Reducible monodromies and prelagrangian tori

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1 Introduction

Definition 1. *A torus T is a 3–manifold M and an open-book (K, θ) on M are transverse to each other if the binding K does not intersect T and the projection θ induces a submersion on T.*

Remark that if (K, θ) is transverse to some torus T then the intersection of T with any page is a system of embeded circles whose isotopy classes are permuted by the monodromy (which is well defined up to isotopy).

Recall a surface S in a contact manifold (M, ξ) is prelagrangian if there is a contact form defining ξ which induces a non-singular closed form on S. The goal of this note is to explain why the following statement is trivial.

Observation. *If ξ is a contact structure supported by (K, θ) on M then any torus transverse to (K, θ) is isotopic to a ξ–prelagrangian torus.*

Note that the isotopy in the proposition does *not* preserve ξ.

Corollary 2. *If τ is a Dehn twist along T then $\tau^*\xi$ is isotopic to ξ.*

2 Open books and relative suspensions

This is a background section on the relation between open book decompositions and relative suspensions of surfaces diffeomorphisms. We explain Giroux's construction of contact structures on relative suspensions using his notion of ideal Liouville domains. In dimension 2 this notion, and symplectic geometry in general, is somehow degenerate but it stills cleans the construction, avoiding in particular any tweaking near the binding.

Definition 3. *Let P be a compact surface with non empty boundary and ω a symplectic form defined on the interior of P. The pair (P, ω) is an ideal Liouville surface if there is an auxilliary 1–form β on the interior of P such that*

- *β is a primitive of ω*
- *for any non-negative function f having ∂P as its regular zero level, the 1–form $f\beta$ extends to a positive contact form on ∂P.*

This 1–form is then called a Liouville form for (P, ω).

Remark 4. *In the above definition, the space of possible auxiliary Liouville forms β is contractible. First remark that the second condition does not depend of the choice of f. Indeed, if f_1 and f_2 are as in the definition then one can check there exists a positive function g such that $f_2 = gf_1$. It follows that, for a fixed β, $f_1 \beta$ extends to a positive contact form if and only if $f_2 \beta$ does. So we can fix an equation f for ∂P and then the condition on β is convex.*

If β is a Liouville form for (P, ω) then there is a collar neighborhood of ∂P identified by $(-\varepsilon, 0] \times \mathbb{S}^1$ with coordinates (s, z) in which $\beta = -\frac{1}{s} dz$. One can then use $f = -s$ as a regular equation of the boundary in this collar. Remark that the ω–dual to β is $X = -s\partial_s$ which is complete on $(-\varepsilon, 0] \times \mathbb{S}^1$ so $(\operatorname{Int} P, \beta)$ is a complete Liouville manifold, in particular its ω-area is infinite. The name ideal Liouville domain refers to ∂P seen as the ideal boundary (at infinity) of $\operatorname{Int} P$.

The contactization of (P, ω) is $P \times \mathbb{R}$ equiped with the contact structure $\ker(\beta + dt)$ where t is the coordinate in $\mathbb{R}$. This definition makes sense along $\partial P \times \mathbb{R}$ because the alternate contact form $f(\beta + dt)$ on the interior extends to $f\beta$ on the boundary. This contact structure is independant of the choice of β up to isotopy relative to the boundary. In the collar defined above, one can use the contact form $f\beta + f dt = dz - s dt$.

Let φ be an exact symplectomorphism of the interior of P. Exactness means there is a function h on P such that $\varphi^*\beta - \beta = dh$. Note that h is constant away from the support of φ. After adding a constant to h, we can assume that it is positive (because P is compact). The suspension of φ is:

$$\Sigma(P, \varphi) := \big(P \times \mathbb{R}\big)/\Phi \text{ where } \Phi(p, t) = (\varphi(p),\, t - h(p))$$

The contact form $\beta + dt$ is invariant under this transformation in the interior and Φ is a translation near the boundary so the contact structure defined on the contactization $P \times \mathbb{R}$ descends to a contact structure ξ_φ on $\Sigma(P, \varphi)$. One also has a submersion θ_Σ from $\Sigma(P, \varphi)$ to $\mathbb{S}^1$ induced by $(p, t) \mapsto \frac{2\pi t}{h}$.

Let D be the open disk of radius $\sqrt{\varepsilon}$, the same ε as in the collar neighborhood above which we now choose so small that h is locally constant in this collar. Also denote by $\dot{D}$ the punctured disk $D \setminus \{0\}$. The relative suspension $\bar{\Sigma}(P, \varphi)$ is obtained from $\Sigma(P, \varphi)$ by gluing $\partial\Sigma \times D$ through the diffeomorphism induced by $\Psi : \partial\Sigma \times \dot{D} \to (-\varepsilon, 0] \times \mathbb{S}^1 \times \mathbb{R}$ with $\Psi(z, r, \theta) = (-r^2, z, \frac{\theta h}{2\pi})$. Note that $\Psi^*(dt + \beta) = \frac{h}{2\pi} d\theta + r^2 d\theta$ (recall h is locally constant in the collar) so ξ_φ has a smooth extension by continuity. Also the fibration of $\Sigma(P, \varphi)$ over the circle smoothly glues to $(z, r, \theta) \mapsto \theta$ in $\partial P \times \dot{D}$. So one gets an open book with binding $K_\Sigma = \partial P \times \{0\}$ and fibration θ_Σ.

All this was the description of what is often called an abstract open book. We want to use this as a model of open books supporting contact structures.

Proposition 5 (Giroux)**.** *If ξ is a contact structure supported by some open book (K, θ) on a closed 3–manifold M then there is an ideal Liouville surface*

(P,ω), an exact symplectomorphism φ and a diffeomorphism Φ from $\bar{\Sigma}(P,\varphi)$ to M such that:

- *Φ maps the binding K_Σ to the binding K*
- *$\theta \circ \Phi = \theta_\Sigma$*
- *$\Phi^*\xi = \xi_\varphi$.*

3 Proof of the observation and its corollary

Let c be the intersection of T with some page of (K,θ). We can use Φ of the previous proposition to identify c with a submanifold in an ideal Liouville surface P. The torus T is isotopic to the image under Φ of the projection of $c \times \mathbb{R}$ in $\Sigma(P,\varphi)$. So it suffices to prove that $c \times \mathbb{R}$ is prelagrangian in the contactization of P. But this is obvious since β induces a closed form on c (c has dimension one!) so $\beta + dt$ induces a closed form on $c \times \mathbb{R}$.

We will now prove the corollary but first recall how neighborhoods of prelagrangian tori look like. We include the proof of the following lemma because we cannot find a proper reference.

Lemma 6. *If α is a closed non-singular 1–form on a torus T then $(T, \ker\alpha)$ is diffeomorphic to the suspension of a rotation in $\mathbb{S}^1$.*

Proof. We choose a base point in $\mathbb{T}^2$. Integration of α on paths between this point and other points gives a map from $\mathbb{T}^2$ to $\mathbb{R}/G$ for some subgroup G of $\mathbb{R}$.

There are two cases. If the cohomology class of α spans a rationnal line then this subgroup is discrete and $\mathbb{R}/G$ is a circle. Because α is non-singular, the kernel of this map has dimension one so it is a submersion. By Ehresmann fibration theorem, it is a locally trivial bundle map and the foliation defined by α is smoothly conjugated to a foliation by circles.

If the cohomology class does not span a rational line then one can approximate α by α' which defines a foliation by circles such that, in some coordinate system (x,y), each circle has constant y coordinate and $\ker\alpha$ is transverse to the vector field ∂_y. Up to rescaling y, α is cohomologous to $dy + \epsilon dx$ for some ε. There exists a function f such that $\alpha = dy + \epsilon dx + df$ and $1 + \partial_y f$ never vanishes (because of the transversality assumption). In this situation, each circle $x =$ *constant* is transverse to $\ker\alpha$ and the Poincar first return map is a rotation of angle $-\epsilon$. Indeed the orbits of $\ker\alpha$ are directed by the vector field $(1+\partial_y f)\partial_x - (\epsilon + \partial_x f)\partial_y$ so integrating the ∂_y component gives $-\epsilon$ (using that f is periodic). So $\mathbb{T}^2$ equiped with the foliation defined by $\ker\alpha$ is diffeomorphic to the suspension of a rotation. □

Lemma 7. *Any prelagrangian torus has a tubular neighborhood diffeomorphic to $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$ with coordinates (x,y,z) in which the contact structure is $\ker\big(\cos(\theta_0 + z)dx - \sin(\theta_0 + z)dy\big)$ for some angle θ_0.*

Proof. Using the previous lemma, we choose coordinates (x, y) on our torus T such that there is a contact form α inducing $\cos(\theta_0)dx - \sin(\theta_0)dy$ on T. Let V be a vector field near T transverse to T and tangent to ξ. The flow of V starting from T is a diffeomorphism on $(-\varepsilon, \varepsilon) \times T$ for some positive ε. We denote by t the time parameter in $(-\varepsilon, \varepsilon)$. The contact structure pulls back to $(-\varepsilon, \varepsilon) \times T$ as $\ker\big(\cos(\theta(x, y, t))dx - \sin(\theta(x, y, t))dy\big)$ for some function θ with $\theta(x, y, 0) = \theta_0$. The contact condition is equivalent to $\partial_t\theta \neq 0$ so that the implicit function theorem allows to use $z = \theta - \theta_0$ as a coordinate instead of t (shrinking ε if needed). □

We can now prove the corollary from the introduction. First notice that the conclusion depends only on the isotopy class of τ. This is indeed what allows this pretty vague statement. Observe now that, in the conclusion of the preceding lemma, the coordinates (x, y) can be changed using any linear transformation, at the only price of changing the angle θ_0. So we can assume that τ is supported in this tubular neighborhood and given by: $\tau(x, y, z) = \tau(x, y+\rho(z), z)$ for some function ρ. Then

$$\tau^*\xi = \ker\Big(\cos(\theta_0 + z)dx - \sin(\theta_0 + z)\big(dy + \rho'(z)dz\big)\Big)$$

It only remains to remark that $\alpha_f := \cos(\theta_0 + z)dx - \sin(\theta_0 + z)dy + f(z)dz$ is a contact form whatever the function f. So we can interpolate linearly and apply Gray's theorem.