[1, \rho] + \mathbb{Z}[1, \rho]j$ of $A$ are the vertices of the 3-3 duoprism honeycomb in the 4-dimensional Euclidean space $\mathbb{H}$. The 9 elements of the set

$$V_{3,3} = \{0, 1, j, 1 + j, \rho, \rho j, 1 + \rho j, j + \rho, \rho(1 + j)\},$$

contained in $\mathcal{O}$, are the vertices of its fundamental 3-3 duoprism $C_{3,3}$, which is a uniform 4-polytope with Schl"afli symbol $\{3\} \times \{3\}$ (the Cartesian product of two equilateral triangles, whose 1-skeleton is given in the picture below).$^{14}$

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$^{13}$The complex obtained as the intersection of $\partial \mathcal{H}_\alpha$ and a sphere with small radius centered at $v_0$.

$^{14}$We refer to Coxeter’s three papers [Cox1, Cox2, Cox3] for references about uniform polytopes, with the help of the numerous and beautiful Wikipedia articles.
The Voronoi cell $C_{6,6}$ of 0 for the lattice $\mathcal{O}$ is the 6-6 duoprism (whose Schlӓfli symbol is $\{6\} \times \{6\}$), which is the Cartesian product of two copies of the Voronoi cell $C_{6}$ of 0 for the hexagonal lattice of the Eisenstein integers in $\mathbb{C}$. The set of vertices of the hexagon $C_{6}$ is $V_{6} = \{ \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{2} \pm \frac{i}{2\sqrt{3}} \}$, so that the set of vertices of $C_{6,6}$ is $V_{6} + jV_{6}$. These 36 vertices of $C_{6,6}$, for example

$$z_{0} = \frac{1}{2} + \frac{i}{2\sqrt{3}} + \frac{j}{2} + \frac{k}{2\sqrt{3}} = (j + \rho)(1 + \rho)^{-1},$$

belong to $A$ and all have reduced norm $\frac{2}{3}$. By Proposition 4.3, $C_{6,6}$ is the projection to $\mathbb{H}$ of the fundamental cell $\Sigma_{\mathcal{O}}$ of SL$_{2}(\mathcal{O})$. The set $V_{3,3}$ of vertices of the 3-3 duoprism $C_{3,3}$ lies in the sphere of radius $\sqrt{\frac{2}{3}}$ centered at $z_{0}$.

Let $u$ and $v$ be either both in $\mathcal{O} \times \mathcal{O}$ or both in $\rho \mathcal{O} \times \mathcal{O}$. The 288 mappings $z \mapsto uzv^{-1}$ and $z \mapsto u \bar{z} v^{-1}$ are Euclidean symmetries of $C_{6,6}$, and they act transitively on the vertices of $C_{6,6}$. With the notation of Lemma 4.4, the stabilizer of $(0,1)$ in SL$_{2}(\mathcal{O})$ contains the subgroup generated by $J$ and the $C_{u,v}$ with $u,v$ in $\mathcal{O} \times \mathcal{O}$. In particular, all the vertices of $X_{\mathcal{O}}$ are in the same orbit under SL$_{2}(\mathcal{O})$, and they all have Euclidean height $\frac{1}{\sqrt{3}}$.

Let $v_{0} = \left( z_{0}, \frac{1}{\sqrt{3}} \right)$, which is the vertex of $\Sigma_{\mathcal{O}}$ whose projection to $\mathbb{H}$ is $z_{0}$.

Let $g : \mathbb{H} \cup \{ \infty \} \to \mathbb{H} \cup \{ \infty \}$ be the homography $z \mapsto \frac{1}{\rho} (z - z_{0})^{-1} + z_{0}$.

**Proposition 4.7.** If $D_{A} = 3$, then the set of $\alpha \in A$ such that $v_{0}$ belongs to the boundary of $B_{\alpha}(\sqrt{3})$ is

$$V = V_{3,3} \cup g(V_{3,3}) \cup \{ \infty, z_{0} \}.$$ 

For every $\alpha \in A$, the point $v_{0}$ of $\mathbb{H}_{R}^{5}$ does not belong to the interior of $B_{\alpha}(\sqrt{3})$.

The second claim implies that when $r < \sqrt{3}$, the family $(B_{\alpha}(r))_{\alpha \in A}$ does not cover $\mathbb{H}_{R}^{5}$. In particular, the inclusions in Proposition 4.1 (2) are also sharp when $D_{A} = 3$.

**Proof.** First observe that $v_{0}$ as well as all the vertices of $\Sigma_{\mathcal{O}}$ are in the horizontal plane $\{(z,t) \in \mathbb{H}_{R}^{5} : t = \frac{1}{\sqrt{3}} \}$, which is the boundary of $B_{\infty}(\sqrt{3})$. 

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For every $\alpha \in A$, recall from Proposition 3.4 (2) that the horoball $B_\alpha(\sqrt{3})$ is the Euclidean ball tangent to $\mathbb{H}$ at $\alpha$ with Euclidean radius $\frac{\sqrt{3} \cdot n(\alpha)}{2}$. Writing $\alpha = pq^{-1}$ with $p, q \in \mathcal{O}$ relatively prime, we have $n(I_\alpha) = n(q)^{-1}$. Thus if $v_0 \in B_\alpha(\sqrt{3})$, then the Euclidean diameter $\sqrt{3} \cdot n(I_\alpha)$ of $B_\alpha(\sqrt{3})$ is at least the Euclidean height $\frac{1}{\sqrt{3}}$ of $v_0$, that is $n(I_\alpha) \geq \frac{1}{3}$. Equality is only possible if $\alpha$ is the vertical projection to $\mathbb{H}$ of $v_0$, that is $\alpha = z_0$. Since $z_0 = (j + \rho)(1 + \rho)^{-1}$ and $j + \rho, 1 + \rho$ are relatively prime (their norms are 2 and 3), we have $z_0 \in A$ and $n(I_{z_0}) = \frac{1}{3}$. Hence the point $v_0$ does belong to the boundary of $B_{z_0}(\sqrt{3})$, and if $\alpha \neq z_0$, then $n(I_\alpha) = 1$ or $n(I_\alpha) = \frac{5}{2}$.

First assume that $n(q) = 1$, or equivalently that $\alpha \in \mathcal{O}$. Then $n(I_\alpha) = 1$, hence $B_\alpha(\sqrt{3})$ is the Euclidean ball of center $(\alpha, \frac{\sqrt{3}}{2})$ and radius $\frac{\sqrt{3}}{2}$, that intersects the horizontal plane at height $\frac{1}{\sqrt{3}}$ in a horizontal ball centered at $(\alpha, \frac{1}{\sqrt{3}})$ and of radius $\sqrt{\frac{2}{3}}$. The 9 vertices of the fundamental 3-3 duoprism $C_{3,3}$ of $\mathcal{O}$ are exactly at this distance from $z_0$, and all other elements of $\mathcal{O}$ are at greater distance from $z_0$. Hence (see the above picture on the left), $v_0$ belongs to the boundary of $B_\alpha(\sqrt{3})$ for every $\alpha \in V_{3,3}$ and $v_0 \notin B_\alpha(\sqrt{3})$ if $\alpha \in \mathcal{O} - V_{3,3}$.

We begin the treatment of the remaining case $n(q) = 2$ by geometric observations. The homography $g : z \mapsto \frac{1}{3}(z - z_0)^{-1} + z_0$ maps $\infty$ to $z_0$, $z_0$ to $\infty$, and the sphere in $\mathbb{H}$ of center $z_0$ and radius $r$ to the sphere in $\mathbb{H}$ of center $z_0$ and radius $\frac{1}{3r}$, for every $r > 0$. In particular, $g$ maps the sphere in $\mathbb{H}$ of center $z_0$ and radius $\frac{1}{\sqrt{3}}$ to itself and the Poincaré extension of $g$ to $\mathbb{H}_R^5$ (again denoted by $g$) fixes $v_0$.

**Lemma 4.8.** The homography $g = g^{-1}$ preserves the family $(B_\alpha(\sqrt{3}))_{\alpha \in A}$ of horoballs.

**Proof.** The homography $g$ is the projective map on $\mathbb{P}^1(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$ induced by

$$M = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & -3z_0 \end{pmatrix} = \begin{pmatrix} 3z_0 & 1 - 3z_0^2 \\ 3 & -3z_0 \end{pmatrix} \in \text{GL}_2(A).$$

Computations (using Mathematica and SAGE) show that $M$ conjugates all the generators of $\text{SL}_2(\mathcal{O})$ given in Lemma 4.4 to elements of $\text{SL}_2(\mathcal{O})$, as follows. We have

$$MJM^{-1} = \begin{pmatrix} 3 + \rho + j + \rho j & 1 - 2\rho - 2j - 2\rho j \\ 4 - j - \rho - \rho j & -3 - \rho - j - \rho j \end{pmatrix},$$
Thus, $M$ preserves the $G$ with reduced norm $\frac{\sqrt{3}}{2}$. As any element cannot have a denominator of norm $g$, since $\sqrt{3}$ above picture). Since the radius of $V$ which satisfies $\frac{\sqrt{3}}{2}$, we have $h(z_0, \beta) \leq \sqrt{34} < \frac{1}{\sqrt{3}}$. Since $g$ fixes $v_0$ and $gB_\beta(\sqrt{3}) = B_{g(\beta)}(\sqrt{3})$ by the above lemma, the element $\alpha = g^{-1}(\beta)$, which satisfies $v_0 \in B_\alpha(\sqrt{3})$ and is outside the ball of center $z_0$ and radius $\frac{1}{\sqrt{3}}$, hence cannot have a denominator of norm $\frac{1}{2}$. Therefore $\alpha$ has denominator 1 and by the previous case, it belongs to $V_{3,3}$ and $v_0$ lies in the boundary of $B_\alpha(\sqrt{3})$. So that $\beta = g(\alpha)$ belongs to $g(V_{3,3})$ and $v_0$ lies in the boundary of $B_\beta(\sqrt{3})$. □

An easy computation gives

$$g(V_{3,3}) = \left\{ \frac{1 + j + \rho j}{2}, \frac{1}{1 - j}, \frac{1 + \rho j}{2}, \frac{1}{1 - j \rho}, \frac{\rho + j}{2}, \frac{1}{\rho - j}, \frac{\rho + j}{2}, \frac{\rho(1 + j)}{2}, \frac{1}{(1 - j) \rho}, \frac{1 + j + \rho j}{2 \rho}, \frac{\rho + j + \rho j}{2}, \frac{\rho + j + \rho j}{2}, \frac{\rho + j + \rho j}{2} \right\}.$$

As any element $\beta$ in $g(V_{3,3})$ is the sum of an element of $\mathcal{G}$ with the inverse of an element of $\mathcal{G}$ with reduced norm 2, we have $n(I_\beta) = \frac{1}{2}$ and the horoball $B_\beta(\sqrt{3})$ has Euclidean radius $\frac{\sqrt{7}}{4}$. This horoball intersects the horizontal plane $\{(z, t) \in \mathbb{H}^2_\mathbb{R} : t = \frac{1}{\sqrt{3}}\}$ in a horizontal ball of Euclidean radius $\frac{1}{\sqrt{6}}$. In particular, the points in $g(V_{3,3})$ are at Euclidean distance $\frac{\sqrt{2}}{2}$ of $z_0$ and the horoballs tangent to $v_0$ are positioned as in the picture above.
By the above proposition, the link of \( v_0 \) in the cellulation of \( \mathbb{H}_\mathbb{R}^5 \) by the Ford-Voronoi cells of \( \mathcal{V} \) has 20 4-cells, which are the intersections of a small sphere centered at \( v_0 \) with the Ford-Voronoi cells \( \mathcal{H}_\alpha \) for \( \alpha \in V = V_{3,3} \cup g(V_{3,3}) \cup \{\infty, z_0\} \). Furthermore, for all \( \alpha \neq \beta \) in \( V \), the horoballs \( B_\alpha(\sqrt{3}) \) and \( B_\beta(\sqrt{3}) \) are tangent at \( v_0 \) if and only if \( \{\alpha, \beta\} \) is one of the 10 pairs \( \{\infty, z_0\}, \{0, \frac{1+\rho j+\rho j^2}{2}\}, \{1, \frac{1+\rho j+\rho j^2}{2}\}, \{\rho, \frac{1+\rho j+\rho j^2}{2}\}, \{j, \frac{1+\rho j+\rho j^2}{2}\}, \{1+j, \frac{\rho+\rho j^2}{2}\}, \{1+\rho j, \frac{\rho+\rho j^2}{2}\}, \{\rho j, \frac{1+\rho j^2}{2}\}, \{1+j+\rho, \frac{\rho+\rho j^2}{2}\} \) and \( \{\rho+j, \frac{1+\rho j^2}{2}\} \). By analyzing the intersections of the horoballs \( B_\alpha(\sqrt{3}) \) contained in the Ford-Voronoi cells incident to \( v_0 \), we find that each Ford-Voronoi cell containing \( v_0 \) intersects 9 others in 4-dimensional cells, that are images under \( \text{SL}_2(\mathbb{O}) \) of the fundamental cell \( \Sigma_\mathcal{V} \), combinatorially equal to the 6-6 duoprism \( C_{6,6} \).

The following graph shows the intersection pattern of the \( \mathcal{H}_\alpha \) for \( \alpha \in V \).

Thus the number of (6-6 duoprismatic) 4-dimensional cells of \( X_\mathcal{V} \) containing \( v_0 \) is exactly \( 90 = (20 \times 9)/2 \), one for each edge of this diagram.

Consider the elements \( g_{\infty,1} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \), \( g_{\infty,2} = \begin{pmatrix} \rho & j \rho \\ 0 & \rho \end{pmatrix} \) and \( h_\infty = \begin{pmatrix} \rho j & 0 \\ 0 & j \end{pmatrix} \) in \( \text{SL}_2(\mathcal{V}) \) inducing respectively the homographies

\[ z \mapsto \rho z \rho + 1, \quad z \mapsto \rho z \rho^{-1} + j \quad \text{and} \quad h_\infty(z) = -\rho j z j. \]

Using the facts that \( z_0 = \frac{1+j+\rho+\rho j}{3} \) and \( \rho j = j \rho^{-1} \), an easy computation shows that they fix \( z_0 \) and \( \infty \), hence fix \( v_0 \) since they preserve the geodesic line between \( z_0 \) and \( \infty \) and the horospheres centered at \( \infty \). Hence \( g_{\infty,1}, g_{\infty,2} \) and \( h_\infty \) belong to\(^{15} \) the stabilizer \( G_{v_0,\infty} \) of \( \infty \) (or equivalently \( z_0 \)) in the stabilizer of \( v_0 \) in \( \text{SL}_2(\mathcal{V}) \). Similar computations give that

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\(^{15}\)and actually generate, though we won’t need this fact
• the group \( G \) generated by \( g_{\infty, 1} \) and \( g_{\infty, 2} \) is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \),
• \( h_\infty \) has order 2 and conjugates \( g_{\infty, 1} \) and \( g_{\infty, 2} \), hence each element of the abelian group \( G \), to its inverse.

Thus the group generated by \( g_{\infty, 1}, g_{\infty, 2} \) and \( h_\infty \) is a semidirect product \( (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \) with 18 elements. The subgroup \( G_{v_0, \infty} \) acts transitively on \( V_{3, 3} \). The graph below shows how the points of \( V_{3, 3} \) are mapped by \( g_{\infty, 1} \) (in continuous green) and \( g_{\infty, 2} \) (in dotted red).

Since the inversion \( g \) conjugates \( g_{\infty, 1} \) and \( g_{\infty, 2} \) to their inverses, the group \( G_{v_0, \infty} \) also acts transitively on \( g(V_{3, 3}) \). By easy computations, the element \( g_\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), inducing the homography \( z \mapsto (1 - z)^{-1} \), is an element of the stabilizer of \( v_0 \) in \( \text{SL}_2(\mathbb{O}) \); it maps \( \infty \) to 0 in \( V_{3, 3} \) and \( \rho j \) in \( V_{3, 3} \) to \( \frac{1 + \rho j}{2} \in g(V_{3, 3}) \), and does not fix \( z_0 \). Since \( G_{v_0, \infty} \) acts transitively on \( V_{3, 3} \) and on \( g(V_{3, 3}) \), it follows that the stabilizer of \( v_0 \) acts transitively on \( V = V_{3, 3} \cup g(V_{3, 3}) \cup \{\infty, z_0\} \).

The dual tiling of the 6-6-duoprismatic tiling of \( \mathbb{H} \) is the 3-3-duoprismatic tiling. Therefore, the link of \( v_0 \) in \( \partial H_\infty \) (hence in all \( \partial H_\alpha \) containing \( v_0 \)) is the 3-skeleton of the dual of the 3-3 duoprism, namely the 3-3 duopyramid, whose Schläfli symbol is \( \{3\} + \{3\} \) and whose symmetry group has order \( 8 \times 3^3 = 72 \). The group generated by \( g_{\infty, 1}, g_{\infty, 2} \) and \( h_\infty \) is a subgroup of index 4 in the full group of symmetries of the link of \( v_0 \) in \( \partial H_\infty \). The link of \( v_0 \) in \( \mathbb{H}^5_\mathbb{R} \) is constructed of 20 copies of the 3-3 duopyramid, that are glued together according to the intersection pattern described above, forming the 4-skeleton of the dual of the birectified 5-simplex.\(^{17}\) This concludes the study of Example 4.6.

5 Waterworlds

Let \( A \) be a definite quaternion algebra over \( \mathbb{Q} \) and let \( \mathcal{O} \) be a maximal order in \( A \). Let \( f \) be an indefinite integral binary Hamiltonian form over \( \mathcal{O} \).

\(^{16}\)Actually, the stabilizer of \( v_0 \) coincides with the group generated by \( g_{\infty, 1}, g_{\infty, 2}, h_\infty \) and \( g_\rho \). It has \( 20 \times 18 = 360 \) elements (the number of 4-cells of the link of \( v_0 \) in the tessellation of \( \mathbb{H}^5_\mathbb{R} \) times the order of the stabilizer of one 4-cell, the one corresponding to \( H_\infty \)). But the full group of symmetries (including orientation-reversing ones) of the link of \( v_0 \) has \( 1440 = 4 \times 360 \) elements, hence half the orientation preserving symmetries of this link are not in \( \text{SL}_2(\mathcal{O}) \).

\(^{17}\)Since the birectified 5-simplex is called the dodecateron and has twelve 4-faces, its dual, which has twenty 4-faces and does not seem to have a name in the literature, could be called the icosateron.
The form $f$ defines a function $F = F_f : \mathbb{P}_r^1(A) \to \mathbb{Q}$ by

$$F([x : y]) = \frac{f(x, y)}{n(\mathcal{O}x + \mathcal{O}y)}.$$

This definition does not depend on the choice of representatives $(x, y) \in A \times A$ of $[x : y] \in \mathbb{P}_r^1(A)$, and $f$ is uniquely determined by its associated function $F$. Note that for every $g \in \text{SL}_2(\mathcal{O})$, the function $F_f \circ g$ associated to the form $f \circ g$ is $F \circ g$ (where we again denote by $g$ the projective transformation of $\mathbb{P}_r^1(A)$ induced by $g$). In particular, $F \circ g = F$ if $g \in \text{SU}_f(\mathcal{O})$. By the finiteness of the class number, there exists $N \in \mathbb{N} - \{0\}$ such that $F$ has values in $\frac{1}{N}\mathbb{Z}$, hence the set of values of $F$ is discrete.

As in [Con] for integral indefinite binary quadratic forms, we will think of $F$ as a map which associates a rational number to (the interior of) any Ford-Voronoi cell. For instance, if $D_A = 2$ and $\mathcal{O}$ is the Hurwitz order, then the values of $F$ on the two Ford-Voronoi cells $\mathcal{H}_\infty, \mathcal{H}_0$ containing the fundamental cell $\Sigma_0$ are $f(1, 0), f(0, 1)$ and the values of $F$ on the 24 Ford-Voronoi cells meeting $\Sigma_0$ in a 3-dimensional cell are $f(u, 1)$ for $u \in \mathcal{O}^\times$ (see the picture below).

Let $m$ be a left fractional ideal of $\mathcal{O}$. For every $s \geq 0$, let

$$\psi_{F, m}(s) = \text{Card } \text{SU}_f(\mathcal{O}) \setminus \{(u, v) \in m \times m : |F(u, v)| \leq s, \mathcal{O}u + \mathcal{O}v = m\},$$

which is the number of nonequivalent $m$-primitive representations by $F$ of rational numbers in $\frac{1}{N}\mathbb{Z}$ with absolute value at most $s$. We showed in [PP2, Theo. 1] and [PP3, Cor. 5.6] that there exists $\kappa > 0$ such that, as $s$ tends to $+\infty$,

$$\psi_{F, m}(s) = \frac{45 D_A \text{ Covol(SU}_f(\mathcal{O}))}{2 \pi^2 \zeta(3) \Delta(f)^2 \prod_{p|D_A} (p^3 - 1)} s^4(1 + O(s^{-\kappa})).$$

Note that the function $F$ takes all signs $0, +, -$. Indeed, it takes positive and negative values since $f$ is indefinite. The values of $F$ are actually positive at the points in $\mathbb{P}_r^1(A)$ in one of the two components of $\mathbb{P}_r^1(\mathbb{H}) - \mathcal{C}_\infty(f)$ and negative at the ones in the other component. But contrarily to the cases of integral binary quadratic and Hermitian forms, all integral binary Hamiltonian forms $f$ over $\mathcal{O}$ represent 0, since by taking a $\mathbb{Z}$-basis of $\mathcal{O}$, the form $f$ becomes an integral binary quadratic form over $\mathbb{Z}$ with 8 variables and all integral binary quadratic forms over $\mathbb{Z}$ with at least 5 variables represent 0, see for instance [Ser1, p. 77].
A Ford-Voronoi cell will be called flooded for \( f \) if the value of \( F \) on its point at infinity is 0. The above discussion says that there are always flooded Ford-Voronoi cells. The flooded Ford-Voronoi cells for \( f \) correspond to Conway's lakes for an isotropic integral indefinite binary quadratic form over \( \mathbb{Z} \), see [Con, page 23]. There were only two lakes, whereas there are now countably infinitely many flooded Ford-Voronoi cells for \( f \), one for each parabolic fixed point of the group of automorphs of \( f \).

**Example 5.1.** Consider the definite quaternion algebra \( A \) with \( D_A = 2 \), \( \mathcal{O} \) the Hurwitz order and a Hamiltonian form \( f \) with \( a(f) = 0 \), \( b = b(f), c = c(f) \in \mathbb{Z} - \{0\} \) such that \( b \) does not divide \( c \) nor \( 2c \). Then \( \mathcal{H}_\infty \) is flooded. Let \( \alpha = xy^{-1} \) with \( x \in \mathcal{O} \) and \( y \in \mathcal{O} - \{0\} \) relatively prime. If \( n(y) \leq 2 \), then the Ford-Voronoi cell \( \mathcal{H}_\alpha \) is not flooded, since otherwise the equation \( b \text{tr}(xy) + c n(y) = 0 \) would imply that \( b \) divides \( c \) or \( 2c \). If \( n(y) > 2 \), then \( n(I_a) = \frac{n(\mathcal{O}x + \mathcal{O}y)}{n(y)} = \frac{1}{n(y)} < \frac{1}{2} \). Hence by Proposition 3.4 (2), we have \( B_n(\sqrt{2}) \cap B_n(\sqrt{2}) = \emptyset \). Therefore \( \mathcal{H}_\alpha \cap \mathcal{H}_\infty = \emptyset \) by Proposition 4.1 (2). This proves that \( \mathcal{H}_\infty \) does not meet any other flooded Ford-Voronoi cell. Thus if the hyperbolic 4-orbifold \( SU_f(\mathcal{O})\backslash \mathcal{E}(f) \) has only one cusp, then the flooded Ford-Voronoi cells are pairwise disjoint.

We have the following analog of the statement of Conway (loc. cit.) that the values of the binary quadratic form along a lake are in an infinite arithmetic progression.

**Proposition 5.2.** Let \( \alpha_0 \in \mathbb{P}_{1}^1(A) \) be such that the Ford-Voronoi cell \( \mathcal{H}_{\alpha_0} \) is flooded for \( f \). If \( \alpha_0 \) belongs to the SL\(_2(\mathcal{O})\)-orbit of \( \infty \), let \( \Lambda_{\alpha_0} = \mathcal{O} \). Otherwise, let

\[
\Lambda_{\alpha_0} = \mathcal{O} \cap \alpha_0^{-1} \mathcal{O} \cap \mathcal{O} \mathcal{O}_0^{-1} \cap \alpha_0^{-1} \mathcal{O} \alpha_0^{-1}.
\]

Then there exists a finite set of nonconstant affine maps \( \{ \varphi_j : \mathbb{H} \to \mathbb{R} : j \in J' \} \) defined over \( \mathbb{Q} \) such that the set of values of \( F \) on the Ford-Voronoi cells meeting \( \mathcal{H}_{\alpha_0} \) is \( \bigcup_{j \in J'} \varphi_j(\Lambda_{\alpha_0}) \).

**Proof.** For every \( \alpha \in \mathbb{P}_{1}^1(A) \), let \( E_\alpha = \{ \beta \in \mathbb{P}_{1}^1(A) - \{\alpha\} : \mathcal{H}_\alpha \cap \mathcal{H}_\beta \neq \emptyset \} \). Note that \( E_{g_\alpha} = g \cdot E_\alpha \) for every \( g \in \text{SL}_2(\mathcal{O}) \), by Proposition 4.1 (1).

First assume that \( \alpha_0 \) belongs to the \( \text{SL}_2(\mathcal{O})\)-orbit of \( \infty \). Then up to replacing \( f \) by \( f \circ g \) for some \( g \in \text{SL}_2(\mathcal{O}) \) such that \( g \cdot \infty = \alpha_0 \), we may hence assume that \( \alpha_0 = \infty \).

Let \( a = a(f), b = b(f), c = c(f) \). Note that \( \mathcal{H}_\infty \) is flooded for \( f \) if and only if \( f(0,1) = 0 \), that is, if and only if \( a = 0 \). We then have \( b \neq 0 \) since \( f \) is indefinite. Hence \( F(E_\infty) = \left\{ \frac{\text{tr}(\overline{n}b) + c}{n(I_a)} : u \in E_\infty \right\} \). Since the stabilizer of \( \infty \) in \( \text{SL}_2(\mathcal{O}) \) acts with finitely many orbits on the cells of \( \partial \mathcal{H}_\infty \), its finite index subgroup \( \mathcal{O} \) acts by translations with finitely many orbits on \( E_\infty \). Hence there exists a finite subset \( J' \) of \( A \) such that \( E_\infty = J' + \mathcal{O} \). Since \( I_{a + o} = I_a \) for all \( a \in A \) and \( o \in \mathcal{O} \), the result follows with \( \varphi_j : u \mapsto \frac{\text{tr}(\overline{n}j + u) + c}{n(I_j)} \) for all \( j \in J' \).

Assume now that \( \alpha_0 \) does not belong to the \( \text{SL}_2(\mathcal{O})\)-orbit of \( \infty \), so that in particular \( \alpha_0 \in A - \{0\} \). Let \( \Gamma_{\alpha_0} \) be the stabilizer of \( \alpha_0 \) in \( \text{SL}_2(\mathcal{O}) \), which acts with finitely many orbits on \( E_{\alpha_0} \). Let \( g = \left( \begin{array}{cc} \alpha_0 & -1 \\ 1 & 0 \end{array} \right) \), which belongs to \( \text{SL}_2(A) \) and whose inverse projectively maps \( \alpha_0 \) to \( \infty \). Then (see for instance [PP2, §5]), \( \Lambda_{\alpha_0} \) is a \( \mathbb{Z} \)-lattice in \( \mathbb{H} \), such that the group of unipotent upper triangular matrices with coefficient \( 1-2 \) in \( \Lambda_{\alpha_0} \) is a finite index subgroup of \( g^{-1} \Gamma_{\alpha_0} g \). A similar argument concludes. \( \Box \)

By **projective real hyperplane** in \( \partial_{\infty}\mathbb{H}^2_{\mathbb{R}} = \mathbb{P}_{1}^1(\mathbb{H}) = \mathbb{H} \cup \{\infty\} \), we mean in what follows the boundary at infinity of a hyperbolic hyperplane in \( \mathbb{H}^2_{\mathbb{R}} \). The ones containing \( \infty = [1 : 0] \)
are the union of \{\infty\} with the affine real hyperplanes in \mathbb{H}. The ones not containing \infty are the Euclidean spheres in the affine Euclidean space \mathbb{H}.

**Lemma 5.3.** The form \(f\) is uniquely determined by the values of its associated function \(F\) at six points in \(\mathbb{P}^1_r(A)\) that do not lie in a projective real hyperplane.

**Proof.** Let \(a = a(f)\), \(b = b(f)\) and \(c = c(f)\). Let first prove that we may assume that the six points in \(A \cup \{\infty\}\) are \(\infty = [1:0]\), 0, \(\alpha_0 = 1\) and \(\alpha_1, \alpha_2, \alpha_3 \in A - \{0\}\).

Note that for all \(x, y \in A\) and \(g \in \text{GL}_2(A)\), if \(g_1, g_2\) are the components of the linear selfmap \(g\) of \(A \times A\), then

\[
F_{f \circ g}([x : y]) = F_f \circ g([x : y]) = \frac{n(g_1(x, y) + g_2(x, y))}{n(x + y)}.
\]

Given six point in \(\mathbb{P}^1_r(A)\) not in a projective real hyperplane of \(\mathbb{P}^1_r(\mathbb{H})\), the first three of them constitute a projective frame of the projective line \(\mathbb{P}^1_r(A)\), hence by the existence part of the fundamental theorem of projective geometry (see [Ber1, Prop. 4.5.10])\(^{18}\), there exists an element \(g \in \text{GL}_2(A)\) mapping them to \(\infty, 0, 1\). The initial claim follows by the above centered formula.

Now, the values of \(F\) at the points \(\infty, 0, \alpha_0, \alpha_1, \alpha_2, \alpha_3\) give a system of six equations on the unknown \(a, b, c\), of the form \(a = A_1\), \(c = A_2\), \(a + tr b + c = A_3\), \(tr(\alpha_i b) = A_{i+3}\) for \(i \in \{1, 2, 3\}\). Thus \(a\) and \(c\) are uniquely determined, and \(b\) belongs to the intersection of four affine real hyperplanes in \(\mathbb{H}\) orthogonal to \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) with equations \(tr(\alpha_i b) = A_i\) for \(i \in \{0, 1, 2, 3\}\). The result follows since if \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) are linearly independent over \(\mathbb{R}\), then for all \(A_0', A_1', A_2', A_3' \in \mathbb{R}\), such an intersection contains one and only one point of \(\mathbb{H}\).

\(\square\)

**Proposition 5.4.** Let \(v\) be a vertex of the spine \(X_{\theta}\). The form \(f\) is uniquely determined by the values of its associated function \(F\) on the Ford-Voronoi cells containing \(v\).\(^{19}\)

**Proof.** A dimension count shows that there are at least six Ford-Voronoi cells meeting at each vertex \(v\) of the spine. Their points at infinity cannot all be on the same projective real hyperplane \(P\), as otherwise the intersection of the equidistant hyperbolic hyperplanes between the pair of them yielding a 4-dimensional cell containing \(v\) would have dimension at least 1 (a germ of the orthogonal through \(v\) to the convex hull of \(P\) in \(\mathbb{H}^5_{\mathbb{R}}\)). The result follows by Lemma 5.3. \(\square\)

The waterworld of \(f\) is

\[
\mathcal{W}(f) = \bigcup_{\alpha \neq \beta \in \mathbb{P}^1_r(A), \ F(\alpha)F(\beta) < 1} \mathcal{H}_\alpha \cap \mathcal{H}_\beta.
\]

Since \(f\) is always isotropic over \(A\), the arguments of Conway and Bestvina-Savin for the anisotropic case no longer apply, and the waterworld of \(f\) could be empty.

**Example 5.5.** The binary Hamiltonian form \(f(u, v) = tr(\overline{u} v)\) is indefinite with discriminant 1. The coefficients of \(f\) are rational integers so it is integral over any maximal order \(\mathcal{O}\) of any definite quaternion algebra \(A\) over \(\mathbb{Q}\). Let us prove that the waterworld \(\mathcal{W}(f)\) is not empty.

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\(^{18}\)which does hold in the noncommutative setting, though the uniqueness part does not.

\(^{19}\)that is, on the points \(\alpha \in \mathbb{P}^1_r(A)\) such that \(v \in \mathcal{H}_\alpha\).
It is easy to check that \( \mathcal{C}_\infty(f) = \{ z \in \mathbb{H} : \text{tr } z = 0 \} \cup \{ \infty \} \). Let \( a \in \mathcal{O} \) be such that \( \text{tr}(a) = 1 \) (which does exist since \( \mathcal{O} \) is maximal, hence \( \text{tr} : \mathcal{O} \to \mathbb{Z} \) is onto). In particular \( a \neq 0 \), \( a \neq -\bar{a} \), and \( a, -\bar{a} \) are in two different components of \( \partial_\infty \mathbb{H}^3_\mathbb{R} - \mathcal{C}_\infty(f) \), so that \( F(a)F(-\bar{a}) < 0 \). Let us prove that \( \mathcal{H}_a \) and \( \mathcal{H}_{-\bar{a}} \) intersect in a 4-dimensional cell of \( X_\mathcal{O} \), which thus belongs to \( \mathcal{W}(f) \). By Proposition 4.1 (2), it is sufficient to prove that \( B_a(1) \) and \( B_{\bar{a}}(1) \) meet. By Theorem 3.5 (1), this is equivalent to proving that \( I_a I_{\bar{a}} = \mathcal{O} (\text{tr } a) \). But this holds since \( \text{tr } a = 1 \) and \( I_b = \mathcal{O} \) when \( b \in \mathcal{O} \).

The figure below illustrates the analogous case of the ocean in \( \mathbb{H}^3_\mathbb{R} \) of the isotropic binary Hermitian form \( f(u, v) = \text{tr}(\bar{u} v) \) considered as an integral form over the Eisenstein integers \( \mathbb{Z}[\frac{1+i\sqrt{3}}{2}] \). The blue hexagons are the components of the ocean of \( f \) in the hyperplane \( \mathcal{C}(f) = \{(z, t) \in \mathbb{H}^3_\mathbb{R} : \text{Im } z = 0 \} \) which is a copy of the (upper halfplane model of the) real hyperbolic plane.

We do not have an example of an empty waterworld and, in fact, it may be that no such example exists. However, the ocean of the isotropic binary Hamiltonian form \( f(u, v) = \text{tr}(\bar{u} v) \) considered over the Gaussian integers \( \mathbb{Z}[i] \) is empty (see the picture below). In order to prove this, let \( \alpha \in \mathbb{Q}(i) \) with \( \text{tr } \alpha \neq 0 \). Note that in the commutative case, \( n(I_\alpha) = n(I_{-\bar{\alpha}}) \), so that the Euclidean balls \( B_\alpha(1) \) and \( B_{-\bar{\alpha}}(1) \) have the same radius. By symmetry, \( \mathcal{C}(f) \) is the equidistant hyperbolic hyperplane of \( B_\alpha(1) \) and \( B_{-\bar{\alpha}}(1) \). Since \( \mathbb{Z}[i] \) is Euclidean, the spine of \( \text{SL}_2(\mathbb{Z}[i]) \) has only one orbit of 2-cells (see [BeS]). Hence the Ford-Voronoi cells \( \mathcal{H}_\alpha \) and \( \mathcal{H}_{-\bar{\alpha}} \) intersect if and only if \( B_\alpha(1) \) and \( B_{-\bar{\alpha}}(1) \) are tangent, that is, if and only if \( B_\alpha(1) \) intersects \( \mathcal{C}(f) \).

Since the hyperbolic 3-orbifold \( \text{SL}_2(\mathbb{Z}[i]) \backslash \mathbb{H}^3_\mathbb{R} \) has only one cusp, there exists \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}[i]) \) such that \( \alpha = g \cdot \infty = ac^{-1} \). Since \( g \cdot (-c^{-1}d, 1) = \infty \), the point \( g \cdot (-c^{-1}d, 1) = (a, \frac{1}{n(c)}) \) is the highest point in \( B_\alpha(1) = gB_\infty(1) \). Thus the Euclidean radius of \( B_\alpha(1) \) is \( \frac{1}{2n(c)} \). As the Euclidean distance of \( \alpha \) from \( \mathcal{C}_\infty(f) \) is \( |\text{tr } \alpha| \), this implies that \( B_\alpha(1) \) intersects \( \mathcal{C}(f) \) if and only if \( |\text{tr } \alpha| \leq \frac{1}{2n(c)} \), that is, if and only if \( \text{tr } a \bar{c} = \pm 1 \). This is impossible since the trace of any Gaussian integer is even.
Proposition 5.6. If the union of the flooded Ford-Voronoi cells does not separate $\mathbb{H}^5_\mathbb{R}$, and in particular if the flooded Ford-Voronoi cells are pairwise disjoint, then the waterworld of $f$ is nonempty.

Proof. The assumption says that the topological space $X = \mathbb{H}^5_\mathbb{R} - \bigcup_{\alpha \in \mathbb{R}^2, F(\alpha) = 0} \mathcal{H}_\alpha$ is connected. If $\mathcal{W}(f) = \emptyset$, then $X = \left( \bigcup_{\alpha \in \mathbb{R}^2, F(\alpha) < 0} \mathcal{H}_\alpha \right) \cup \left( \bigcup_{\alpha \in \mathbb{R}^2, F(\alpha) > 0} \mathcal{H}_\alpha \right)$ would be a partition into two nonempty (since $f$ is indefinite) locally finite, hence closed, unions of closed polyhedra, contradicting the connectedness of $X$. \hfill \square

We introduce two variants of $\mathcal{W}(f)$. The sourced waterworld $\mathcal{W}_+(f)$ of $f$ is the union of its waterworld and of its flooded Ford-Voronoi cells

$$\mathcal{W}_+(f) = \mathcal{W}(f) \cup \bigcup_{\alpha \in \mathbb{R}^2, F(\alpha) = 0} \mathcal{H}_\alpha.$$ 

The coned-off waterworld $\mathcal{CW}(f)$ of $f$ is obtained from $\mathcal{W}(f)$ by adding geodesic rays from its boundary points to the points at infinity of the corresponding flooded Ford-Voronoi cells

$$\mathcal{CW}(f) = \mathcal{W}(f) \cup \bigcup_{\alpha \in \mathbb{R}^2, x \in \mathcal{W}(f) \cap \mathcal{H}_\alpha : F(\alpha) = 0} [x, \alpha[.$$ 

Both the waterworld $\mathcal{W}(f)$, the sourced waterworld $\mathcal{W}_+(f)$ and the coned-off waterworld $\mathcal{CW}(f)$ of $f$ are invariant under the group of automorphs $SU_f(\mathcal{O})$ of $f$.

Proposition 5.7. The quotient $SU_f(\mathcal{O}) \setminus \mathcal{W}(f)$ is compact, and the set of flooded Ford-Voronoi cells consists of finitely many $SU_f(\mathcal{O})$-orbits.

Proof. The points at infinity of the flooded Ford-Voronoi cells are the parabolic fixed points of $SL_2(\mathcal{O})$ contained in $\mathcal{C}_\infty(f)$, hence are the parabolic fixed points of the group of automorphs $SU_f(\mathcal{O})$. Since $SU_f(\mathcal{O})$ is a lattice in the real hyperbolic $4$-space $\mathcal{C}(f)$, the quotient $SU_f(\mathcal{O}) \setminus \mathcal{C}(f)$ has only finitely many cusps. This proves the second claim.

Let $\alpha, \beta \in \mathbb{R}^2$ be such that $F(\alpha)F(\beta) < 0$ and the intersection $\mathcal{H}_{\alpha} \cap \mathcal{H}_{\beta}$ is nonempty. Then the intersection $B_{\alpha}(\sqrt{D_\alpha}) \cap B_{\beta}(\sqrt{D_\beta})$ is nonempty by Proposition 4.1 (2), hence the hyperbolic distance between the horoballs $B_{\alpha}(1)$ and $B_{\beta}(1)$ is at most $\ln D_\alpha$. By Remark 3.6, we hence have $\frac{n(\alpha - \beta)}{n(I_{\alpha}I_{\beta})} \leq D_\alpha$.

Let $a = a(f), b = b(f), c = c(f)$ and $\Delta = \Delta(f)$. Write $\alpha = [x : y]$ and $\beta = [u : v]$ with $x, y, u, v \in \mathcal{O}$ and $y, v \in \mathbb{R}$. Note that

$$\left( \begin{array}{cc} x & u \\ y & v \end{array} \right) * \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} x & u \\ y & v \end{array} \right) = \left( \begin{array}{cc} f(x, y) & z \\ y^2 & f(u, v) \end{array} \right),$$

for some $z \in \mathcal{O}$. Since $y, v \in \mathbb{R}$, an easy computation of Dieudonné determinants thus gives

$$|n(z) - f(x, y)f(u, v)| = n(xv - uy) \Delta.$$ 

Hence $0 \leq -f(x, y)f(u, v) \leq n(z) - f(x, y)f(u, v) = n(xv - uy) \Delta$ and

$$0 \leq -F(\alpha)F(\beta) = \frac{-f(x, y)f(u, v)}{n(\mathcal{O}x + \mathcal{O}y) n(\mathcal{O}u + \mathcal{O}v)} \leq \frac{n(\alpha - \beta)}{n(I_{\alpha})n(I_{\beta})} \Delta \leq D_\alpha \Delta.$$
Since the set of values of $F$ is discrete in $\mathbb{R}$, this implies that $F$ takes only finitely many values on the Ford-Voronoi cells that intersect $\mathcal{W}(f)$.

Given any vertex $v \in \mathcal{W}(f)$, for every $g \in \text{SL}_2(\mathcal{O})$, if $F(\alpha) = F(g \cdot \alpha)$ for all $\alpha \in A$ such that the Ford-Voronoi cell $\mathcal{H}_\alpha$ contains $v$, then $f = f \circ g$ by Proposition 5.4. Since there are only finitely many orbits of $\text{SL}_2(\mathcal{O})$ on the vertices of the spine $X_\mathcal{O}$ and since $F$ takes only finitely many values on the Ford-Voronoi cells meeting the waterworld $\mathcal{W}(f)$, this implies that $\text{SU}_f(\mathcal{O})$ has only finitely many orbits of vertices in $\mathcal{W}(f)$. The result follows. □

Note that even if the waterworld $\mathcal{W}(f)$ could be empty, since the flooded Ford-Voronoi cells only have their points at infinity on the 3-sphere $\mathcal{C}_\infty(f)$ in $\mathbb{P}_1^1(\mathbb{H})$ and by the co-compactness of the action of $\text{SL}_2(\mathcal{O})$ on its spine $X_\mathcal{O}$, there exist a positive constant and finitely many pairs $\{\alpha, \beta\} \in A$ such that, for all indefinite integral binary Hamiltonian forms $f$ over $\mathcal{O}$ up to the action of $\text{SL}_2(\mathcal{O})$, the distance between $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ is at most this constant and $F(\alpha)F(\beta) < 0$. The above arguments hence allow to give another proof of Corollary 25 in [PP2], saying that the number of $\text{SL}_2(\mathcal{O})$-orbits in the set of indefinite integral binary Hamiltonian forms over $\mathcal{O}$ with given discriminant is finite.

We can now state and prove the main result of this paper that implies Theorem 1.2 in the Introduction.

**Theorem 5.8.** For every indefinite integral binary Hamiltonian form $f$ over $\mathcal{O}$, the closest point mapping $\pi : \mathcal{W}_+(f) \to \mathcal{C}(f)$ is a proper $\text{SU}_f(\mathcal{O})$-equivariant homotopy equivalence. If the flooded Ford-Voronoi cells for $f$ are pairwise disjoint, then the closest point mapping $\pi : \mathcal{C}(f) \to \mathcal{C}(f)$ is a $\text{SU}_f(\mathcal{O})$-equivariant homeomorphism and its restriction to the waterworld $\mathcal{W}(f)$ is a $\text{SU}_f(\mathcal{O})$-equivariant homeomorphism onto a contractible 4-manifold with a polyhedral boundary component homeomorphic to $\mathbb{R}^3$ contained in every flooded Ford-Voronoi cell.

**Proof.** The $\text{SU}_f(\mathcal{O})$-equivariance properties are immediate. We will subdivide this proof into several steps. Unless otherwise stated, polyhedra are compact and convex.

**Claim 1.** The restriction of $\pi$ to any cell of $\mathcal{W}(f)$ is a homeomorphism onto its image, which is a hyperbolic polyhedron in the hyperbolic hyperplane $\mathcal{C}(f)$. The restriction of $\pi$ to any flooded Ford-Voronoi cell $\mathcal{H}_\alpha$ of $f$ is a proper map onto a noncompact convex hyperbolic polyhedron in $\mathcal{C}(f)$ containing $B_\alpha(1) \cap \mathcal{C}(f)$ and contained in $B_\alpha(\sqrt{D_A}) \cap \mathcal{C}(f)$. If the flooded Ford-Voronoi cells for $f$ are pairwise disjoint, then the restriction of $\pi$ to any cell in the boundary of a flooded Ford-Voronoi cells for $f$ is a homeomorphism onto its image, which is a hyperbolic polyhedron in the hyperbolic hyperplane $\mathcal{C}(f)$.

**Proof.** If $P, P'$ are hyperbolic hyperplanes in $\mathbb{H}_R^3$ that do not intersect perpendicularly, then the closest point mapping from $P$ to $P'$ is a homeomorphism onto a convex open subset of $P'$, which maps any hyperbolic polyhedron of $P$ to a hyperbolic polyhedron of $P'$ (see for instance [BeS] for a proof).

Any 4-dimensional cell, hence any cell, of $\mathcal{W}(f)$ is a hyperbolic polyhedron in the equidistant hyperbolic hyperplane

$$\mathcal{S}_{\alpha, \beta} = \{x \in \mathbb{H}_R^3 : d_\alpha(x) = d_\beta(x)\}$$

for some $\alpha \neq \beta$ in $\mathbb{P}_1^1(\mathcal{A})$ with $F(\alpha)F(\beta) < 0$. Note that $\mathcal{S}_{\alpha, \beta}$ is not perpendicular to $\mathcal{C}(f)$, otherwise $\alpha$ and $\beta$, which are the points at infinity of a geodesic line perpendicular
to \( L_{\alpha,\beta} \), would belong to the closure of the same component of \( \partial_\infty \mathbb{H}^5_\mathbb{R} - F_\infty(f) \), which contradicts the fact that \( F(\alpha)F(\beta) < 0 \). Hence the first part of the claim follows from the preliminary remark.

The closest point mapping from a horoball \( H \) to a hyperbolic hyperplane \( P \) passing through the point at infinity of \( H \) is a proper map (since the intersection of \( H \) with any geodesic line not passing through its point at infinity is compact), whose image is \( H \cap P \), and which maps the geodesic segment between two points to the geodesic segment between their images. The second part of the claim hence follows from Proposition 4.1 (2).

If the flooded Ford-Voronoi cells for \( f \) are pairwise disjoint, any 4-dimensional cell, hence any cell, in the boundary of a flooded Ford-Voronoi cell for \( f \) is a hyperbolic polyhedron in the hyperbolic hyperplane \( L_{\alpha,\beta} \) for some \( \alpha \neq \beta \) in \( \mathbb{P}^1_\mathbb{R}(A) \) with \( F(\alpha) = 0 \) and \( F(\beta) \neq 0 \). Note that \( L_{\alpha,\beta} \) is again not perpendicular to \( C(f) \), otherwise \( \alpha \) and \( \beta \) would both belong to \( C_\infty(f) \), and the Ford-Voronoi cells \( H_\alpha \) and \( H_\beta \) would both be flooded for \( f \) and not disjoint. The last part of the claim follows.

Claim 2. Any 3-dimensional cell \( \sigma \) of \( W(f) \) not contained in a flooded Ford-Voronoi cell for \( f \) belongs to an even number of 4-dimensional cells of \( W(f) \). If the flooded Ford-Voronoi cells for \( f \) are pairwise disjoint, then any 3-dimensional cell \( \sigma' \) of \( W(f) \) contained in a flooded Ford-Voronoi cell for \( f \) belongs to an odd number of 4-dimensional cells of \( W(f) \).

Proof. The link of \( \sigma \) in the Ford-Voronoi cellulation of \( \mathbb{H}^5 \) is a circle, subdivided into closed intervals with disjoint interiors, each one of them contained in some nonflooded Ford-Voronoi cell, on which the sign of \( F \) is either + or −. In such a cyclic arrangement of signs, the number of sign changes is even.

Similarly, the link of \( \sigma' \) is subdivided into at least 3 closed intervals with disjoint interiors carrying a sign +, 0, −. By the assumptions, exactly one of them, denoted by \( I_0 \), belongs to a flooded Ford-Voronoi cell \( H_{\alpha_0} \) for some \( \alpha_0 \in \mathbb{P}^1_\mathbb{R}(A) \), that is, carries the sign 0. Assume for a contradiction that the two intervals adjacent to \( I_0 \) carry the same sign. Let \( \beta_1, \beta_2 \in \mathbb{P}^1_\mathbb{R}(A) \) be such that \( H_{\alpha_0} \cap H_{\beta_1} \) and \( H_{\alpha_0} \cap H_{\beta_2} \) are the 4-dimensional cells corresponding to the endpoints of \( I_0 \). Note that the points at \( +\infty \) of the geodesic lines starting from a given point \( \alpha_0 \) of \( C_\infty(f) \), passing through a geodesic line both of whose endpoints \( \beta_1, \beta_2 \) are contained in the same component \( C \) of \( \partial_\infty \mathbb{H}^5_\mathbb{R} - F_\infty(f) \) also belong to \( C \). Hence all intervals in the link of \( \sigma' \) carry the same sign, which contradicts the fact that \( \sigma' \) belongs to \( W(f) \). As for \( \sigma \), this proves that the number of sign changes between + and − in the link of \( \sigma' \) is odd.

Claim 3. If \( \sigma \) and \( \tau \) are distinct 4-dimensional cells of \( W(f) \) or flooded Ford-Voronoi cells for \( f \), then \( \pi(\sigma) \) and \( \pi(\tau) \) have disjoint interiors.

Proof. Note that no 4-dimensional cell of \( W(f) \) is contained in a flooded Ford-Voronoi cell for \( f \).

For a contradiction, assume that a point \( p \in C(f) \) is contained in the interior of both \( \pi(\sigma) \) and \( \pi(\tau) \) and, up to moving it a little bit, is not in the (measure 0) image by \( \pi \) of the codimension 1 skeleton of \( X_\theta \). Let \( \ell \) be the geodesic line through \( p \) perpendicular to \( C(f) \), meeting \( \sigma \) and \( \tau \) at interior points \( x \) and \( y \) respectively. Since the cell complex \( X_\theta \) is locally finite, we may assume that the geodesic segment \([x, y]\) does not meet any other 4-dimensional cell of \( W(f) \) or flooded Ford-Voronoi cell for \( f \) than \( \sigma \) and \( \tau \).

Assume for a contradiction that \([x, y]\) is contained in \( \sigma \cup \tau \). Then \( \sigma \) and \( \tau \) are flooded Ford-Voronoi cells, meeting in a 4-dimensional cell \( C \), which is crossed transversally by \([x, y]\)
since $\ell$ does not meet the 3-skeleton of $X_\sigma$. Since $\sigma, \tau$ are flooded, their points at infinity $\alpha, \beta \in \mathbb{P}_4^1(A)$ belong to $C_\infty(\mathcal{C})$. Hence the hyperbolic hyperplane $S_{\alpha, \beta}$ equidistant to $\alpha$ and $\beta$, which contains $\sigma$, is perpendicular to $C(f)$. In particular, $\ell$, which is perpendicular to $C(f)$, is contained in the closure of one of the two connected component of $\mathbb{H}_R^5 - S_{\alpha, \beta}$. This contradicts the fact that $\ell$ meets transversally $C$.

Hence $[x, y]$ is not contained in $\sigma \cup \tau$. Let $]x', y'[ = [x, y] - (\sigma \cup \tau) \cap [x, y]$ with $x, x', y', y$ in this order on $[x, y]$, so that $]x', y']$ is contained in a Ford-Voronoi cell $\mathcal{H}_\alpha$ for some $\alpha \in \mathbb{P}_4^1(A)$. Let $\sigma'$ and $\tau'$ be the 4-dimensional cells of $X_\sigma$ containing $x'$ and $y'$ respectively (note that for instance $x = x'$ and $\sigma = \sigma'$ if $\sigma$ is a 4-dimensional cell of $\mathcal{W}(f)$, but $x \neq x'$ if $\sigma$ is a flooded Ford-Voronoi cell).

**Lemma 5.9.** If $\ell$ is a geodesic line in $\mathbb{H}_R^5$ perpendicular to $C(f)$, oriented such that $\ell(\pm \infty) \in \{(x : y) \in \mathbb{P}_4^1(\mathbb{H}) : \pm f(x, y) > 0\}$, if $\ell$ meets transversally at a point $z$ the interior of a 4-dimensional cell $\mathcal{H}_\alpha \cap \mathcal{H}_\beta$ of $X_\sigma$ with $F(\alpha_-) \leq 0$ and $F(\alpha_+) \geq 0$ and $(F(\alpha_-), F(\alpha_+)) \neq (0, 0)$, then a germ of $\ell$ at $z$ pointing towards $\ell(\pm \infty)$ is contained in $\mathcal{H}_{\alpha, \beta}$.

**Proof.** The proof of Claim 2 page 12 of [BeS] applies. \qed

Now this lemma implies that, since the two germs of the segment $[x', y']$ at its endpoints have opposite direction, the sign of $F(\alpha)$ should be both positive and negative, a contradiction. \qed

**Claim 4.** No 3-dimensional cell of $\mathcal{W}(f)$ is contained in two distinct flooded Ford-Voronoi cells. Any 3-dimensional cell $\sigma$ of $\mathcal{W}(f)$ not contained in a flooded Ford-Voronoi cell for $f$ belongs to exactly two 4-dimensional cells $\tau$ and $\tau'$ of $\mathcal{W}(f)$, and $\pi$ embeds their union. Any 3-dimensional cell $\sigma$ of $\mathcal{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathcal{H}_\alpha$ for $f$ belongs to exactly one 4-dimensional cell $\tau$ of $\mathcal{W}(f)$, and $\pi$ embeds the union of $\tau$ and $\tau' = \bigcup_{x \in \sigma} [x, \alpha]$.

**Proof.** Assume for a contradiction that $\sigma$ is a 3-dimensional cell of $\mathcal{W}(f)$ contained in the flooded Ford-Voronoi cells $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ with $\alpha \neq \beta \in \mathbb{P}_4^1(A)$. Let $\tau$ be a 4-dimensional cell of $\mathcal{W}(f)$ containing $\sigma$. Then the interiors of the images by $\pi$ of $\tau$ and either $\mathcal{H}_\alpha$ or $\mathcal{H}_\beta$ are not disjoint, which contradicts Claim 3.

Three $n$-dimensional polytopes in $\mathbb{H}_R^5$ having a common codimension 1 face cannot have pairwise disjoint interiors, so that the claims on the number of 4-dimensional cells of $\mathcal{W}(f)$ containing $\sigma$ follows from Claim 3. Since the polyhedra $\pi(\tau)$ and $\pi(\tau')$ are convex, the result follows. \qed

**Claim 5.** For every 2-dimensional cell $\sigma$ of $\mathcal{W}(f)$ not contained in a flooded Ford-Voronoi cell for $f$, the link of $\sigma$ in $\mathcal{W}(f)$ is a circle and the union of the 4-dimensional cells of $\mathcal{W}(f)$ containing $\sigma$ embeds in $C(f)$ by $\pi$. If the flooded Ford-Voronoi cells for $f$ are pairwise disjoint, for every 2-dimensional cell $\sigma'$ of $\mathcal{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathcal{H}_\alpha$, the link of $\sigma'$ in $\mathcal{W}(f)$ is an interval and the union of the 4-dimensional cells of $\mathcal{W}(f)$ containing $\sigma'$ and of the geodesic rays $[x, \alpha]$ for $x$ in the two 3-dimensional cells of $\mathcal{W}(f) \cap \partial \mathcal{H}_\alpha$ containing $\sigma'$ embeds in $C(f)$ by $\pi$.

**Proof.** By Claim 4, the link $Lk(\sigma)$ of $\sigma$ in $\mathcal{W}(f)$ is a disjoint union of circles. Each component of $Lk(\sigma)$ corresponds to a finite set of 4-dimensional cells cyclically arranged around $\sigma$. By Claim 4 again, their images by $\pi$ are not folded, hence are cyclically arranged around $\pi(\sigma)$. If $Lk(\sigma)$ was not connected, the image of two 4-dimensional cells of $\mathcal{W}(f)$
by π would have intersecting interiors, contradicting Claim 3. An analogous proof gives that the link of σ′ in CW(f) is a circle.

**Claim 6.** For every 1-dimensional cell σ of W(f) not contained in a flooded Ford-Voronoi cell for f, the link of σ in W(f) is a 2-sphere and the union of the 4-dimensional cells of W(f) containing σ embeds in C(f) by π. If the flooded Ford-Voronoi cells for f are pairwise disjoint, for every 1-dimensional cell σ′ of W(f) contained in a flooded Ford-Voronoi cell Hα, the link of σ′ in W(f) is a 2-disc and the union of the 4-dimensional cells of W(f) containing σ′ and of the geodesic rays [x, α] for x in any 3-cell of W(f) ∩ ∂Hα containing σ′ embeds in C(f) by π.

**Proof.** By Claim 5, the links of the vertices of the link Lk(σ) of σ in W(f) are circles, hence Lk(π(σ)) is a compact surface, mapping locally homeomorphically to Lk(π(σ)) by π, which is a 2-sphere. Hence Lk(π(σ)) is a union of 2-spheres, again with only one of them by Claim 3. The proof that the link of σ′ in CW(f) is a 2-sphere is similar.

**Claim 7.** For every vertex v of W(f) not contained in a flooded Ford-Voronoi cell for f, the link of v in W(f) is a 3-sphere and the union of the 4-dimensional cells of W(f) containing v embeds in C(f) by π. If the flooded Ford-Voronoi cells for f are pairwise disjoint, for every vertex v′ of W(f) contained in a flooded Ford-Voronoi cell Hα, the link of v′ in W(f) is a 3-disc and the union of the 4-dimensional cells of W(f) containing v′ and of the geodesic rays [x, α] for x in any 3-cell of W(f) ∩ Hα containing v′ embeds in C(f) by π.

**Proof.** The proof is similar to the previous one.

Now, the properness of π : W+(f) → C(f) follows from the fact that π is SUf(O)-equivariant, that SUf(O) acts cocompactly on W(f) and with finitely many orbits on the set of flooded Ford-Voronoi cells by Proposition 5.7, and from its properness when restricted to each flooded Ford-Voronoi cell (see Claim 1).

Claim 7 proves that when the flooded Ford-Voronoi cells for f are pairwise disjoint, the map π : CW(f) → C(f) is a proper local homeomorphism between locally compact spaces, hence is a covering map. Since C(f) is simply connected, π is hence a homeomorphism on each of the connected components of CW(f). But since π is injective outside the codimension 1 skeleton by Claim 3, it follows that CW(f) is connected and π is a homeomorphism. This concludes the proof of Theorem 5.8.
A An algebraic description of the distance to the cusps

Let \( A \) be a definite quaternion algebra over \( \mathbb{Q} \) and let \( \mathcal{O} \) be a maximal order in \( A \). In this independent appendix, following Mendoza [Men] in the Hermitian case, we give an algebraic description of the distance functions \( d_\alpha \) to the rational points at infinity \( \alpha \in \mathbb{P}_1^1(A) \), defined just before Proposition 3.3.

An \( \mathcal{O} \)-flag is a rank one\(^{20} \) right \( \mathcal{O} \)-submodule \( L \) of the right \( \mathcal{O} \)-module \( \mathcal{O} \times \mathcal{O} \) such that the quotient \( (\mathcal{O} \times \mathcal{O})/L \) has no torsion. We denote by \( \mathcal{F}_\mathcal{O} \) the set of \( \mathcal{O} \)-flags.

For all right \( \mathcal{O} \)-submodules \( M \) of \( A \times A \) and \( v \in A \times A - \{0\} \), let us define

\[
M_v = \{ x \in A : vx \in M \}.
\]

Note that for every \( \lambda \in A - \{0\} \), we immediately have

\[
\lambda M_v \lambda = M_v.
\]

Example A.1. Recall that the inverse \( I^{-1} \) of a left fractional ideal \( I \) of \( \mathcal{O} \) is the right fractional ideal of \( \mathcal{O} \)

\[
I^{-1} = \{ x \in A : IxI \subset I \}.
\]

It is well known and easy to check that for every \( a, b \in \mathcal{O} \), if \( ab \neq 0 \), then

\[
(\mathcal{O}a + \mathcal{O}b)^{-1} = a^{-1}\mathcal{O} \cap b^{-1}\mathcal{O}.
\]

We claim that if \( v = (a, b) \), then

\[
(\mathcal{O} \times \mathcal{O})_v = (\mathcal{O}a + \mathcal{O}b)^{-1}.
\]

Indeed, if \( a, b \neq 0 \), then by Equation (12)

\[
(\mathcal{O} \times \mathcal{O})_v = \{ x \in A : (ax, bx) \in \mathcal{O} \times \mathcal{O} \} = a^{-1}\mathcal{O} \cap b^{-1}\mathcal{O} = (\mathcal{O}a + \mathcal{O}b)^{-1}.
\]

The result is immediate if \( a = 0 \) or \( b = 0 \).

Proposition A.2. (1) For all right \( \mathcal{O} \)-submodule \( M \) of \( A \times A \) and \( v \in A \times A - \{0\} \), the subset \( M_v \) of \( A \) is a right fractional ideal of \( \mathcal{O} \).

(2) For every \( v \in A \times A - \{0\} \), the subset \( v(\mathcal{O} \times \mathcal{O})_v \) of \( \mathcal{O} \times \mathcal{O} \) is an \( \mathcal{O} \)-flag.

(3) For all \( \mathcal{O} \)-flags \( L \) and all \( v \in L - \{0\} \), we have

\[
L = v(\mathcal{O} \times \mathcal{O})_v.
\]

(4) The map \( \text{SL}_2(A) \times \mathcal{F}_\mathcal{O} \to \mathcal{F}_\mathcal{O} \) defined by

\[
(g, L) \mapsto (gv)(\mathcal{O} \times \mathcal{O})_{gv}
\]

for any \( v \in L - \{0\} \) is an action on the set \( \mathcal{F}_\mathcal{O} \) of \( \mathcal{O} \)-flags of the group \( \text{SL}_2(A) \).

(5) The map \( \Theta' : \mathbb{P}_1^1(A) \to \mathcal{F}_\mathcal{O} \) defined by \( [a : b] \mapsto (a, b)(\mathcal{O} \times \mathcal{O})_{(a,b)} \) is a \( \text{SL}_2(A) \)-equivariant bijection.

\(^{20}\) in the sense that \( LA \) is a line in the \( A \)-vector space \( A \times A \)
Proof. (1) This follows immediately from the fact that $M$ is stable by addition and by multiplications on the right by the elements of $\mathcal{O}$.

(2) Let $L = v(\mathcal{O} \times \mathcal{O})_v \subset vA$. Then $L$ is contained in $\mathcal{O} \times \mathcal{O}$ by the definition of $(\mathcal{O} \times \mathcal{O})_v$ and is a right $\mathcal{O}$-submodule of $\mathcal{O} \times \mathcal{O}$ by Assertion (1). Since $v \neq 0$, note that $(\mathcal{O} \times \mathcal{O})_v$ is a nonzero right fractional ideal, so that $L \neq 0$ and $L$ has rank one.

Assume that $w \in \mathcal{O} \times \mathcal{O}$ has its image in $(\mathcal{O} \times \mathcal{O})/L$ which is torsion. Then there exists $y \in \mathcal{O} \setminus \{0\}$ and $x \in A$ such that $wy = vx$. Hence $w = vxy^{-1}$. Since $w \in \mathcal{O} \times \mathcal{O}$, this implies that $xy^{-1} \in (\mathcal{O} \times \mathcal{O})_v$, so that $w \in L$, and the image of $w$ in $(\mathcal{O} \times \mathcal{O})/L$ is zero.

(3) As $L$ has rank 1 and $v \in L - \{0\}$, we have $L \subset vA \cap (\mathcal{O} \times \mathcal{O}) = v(\mathcal{O} \times \mathcal{O})_v$. Conversely, for every $x \in (\mathcal{O} \times \mathcal{O})_v$, so that $vx \in \mathcal{O} \times \mathcal{O}$, let us prove that $vx \in L$. Since $x \in A$ which is the field of fractions of $\mathcal{O}$, there exists $y \in \mathcal{O}$ such that $xy \in \mathcal{O}$. Hence $(vx)y = v(xy)$ belongs to $L$, since $v \in L$ and $L$ is a right $\mathcal{O}$-module. In particular, the image of $vx$ in $(\mathcal{O} \times \mathcal{O})/L$ is torsion. Since $L$ is an $\mathcal{O}$-flag, this implies that this image is zero, as wanted. This proves that $v(\mathcal{O} \times \mathcal{O})_v$ is contained in $L$, hence is equal to $L$ by the previous inclusion.

(4) Let us prove that this map is well defined. If $v, w \in L - \{0\}$, since $L$ has rank one, there exists $x \in A - \{0\}$ such that $w = vx$, thus, for every $g \in SL_2(A)$, by the linearity on the right of $g$ and by Equation (11), we have

$$ (gw)(\mathcal{O} \times \mathcal{O})_{gw} = (gv)x(\mathcal{O} \times \mathcal{O})_{(gv)x} = (gv)(\mathcal{O} \times \mathcal{O})_{gv} . $$

The fact that this map is an action is then immediate: for all $g, g' \in SL_2(A)$ and $L \in \mathcal{F}_{\mathcal{O}}$, let $v \in L - \{0\}$ and $\lambda \in A$ be such that $gv\lambda \in (gv)(\mathcal{O} \times \mathcal{O})_{gv} \setminus \{0\}$; then using twice Equation (11) and the linearity, we have

$$ g'(gL) = g'(gv(\mathcal{O} \times \mathcal{O})_{gv}) = g'(gv\lambda(\mathcal{O} \times \mathcal{O})_{gv\lambda}) = g'(gv\lambda)(\mathcal{O} \times \mathcal{O})_{g'gv\lambda} $$
$$ = (g'gv\lambda(\mathcal{O} \times \mathcal{O})_{(g'g)\lambda}v) = (g'gv(\mathcal{O} \times \mathcal{O})_v = (g'gL) . $$

(5) For every $\alpha = [a : b] \in \mathbb{P}^1(A)$, the subset $(a, b)(\mathcal{O} \times \mathcal{O})_{(a, b)}$, which is an $\mathcal{O}$-flag by Assertion (2), does not depend on the choice of homogeneous coordinates of $\alpha$ by Equation (11). Hence the map $\Theta'$ is well defined, and equivariant by the definition of the action of $SL_2(A)$ on $\mathcal{F}_{\mathcal{O}}$.

The fact that $\Theta'$ is onto follows from Assertion (3). Clearly, it is one-to-one since if $(a, b)(\mathcal{O} \times \mathcal{O})_{(a, b)} = (c, d)(\mathcal{O} \times \mathcal{O})_{(c, d)}$, then there is $\lambda \in A - \{0\}$ such that $(a, b) = (c, d)\lambda$.

Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a positive definite binary Hamiltonian form and let $L$ be a rank one right $\mathcal{O}$-submodule of $\mathcal{O} \times \mathcal{O}$. Then $L$ is a rank 4 free $\mathbb{Z}$-submodule of $\mathbb{H} \times \mathbb{H}$, and we denote by $(L)_\mathbb{R}$ the 4-dimensional real vector subspace of $\mathbb{H} \times \mathbb{H}$ generated by $L$, endowed with the restriction of the scalar product $\langle \cdot, \cdot \rangle_f$ on $\mathbb{H} \times \mathbb{H}$ defined by $f$, hence with the induced volume form. Recall that for all $z, z' \in \mathbb{H} \times \mathbb{H}$, we have

$$ \langle z, z' \rangle_f = \frac{1}{2} \left( f(z + z') - f(z) - f(z') \right) . \quad (14) $$

We define the covolume of $L$ for $f$ as

$$ \text{Covol}_f L = \text{Vol}(L)_\mathbb{R}/L . $$

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Recall\(^{21}\) that if \( G = (e_i, e_j)_f \) \(_{1 \leq i, j \leq 4}\) is the Gram matrix of a \( \mathbb{Z}\)-basis \((e_1, e_2, e_3, e_4)\) of \( L\) for the scalar product \( (\cdot, \cdot)_f\), then

\[
\text{Covol}_f L = (\det G)^{\frac{1}{2}}.
\]

**Theorem A.3.** For all \( x \in \mathbb{H}_5^\mathbb{R} \) and \( \alpha \in \mathbb{P}^1(A) \), we have

\[
d_{\alpha}(x) = \frac{2}{\sqrt{D_A}} \left( \text{Covol}_{\Theta(x)} \Theta'(\alpha) \right)^{\frac{1}{2}}.
\]

**Proof.** Fix \( a, b \in \mathcal{O} \) such that \( \alpha = [a : b] \). Let \( f = \Theta(x), L = \Theta'(\alpha) = (a, b)(\mathcal{O} \times \mathcal{O})_{(a,b)} \) and \( L' = (a, b)\mathcal{O} \). Since \( a, b \in \mathcal{O} \), we have \( \mathcal{O} \subset (\mathcal{O} \times \mathcal{O})_{(a,b)} \), hence \( L' \) is a finite index \( \mathbb{Z}\)-submodule in \( L \). Furthermore, by Equation (13) and the relation (see Equation (2)) between the norm and reduced norm of a left integral ideal of \( \mathcal{O} \), we have

\[
[L : L'] = [(\mathcal{O} \times \mathcal{O})_{(a,b)} : \mathcal{O}] = [(\mathcal{O}a + \mathcal{O}b)^{-1} : \mathcal{O}] = [\mathcal{O} : \mathcal{O}a + \mathcal{O}b] = n(\mathcal{O}a + \mathcal{O}b)^2.
\]

Let \((x_1, x_2, x_3, x_4)\) be a \( \mathbb{Z}\)-basis of \( \mathcal{O} \), so that \((a, b)x_i) \) \(_{1 \leq i \leq 4}\) is a \( \mathbb{Z}\)-basis of \( L' \). Using Equation (14) and the fact that \( f((u, v)\lambda) = n(\lambda)f(u, v) \) for all \( u, v, \lambda \in \mathbb{H} \), we have for \( 1 \leq i, j \leq 4 \),

\[
(a, b)x_i, (a, b)x_j)_f = \frac{1}{2} \left( f( (a, b)(x_i + x_j) ) - f((a, b)x_i) - f((a, b)x_j) \right)
\]

\[
= \frac{f(a, b)}{2} (n(x_i + x_j) - n(x_i) - n(x_j)) = \frac{f(a, b)}{2} \text{tr}(x_i x_j).
\]

Note that \( (u, v) \mapsto \frac{1}{2} \text{tr}(\overline{u}v) \) is the standard Euclidean scalar product on \( \mathbb{H}\) (making the standard basis \((1, i, j, k)\) orthonormal), hence \( \left( \frac{1}{2} \text{tr}(x_i x_j) \right) \) \(_{1 \leq i, j \leq 4}\) is the Gram matrix of the \( \mathbb{Z}\)-lattice \( \mathcal{O} \) in the Euclidean space \( \mathbb{H} \). Therefore, by Equation (15) and by [KO, Lem. 5.5], we have

\[
\left( \det \left( \text{tr}(x_i x_j) \right) \right)_{1 \leq i, j \leq 4}^{\frac{1}{2}} = (2^4)^{\frac{1}{2}} \text{Vol}(\mathbb{H}/\mathcal{O}) = 4 \frac{D_A}{4} = D_A.
\]

Thus using Equations (15), (16) and (17), we have

\[
\text{Covol}_f(L) = \frac{1}{[L : L']} \text{Covol}_f(L') = \frac{1}{[L : L']} \left( \det \left( (a, b)x_i, (a, b)x_j \right)_f \right)_{1 \leq i, j \leq 4}^{\frac{1}{2}}
\]

\[
= \frac{1}{[L : L']} \left( \frac{f(a, b)}{2} \right)^2 \left( \det \left( \text{tr}(x_i x_j) \right) \right)_{1 \leq i, j \leq 4}^{\frac{1}{2}} = \frac{D_A}{4} \frac{f(a, b)}{n(\mathcal{O}a + \mathcal{O}b)^2}.
\]

By Proposition 3.3 (2), this proves Theorem A.3. \( \square \)

**References**


\(^{21}\)See for instance [Ber2, Vol 2, prop. 8.11.6].


