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Centre de mathématiques Laurent Schwartz (UMR 7640 CNRS)

Mémoire présenté pour l'obtention du

Diplôme d'habilitation à diriger les recherches

Discipline : Mathématiques

par

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Quelques applications de la convexité en topologie de contact

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Date de soutenance : 12 décembre 2016

Composition du jury :

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Contents

Prologue

En 1991, Y. Eliashberg et M. Gromov dégagent la notion de convexité en géométrie symplectique en tant qu'objet d'étude à part entière. La dernière section de leur article concerne une proposition de définition des structures de contact convexes. Voici la toute dernière phrase de cet article :

> « We think that the understanding of contact convexity is essential for the symplectic geometry.» (Eliashberg et Gromov 1991)

Puis, au moins provisoirement, ils passent le relai à un étudiant en thèse qui rassemble ses premières réflexions sur le sujet dans un article dont voici la toute première phrase :

> « Cet article aborde l'étude de la convexité en géométrie de contact, telle qu'elle a été définie dans [EG] : une structure, symplectique ou de contact, est dite convexe si elle est conformément invariante par le gradient d'une fonction de Morse propre.» (Giroux 1991)

Le but de ce mémoire est de décrire mes travaux mathématiques dans le contexte des vingt-cinq années de convexité en topologie de contact qui ont suivi ces deux citations.

Structure du mémoire

Ce mémoire se décompose en quatre ensembles de styles assez différents. Le premier chapitre est une introduction très rapide à la topologie de contact en général et à mes travaux en particulier. Son but est d'amener un lecteur qui ne sait pas ce qu'est une variété de contact (mais qui sait ce qu'est une variété) à avoir une idée du thème de mes travaux, avec un énoncé par article depuis ma thèse.

La partie I est une présentation (biaisée) de l'histoire de la convexité en topologie de contact et ses interactions avec le reste la topologie différentielle et symplectique.

La partie II est une description plus précise de mes travaux avec des énoncés techniques, des plans de démonstrations et quelques indications de développements possibles. Les chapitres qui la constituent peuvent être lus (presque) indépendamment les uns des autres.

La partie III décrit mes travaux en cours, plus ou moins avancés selon les chapitres. Cette partie ne contient pas de théorème mais des questions qui me fascinent.

Remerciements

Les travaux décrits dans ce mémoire existent avant tout grâce à l'organisation du service public d'enseignement supérieur et de recherche en mathématiques en France. Après des études financées par l'État en classe préparatoires et à l'École Normale Supérieure de Lyon, j'ai, quelques mois avant ma soutenance de thèse, obtenu un poste d'agrégé préparateur à l'ÉNS Lyon pour trois ans. Ce poste m'a permis de préparer mon manuscrit et ma soutenance puis de continuer mes travaux sans aucun souci pour l'avenir à moyen terme. Environ six mois après ma soutenance de thèse, j'ai obtenu un contrat de sérénité à durée indéterminée en tant que maître de conférence à l'Université Paris Sud à Orsay. J'y ai passé quatre ans avant de me voir offrir des conditions de travail encore meilleures au sein de l'École polytechnique à Palaiseau. Les trois institutions citées, et plus particulièrement leurs laboratoires *Unité de Mathématiques Pures et Appliquées*, *Laboratoire de Mathématiques d'Orsay* et *Centre de Mathématiques Laurent Schwartz* m'ont offert une ambiance chaleureuse et un support en secrétariat, gestion et informatique bon ou excellent. En ces jours de diminution du nombre de postes permanents ouverts au concours, je crois qu'il est important de rappeler cette histoire.

D'un point de vue mathématique, ce mémoire expose amplement la façon dont mes travaux reposent sur les contributions fondamentales de Mikhaïl Gromov, Yakov Eliashberg et Emmanuel Giroux. J'ai eu la chance d'avoir Emmanuel Giroux comme directeur de thèse et je continue à l'en remercier. J'ai aujourd'hui l'honneur d'avoir Yakov Eliashberg comme rapporteur de ce mémoire et je le remercie d'avoir accepté d'y consacrer du temps. D'une façon moins visible mais néanmoins cruciale, les autres membres du jury sont eux aussi des témoins et des acteurs de ma formation et de mon travail. J'ai rencontré Jean-Yves Welschinger à l'ÉNS Lyon où j'ai suivi un de ses cours sur les courbes holomorphes ainsi que de nombreux exposés en groupe de travail et séminaires. J'ai rencontré Alexandru Oancea brièvement à l'ÉNS Lyon et lors du premier *week-end Goutelas* puis surtout lors des ateliers *Symplectic Field Theory* à Leipzig où j'ai aussi fait la connaissance de Frédéric Bourgeois. Tous deux m'ont apporté à la fois de lumineuses explications lors de leurs exposés et des encouragements par l'intérêt porté à mes travaux. Cet intérêt culmine aujourd'hui avec la rédaction de rapports sur ce mémoire pour lesquels je les remercie vivement. J'ai rencontré Claude Viterbo grâce au projet ANR Symplexe, porté par Étienne Ghys, et au séminaire Symplect'X. Dans ces deux cadres il jouait, et joue toujours pour le séminaire, un rôle important d'animation de la communauté et d'encouragement des jeunes. Hansjörg Geiges est le premier mathématicien (ex æquo avec Paolo Lisca) à m'avoir invité à exposer mon travail hors de France à l'automne 2007. Il a continué à se tenir au courant de mes progrès depuis et je le remercie d'apporter à ce jury un regard extérieur au système universitaire français. Pierre Pansu a été le premier Orséen à se pencher sur mon travail avant mon recrutement. Il m'a ensuite beaucoup appris comme directeur du département de mathématiques lorsque j'étais dans la commission des services puis beaucoup écouté et encouragé lors de mes tentatives de lier topologie de contact et géométrie de Carnot-Carathéodory. C'est pourquoi je suis heureux qu'il ait accepté d'apporter à ce jury un regard extérieur à la topologie symplectique et de contact.

Au-delà de mon directeur de thèse et des membres du jury, il est impossible de ne pas mentionner l'influence exercée par la vision et l'enthousiasme d'Étienne Ghys, Jean-Claude Sikorav et Vincent Colin. D'une façon manifeste, mon travail doit aussi beaucoup à mes collaborateurs John Etnyre, Rafał Komendarczyk, Klaus Niederkrüger et Chris Wendl (en plus d'Emmanuel Giroux déjà cité). Il faut ajouter à cette liste Paolo Ghiggini et Sylvain Courte que seules des fluctuations quantiques inexpliquées empêchent, pour l'instant, d'être officiellement mes collaborateurs. Pour finir cette énumération tautologique, je signale qu'Élisabeth, Cyrille, Julie et Marie me sont indispensables pour toute activité, mathématique ou non.

1. Introduction

1.1. Exemples et définition des structures de contact

Les structures de contact sont des champs d'hyperplans qui apparaissent naturellement au bord de variétés holomorphes ou symplectiques sous certaines hypothèses de «convexité». Sur le bord V d'un domaine convexe (au sens le plus élémentaire du terme) dans $\mathbb{C}^n$, le champ d'hyperplans considéré est la donnée, pour tout v dans V, de l'unique hyperplan complexe ξ inclus dans $T_v\partial V$. D'un point de vue symplectique, on peut voir ξ comme le noyau de la restriction à V de $\iota_X\omega$ où ω est la forme symplectique standard sur $\mathbb{C}^n$ et X est un champ de vecteurs radial transverse à V. Les notions de convexité holomorphe et symplectique sont des généralisations de cet exemple ne faisant intervenir que la structure holomorphe (resp. symplectique). Par exemple, on trouve des structures de contact à l'infini des variétés des Stein, les variétés complexes qui admettent un plongement holomorphe propre dans $\mathbb{C}^n$. On en trouve aussi sur le fibré cotangent unitaire d'une variété quelconque, en lien avec la structure symplectique de Liouville.

À dimension fixée, tous les champs d'hyperplans mentionnés ci-dessus sont localement isomorphes et on dispose d'une caractérisation simple de ce modèle local, indépendamment de l'existence d'une variété dont le bord porte le champ d'hyperplans. On peut donc définir les structures de contact «abstraitement» comme les champs d'hyperplans admettant ce modèle au voisinage de tout point. La figure 1.1 représente un modèle local en dimension 3. Le modèle local est caractérisé par le théorème de Darboux : un

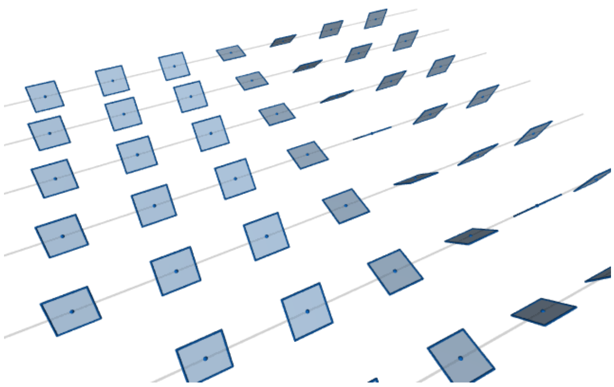


FIG. 1.1. : Modèle de structure de contact en dimension 3. Ce champ de plans est invariant par translation dans la direction qui n'est pas dessinée.

champ d'hyperplans coorientable sur une variété de dimension $2n+1$ est une structure de contact s'il admet une équation α qui est une 1-forme pour laquelle $\alpha \wedge d\alpha^n$ est une forme volume. Dans ce cas toutes ses équations vérifient cette condition. La topologie de contact est l'étude des propriétés globales de ces structures, considérées modulo isomorphisme. Par isomorphisme entre ξ_1 et ξ_2, on entend un difféomorphisme de la variété ambiante qui pousse ξ_1 sur ξ_2. S'il existe un tel difféomorphisme qui est de plus homotope à l'identité alors on dit que ξ_1 et ξ_2 sont isotopes.

Avant de discuter de propriétés possibles pour une classe d'isomorphisme ou d'isotopie de structures de contact, on note que chaque structure de contact est accompagnée d'une classe privilégiée de champs de vecteurs, appelés ses champs de Reeb. Dans le cas du fibré unitaire tangent, les flots géodésiques en sont des exemples. Plus généralement, on les trouve comme restriction d'un flot hamiltonien à tout niveau d'énergie vérifiant l'hypothèse de convexité adéquate. Dans le contexte de l'analyse complexe, si φ est une fonction strictement pluri-sous-harmonique alors les niveaux réguliers de φ sont des variétés de contact pour lesquelles $i\nabla\varphi$ est parallèle à un champ de Reeb.

1.2. Les questions primaires

La présentation ci-dessus fait émerger trois grandes questions (ou classes de questions) que l'on peut appeler primaires car elles découlent directement des définitions. Elles sont bien sûr toutes de nature globales puisque deux structures de contact en dimension fixée sont toujours localement isomorphes.

La question des remplissages Existe-t-il une variété symplectique ou complexe ayant pour bord naturel une variété de contact donnée ? Si oui, en existe-t-il beaucoup qui soient essentiellement différentes ? Une version relative de cette question s'intéresse aux cobordismes entre variétés de contact. Pour les variétés de contact de dimension 3 il existe aussi une notion de remplissage symplectique faible. Les remplissages symplectiques venant de la géométrie complexe sont toujours forts.

La question de la classification Une variété de dimension impaire admet-elle toujours une structure de contact ? Si elle en admet, peut-on les classifier à isomorphisme près ? Cette question comporte un versant purement homotopique sous la forme d'obstructions homotopiques à l'existence et, dans les cas d'existence, d'un «foncteur d'oubli» à valeur dans les classes d'homotopie de champs d'hyperplans munis de structures complexes. La question peut ainsi être formulée plus précisément comme l'étude de ce qui est oublié par ce foncteur, ce qui n'est pas vu par la théorie homotopique. Un autre versant de la question concerne le groupe des automorphismes d'une structure de contact. Il s'agit d'un «groupe de Lie de dimension infinie» dont on peut chercher à comprendre la structure algébrique, la topologie et la géométrie.

La question dynamique Quelles sont les propriétés dynamiques communes à tous les champs de Reeb ? Est-ce que les champs d'une structure de contact donnée ont des propriétés communes supplémentaires ? Quelles informations la donnée d'un champ de Reeb permet d'obtenir sur une structure de contact ?

A priori, les trois questions ci-dessus sont complètement indépendantes. Cependant, les réponses sont étonnamment imbriquées et certaines classes de structures de contact ont des propriétés spéciales vis-à-vis des trois questions.

Le premier exemple de telle classe est apparu en dimension 3 dès la fin des années 80. On dit qu'une structure de contact en dimension 3 est vrillée si elle contient un disque vrillé, c'est à dire un disque plongé de sorte que son espace tangent coïncide avec la structure de contact le long du bord. Ces structures ne sont jamais remplissables, même faiblement. Elles forment une classe de structures de contact flexibles : il n'y a aucune obstruction à leur existence et la restriction du foncteur d'oubli à cette classe n'oublie rien, c'est-à-dire que deux structures de contact vrillées qui sont homotopes parmi les champs de plans sont homotopes parmi les structures de contact. Enfin tout champ de Reeb d'une structure de contact vrillée admet une orbite périodique contractile. De plus, la démonstration de cette dernière propriété passe par la symplectisation de la variété de contact, qu'on peut voir comme un cobordisme symplectique de la variété vers elle-même, et les courbes (pseudo)-holomorphes, et donc la géométrie (presque)-complexe, y jouent un rôle crucial. La symplectisation de (M, ξ) est l'ensemble des covecteurs λ dans T^*M ayant pour noyau ξ (avec une coorientation prescrite). C'est une sous-variété symplectique de T^*M.

Les structures de contact vrillées forment donc une classe très bien comprise et remarquablement homogène. Par contraste, la classe complémentaire, celle des structures dites tendues, est beaucoup plus mystérieuse et hétérogène. On sait en particulier que les structures de contact tendues sont rares. Elles sont parfois remplissables et parfois non. En général elles ne sont pas flexibles et elle peuvent admettre des champs de Reeb sans orbite périodique contractile.

Exemple emblématique. *Sur le tore $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, on peut définir, pour chaque entier strictement positif n, la structure de contact*

$$\xi_n = \ker(\cos(2n\pi z)dx - \sin(2n\pi z)dy).$$

Ces structures sont toutes tendues et :

- *elles ne sont pas flexibles car deux à deux non isomorphes bien qu'homotopes parmi les champs de plans.*
- *elles ne sont pas fortement symplectiquement remplissables à l'exception de ξ_1*
- *elles admettent toutes des champs de Reeb sans orbite périodique contractile.*

Certaines propriétés apparaissant dans l'exemple précédent ont peu à peu reçu comme explication l'existence de «domaines de torsion de Giroux»: des plongements de tores épais $\mathbb{T}^2 \times [0, 2n\pi]$ munis de la structure de contact $\ker(\cos(z)dx - \sin(z)dy)$. Ceux-ci permettent de définir la torsion de Giroux d'une variété de contact de dimension 3 comme le supremum des entiers n pour lesquels un tel plongement existe ou bien zéro s'il n'en existe aucun. Les structures vrillées sont toujours de torsion infinie, toute torsion non-nulle est une obstruction à la remplissabilité forte et, à torsion finie fixée, il n'y a sur chaque variété qu'un nombre fini de classes d'isomorphismes de structures de contact.

Pour finir ce bref tour d'horizon, signalons que, comme indiqué dans la question dynamique, on peut s'interroger sur les propriétés communes à tous les champs de Reeb d'une structure de contact donnée. En dimension quelconque, on dispose d'une théorie qui décrit une manière de «compter» les orbites périodiques d'un champ de Reeb d'une façon qui ne dépende que de la structure de contact. Plus précisément, pour toute structure de contact ξ, on peut construire une algèbre, l'homologie de contact de ξ, définie à partir des orbites d'un champ de Reeb mais qui ne dépend que de ξ. Il s'agit d'une situation analogue à celle de l'homologie simpliciale d'une triangulation qui ne dépend que de la variété sous-jacente. Cette théorie fait partie de la grande famille des homologies de Floer.

1.3. La correspondance de Giroux

Assez indépendamment de l'étude de classes remarquables de structures de contact, la topologie de contact a progressé de façon spectaculaire lorsque Emmanuel Giroux a décrit une correspondance inattendue entre les structures de contact et les décompositions en livres ouverts symplectiques.

Une décomposition en livre ouvert d'une variété V est un couple (K, θ) où K est une sous-variété de codimension 2 et θ est une fibration du complémentaire de K sur $\mathbb{S}^1$ qui se comporte joliment près de K. La sous-variété K est appelée reliure du livre ouvert et les fibres de θ sont appelées pages. Ces structures sont toujours isomorphes au résultat de la construction suivante. Partant d'une variété Σ à bord non vide (qui sera la page) et d'un difféomorphisme φ relatif au bord de Σ, on peut épaissir Σ en $\Sigma \times [0, 1]$, recoller $\Sigma \times \{0\}$ et $\Sigma \times \{1\}$ en utilisant φ puis coller $\partial\Sigma \times D^2$ le long de $\partial\Sigma \times \mathbb{S}^1$. On obtient alors un livre ouvert avec $K = \partial\Sigma \times \{0\}$ et θ provenant de la projection de $\Sigma \times [0, 1]$ sur le second facteur.

Giroux montre que, partant d'une page Σ munie d'une forme symplectique exacte à bord convexe et d'un difféomorphisme symplectique φ relatif au bord, one obtient une variété qui porte une structure de contact étroitement liée au livre ouvert : la reliure est une sous-variété de contact et il existe un champ de Reeb transverse aux pages. Bien plus, il démontre que toutes les classes d'isotopies de structures de contact s'obtiennent ainsi et qu'on peut imposer à la page d'être un domaine de Stein. En dimension 3, il parvient même à décrire comment sont reliés les couples (Σ, φ) donnant lieu à la même classe.

Cette correspondance introduit un point de vue très différent sur les structures de contact. Elle permet de traduire toutes les questions portant sur les structures de contact en questions portant sur les difféomorphismes symplectiques (tout cela à isotopie près). Ceci est particulièrement efficace en dimension 3, puisque les variétés symplectiques concernées sont alors de dimension 2 donc beaucoup plus accessibles.

Cette théorie de Giroux des livres ouverts est le sommet d'un programme débuté en 1991 : l'étude de la convexité en topologie de contact. On dit qu'une structure de contact est convexe si elle est invariante par un flot de gradient d'une fonction de Morse. L'existence d'un livre ouvert portant la structure de contact est équivalente à sa convexité. En plus de cet aspect global, de nombreux lemmes semi-locaux ont eu une influence décisive sur la topologie de contact en dimension 3. Ces lemmes portent sur la géométrie de contact d'un niveau régulier d'une fonction ayant un gradient de contact, ou sur le passage des valeurs critiques. J'ai beaucoup travaillé à les exposer, dans ma thèse d'abord mais ensuite surtout lors d'écoles thématiques, à La Llagonne en janvier 2008, à Nantes en juin 2011 et à Bonn en juillet 2013 (mon cours à l'école d'été de Londres en juillet 2014 concerne un autre sujet). Des notes abondamment illustrées de mon cours à Nantes sont publiées dans Massot 2014. La partie I du présent mémoire raconte l'histoire de cette théorie.

1.4. Structures de contact en grande dimension

La topologie de contact en grandes dimensions est beaucoup moins bien comprise qu'en dimension 3 mais progresse rapidement ces dernières années. En particulier il existe maintenant une définition des structures de contact vrillées en dimension quelconque et une démonstration de leur flexibilité.

L'article Massot, Niederkrüger et Wendl 2013 vise à démontrer l'existence en grandes dimensions d'un certain nombre de phénomènes connus en dimension 3 et décrit dans la section 1.2. Le premier outil introduit est une notion de remplissage symplectique faible en grandes dimensions. Outre l'intérêt intrinsèque d'avoir une notion plus faible mais toujours non triviale, cette étude permet de démontrer indirectement que certaines variétés de contact ne sont pas fortement (ou a fortiori holomorphiquement) remplissables. La stratégie consiste à montrer que tout remplissage fort pourrait être étendu par un cobordisme symplectique en remplissage faible d'une nouvelle variété de contact qui est vrillée.

On ouvre ainsi la possibilité de montrer que certaines structures de contact ne sont pas fortement remplissables bien que n'ayant pas d'autre caractéristique des structures vrillées puisqu'elles ne sont pas flexibles et ont des champs de Reeb sans orbite périodique contractile. En cela la référence évidente en dimension 3 est l'exemple emblématique décrit dans la section 1.2. Pour construire des exemples analogues en grandes dimensions, nous faisons un détour par les groupes de Lie et la théorie des nombres. Avant l'énoncé, rappelons qu'un sous-corps de $\mathbb{C}$ contient nécessairement $\mathbb{Q}$ et que son degré est, par définition, sa dimension en tant que $\mathbb{Q}$-espace vectoriel. Un corps de nombres est un sous-corps de $\mathbb{C}$ qui est de degré fini. Le corps $\mathbb{R}$ contient des corps de nombres de degré arbitrairement grand.

Theorem 1.1. *À tout corps de nombres $\Bbbk$ de degré d, on peut associer canoniquement une variété de contact $M_{\Bbbk}$ de dimension $2d-1$. Si $\Bbbk$ peut se plonger dans $\mathbb{R}$ alors $M_{\Bbbk}$*

porte deux 1–formes α_+ et α_- telles que la formule :

$$\xi_n := \ker\left(\frac{1+\cos ns}{2}\alpha_+ + \frac{1-\cos ns}{2}\alpha_- + \sin ns\, dt\right)$$

(avec $(s,t) \in \mathbb{T}^2$) définisse, pour $n \in \mathbb{N}^*$, une famille de structures de contact sur $\mathbb{T}^2 \times M_\Bbbk$ ayant les propriétés suivantes :

1. *elles ne sont pas flexibles car elles sont homotopes mais pas isomorphes ;*
2. *seule ξ_1 est fortement remplissable ;*
3. *elles admettent toutes un champ de Reeb sans orbite périodique contractile.*

On retrouve l'exemple emblématique de la dimension 3 pour $\Bbbk = \mathbb{Q}$. On a ainsi obtenu l'existence de domaines généralisant les domaines de torsion de Giroux en grandes dimensions. Notons que, dans le premier point du théorème, on distingue les classes d'isomorphisme par un calcul d'homologie de contact, donc via un invariant dynamique.

Au fil de la construction de ces exemples, on répond à la question de l'existence de variétés symplectiques exactes dont le bord n'est pas connexe mais dont chaque composante est convexe (au sens symplectique du terme). De tels exemples ne peuvent pas exister dans la catégorie des variétés de Stein et, dans le cadre plus large des variétés symplectiques, les seuls exemples compacts connus étaient en dimension 4 et 6. Dans notre travail, ces variétés symplectiques exotiques apparaissent plongées dans les variétés de contact étudiées. On peut les qualifier d'internes pour les distinguer des variétés «externes» apparaissant comme remplissages ou cobordismes. Traditionnellement les résultats utilisant les variétés symplectiques internes et externes étaient essentiellement disjoints, au moins si l'on exclue le cas de la dimension 3 pour laquelle les variétés internes sont des surfaces donc des variétés symplectiques très dégénérées. Ici on a une interaction directe entre la présence d'exemples internes exotiques et la non-existence de variété symplectique externe.

1.5. Invariants d'Ozsváth-Szabó

Une des application majeures de la théorie des livres ouverts en dimension 3 est la construction de l'invariant d'Ozsváth–Szabó. Il s'agit d'un invariant de classe d'isotopie de structures de contact à valeurs dans l'homologie d'Heegaard–Floer de la variété. Cette homologie associe à toute variété de dimension 3 des groupes abéliens par une construction qui mélange des aspects analytiques, topologiques et combinatoires. Dès le début, il a été démontré que l'invariant d'Ozsváth–Szabó des structures de contact vrillées est trivial, ce qui n'est pas surprenant au vu de leur flexibilité : il n'y a rien de subtil à voir donc les invariants subtils ne voient rien. Plus généralement, cet invariant s'annule pour les structures de contact dont la torsion de Giroux est strictement positive. Au contraire, il n'est jamais nul pour une structure de contact symplectiquement remplissable, on retrouve ainsi la non-existence de remplissages pour les structures vrillées ou, plus généralement, de torsion strictement positive. Par contraposition, la propriété d'annulation pour les structures vrillées a permis de montrer que de nombreuses structures de contact étaient tendues alors qu'aucun autre outil ne semblait en mesure de le faire. De même l'invariant d'Ozsváth–Szabó est le seul outil connu pour distinguer certaines classes d'isotopies de structures de contact.

Malgré les succès de l'invariant d'Ozsváth–Szabó, son contenu topologique et ses relations avec d'autres informations restent assez mystérieux. Comme indiqué plus haut, on savait que les structures de contact dont la torsion de Giroux est strictement positive ont un invariant d'Ozsváth–Szabó trivial et la réciproque semblait possible. Dans mon article Massot 2012, j'ai démontré que la réalité est plus subtile.

Theorem 1.2. *Il existe une infinité de variétés de dimension 3 portant chacune une infinité de classes d'isotopie de structures de contact tendues dont l'invariant d'Ozsváth–Szabó est trivial bien que leur torsion de Giroux soit nulle.*

Il existe en fait plusieurs variantes de l'homologie d'Heegaard–Floer, en particulier liées au choix d'un anneau de coefficients pour les complexes de chaînes. Le choix de l'anneau $\mathbb{Z}/2$ est techniquement le plus léger et il était suffisant pour toutes les applications connues en topologie de contact. De plus la version à coefficients dans $\mathbb{Z}$ souffre d'une ambigüité de signe dont on ne savait pas si elle était un artefact de difficultés techniques ou quelque chose de plus essentiel.

Sur le tore $\mathbb{T}^3$, il existe un invariant de classes d'isotopies de structures de contact introduit par Giroux. J'ai montré dans ibid. comment il est lié à l'invariant d'Ozsváth–Szabó. En plus d'éclairer le contenu topologique de dernier, cela a donné un premier exemple de variété sur laquelle sa version à coefficient dans $\mathbb{Z}$ permet de distinguer une infinité de classes d'isotopies de structures de contact, au contraire de sa version à coefficients dans $\mathbb{Z}/2$ qui vit dans un $\mathbb{Z}/2$-espace vectoriel de dimension finie, donc un ensemble fini. Enfin l'ambigüité de signe est très naturelle dans cet exemple puisque l'invariant de Giroux est la classe d'homologie dans $H_2(\mathbb{T}^3)$ de certains tores n'ayant pas d'orientation naturelle.

Le thème majeur de ibid. est donc les interactions entre différents types d'invariants de classes d'isotopies de structures de contact. Bien que s'appuyant sur la classification de certaines structures de contact tendues, les démonstrations de cet article sont essentiellement combinatoires et algébriques. Certains des résultats obtenus ont depuis été redémontrés par d'autres, de façon encore plus algébrique ou bien via de la topologie algébrique plus classique. Cette diversité d'approches montre bien la richesse des objets étudiés.

1.6. Groupes d'automorphismes en géométrie de contact

Comme en présence de tout type de structure mathématiques, il est naturel de s'intéresser au groupe des symétries d'une structure de contact. Du fait de l'absence de rigidité locale, ces groupes sont de dimension infinie et agissent transitivement sur les n-uplets de points pour tout n. En ce sens ils sont bien plus proche du groupe de tous les difféomorphismes que d'un groupe de Lie. Ainsi la comparaison la plus tentante se fait avec la théorie des groupes de difféotopies (mapping class groups). Ainsi on notera $\pi_0\mathcal{D}(\xi)$ le groupe des classes d'isotopies de difféomorphismes préservant une structure de contact ξ. À l'intérieur de ce groupe, on s'intéresse plus particulièrement au noyau de $\pi_0\mathcal{D}(\xi) \to \pi_0\mathcal{D}$, le sous groupe des classes qui sont topologiquement triviales. C'est ce dernier sous-groupe qui est le plus intéressant, il mesure les obstructions à relier deux difféomorphismes de contact par un chemin de tels difféomorphismes lorsqu'un chemin existe parmi les difféomorphismes quelconques.

Dans l'article Giroux et Massot 2015, nous explorons les liens généraux entre les groupes de difféotopies de contact et la classification des structures de contact et nous calculons ce groupe pour quelques exemples de variétés de contact.

Les résultats généraux découlent de l'existence de fibrations localement triviales liant des espaces de difféomorphismes, de structures de contact et de plongements de surfaces. La fibration la plus facile à décrire est la suivante. On fixe une structure de contact ξ sur une variété V compacte sans bord. On note Ξ l'espace des structures de contact sur V. L'application $\pi : \mathcal{D}(V) \to \Xi$ qui envoie φ sur $\varphi_*\xi$ est une fibration localement triviale. La suite exacte en homotopie provenant de cette fibration montre en particulier l'isomorphisme

$$\ker(\pi_0\mathcal{D}(\xi) \to \pi_0\mathcal{D}) \simeq \pi_1(\Xi, \xi)/\pi_1(\mathcal{D}(V), \mathrm{Id}).$$

Nos résultats spécifiques concernent les structures de contact qui sont tangentes aux fibres d'un fibré en cercles au-dessus d'une surface. Ils reposent sur les considérations générales mentionnées ci-dessus et sur les techniques topologiques introduites par Giroux.

Theorem 1.3. *Soit V une variété fibrée en cercles au-dessus d'une surface orientable compacte sans bord et de genre strictement positif. Si ξ est une structure de contact tangente aux fibres de V et qui fait d tours le long de chaque fibres par rapport aux fibres voisines alors $\ker(\pi_0\mathcal{D}(\xi) \to \pi_0\mathcal{D})$ est isomorphe à $\mathbb{Z}/d\mathbb{Z}$. De plus le groupe fondamental de l'espace des structures de contact isotopes à ξ est infini cyclique.*

Dans l'article Massot et Niederkrüger 2016, nous étendons une partie de cette étude en dimension plus grande, en utilisant cette fois des techniques de courbes holomorphes et en s'appuyant sur les constructions de Massot, Niederkrüger et Wendl 2013 évoquées dans la Section 1.4.

Theorem 1.4. *Soit $\Bbbk$ un corps de nombres admettant un plongement réel et soient $(\mathbb{T}^2 \times M_\Bbbk, \xi_n)$ les variétés de contact du Théorème 1.1. Pour tout $n > 1$ et tout $1 \leq m < n$, la transformation de contact*

$$\Psi_{n,m} \colon (\mathbb{T}^2 \times M_\Bbbk, \xi_n) \to (\mathbb{T}^2 \times M_\Bbbk, \xi_n),\ (s, t, \theta) \mapsto \left(s + \tfrac{2\pi m}{n}, t, \theta\right)$$

est topologiquement isotope à l'identité mais pas parmi les transformations de contact.

1.7. Interactions avec la géométrie riemannienne

En dimension trois, l'hétérogénéité de la classe des structures de contact tendues pousse à chercher des sous-classes privilégiées. J'ai exploré plusieurs classes définies en terme de géométrie riemannienne. Le but est de trouver des conditions géométriques assurant des propriétés topologiques, en particulier l'absence de disque vrillé. Il s'agit d'un champ de recherches très ouvert.

La première piste suivie était suggérée depuis longtemps, en particulier par Étienne Ghys. Il s'agit de l'étude des structures de contact dites géodésibles, c'est à dire pour lesquelles il existe une métrique telle que toute géodésique qui part tangente à la structure de contact le reste à jamais. Il s'agit d'une relation locale entre la métrique et la structure de contact qu'on peut exprimer comme l'annulation d'une «seconde forme fondamentale», comme dans le cas des sous-variétés. Durant ma thèse j'ai étudié cette classe. Des travaux antérieurs rendaient cette étude essentiellement équivalente à celle des structures de contact transversales aux fibres d'une variété de Seifert. On peut en déduire que les structures de contact concernées sont toutes tendues et symplectiquement remplissables. En particulier la condition riemannienne locale qui est imposée entraîne, par une voie assez indirecte, des propriétés globales de la structure de contact et des interactions avec la géométrie symplectique, cf. Massot 2008a.

J'ai ensuite exploré d'autres interactions avec la géométrie riemannienne en essayant de mélanger plus intimement les techniques provenant des différentes branches. Le contexte de cet étude est celle des métriques riemanniennes «compatibles» avec une structure de contact ξ, ce qui signifie qu'il existe une structure complexe J sur ξ, une équation α de ξ et une constante θ' telles que $g = 1/\theta' d\alpha(\pi\cdot, J\pi\cdot) + \alpha^2$ où π est la projection sur ξ parallèlement à l'unique champ de Reeb α sur lequel α vaut un. Le résultat principal de Etnyre, Komendarczyk et Massot 2012 est un analogue du théorème de la sphère pour la géométrie de contact en dimension 3.

Theorem 1.5. *Soit (M, ξ) une variété de contact de dimension 3 et g une métrique riemannienne complète sur M qui est compatible avec ξ. Si la courbure sectionnelle de g est strictement positive et $4/9$–pincée (son minimum est au moins $4/9$ fois son maximum) alors le revêtement universel de (M, ξ) est isomorphe à $\mathbb{S}^3$ munie de sa structure de contact standard.*

L'intérêt principal de ce travail, à mon avis, réside moins dans l'énoncé que dans la façon dont différents outils s'agencent dans la démonstration. Quitte à passer au revêtement universel, on peut supposer que M est simplement connexe. Le théorème de la sphère classique montre alors que M est difféomorphe à $\mathbb{S}^3$ (sans passer par Perelman). Les techniques topologiques d'Eliashberg garantissent que toutes les structures de contact tendues sur $\mathbb{S}^3$ sont isomorphes, il suffit donc de démontrer l'absence de disque vrillé. La géométrie riemannienne développée pour montrer le théorème de la sphère classique fournit un recouvrement par deux boules riemanniennes plongées dont l'une est convexe. Puis on montre, en utilisant des calculs de géométrie différentielle et des techniques analytiques de courbes pseudoholomorphes, que la boule convexe est tendue et que tout disque vrillé pourrait être poussé dans l'autre boule. On utilise alors un peu de géométrie et des techniques topologiques de Giroux pour montrer que la présence d'un disque vrillé dans une boule riemannienne force l'existence de tels disques arbitrairement près du bord, en contradiction avec l'étude de la boule convexe.

Les méthodes topologiques intervenant dans la démonstration que l'on vient d'esquisser n'ont pas d'analogue en dimension supérieure. Cependant la partie consistant à contrôler la structure de contact dans des boules de taille contrôlée a un sens en toute dimension. Dans Etnyre, Komendarczyk et Massot 2016, nous étudions cette question par deux techniques très différentes. La première affine l'étude du lien entre convexités riemannienne et complexe entamée dans Etnyre, Komendarczyk et Massot 2012 et utilise les courbes pseudoholomorphes pour garantir l'absence d'analogues du disque vrillé en grande dimension. Cependant la technologie actuelle ne permet pas d'affirmer que les boules ainsi contrôlées se plongent dans le modèle local de structure de contact sur $\mathbb{R}^{2n+1}$. Pour obtenir de telle garanties (pour des boules plus petites) on utilise des méthodes purement géométriques basées sur les champs de Jacobi. La version la plus lisible du résultat (mais pas la plus fine) est l'énoncé suivant, dans lequel intervient aussi le tenseur de Nijenhuis $[J, J]$, une obstruction à provenir d'un plongement dans une variété holomorphe.

Theorem 1.6. *Soit (M, ξ) une variété de contact de dimension $2n + 1$ munie d'une métrique riemannienne complète compatible avec ξ: $g = 1/\theta' d\alpha(\pi\cdot, J\pi\cdot) + \alpha^2$. Si la courbure sectionnelle de g est comprise entre $-K$ et K pour une constance strictement positive K alors toute boule riemannienne se plonge dans $\mathbb{R}^{2n+1}$ muni de sa structure de contact standard tant que son rayon est inférieur à*

$$r = \min\left(\frac{inj(g)}{2}, \frac{1}{208n^2 \max(\sqrt{K}, \|[J, J]\|, \theta')}\right).$$

Dans ce résultat le fait important n'est pas tant la valeur précise de la borne que le fait qu'elle est explicite et ne dépend que des longueurs obtenues à partir de K, θ' et $[J, J]$ et de la dimension.

Part I.

A history of convexity in contact topology

Contents

Caveat

This part is a biased survey of convexity in contact topology, and its interactions with other aspects of the theory. This order is roughly chronological but not completely, partly because a strict chronological order would cut the story in too many pieces, and partly because it was sometimes convenient to explain things from a modern perspective. The most striking example is probably the appearance of ideal Liouville domains in Sections 2.7 and 3.1.2, fifteen years early. We think this notion is really helpful and not yet used enough.

An important bias comes from the fact that this part is meant to motivate my own work, described in the following part. Hence I emphasize the most relevant aspects from this perspective, especially the history of Giroux torsion and symplectic cobordism constructions as well as the study of thickened spheres and tori. For instance the whole subject of Legendrian and transverse knots benefited strongly from contact convexity, but does not feature prominently in my work, so it only makes a brief appearance here. Also, the aspects that interact strongly with convexity but are not strictly part of the story are treated more briefly, especially holomorphic curves theory and gauge theory. Of course the level of details is also influenced by my level of understanding (for instance I could not give many more details about gauge theory) and what I feel is the general level of understanding (for instance it seems many people do not have a clear vision of where are the difficulties in Giroux's proof of the open book decomposition theorem in dimension 3). For all these reasons, the level of detail oscillates quite a lot.

This discussion involves many geometrical objects whose definitions are not always laid out in a dedicated paragraph. The index on Page 187 should be helpful here.

There are no full proofs in this historical survey. Instead I tried to give hints of why things are true, and how different ideas interact. I also tried to state what I think are the starting points of each result. These are called "key observations" in the text and are almost always easy to explain, or at least motivate. Of course many theorems described are deep and technically difficult, I certainly do not mean that the "key observations" are all it takes to prove them.

2. Prehistory

This chapter explains what came before contact convexity was defined, from the earliest non-tautological constructions of contact manifolds to the advent of holomorphic curves in symplectic manifolds and symplectic convexity.

2.1. Early constructions

2.1.1. Principal circle bundles

After the examples mentioned in Chapter 1, the first non-trivial examples of contact manifolds were constructed (and characterized) in Boothby and H.-C. Wang 1958. A *Boothby-Wang structure* is a contact structure ξ on a principal $\mathbb{S}^1$ bundle $V \to B$ which is $\mathbb{S}^1$-invariant and transverse to fibers. This means that the vector field generating the $\mathbb{S}^1$ action is Reeb for ξ or, equivalently, that $\xi = \ker \alpha$ for some connection 1-form α.

Key observation 2.1. *A connection 1-form α on a $\mathbb{S}^1$-bundle $V \to B$ is contact if and only if the associated curvature 2-form on B is symplectic.*

Indeed the tangent space to B is isomorphic to $\ker \alpha$ at any point above it, and the curvature form is the projection of the restriction of $d\alpha$ to $\ker \alpha$. Conversely one can start with a symplectic form ω on B such that the cohomology class $[\omega]/2\pi$ admits a lift to $H^2(B; \mathbb{Z})$, and build B and α from there. These circle bundles will come back in Section 2.6.2 and Section 5.1.1.

Of course the space of contact connection 1-forms on such a V is convex hence the Boothby-Wang contact structure is determined by the $\mathbb{S}^1$-bundle structure uniquely up to $\mathbb{S}^1$-invariant isotopy.

Lutz 1977 dropped the transversality condition from Boothby and H.-C. Wang 1958 to study all $\mathbb{S}^1$-invariant contact structures on 3-dimensional circle bundles. His motivation was to exhibit manifolds carrying infinitely many pairwise non-isomorphic contact structures.

Key observation 2.2. *On a principal $\mathbb{S}^1$-bundle $\pi\colon V \to B$ over a compact orientable surface, let ξ be any $\mathbb{S}^1$-invariant contact structure. The set Γ of points b in B such that ξ is tangent to $\pi^{-1}(b)$ is a multi-curve on B: a collection of smooth properly embedded curves.*

Indeed, if X is the vector field generating the $\mathbb{S}^1$ action and α is an invariant contact form then $\alpha(X)$ is an invariant function whose zero set projects to Γ, and the contact condition ensures that it is a submersion.

The multi-curve Γ from Key observation 2.2 is now called a *dividing set* for B because it divides the base B into regions $B_\pm$ over which ξ is positive or negatively transverse to fibers. Conversely, Lutz 1977 proved that any properly embedded multi-curve in B which divides it into two subsurfaces is the dividing set of an $\mathbb{S}^1$-invariant contact structure. In addition, two invariant contact structures which agree near ∂V and have the same dividing set are isotopic relative to ∂V, and this condition is also necessary if one restricts to $\mathbb{S}^1$-equivariant isotopies. This discussion can clearly be seen as an early precursor of the theory of ξ-convex surfaces discussed in Section 3.1.2, although it seems that Giroux was unaware of this paper when he wrote Giroux 1991.

2.1.2. The Thurston-Winkelnkemper construction

W. P. Thurston and Winkelnkemper 1975 is another early paper constructing contact structures which turned out to be important in retrospect. The goal of that paper was to reprove that any closed orientable 3-manifold carries a contact structure. Inspired by constructions of foliations (before Thurston revolutionized this subject), the starting point of that paper is an open book decomposition of the manifold.

Recall that an *open book decomposition* of a closed manifold V is a pair (K, θ) where K is a codimension 2 submanifold of V called the *binding*, θ is a locally trivial smooth fibration from $V \setminus K$ to $\mathbb{S}^1$, and there is a neighborhood $K \times \mathbb{D}^2$ of K in V where θ is the angular coordinate in $\mathbb{D}^2$. Each fiber $\theta^{-1}(*)$ has closure $\theta^{-1}(*) \cup K$ which is called a *page* of the open book. In dimension 3, a link is called fibered if it is the binding of an open book.

Let P be the page of an open book and denote by $L(P)$ the space of 1-forms β on P that are positive on ∂P and such that $d\beta$ is symplectic. These forms will be called Liouville forms (there will be many more starting from Section 2.6.2).

Key observation 2.3. *If* $\dim(P)2$*, the space* $L(P)$ *of Liouville forms on* P *is non-empty and convex.*

Indeed, one can start with a model near ∂P and extend it in a random way to get a 1-form β_1. Stokes' theorem ensures that $\int d\beta_1$ is positive and we can choose a (non-exact) area form ω which coincides with $d\beta_1$ in the model and has the same integral over P. Poincaré and de Rham then offer some β_2 vanishing along ∂B, so that $\omega - d\beta_1 = d\beta_2$, hence $\beta = \beta_1 + \beta_2$ is Liouville. Convexity of these conditions is obvious. This observation allows to construction a 1-form α away from the binding of an open book that induces Liouville forms on each page. Then one can add a large multiple of the 1-form defining the fibration to $\mathbb{S}^1$ to get a contact form away from the binding. In a neighborhood $K \times \mathbb{D}^2$ of the binding, the resulting contact structure is transverse to each $K \times \{z\}$ –where defined– and this allowed ibid. to extend it.

2.2. Riemannian metrics on contact manifolds

2.2.1. Instantaneous rotation

Interaction between Riemannian metrics and contact structures were studied at least as early as Chern and Hamilton 1985. The case of foliations was studied even before, and some notions actually apply to any kind of plane fields.

One can first observe how to detect if a plane field ξ is contact using any Riemannian metric on a 3–manifold. Recall that the Frobenius integrability criterion says that ξ is integrable if and only if the flow of a (local) non-zero vector field tangent to ξ preserves ξ. Given a Riemannian metric g one can (locally) choose an oriented orthonormal moving frame u, v, n where u, v is an oriented basis for ξ and n is a unit normal vector to ξ. Denote by ϕ_t the flow of u and define $\theta(t)$ to be the angle between $(\phi_{-t})_* v$ and n. That is

$$\cos\theta(t) = \frac{g((\phi_{-t})_*(v), n)}{\|(\phi_{-t})_*(v)\|}.$$

One may compute that $\theta'(0) = -g([u, v], n)$. Setting $\alpha(\cdot) = g(n, \cdot)$, one can then characterize $\theta'(0)$ by $\alpha \wedge d\alpha = \theta'(0)\mathrm{vol}_g$. In particular, $\theta'(0)$ depends on g and ξ but not on u, v.

The function $\theta'(0)$ is denoted by by θ' and called the *instantaneous rotation* of ξ with respect to g. The Frobenius condition implies that ξ is a (positive) contact structure if and only if $\theta' > 0$.

2.2.2. Second fundamental form

In analogy with foliations, any hyperplane field has a second fundamental form. This notion goes back at least as far as Reinhart 1977. The second fundamental form of ξ is the quadratic form on ξ defined as follows: for vectors u and v in $\xi_p = T_pM \cap \xi$,

$$\mathrm{II}(u, v) = \frac{1}{2}\left\langle \nabla_u v + \nabla_v u, n\right\rangle, \tag{2.1}$$

where n is the oriented unit normal to ξ. (Note that u and v need to be extended to vector fields tangent to ξ in a neighborhood of p to compute $\mathrm{II}(u, v)$, but the value of $\mathrm{II}(u, v)$ is independent of this extension *i.e.* II is tensorial.)

One has the following geometric interpretation of II (see Giroux 1994a; Massot 2008b). For any point $p \in M$ and a unit vector $v \in \xi$ let P_v denote the plane spanned by v and the oriented unit normal to ξ. There is a neighborhood N of the origin in T_pM such that the exponential map pulls ξ back to a plane field $(\exp|_N)^*\xi$ that is transverse to $P_v \cap N$. This plane field induces a foliation on $P_v \cap N$ and $\mathrm{II}(v, v)$ is the curvature of the leaf of this foliation through the origin (measured by the flat metric on T_pM given by g_p).

The following curvatures derive from II. The *extrinsic curvature*, K_e, of ξ is the determinant of the quadratic form II with respect to g. The *mean curvature*, H, of ξ is the mean of the eigenvalues of II. It is clear that when ξ is an integrable plane field, then II is the standard second fundamental form of the leaves of the foliation associated to ξ and the quantities K_e and H are their classical curvatures. In all cases, II vanishes if and only if the plane field is *totally geodesic*: any geodesic which starts tangent to ξ stays tangent forever. In the standard Euclidean $\mathbb{R}^3$, the integrable plane field $\ker dz$ is obviously totally geodesic, but the same holds for the contact structure $\ker(dz + r^2 d\theta)$.

2.2.3. Compatibility

A metric g and a contact structure ξ are *compatible* if there is a contact form α for ξ and a complex structure J on ξ such that $g(u, v) = \frac{1}{\theta'} d\alpha(u, Jv)$ for all $u, v \in \xi$, where θ' is some positive constant, and for which the Reeb vector field R_α is the unit normal to ξ.

Compatible metrics metrics always exist in abundance, one can start with α, a suitable J, and θ', and build g from there. The canonical Riemannian metric on $\mathbb{S}^{2n-1}$ is compatible with the standard contact structure. Allowing that θ' be any constant rather than fixing one gives a class of compatible metrics that is stable under homothety. This is slightly different from Chern and Hamilton 1985 where $\theta' = 2$.

Key observation 2.4. *If g is compatible with α, then R_α is a totally geodesic and divergence free vector field.*

This follows from local tensor computations. In the sphere example, R_α is the Hopf vector field: $R_\alpha(z) = iz$ in $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. As a consequence of Key observation 2.4, the mean curvature H of ξ with respect to a g always vanishes.

In dimension 3, the geometric setup simplifies and one can check that a contact form α and a Riemannian metric g on a 3–manifold M are compatible if and only if $\|\alpha\| = 1$ and $*d\alpha = \theta' \alpha$ for some positive constant θ', where $*$ is the Hodge operator coming from g and the orientation. It is equivalent to saying that the instantaneous rotation of ξ with respect to g is constant and the Reeb vector field R_α is unit length and orthogonal to ξ.

A more important specificity of dimension 3 is that, given any metric g_ξ on a contact structure ξ, there is a canonical way to extend it to all of M so that it is compatible with ξ. The metric g_ξ induces an area form on ξ. Since, for any contact form α_0 and positive function f, $d(f\alpha_0)|_\xi = f(d\alpha_0)|_\xi$, there is a unique function f such that $d(f\alpha_0)|_\xi$ agrees with the area form given on ξ by g_ξ. Setting $\alpha = f\alpha_0$ allows to extend g_ξ to M by demanding that the Reeb vector field R_α is of unit length and orthogonal to ξ.

The study of compatible metrics can be used in fluid mechanics, plasma physics and other subjects, see for example Etnyre and Ghrist 2000. In addition, it has produced a great many questions from the Riemannian geometry perspective, especially in constructions of Einstein metrics, see Blair 2002; Boyer and Galicki 2008.

The definition above clearly shows that compatible metrics are also linked with complex geometry. They naturally appear on the boundary of some domains in complex manifolds. Studying this relation is part of (strictly pseudoconvex) *CR geometry*, where the basic object is a pair (ξ, J) where ξ is a contact structure and J is a complex structure on ξ, which is *not* seen as an auxiliary structure. In order to have a chance of coming from a complex manifold, such a pair needs to satisfy some infinitesimal constraint. Recall the *Nijenhuis torsion* $[T, T]$ of a $(1, 1)$–tensor field T is a skew–symmetric tensor field of type $(1, 2)$ defined as

$$[T, T](X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY]. \tag{2.2}$$

A pair (ξ, J) is a *strictly pseudoconvex integrable CR–structure*) if and only if J is tamed by CS_ξ –ie $d\alpha(v, Jv) > 0$ for all non-zero v in ξ– and the Nijenhuis torsion $[J, J]$ of J on ξ vanishes. One can check that $[J, J]$ is well defined as a $(1, 2)$–tensor field on ξ, and it automatically vanishes in dimension three.

Compatible metrics will be the main objects of Chapter 8.

2.3. Flexibility and rigidity

The *h-principle* question can be vaguely formulated as follows. Given a geometric construction problem, suppose there is no homotopic obstruction to its solution, does there exist a solution? If yes then the problem is said to satisfies the h-principle or to be on the flexible side. If not then the problem is on the rigid side. This dichotomy appeared gradually in immersion theory between 1950 and 1970 but really took off when Gromov systematized, and expanded, and invented, several general ways of proving h-principles, see Gromov 1986; Eliashberg and Mishachev 2002.

From the h-principle point of view, the subject of symplectic and contact geometry is very interesting because it features problems on both sides and the border is surprisingly rich. In contact geometry, the first rigidity result is Bennequin's inequality from Bennequin 1983: for any Legendrian knot L in $\mathbb{R}^3$ or $\mathbb{S}^3$ equipped with its standard contact structure ξ

$$\mathrm{tb}(L) + |r(L)| \leq -\chi(L).$$

In this inequality, $\mathrm{tb}(L)$ is the *Thurston-Bennequin invariant*, it measures how many turns ξ rotates along L compared to the tangent space of any Seifert surface of L (i.e. an embedded surface whose boundary is L). More generally one can define a Thurston-Bennequin invariant for any Legendrian knot endowed with a framing, a trivialization of its normal bundle. The *rotation number* $r(L)$ is the degree of the oriented tangent to L in ξ measured against a global trivialization of ξ. And $\chi(L)$ is the maximal Euler characteristic of a Seifert surface of L.

Bennequin reduced this inequality to analogous inequalities, first for knots transverse to the contact structure (by pushing slightly the Legendrian knot by the flow of a Legendrian vector field transverse to it), and then for braids, after proving that any transverse knot can be braided. Note that a braid is a knot (or link) that is positively transverse to all pages of the canonical open book of $\mathbb{S}^3$, whose pages are disks and whose binding is the unknot. In order to prove the inequality for braids, Bennequin first proved that any Seifert surface with minimal genus bounding a given braid Γ is isotopic, relative to Γ, to what he calls a *Markov surface* Σ: a surface transverse to the binding of the canonical open book and whose tangencies with pages are all saddles. He then studied the family of intersections between Σ and pages, and the foliation printed by pages on Σ. Bennequin's inequality was then proved as a combinatorial constraint on these objects. Somewhat related constraints reappeared in Eliashberg 1992; Giroux 2000.

Bennequin's inequality proves in particular that there can be no embedded disk in the standard contact $\mathbb{R}^3$ looking like Figure 2.1. Since Eliashberg 1989, such disks and

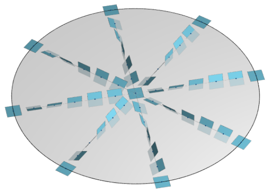


Figure 2.1.: An overtwisted disk

the contact manifolds containing them are called *overtwisted* and characterize a flexible class of contact manifolds.

The main topological tool to study contact 3-manifold is the **characteristic foliation**. The characteristic foliation of a surface S in a contact 3–manifold (V, ξ) is the singular foliation ξS tangent to $TS \cap \xi$, see Figure 2.2. This foliation completely determines

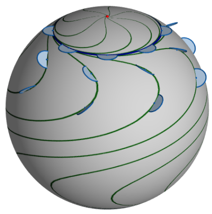


Figure 2.2.: A characteristic foliation, showing a singular point on top, a closed leaf and spiraling.

the germ of ξ along S. In higher dimension, the characteristic foliation is the singular foliation tangent to the CS_ξ-orthogonal of TS.

The main result of ibid. states that two overtwisted contact structures that are homotopic among plane fields are homotopic among contact structures. Our goal here is to describe the main geometric construction because a related phenomenon will be crucial in Chapter 8. But we need to explain (very roughly) its context. Starting with a homotopy of plane fields between two contact structures, one can first ensure that there is a fixed disk as in Figure 2.1 all along the homotopy. Then one uses ideas coming from Thurston's study of foliations to get a homotopy of contact structures outside a ball, and make sure that, near the boundary of this ball the plane field is "almost constant". After taking an embedded connected sum of this ball and a neighborhood of an overtwisted disk, one gets to the situation of the following observation.

Key observation 2.5. *Let $\mathcal{F}$ be a singular foliation printed by a plane field on a sphere which has exactly two singularities $p_\pm$ where $\xi = \pm TS$ and ξ is cooriented towards p_+ along all closed leaves in ξS except for two closed leafs near p_-. There a contact structure on a ball which prints $\mathcal{F}$ on the boundary. This contact structure is sufficiently canonical to be built continuously with respect to any number of parameters.*

Indeed, assume for instance that we try to fill the singular foliations of Figure 2.3. Then one can use the restriction of $\ker(\cos(r)dz + r\sin(r)d\theta)$ to the ball obtained by

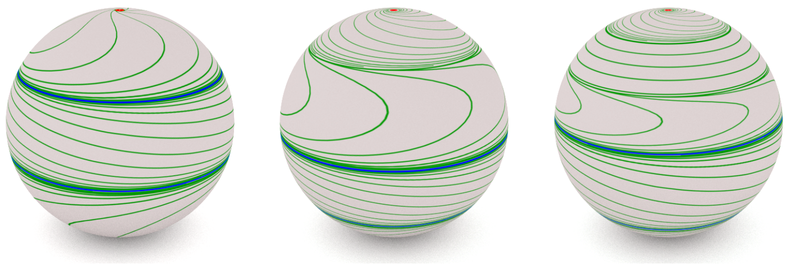


Figure 2.3.: A typical movie of foliations $\xi_s B'$. First B has a North-South dynamics hence contributes nothing, and we see only the foliation of a neighborhood of the overtwisted disk. Then a closed leaf appears in $\xi_s B$ then a whole interval of them. Then it disappears the same way it came.

rotating the half disks of Figure 2.4 around the z-axis.

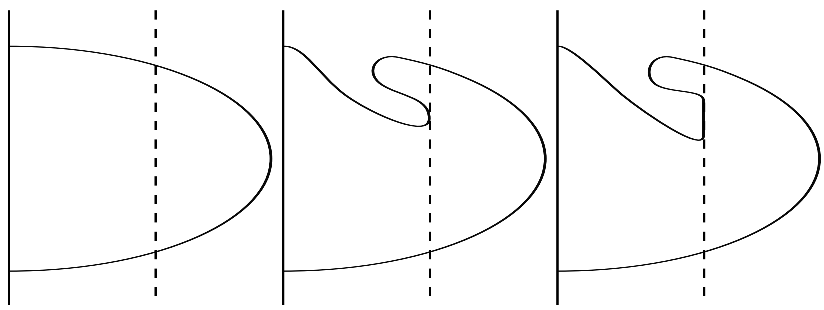


Figure 2.4.: The family of spheres corresponding to Figure 2.3. The solid vertical line is the rotation axis. The dashed one indicates the cylinder where ξ is horizontal again.

2.4. Holomorphic curves in symplectic manifolds

Despite the early successes of variational methods, Eliashberg's C^0-rigidity of symplectic diffeomorphisms (discussed in Section 5.3.3), and Bennequin's theorem, one can say that the rigid side of symplectic topology took off with the introduction of holomorphic curves in Gromov 1985. Gromov sought to extend to symplectic manifolds the usefulness of holomorphic curves on complex manifolds, especially Kähler manifolds (that are both complex and symplectic). The natural object from the Kähler point of view is a pair (ω, J) where ω is symplectic and J is an almost complex structure compatible with ω, ie. $\omega(\cdot, J\cdot)$ is a Riemannian metric. From the point of view of Key observation 2.8 below, it is natural to relax this condition and ask that J is *tamed* by ω, ie. ω is positive on (oriented) complex lines: for every non-zero vector v, $\omega(v, Jv) > 0$.

Key observation 2.6. *On every symplectic manifold (X, ω), the space of almost complex structures tamed by ω is non-empty and contractible. The same holds for compatible almost complex structures.*

Indeed, if we fix an auxiliary Riemannian metric g then $\omega = g(A\cdot, \cdot)$ for some skew-symmetric endomorphism A, and $J_0 = (\sqrt{AA^*})^{-1}A$ is an almost complex structures compatible with ω. Other almost complex structures tamed by ω can be analysed using the Cayley transform $J \mapsto (J + J_0)^{-1}(J - J_0)$ which sends tamed and contractible complex structures to sections of bundles with contractible fibers (this proof is not quite the one in ibid., it comes from Sévennec).

The next crucial point is that holomorphic curves live in finite dimensional families.

Key observation 2.7. *Holomorphic curves that are closed or have suitable boundary or asymptotic constraints are solutions of a Fredholm elliptic PDE problem.*

Here Gromov was inspired by the Bers-Vekua theory of equations related to the Cauchy-Riemann or Beltrami operators. A key aspect is that, in order to study the local properties of the holomorphic curves equation, one can zoom near any point until one barely sees the difference with an integrable complex structure.

Inspiration for the next ingredient came from algebraic geometry and also Sacks and Uhlenbeck 1981 studying minimal surfaces.

Key observation 2.8. *Spaces of holomorphic curves with bounded area for a tamed almost complex structures can be compactified using other spaces of holomorphic curves.*

Indeed the taming condition allows to control the L^2 norm of the gradient of the derivative of a holomorphic curve by its area. Together with C^0 bounds, and elliptic regularity, this is almost sufficient to control everything. Compactness can fail only because energy concentrates more and more near finitely many points to create broken holomorphic curves.

Gromov 1985 contains many spectacular results obtained in the general setup roughly sketched above but, for our purposes, the main ones concern the definition of the *Gromov width*: the supremum of radii of Darboux balls that can be symplectically embedded into a given manifold which later inspired the definition of Giroux torsion ; the absence of exact Lagrangian submanifolds in $\mathbb{C}^n$ which will come back in Chapter 9 ; tightness of fillable contact manifolds explained in Section 2.7.2 ; and ideas about foliation by spheres which will appear in Sections 3.5 and 4.3.

2.5. Floer homologies

After the Big Bang of ibid., the next revolution came from Floer 1988c introducing *Floer theory*, which we need to briefly describe. The story starts with Morse theory on (finite dimensional) closed manifolds. Given a Morse function f with distinct critical value, the homotopy type of sublevel sets $\{f \leq a\}$ changes in a controlled way when a moves through a critical value, and this allows to tie the topology of the manifold to critical point theory. This is still useful in infinite dimension when the index of critical points is finite, the main example arises in Riemannian geometry with the energy function, which characterizes geodesics. Several problems in symplectic geometry are also characterized by some variational property. The most emblematic one is existence of contractible periodic orbits of a Hamiltonian vector field X_H. Those are critical points of the *action functional* on the space of loops $\gamma\colon \mathbb{S}^1 \to X$ having some extension $u\colon \mathbb{D}^2 \to X$, for a symplectically aspherical manifold (X, ω):

$$\boldsymbol{\mathcal{A}}_H(\gamma) = -\int_{\mathbb{D}^2} u^*\omega - \int_{\mathbb{S}^1} H_t(\gamma(t))\, dt.$$

This action functional (or its opposite) does not have finite index critical points. However, it was realized by Thom 1949; Smale 1960; Witten 1982 that homological consequences of Morse theory do not really need the detailed description of topology changes at critical points, they only depend on spaces of gradient trajectories between critical points. This observation is not enough to handle the case of the action functional tough, because the natural candidate gradient flow is not well defined (at least not in useful function spaces). Floer overcame all these difficulties.

Key observation 2.9. *Although both the negative and the positive eigenspace of the Hessian of the action functional at a critical point are infinite dimensional, and its gradient has no flow, there are well-defined finite dimensional spaces of trajectories between any two given critical points.*

Part of the explanation is that, although indices are infinite, there is a notion of finite relative indices, which play the role of the difference of indices between two critical points. Regarding the absence of flow, the key paradigm shift is to stop looking for solutions of an evolution problem, but rather consider the complete trajectory as solution of an elliptic PDE, which is more or less directly a holomorphic curve equation.

This allows to use variations on Gromov's compactification of spaces of holomorphic curves by other spaces of holomorphic curves, with additional inspiration from the behavior of spaces of gradient trajectories in finite dimensional manifolds. In contrast with the closed curve theory, the compactification is obtained by adding codimension one pieces (instead of codimension 2), and this gives rise to homological theories, as in Morse theory.

Key observation 2.10. *Suppose we have a set S, and manifolds $\mathcal{M}(x, y)$ for each pair of objects x and y in S. On the $\mathbb{Z}/2$ vector space spanned by S, let ∂ be the linear map* *sending x in S to the sum of $|\mathcal{M}(x,y)|y$ over all y such that $\mathcal{M}(x,y)$ has dimension zero. Assume that, whenever $\mathcal{M}(x,y)$ has dimension 1, it can be compactified to a manifold with boundary by adding the union of all $\mathcal{M}(x,z) \times \mathcal{M}(z,y)$ such that $\mathcal{M}(x,z)$ and $\mathcal{M}(z,y)$ have dimension zero. Then $\partial \circ \partial$ vanishes so that it defines a homology vector space.*

Indeed one then have:

$$\partial\partial x = \sum_y \left(\sum_z |\mathcal{M}(x,z)|\,|\mathcal{M}(z,y)| \right) y$$

and, by assumption, the coefficient appearing is the number of boundary components of a compact 1-manifolds, which vanishes modulo two. With more work, one can often replace $|\mathcal{M}(x,y)|$ with signed count of elements and get a homology theory over $\mathbb{Z}$ coefficients.

Early applications of Floer theory dealt with periodic orbits of Hamiltonian flows in Floer 1988c, intersection between Lagrangian submanifolds in Floer 1988b and 3-manifold topology in Floer 1988a.

2.6. Complex and symplectic convexity

2.6.1. From periodic orbits to convexity

Some notion of convexity for the boundary of a symplectic manifold first arose in the study of periodic orbits of Hamiltonian systems.

Key observation 2.11. *If two autonomous Hamiltonians on a symplectic manifold admit the same hypersurface S as a regular level set then their flows on S are the same up to parametrization.*

Indeed if two functions H and K have S as a common regular level set then their differential are colinear along S.

In Weinstein 1979, the author tried to understand what kind of intrinsically symplectic geometric feature of hypersurfaces allowed to prove existence of periodic orbits on a fixed energy level, especially in Rabinowitz 1979:

> “ Our principal discovery is that a simple geometric feature is common to all the situations in which the existence of periodic orbits has been proven by variational methods.” (Weinstein 1979)

Weinstein then defines *contact type hypersurfaces* in a symplectic manifold and characterize them by what is now the standard definition: S is a contact type hypersurface in (X, ω) if, near S, there is vector field Z transverse to S and dilating ω: $\mathcal{L}_Z\,\omega = \omega$.

Such fields Z were later called *Liouville fields*, because a canonical example is the radial field in the cotangent fibers which is transverse to sphere bundles. The other canonical example is the radial vector field in a symplectic vector space. In both cases, convex subset containing the origin (in fibers) have contact type boundary, so we have a first hint of the etymology of “symplectic convexity”.

In this situation, the common line field of Key observation 2.11 is the line field spanned by the Reeb vector field of the contact form $\iota_Z\omega|_{TS}$. This paper ends with the famous statement of the Weinstein conjecture (with an extra $H^1 = 0$ hypothesis which is now believed to be unnecessary caution): Reeb vector fields on closed contact manifolds should always have closed orbits.

Fifteen years later, a lot of further work on periodic orbits in Hamiltonian and Reeb dynamics yielded further insight on this topic. In particular, some results gave existence of periodic orbits on almost every energy level. In the contact type case, this is enough to get existence on a fixed level because the dynamics on levels transverse to a given Liouville field are conjugated. Hofer and Zehnder 1994 then defined *stable hypersurfaces* to be those where this last property hold and remarked that contact type hypersurfaces are not the only examples. This will play a role in Chapter 7.

2.6.2. From Stein to Weinstein

From the point of view of holomorphic curves described in Section 2.4, it is natural to consider how notions of convexity in complex geometry can be adapted to symplectic manifolds.

Before turning to convexity in complex geometry, we need to recall some notions in Riemannian geometry (which will also play a key role in Chapter 8). The most direct analog of elementary convexity in Riemannian geometry is to say that a subset A of a Riemannian manifold M is convex if, between every pair of points p and q in M, there is a geodesic $\gamma \subset A$ which is the only length minimizing geodesic between p and q in M (see e.g. Chavel 2006). Since we are interested in contact boundaries of symplectic manifolds, we assume that A is a codimension 0 submanifold with smooth boundary, and turn to some weaker convexity requirement about the boundary of A. We say that ∂A is convex if every geodesic γ which is tangent to ∂A, say at time 0, is locally outside A for times close to 0. More precisely, if f is a regular equation of ∂A which is positive outside A then $(f \circ \gamma)''(0)$ should be positive, i.e. the restriction of f to the geodesic γ should be subharmonic. Equivalently, the second fundamental form II of ∂A with outward coorientation should be positive definite. If f is normalized to have $\|\nabla f\| = 1$ along ∂A then $\mathrm{II} = \operatorname{Hess} f|_{T\partial A}$ and, maybe after post-composing f with a suitable convex function from $\mathbb{R}$ to $\mathbb{R}$, convexity of ∂A is equivalent to positivity of $\operatorname{Hess} f$ along ∂A, including the transverse direction. In other words, the restriction of f to any geodesic should be subharmonic.

In complex geometry, we want holomorphic curves to play a role which is at least partially analogous to geodesics. So we are interested in domains A (say in $\mathbb{C}^n$ for simplicity) such that any holomorphic curve tangent to ∂A is locally outside A. In order to understand what this implies on the 2-jet of ∂A, recall that holomorphic curves are area minimizing and their tangent space (at immersed points) is a complex line. Hence, for every vector field V tangent to an immersed holomorphic curve C and vector field Y orthogonal to C, $\langle D_V V + D_{iV} iV, Y\rangle = 0$ (see e.g. Gallot, Hulin, and Lafontaine 2004, Section 5.C.1). The simplest case is when ∂A contains an immersed holomorphic curve passing through some point p. Then we learn that, at p, the mean curvature $\mathrm{II}(v)+\mathrm{II}(iv)$ vanishes for some non-zero v in $\xi = T\partial A \cap iT\partial A$ which is the union of all complex lines contained in the real subspace $T\partial A$ (and also the maximal complex subspace of $T\partial A$). On this complex subspace we saw the role of the quadratic form $L_{\partial A}(v) = \mathrm{II}(v) + \mathrm{II}(iv)$ called the (normalized) *Levi form* of ∂A. The cooriented hypersurface ∂A is called *i-convex* if $L_{\partial A}$ is a positive definite quadratic form.

Turning to the functional point of view, we see that ∂A is i-convex if its regular equations are subharmonic when restricted to any holomorphic curve tangent to ∂A. Again, composing with a suitable convex function allows to get this condition for all holomorphic curves near ∂A. From an Euclidean perspective this is written as positivity of $L_f(v) = \mathrm{Hess}\, f(v,v) + \mathrm{Hess}\, f(iv,iv)$ but it is better to rewrite $L_f(v) = i\partial\bar\partial f(v,iv) = -d(df \circ i)(v,iv)$. A short computation proves that, if φ is holomorphic then $L_{f\circ\varphi}(x)(v) = L_f(\varphi(x))(\varphi'(x)v)$ so positivity of L_f is an intrinsic condition in complex geometry. We record all this discussion in the following.

Key observation 2.12. *The notion of i-convex hypersurfaces and functions is invariant under biholomorphic maps. A holomorphic curve in a domain A cannot be tangent to the boundary ∂A if the latter is i-convex.*

We now discuss one way this notion of i-convexity is important from the complex point of view. In complex analysis, a special role is played by complex manifolds that have "many holomorphic functions". Those complex manifolds are the one where most of one-dimensional complex analysis results carry through, see the introduction of Grauert and Remmert 1979.

The most obvious example is $\mathbb{C}^n$ and its analytic submanifolds. After a long quest (summarized in a symplectic friendly way in Eliashberg and Cieliebak 2012, Chapters 2 and 5), it was proved that they are essentially the only ones.

We want to explain how this is linked to convexity. Again for simplicity we mention only open subsets of $\mathbb{C}^n$. An real valued function φ on an open subset Ω in $\mathbb{C}^n$ allows to endow the trivial line bundle $\Omega \times \mathbb{C}$ with the hermitian metric $\|\cdot\|_\varphi = e^{-\varphi}\|\cdot\|_{euc}$. If φ is exhausting and i-convex then the Chern connexion of $\|\cdot\|_\varphi$ has positive curvature. Hörmander's L^2 theory of $\bar\partial$ then allows to produce holomorphic functions embedding Ω as a proper analytic submanifold of some $\mathbb{C}^N$. We will use this as a definition.

Definition 2.13. *A Stein manifold is a complex manifold that admits an exhausting i-convex function.*

The link with the discussion of hypersurfaces is that the boundary of the corresponding disk bundles $\{(z,v); \|v\|_\varphi \leq r\}$ are then i-concave (i.e. its complement has i-convex boundary). This is consistent with the existence of holomorphic sections since those provide many (local) holomorphic curves tangent to the boundary of disk bundles from the inside. This story is parallel to the Kodaira embedding theorem where positive curvature of a line bundle over a *closed* manifold allows to build enough holomorphic sections to get an embedding into some projective space.

These positive curvature ideas will come back to the symplectic world in Section 5.1.1, but here we only want to see how this influences the definition of convex symplectic manifolds.

Let φ be an i-convex function on a Stein manifold. An equivalent reformulation of i-convexity is that the 2-form $\omega_\varphi := -d(d\varphi \circ i)$ is non-degenerate. So we get an exact symplectic form on X. In addition we get a Riemannian metric $g_\varphi := \omega_\varphi(\cdot, J\cdot)$ and the gradient of φ with respect to g_φ is a Liouville vector field.

One of the insights of Eliashberg and Gromov 1991 is that different exhausting i-convex functions on the same Stein manifold yield isomorphic symplectic structures (it basically comes from convexity of the i-convexity condition but one must be careful with critical points that escape towards infinity). This common isomorphism class should see at least some complex geometry, and provide interesting examples of symplectic manifolds.

The first result in this direction is a very geometric new proof of the Lefschetz-Thom-Andreotti-Frankel theorem: an i-convex Morse function on a complex manifold X has critical points of index at most $\dim_{\mathbb{C}} X$. Note that i-convexity is an open condition in C^2-topology so assuming Morse singularity does not cost anything.

Key observation 2.14. *Let $(X, d\lambda)$ be an exact symplectic manifold and let Z be the vector field $d\lambda$-dual to λ: $\iota_Z d\lambda = \lambda$. The form λ vanishes on the stable manifold $W^s(Z, p)$ of any hyperbolic fixed point p of Z.*

Indeed the flow of Z exponentially dilates $\omega = d\lambda$ since $\mathcal{L}_Z\omega = \omega$ and any vector tangent to such a stable manifold is crushed by the differential of this flow. In particular $W^s(X, p)$ has dimension at most $\dim_{\mathbb{R}}(X)/2$. Applying this observation to the gradient of a J-convex function φ with respect to g_φ yields the announced constraint on Morse indices.

Definition 2.15. *A Weinstein manifold is a symplectic manifold (W, ω) equipped with an exhausting Morse function φ and a Liouville pseudogradient Z: $d\varphi(Z) \geq c\|d\varphi\|$ and $\mathcal{L}_Z\omega = \omega$. A Liouville manifold is an exact symplectic manifold (W, ω) with equipped with some complete Liouville vector field Z for which there an exhaustion by compact domains W_k whose boundaries are transverse to Z.*

2.6.3. Morse theory and Weinstein handles

In Section 2.6.2, Key observation 2.14 about stable manifolds of Liouville vector fields was not fully used. There we used isotropy but not the stronger conclusion that the Liouville form vanishes (which follows from isotropy and the fact that the Liouville field is tangent to the stable manifold). One extra information coming from this strengthening is that such stable manifolds intersects regular level sets in submanifolds which are isotropic for the contact structure induced by the Liouville form. Using standard neighborhood techniques, one can then prove that a neighborhood of a stable manifold cut off by a regular level set is a *Weinstein handle*, introduced in Eliashberg 1990b; Weinstein 1991.

In particular all Weinstein handles in dimension $2n$ have index at most n. This is a strong constraint on the topology of a Weinstein manifold. In particular it implies vanishing homology starting from dimension $n+1$. Another topological constraint comes from the existence of an almost complex structure, which can be checked by obstruction theory. Eliashberg 1990b proved that, starting from dimension 6, those two necessary conditions for existence of a Weinstein structure are sufficient and one can even build a corresponding integrable complex structure. In dimension 4 there are additional framing conditions for 2-handles. In all cases this is an example of validity of the h-principle without parameter. Adding parameters is not possible in general if there are handles of index n, but see Section 5.2.2 for important special cases.

The next natural question was to understand whether similar topological constraints existed for Liouville manifolds. This was settled in dimension 4 by McDuff 1991a which constructed a Liouville form on $\Sigma \times [-1, 1]$ for any closed orientable surface Σ with genus at least 2. This construction had no obvious analog in higher dimensions and prompted the following question.

> “ Does there exist a complete convex symplectic manifold of dimension $2n > 4$ with non-trivial $(2n-1)$-dimensional homology? ” (Eliashberg and Gromov 1991, Question 3.2.B)

This question was quickly answered in dimension 6 by Geiges 1994 but then had stalled for almost twenty years. It will be answered in every dimension in Chapter 7.

2.7. Fillable contact manifolds

2.7.1. Contact manifolds at infinity

In a Weinstein manifold (X, ω, Z, φ), every regular sublevel set of φ has contact type boundary, the contact structure being the kernel of the restriction of $\lambda = \iota_Z\omega$. If we assume that φ has only finitely many critical points, then eventually all those contact manifolds are isomorphic, and one would like to see this all those manifolds as incarnations of a single contact manifold living “at infinity of X”.

However we would like everything but ω to play only an auxiliary role. For a long time it was an open question whether the symplectic structure alone would be enough to recover a contact manifold at infinity. This changed with the breakthrough of Courte 2014. In every dimension at least 6 there are examples of Liouville (or even Stein) manifolds admitting a pair of complete finite type Liouville vector fields whose associated contact manifolds at infinity are not even homeomorphic.

There is however a better solution to this “problem” than fixing a Liouville form. This solution was invented by Giroux (at a time when he probably suspected the existence of the *Courte phenomenon* but before Courte actually exhibited it).

Definition 2.16. *Let Σ be a compact manifold with boundary and ω a symplectic form on the interior $\mathring{\Sigma}$ of Σ. The pair (Σ, ω) is an ideal Liouville domain if there exists an auxiliary 1-form β on $\mathring{\Sigma}$ such that:*

- *$d\beta = \omega$ on $\mathring{\Sigma}$,*

- *For any smooth function $f\colon \Sigma \to [0, \infty)$ with regular level set $\partial\Sigma = f^{-1}(0)$, the* 1*-form $f\beta$ extends smoothly to $\partial\Sigma$ such that its restriction to $\partial\Sigma$ is a contact form.*

In this situation, β is called a Liouville form for (Σ, ω).

One important point of the above definition, compared to the definition in the previous section, is that the primitive (or its dual vector field) is not part of the data. And still we have the following result from Courte 2015, refining an earlier result by Giroux.

Key observation 2.17. *The space of Liouville forms of an ideal Liouville domain (Σ, ω) is contractible and there exists a single contact structure ξ on $\partial\Sigma$ (not only an isotopy class thereof) such that, for every regular equation f of $\partial\Sigma$ and every Liouville form β, $f\beta$ restricts to $\partial\Sigma$ as an equation for ξ.*

Indeed one can check that $\xi = \ker(f^2\omega|_{T\partial\Sigma})$ for any equation f, and the other properties are obvious.

So the minimal relevant information (in addition to ω) is not contained in a choice of β but rather in the diffeomorphism type of the compactification going from $\mathring{\Sigma}$ to Σ.

2.7.2. Holomorphic curves and flavors of convexity

In the setup of compact symplectic manifolds with contact type boundary or ideal Liouville domains, one can choose an almost complex structure compatible with the symplectic structure, and such that holomorphic curves cannot escape through the boundary. Indeed one can build a compatible J so that there is a collar neighborhood $(-1, 0] \times \partial X$ where the projection onto $(-1, 0]$ is J-convex (as defined in Section 2.6.2).

Eliashberg and Gromov 1991 analysed the symplectic origin of this construction as follows. Every co-oriented contact structure ξ carries a natural conformal class CS_ξ of symplectic structures: if λ is any contact form for ξ, then $d\lambda|_\xi$ defines a symplectic bundle structure that is independent of the choice of λ up to scaling. If (W, ω) is a symplectic manifold and $V = \partial W$ carries a positive contact structure ξ, one says that ω *dominates* ξ if the restriction $\omega|_\xi$ belongs to CS_ξ. This is obviously the case if V is a contact type hypersurface.

In dimension three this domination condition is strictly weaker than the contact type condition. However, McDuff 1991a, Lemma 2.1 states that, from dimension 5 upward, the domination condition already implies that (W, ω) is a *strong* filling. Even without this proof, one can see that the domination condition on ω is open in dimension 3 but not in higher dimensions. Then higher dimensional contact topology lacked a definition of weak fillings for twenty years, until the results surveyed in Chapter 7. In the mean time, terminology stabilized in dimension 3 as follows: (W, ω) is a *strong symplectic filling* of $(\partial W, \xi)$ is there is a primitive λ of ω defined near ∂W, whose ω-dual vector field points outward and such that $\xi = \ker \lambda|_{T\partial W}$, an *exact or Liouville filling* if such a λ exists on the whole W and a *weak filling* if ω dominates ξ. All these condition are a priori weaker than existence of a Weinstein or Stein filling. The latter two conditions are equivalent according to Eliashberg 1990b, see Section 2.6.3.

All those fillability conditions imply tightness. This was proved in Gromov 1985, Section 2.4.D_1' –at least under the extra assumption that (W, ω) is *symplectically aspherical*: for all $v : \mathbb{S}^2 \to W$, $\int v^*\omega = 0$. We now sketch the argument since it will play a major role in Chapters 7 and 10.

Key observation 2.18. *Let S be a surface in a contact 3-manifold (V, ξ). We fix a trivialization $S\xi \simeq \mathbb{R} \times M$. Assume that ξS has an elliptic singularity at some point p, i.e. we have the situation at the center of Figure 2.1. Then there is an almost complex structure J on $\mathbb{R}\times M$ and a family of Bishop disks: J-holomorphic disks $u_c : D^2 \to \mathbb{R}\times M$, $c \in [0, 1)$ such $u_c(\partial D^2) \subset \{0\} \times S$ goes once around the singularity, and u_c converges uniformly to p when c goes to one.*

Indeed one can consider the model case where V is the unit sphere in $\mathbb{C}^2$, S is a neighborhood of $p = (0, 1)$ in $V \cap \{\Im z_2 = 0\}$, and the image of u_c is $V \cap \{\Re z_2 = c, \Im z_2 = 0\}$.

Key observation 2.7 applies in this case to prove that, after perturbation of J and removing reparametrization degrees of freedom, J-holomorphic curves homotopic to u_c (with the same boundary condition) form a 1-dimensional manifold $\mathcal{M}$. One then proves the existence of a neighborhood U of $(0, p)$ such that any J-holomorphic disk $u : (D^2, \partial D^2) \to (S\xi, \{0\} \times S)$ which intersects U is, up to parametrization, one of the u_c's. The uniqueness ensures that the moduli space $\mathcal{M}$ has at least one end. Assume for contradiction that (V, ξ) is overtwisted and filled by some (W, ω). Then we can use the center of an overtwisted disk to start a Bishop family. Almost complex convexity of ∂W guarantees these holomorphic disks cannot escape W and their boundary stays transverse to ξ. The geometry of the setup also ensures an energy bound so that Key observation 2.8 ensures that the Bishop family is the only possible end of this family of holomorphic curves. This is a contradiction since there is no 1-dimensional manifold with exactly one end.

The other early highlight of the study of symplectic fillings is ibid., Section 2.4.A_2'' proving that any symplectic manifold containing no symplectic 2-sphere and having an end symplectomorphic to the standard $\mathbb{R}^4$ is globally symplectomorphic to it. This can be seen as a result about the classification of symplectic fillings of the standard contact structure on $\mathbb{S}^3$. It was proved using holomorphic spheres in a compactification containing two spheres whose union has a neighborhood isomorphic to a neighborhood of $(\mathbb{S}^2 \times \{*\}) \cup (\{*\} \times \mathbb{S}^2)$ in $\mathbb{S}^2 \times \mathbb{S}^2$.

This filling classification theorem was improved using holomorphic disks in Eliashberg 1990a, Theorem 5.1 which removed the hypothesis that the contact structure is standard. This information then became part of the conclusion, ibid., Corollary 5.3. Note that it was not known at the time that all tight contact structures on $\mathbb{S}^3$ are isotopic to the standard one. Actually several tools from ibid. were generalized from the setup of fillable contact structure to tight ones in Eliashberg 1992, and this was important in order to get uniqueness of tight contact structures up to isotopy on $\mathbb{S}^3$.

3. Contact convexity

3.1. Convex contact structures

3.1.1. From symplectic convexity to contact convexity

We have seen what led to the definition of a Liouville manifold as a symplectic manifold (W, ω) admitting a complete vector field X dilating the symplectic form: $\mathcal{L}_X \omega = \omega$. Such a manifold is called Weinstein if in addition X is pseudogradient for some exhausting Morse function and, in that case we get a nice symplectic handle decomposition.

We are now ready to discuss the last section of Eliashberg and Gromov 1991. The authors wanted to extend the above definitions to the case of contact manifolds. In order to get a definition which depends on a contact structure only and not a choice of contact form, they consider contact vector fields as the analogue of a Liouville vector field. Since contact vector field always exist, they skipped the Liouville case, and defined a contact manifold to be convex if there is a contact vector field which is pseudogradient for some exhausting Morse function.

They noticed immediately that the standard sphere is convex and asked the following questions:

> “ Do there exist non-convex contact manifolds? If they do, what is their relationship with convex symplectic manifolds? For instance, let (V, ξ) be a fillable contact manifold. Is it convex? Is the converse true? ” (ibid.)

The “most fillable” contact manifold V would be a regular level set of a Weinstein structure (W, λ, φ). But, the Morse function φ restricts to a constant function on V, and this does not seem to be a very good starting point to prove that the induced contact structure is convex…

Faced with this obvious problem, Giroux turned the question inside out. Starting with a convex contact manifold (V, ξ), he exhibited convex symplectic manifolds *inside* V.

3.1.2. Convex hypersurfaces

Let (V, ξ) be a convex contact manifold with Morse function f and a contact pseudogradient X. Regular level sets of f are hypersurfaces transverse to X along which we want to attach some model handles. This motivates the following definition where we do not impose existence of any globally defined (f, X) pair.

Definition 3.1 (Giroux 1991)**.** *A hypersurface S in a contact manifold (V, ξ) is ξ–convex if it is transverse to a contact vector field or, equivalently, if it has a so called* *homogeneous neighborhood: a tubular neighborhood $S \times \mathbb{R}$ where the restriction of ξ is $\mathbb{R}$–invariant.*

For instance, in $(\mathbb{R}^{2n+1}, \ker(dz + \sum x_i dy_i - y_i dx_i))$, any round sphere around the origin is ξ-convex since it is transverse to the contact vector field $2\partial_z + \sum x_i \partial_{x_i} + y_i \partial_{y_i}$. In $(\mathbb{T}^3, \ker(\cos(z)dx - \sin(z)dy))$, the tori $\{x = \mathrm{cst}\}$ are ξ-convex because they are transverse to ∂_x.

In a homogeneous neighborhood $S \times \mathbb{R}$ of S, ξ admits a contact form $\alpha = udt + \beta$ where t is the coordinate in $\mathbb{R}$ so that $X = \partial_t$ in $S \times \mathbb{R}$, $u = \alpha(X)$ is a function on S, and β is the restriction of α to S (neither u nor β depend on t). The set $\Gamma_X = \{u = 0\} \subset S$ is called the *dividing set* of S associated to X. It is the intersection of S with the *characteristic hypersurface* of X: $\Sigma_X = \{X \in \xi\}$. Generically Σ_X is cut out transversely by the equation $\iota_X \alpha = 0$ and we will always assume this property in the following.

The following is Giroux 1991, Proposition 3.4(i) rewritten in the modern language of ideal Liouville domains.

Key observation 3.2. *Let $S_\pm$ be the closure of $\{\pm u > 0\}$ in S. Both S_+ and S_- are (maybe disconnected) domains in S with smooth boundary Γ_X and $(S_\pm, d(\pm\beta/u))$ are $2n$-dimensional ideal Liouville domains.*

Indeed, one compute that $\alpha \wedge d\alpha^n = \theta \wedge dt$ where $\theta := (d\beta)^{n-1} \wedge (ud\beta + n\beta \wedge du)$ is therefore a volume form on S. Along Γ_X, θ becomes $n\beta \wedge d\beta^{n-1} \wedge du$ so we learn that Γ_X is transversely cut-out by u and that β induces a contact form on Γ_X. Outside Γ_X one compute that $\theta = u^{n+1} d(\beta/u)^n$ and the observation follows.

Conversely, in order to prove that a hypersurface S is ξ-convex with dividing set $\Gamma = \{u = 0\}$ for some function $u \colon S \to \mathbb{R}$, it is enough to find a contact form α whose restriction β to S leads to ideal Liouville domain structures $(S_\pm, d(\beta/u))$. Away from Γ we can assume $u = \pm 1$ and we are led to the following question: given a manifold P and a 1-form β on P, does there exist a positive function f such that $d(f\beta)^n$ is a volume form? Let Ω be an auxiliary volume form on P and let Y be the unique vector field on P such that $\iota_Y(\beta \wedge d\beta^{n-1}) = \Omega$. Note that when β is the restriction of a contact form to P then Y directs the characteristic foliation $\ker(d\beta)$ which depends only on the contact structure. In the general case we can still see that multiplying β by f multiplies Y by $h := f^n$. And we can compute $d(f\beta)^n = (dh(Y) + h \operatorname{div} Y)\Omega$. So f exists if and only if there exists a function $h \colon P \to \mathbb{R}$ such that

$$h > 0 \quad \text{and} \quad \pm(dh(Y) + h \operatorname{div} Y) > 0. \tag{3.1}$$

This problem can be obstructed by the dynamics of Y and its divergence, e.g. if Y has vanishing divergence along a closed orbit γ then a solution h would have no critical point on the circle γ. On the other hand if Y has an orbit γ travelling between two singular points p and q and $\operatorname{div} Y$ is positive at p and q then there is a solution along γ: any positive function h which grows sufficiently fast along γ will do. Similar considerations prove that the region near Γ is not a problem as long as the characteristic foliation transversely exits $\{u > 0\}$ to enter $\{u < 0\}$.

This discussion is especially useful in dimension 3 because it uncovers a very convenient low dimension degeneracy: every closed surface can be approximated in C^∞ topology by a ξ convex one. Here, the key feature of dimension two is the fact proved in Peixoto 1962 that any vector field on a surface can be approximated by a vector field whose dynamics is sufficiently simple to solve Equation (3.1) (see Section 3.2.3 for more discussion of obstructions in this dimension).

For instance, in $(\mathbb{T}^3, \ker(\cos(z)dx - \sin(z)dy))$, the torus $\{z = 0\}$ is not ξ-convex. However it can be perturbed to $T_{k,\varepsilon} = \{z = \varepsilon \sin(kx)\}$ which is ξ-convex. Indeed for small ε, the characteristic foliation of $T_{k,\varepsilon}$ has $2k$ closed leaves and all other leaves spiral between closed leaves, without Reeb component. This foliation is divided by $2k$ curves parallel to the closed leaves, which cut the surface into annuli where the foliation alternatively contracts or expands an area form.

It is also useful to consider surfaces with Legendrian boundary. The approximation property still holds, provided the Thurston-Bennequin invariant of the boundary is non-positive, and after some C^0-small isotopy near the boundary. On the other hand Mori 2011 proved that ξ-convex approximations of hypersurfaces do not always exist in higher dimensions.

In order to analyse contact structures near a ξ-convex hypersurface, it is fruitful to turn the construction inside out and start with an ideal Liouville domain (Σ, ω). Any choice of Liouville form γ and function u which is positive on $\mathring{\Sigma}$ and vanishes transversely along $\partial\Sigma$ give a contact structure $\ker(udt + u\gamma)$ on $\Sigma \times \mathbb{R}$. Over the interior, of Σ we can divide this contact form by u to get the usual contactization $\ker(dt + \gamma)$ whereas, along $\partial\Sigma$, we get $\ker(u\gamma) = \xi_{\partial\Sigma} \oplus T\mathbb{R}$. Because the space of choices (u, γ) is contractible and the result along the boundary depends on no choice, we have a notion of contactization of an ideal Liouville domain which is well defined modulo isotopy relative to the boundary.

Key observation 3.2 can then be rephrased as saying that $S \times \mathbb{R}$ is made of contactizations of ideal Liouville domains glued along their boundary. Because of the isotopy result of the previous paragraph, we get a version of Giroux's realization lemma: starting from an invariant contact structure $\xi = \ker(udt + \beta)$ on $S \times \mathbb{R}$, for any other choice $u'dt + \beta'$ with the same ideal Liouville domains ($\{u' = 0\} = \{u = 0\}$ and $d(\beta'/u') = d(\beta/u)$ elsewhere), there is a compactly supported isotopy δ in $S \times \mathbb{R}$ such the germ of ξ along $\delta_1(S)$ is the same as the germ of ξ' along S.

This is especially powerful in dimension 3 because the relevant ideal Liouville domains become surfaces, so they are completely determined by their topology, and the only remaining condition in the above paragraph is equality of the dividing sets. In this dimension we can rewrite the preceding discussion in the following terms.

Definition 3.3. *A singular foliation $\mathcal{F}$ of a surface S is divided by a multi-curve Γ if there is some area form Ω on S and a vector field Y directing $\mathcal{F}$ such that the divergence of Y does not vanish outside Γ –we set $S_\pm = \{p \in S;\ \pm\operatorname{div}_\Omega Y(p) > 0\}$– and the vector field Y goes transversely out of S_+ and into S_- along Γ.*

The dividing set Γ_X introduced above indeed divided ξS and existence of a dividing set for ξS is also sufficient for ξ-convexity as we have seen. The fact that the dividing set essentially determines the germ of contact structure is then precisely stated as the following observation, which is also valid parametrically.

Key observation 3.4 (Realization Lemma)**.** *Let S be a ξ–convex surface divided by some multi-curve Γ. For any singular foliation $\mathcal{F}$ divided by Γ, there is an isotopy δ_t with support in an arbitrarily small neighborhood of S and such that $\xi' = \delta_1^*\xi$ satisfies $\xi' S = \mathcal{F}$. Equivalently, one has $\xi\delta_1(S) = \delta_1(\mathcal{F})$.*

3.1.3. Characteristic hypersurfaces and convex contact structures

The above discussion holds for a hypersurface transverse to any contact vector field. Note however that, for a general contact vector field, it could be that no closed transverse hypersurface exists, this happens to every Reeb fields.

We now come back to the special case of pseudogradients. Suppose X is a contact vector field on (V, ξ) which is pseudogradient for some Morse function $f : V \to \mathbb{R}$. In particular there is a function $c : V \to \mathbb{R}$ such that $\mathcal{L}_X \alpha = c\alpha$. Let β be the 1-form induced by α on Σ_X.

Key observation 3.5. *The vector field X is tangent to its characteristic hypersurface Σ_X, and directs its characteristic foliation. In particular all critical points of f belong to Σ_X. At each critical point of f, $\mathcal{L}_{X|_{\Sigma_X}} d\beta = c d\beta$ and the eigenvalue of the linearization of X whose eigendirection is transverse to Σ_X has the sign of c. If i denotes the index of f at p then the index of $f|_{\Sigma_X}$ at p is i if $i \leq n$ and $i - 1$ otherwise.*

Indeed the flow of X preserves ξ and X itself so it preserves Σ_X and one easily check that it is $d\beta$-orthogonal to $\xi \cap T\Sigma_X$. At each critical point of f we have $\xi = T\Sigma_X$ hence $d\beta$ is non-degenerate. The index discussion is then parallel to the one concerning Weinstein manifolds in Key observation 2.14.

This parallel is not a coincidence. Assume that f is ordered (i.e. $f(p) < f(p')$ whenever p and p' are critical points and $\mathrm{ind}(p) < \mathrm{ind}(p')$). Let $\Sigma = f^{-1}(a)$ be a regular level set of f separating critical points of index n and $n + 1$. We will see that Σ_X is then a ξ-convex surface divided by $\Sigma_X \cap \Sigma$ into $(\Sigma_X)_\pm = \{\pm(f - a) > 0\}$. Let Y be the restriction of X to Σ_X. The above observation implies that $\pm \operatorname{div} Y$ is positive at each singular point in $(\Sigma_X)_\pm$ (recall that the divergence does not depend on the volume form at singular point so we can use $\pm d\beta^n$ there). In addition, Y has no closed orbit, or any other form of recurrence, and orbits leaving critical points in $(\Sigma_X)_+$ either go to $(\Sigma_X)_-$ or to another critical point with positive divergence. Hence the discussion surrounding Equation (3.1) applies, and $\Sigma_X \setminus \Sigma$ is Weinstein.

From a topological point of view, the pair (f, X) gives a simultaneous handle decomposition of V and Σ_X (where indices in V and Σ_X are shifted by one for high indices). One says that Σ_X is an f-essential hypersurface. The main result of Giroux 1991 is a converse in dimension 3: starting with a Morse function f, and a pseudogradient X, it modifies (f, X) until there is an f-essential surface Σ, and it builds a contact structure invariant under the flow of X (and such that $\Sigma = \Sigma_X$). Historically the first goal of the realization lemma (Key observation 3.4) was thus to inductively modify the contact structure near a regular level in order to be able to extend it to the next handle. Key observation 3.5 also implies that Σ_X cuts V into handlebodies, i.e. each component of the complement of Σ_X retracts onto a 1-complex, and that Σ_X cuts each regular level S of f into subsurfaces $S_\pm$ having the same Euler characteristic.

The modification of (f, X) to ensure existence of Σ is itself non-trivial. There is no obstruction coming from handles of index 0 and 1. A 2-handle can be attached along a circle only if this circle intersects exactly twice the dividing set of the attaching level set. When this cannot be arranged by isotopy, Giroux inserts a finite sequence of pair of critical points which are topologically in elimination position. The topology of the manifold does not change but the dividing set undergoes controlled surgery. This is the first instance of what later became bypass attachements (see Section 3.2.3). Finally, 3-handles can be attached only to a sphere with connected dividing set and this is arranged by the same trick (applied somehow backwards).

The story of this construction and the mysterious definition of f-essential hypersurfaces will be taken up in Section 3.3, from the perspective of open book decompositions.

3.2. Topological methods in dimension 3

3.2.1. Early results

Shortly after ibid., Eliashberg 1992 completed the foundations to study tight contact structures on 3-manifolds by proving uniqueness of tight contact structures on balls and the *Eliashberg-Bennequin inequality* on Euler class of a tight contact structure ξ evaluated against a surface S:

$$\begin{aligned} |\langle e(\xi), [S]\rangle| &\leq -\chi(S) \quad \text{if } S \text{ has positive genus} \\ \langle e(\xi), [S]\rangle &= 0 \quad \text{if } S \text{ is a sphere.} \end{aligned}$$

Although that paper "got inspiration from the work Giroux 1991" it uses barehanded manipulations of characteristic foliations instead of the technology of ξ-convex surfaces. The key point is the *elimination lemma* which states that, whenever two singularities of ξS have the same sign and are related by a regular leaf γ of ξS they can be eliminated as in Figure 3.1 by a C^0-small isotopy of S supported arbitrarily close to γ. This lemma

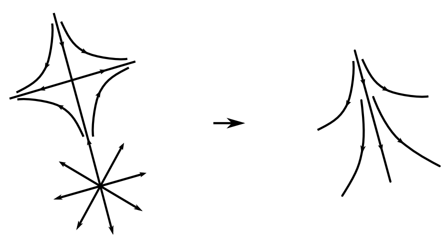


Figure 3.1.: Elimination of a pair of singular points.

of Giroux (with contributions by Fuchs) is of course inspired by Smale's elimination lemma in classical Morse theory, but also by earlier work of Eliashberg on surfaces inside complex surfaces.

Contrasting with the explicit use of the elimination lemma, the Eliashberg-Bennequin inequalities were reproved using the technology of ξ-convex surfaces as soon as Giroux 1993, and what is now called (half of) the *Giroux criterion.*

Key observation 3.6. *Let S be a closed ξ-convex surface and let Γ be a dividing set for S. If a component of $S \setminus \Gamma$ is a disk then the realization lemma can be used to exhibit an overtwisted disk, unless S is a sphere and Γ is connected.*

Indeed it is sufficient to find a characteristic foliation divided by Γ and exhibiting an overtwisted disk. Let S' be a component of $S \setminus \Gamma$ which is a disk and denote by γ its boundary. Let S'' be the other component containing γ in its boundary. Since Γ is not connected, S'' has more boundary components. Using this, one can construct a foliation $\mathcal{F}$ on S which is divided by Γ, has a circle of singularities L in S'', is radial inside a disk bounded by L and coincides with ξS outside $S' \cup S''$, see Figure 3.2.

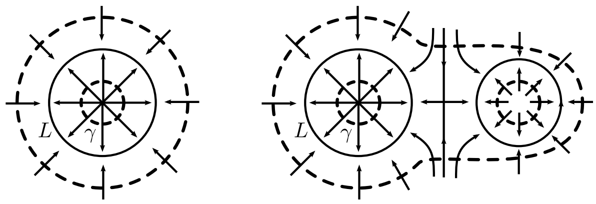


Figure 3.2.: Characteristic foliations for Key observation 3.6. The dividing set Γ is dashed. On the left-hand side one has the simplest case when S'' is an annulus. On the right hand side one sees a possible foliation when S'' has one more boundary component (on the right). Note that the disk bounded by the small component of Γ on the right may contain more components of Γ. The extension to more boundary components uses the same idea.

Hence both subsurfaces S_+ and S_- (defined in Section 3.1.2) have non-positive Euler characteristic. Since $\langle e(\xi), [S] \rangle = \chi(S_+) - \chi(S_-)$ and ξ-convex surfaces are generic, this observation can be used to evaluate the Euler class $e(\xi)$ against any homology class of surface and get the inequality.

The classification of tight contact structures on balls (hence also on $\mathbb{S}^3$) was also based on the elimination lemma, keeping track of singularities in a way reminiscent of Bennequin 1983; Eliashberg 1990a. Besides its role as the fundamental building block of almost every later classification result, this result played a key role in Colin 1999b which proved C^0 stability of contact structures on 3-manifolds: for every contact structure ξ on a closed 3-manifold, every contact structure sufficiently C^0-close to ξ is isotopic to it (not necessarily among C^0-close contact structures a priori). Recently I found a new proof of this stability result using existence of supporting open book decomposition (a result which was not yet available in 1999).

In the particular case of tori in tight contact manifold, the Eliashberg-Bennequin inequality, seen as in Key observation 3.6, only leaves as possible dividing sets collections of $2k$ isotopic essential curves. The realization lemma then allows to get a non-singular characteristic foliation which has $2k$ closed leaves and all other leaves spiral from one to another, as in the example of Section 3.1.2.

This was the crucial ingredient in Giroux 1994b to turn rigidity results for Lagrangian in cotangent bundle into rigidity results for tight contact structures on $\mathbb{T}^3$ (the use of Lagrangian rigidity builds on ideas from Eliashberg 1991a).

This strategy gives a special role to surfaces in contact manifold that can be lifted to Lagrangian surfaces in the symplectization. In any dimension, for any submanifold $P \overset{\iota}{\hookrightarrow} (M, \xi)$, a *lift* of P is a submanifold L in $S\xi$ which is transverse to the $\mathbb{R}$-action and projects onto P. Any lift of P can be seen as $\alpha(P)$ for some contact form α. Because the Liouville form λ has the tautological property $\alpha^*\lambda = \alpha$ for any α, we get that $\alpha(P)$ is an isotropic submanifold of $S\xi$ if and only if $\iota^*\alpha$ is closed. This motivates the following definition (attributed to Bennequin in Eliashberg, Hofer, and Salamon 1995): an $(n+1)$-dimensional submanifold $P \subset M$ is *pre-Lagrangian* if there is a contact form for ξ whose restriction to P is closed.

A pre-Lagrangian submanifold is everywhere transverse to the contact structure since otherwise its tangent space would be an $(n+1)$-dimensional isotropic subspace of ξ. In the 3-dimensional case, this implies that closed orientable pre-Lagrangian surfaces are tori. One easily prove they have a tubular neighborhood isomorphic to $(\mathbb{T}^2 \times [-\varepsilon, \varepsilon], \ker(\cos(z - z_0)dx - \sin(z - z_0)dy))$ where (x, y) is in $\mathbb{T}^2$ and z in $[-1, 1]$.

One of the key lessons of Eliashberg 1991a; Giroux 1994b is that, while pre-Lagrangian tori exits near any curve transverse to the contact structures, *incompressible* ones are rare and play an important structural role.

At least as early as July 1994 (according to Thomas, Eliashberg, and Giroux 1996) Giroux observed the following rigidity result, and its consequences discussed below, that eventually got published in Giroux 1999.

Key observation 3.7. *In $\mathbb{T}^2 \times \mathbb{R}$ equipped with an $\mathbb{R}$-invariant tight contact structure, all regular closed leaves of the characteristic foliation of an incompressible torus are isotopic.*

Key observation 3.2 ensures that the dividing set Γ associated to the splitting $\mathbb{T}^2 \times \mathbb{R}$ is made of $2k$ parallel essential circle. Let L be a regular closed leaf of ξT for some incompressible torus T in $\mathbb{T}^2 \times \mathbb{R}$. In particular T and ξ endow L with the same framing. Classical topology ensures that T is isotopic to $\mathbb{T}^2 \times \{0\}$. If the homology class of the projection of L onto $\mathbb{T}^2 \times \{0\}$ were not the one of the components of Γ then one could embed $\mathbb{T}^2 \times \mathbb{R}$ into the standard $\mathbb{R}^2 \times \mathbb{S}^1$ in such a way that L would bound a disk transverse to the image of T. This would contradict Bennequin's theorem.

This argument was among the first of a series of geometric arguments reducing many rigidity results to Bennequin's foundational results. In particular, after two more iterations, the above reasoning became the final form of the so-called *semi-local Bennequin inequality* of Giroux 2001a. Let ξ be a $\mathbb{R}$-invariant contact structure on $U = F \times \mathbb{R}$

where F is a closed oriented surface of positive genus ; let C be a simple closed curve on $F = F \times \{0\}$, and Γ a dividing set for ξF. For any isotopy φ in U which brings C to a Legendrian curve $L = \varphi_1(C)$, the Thurston-Bennequin invariant of L, compared to $\varphi_1(F)$, is at most $-\mathrm{i}(C, \Gamma)/2$ where $\mathrm{i}(C, \Gamma)$ is the minimal number of intersection points between Γ and a curve isotopic to C. This inequality is sharp.

Right now we only want to explain how to deduce from Key observation 3.7 the fact that, in

$$(\mathbb{T}^3, \xi_n = \ker(\cos(nz)dx - \sin(nz)dy))$$

all incompressible pre-Lagrangian tori are isotopic to the obvious ones $\{z = *\}$. This result will play an important role in Chapters 6 and 9. Let T be an incompressible pre-Lagrangian torus. Again, classical topology ensure that T is isotopic to some torus T_0 which is linear (i.e. T_0 is the image of an affine subspace of $\mathbb{R}^3$ in $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$). One observes that, if T_0 is not a constant z torus, there is an $\mathbb{S}^1$ action preserving the contact structure and transverse to T_0. Hence $\mathbb{T}^3$ is covered by $T_0 \times \mathbb{R}$ equipped with a tight $\mathbb{R}$-invariant contact structure and T lifts to a pre-Lagrangian torus there. Because of the local model described above, there are several isotopy classes of regular closed characteristics of tori close to T, contradicting Key observation 3.7.

Note that the same observation also allows to prove that ξ_n is not isomorphic to ξ_m unless $n = m$. Assume that some diffeomorphism φ sends ξ_n to ξ_m. The preceding paragraph ensures that φ preserves the homology class of $\{z = *\}$ (defined up to sign). One explicitly check that this implies φ is smoothly isotopic to a diffeomorphism ψ which preserves ξ_m. Hence we can replace φ by $\varphi \circ \psi^{-1}$ and assume that φ is smoothly isotopic to the identity. It can then be lifted to $\mathbb{R}^2 \times \mathbb{S}^1$ where z remains in $\mathbb{S}^1$ but x and y are now real valued. The complement of $\{0\} \times \mathbb{S}^1$ is again some thickened torus with an invariant contact structure. The values of n and m control the isotopy class of regular closed leaves of incompressible tori, so we can apply Key observation 3.7 again to get $n = m$.

Another important early result about these contact structures on $\mathbb{T}^3$ had to do with symplectic fillings. As discussed in Section 2.7.2, Gromov 1985 proved that any symplectic manifold containing no symplectic 2-sphere, and symplectomorphic to the standard $\mathbb{R}^4$ at infinity, is globally symplectomorphic to it. This was used in Eliashberg 1996 which described a symplectic cobordism between $(\mathbb{T}^3, \xi_n)$ and a disjoint union of n copies of $\mathbb{S}^3$ with its standard contact structure. This cobordism could be glued to a strong symplectic filling of $(\mathbb{T}^3, \xi_n)$ to get, after some more work, a contradiction to Gromov's result when n is greater than one. This proves that only ξ_1 admits a strong symplectic filling.

Gluing a symplectic filling is always possible for strong fillings but not always for weak fillings. And indeed Giroux 1994b constructed many weak symplectic fillings of $(\mathbb{T}^3, \xi_n)$ for all n. Each ξ_n is isotopic to $\ker(dz + \cos(nz)dx - \sin(nz)dy)$ which is transverse to ∂_z. The later spans the kernel of the restriction to $\mathbb{T}^3$ of any product symplectic form on $\mathbb{T}^2 \times \mathbb{D}^2$. Actually one can replace $\mathbb{D}^2$ by any surface whose boundary is $\mathbb{S}^1$ to get infinitely many weak symplectic fillings, none of which contains symplectic spheres.

3.2.2. The Eliashberg-Thurston approximation theorem

In the early 90's, an important source of interesting contact structures came from the approximation theorem in Eliashberg and W. P. Thurston 1998, stating that a C^2 foliation by surfaces on a closed 3-manifold can be approximated in C^0 topology by a contact structure with either orientation, unless it is the foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ by spheres. If the foliation is taut then the corresponding contact structure is tight because of the Eliashberg-Gromov tightness criterion and the following observation.

Key observation 3.8. *If $\mathcal{F}$ is taut then $V \times [-1, 1]$ has a symplectic structure Ω with weakly convex boundary. Any pair $\xi_\pm$ of positive and negative contact structures sufficiently C^0-close to $\mathcal{F}$ is weakly filled by Ω.*

Indeed, taut foliations on V are characterized by the existence of a closed 2–form ω whose restriction to each leaf is non-degenerate. Then we can use $\Omega = \omega + \varepsilon d(t\alpha)$ where α is a 1-form whose kernel is tangent to $\mathcal{F}$.

We will see that this result, combined with techniques originating in Gabai 1983 to construct taut foliations, had a deep impact on the developments of gluing techniques and topological applications. About the proof we only mention what is probably the starting point of this story. It uses an important intermediate class of geometric object introduced in Altschuler 1995; Eliashberg and W. P. Thurston 1998: a positive confoliation on an oriented 3-manifold is a plane field $\xi = \ker\alpha$ with $\alpha \wedge d\alpha \geq 0$.

Key observation 3.9. *Let γ be a closed curve inside a leaf of some C^k foliation $\mathcal{F}$. If γ has non-trivial linear holonomy then $T\mathcal{F}$ can be perturbed in C^k topology to a positive confoliation ξ which is contact is a neighborhood of γ and coincides with $T\mathcal{F}$ outside a slightly larger neighborhood. In coordinates $(x, y, z) \in \mathbb{S}^1 \times [-1, 1]^2$ such that $\gamma = \mathbb{S}^1 \times \{(0, 0)\}$ and ∂_y is tangent to $\boldsymbol{\mathcal{F}}$, one can push down $\boldsymbol{\mathcal{F}}$ in $\mathbb{S}^1 \times [-1 + \varepsilon, -1 + 3\varepsilon] \times [-1 + \varepsilon, 1 - \varepsilon]$ and replace it by a contact structure tangent to ∂_y in $\mathbb{S}^1 \times [-1 + 3\varepsilon, 1 - 3\varepsilon] \times [-1 + \varepsilon, 1 - \varepsilon]$, see Figure 3.3.*

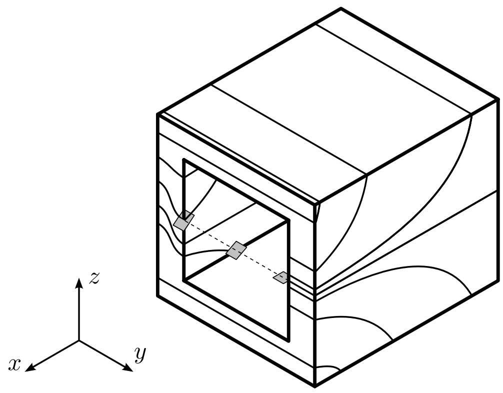


Figure 3.3.: Eliashberg-Thurston perturbation near a holonomy curve

Indeed, we can assume that $T\mathcal{F} = \ker(dz + zdx)$ and push it by $\varphi(x, y, z) = (x, y, z - \rho(y, z))$ where $\rho \geq 0$ is C^{k+1}-small, has support in $[-1+\varepsilon, 1-\varepsilon]^2$, is positive along $\{-1+3\varepsilon\} \times [-\varepsilon, \varepsilon]$ and $\partial_y \rho$ has support in $[-1+\varepsilon, -1+2\varepsilon] \times [-1+\varepsilon, 1-\varepsilon]$. So near $\{-1+3\varepsilon\} \times [-\varepsilon, \varepsilon]$ we have $\varphi_*(dz + zdx) = dz + (z - \rho)dx$ with $z - \rho < z$ so we can extend it by $dz + (z - \chi(y)\rho(-1+3\varepsilon, z))dx$ with $\chi'(y) \leq 0$ and $\chi'(y) < 0$ near γ.

3.2.3. Bifurcations and bypasses

Births, deaths, and crossings

Although individual ξ-convex surfaces can be used to distinguish isotopy classes of contact structures (e.g. as in Section 3.2.1), full classification results require to understand sweeping families of surfaces $S_t := S \times \{t\}$ in $S \times [0, 1] \subset V$, where one expects ξ-convexity to fail at least for a finite number of surfaces S_t. In this discussion all surfaces and contact structure are cooriented and the orientation of the ambient manifold is prescribed by the contact structure. In particular all characteristic foliations and their transversal arcs are oriented. Singular points also get a sign: they are points where TS coincides ξ, and are positive if those planes have the same coorientation, negative otherwise.

A special feature of families of foliations coming from sweeping surfaces in contact manifolds is that certain bifurcations are forced by the contact condition. The easiest example, which will play a major role in Chapter 8, is the *birth/death lemma*, Giroux 2000, Lemma 2.12. We will explain a weak version of this lemma, which is sufficient for our purposes.

Key observation 3.10. *Let L be regular close leaf of $S_0 := S \times \{0\}$ in $S \times [-1, 1]$. Assume that L is weakly degenerate: the first return map of ξS_0 on some arc C transverse to L is tangent to the identity at order one at $C \cap L$ but its second derivative is positive. Then there is an annulus A around L and some positive ε such that there is no closed leaves in A for t in $(-\varepsilon, 0)$ and at least two for t in $(0, \varepsilon)$.*

Indeed, the middle picture in Figure 3.4 shows a positive degenerate orbit L in ξS_0. Let A be a small annulus around L. Along L, the slope of ξS_0 is zero and it is positive in $A \setminus L$. Because ξ is a positive contact structure this slope decreases when t increases (maybe after some modification of the product structure of $S \times [-1, 1]$). So, for $t < 0$ it was everywhere positive in A, and there were no closed leaf at all in A. For $t > 0$, the slope becomes negative along L, and stays positive along the boundary of A. Then the complement of L in A is made of two (half-open) annuli whose boundaries are transverse to ξS, see Figure 3.4. The Poincaré-Bendixson theorem guaranties that each of these two sub-annuli contain at least one closed leaf for $t > 0$ sufficiently close to 0.

We now turn to the second important cause of non-convexity.

Key observation 3.11. *Let S be a ξ-convex surface, let Γ be the dividing set associated to some contact vector field transverse to S and let $S_\pm$ be the subsurfaces bounded by Γ as in Key observation 3.2. All positive (resp. negative) singularities of ξS are in S_+ (resp. S_-). There is no leaf connecting a negative singularity to a positive one.*

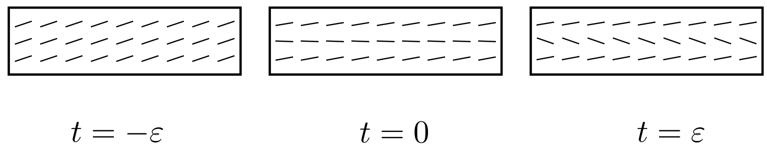


Figure 3.4.: Birth of at least a pair of periodic orbits. The annulus A is obtained by gluing left and right. The circle L is at mid-height of each annulus.

Indeed, any $\mathbb{R}$-invariant plane field on $S \times \mathbb{R}$ can be written as $\ker(udt + \beta)$ for some function u and 1-form β on S and t is the $\mathbb{R}$ coordinate. Such a plane field is a positive contact structure if and only if:

$$u d\beta + \beta \wedge du > 0$$

At any singular point p of ξS, this condition becomes $ud\beta(p) > 0$ and $d\beta$ defines the orientation of ξ. So singularities are positive in S_+ and negative in S_-. The last assertion then follows from the fact that leaves are transverse to Γ and always exit S_+ to enter S_-.

In the characteristic foliation of a surface, a *retrograde connection* is a leaf which goes from a negative singularity to a positive one. The discussion above proves that ξ–convex surfaces have no retrograde connections.

ibid., Lemma 2.14 is the *crossing lemma* which asserts that a retrograde connection is always isolated, and always looks like Figure 3.5 rather than the other way around.

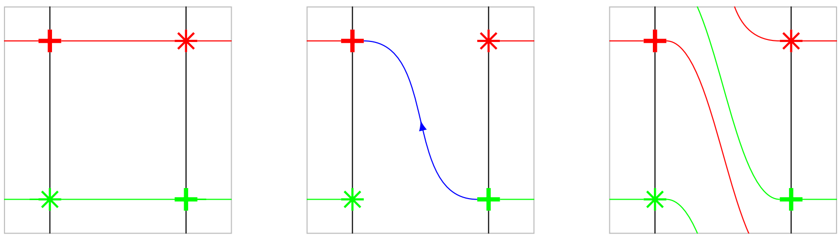


Figure 3.5.: Retrograde saddle connection in a thickened torus $\mathbb{T}^2 \times [-1, 1]$. The left-hand side and right-hand side picture show $\xi\mathbb{T}^2_{-1}$ and $\xi\mathbb{T}^2_1$ which are topologically stable. The middle picture has a retrograde saddle connection. The contact condition forces this orbit to turn to its right when t increases.

The bifurcation lemmas (birth/death and crossing), together with generic properties of one-parameter families of contact structures, the elimination lemma, and suitable versions of the realization lemma with parameters, can be applied to foliations by spheres to reprove Bennequin and Eliashberg's foundational results about tightness of the model contact structures and uniqueness of tight contact structures on balls.

We now explain a cartoon version of Giroux's proof of Bennequin's theorem because it was a very important inspiration for a key ingredient of Chapter 8. The first point is to restate the theorem in a way parallel to Reeb's theorem which characterizes the sphere in term of Morse functions. Giroux's version of Bennequin's theorem is indeed: if a contact structure ξ on a closed manifold is invariant under the flow of a vector field which is gradient for some Morse function f admitting only two critical points then ξ is tight. One can perturb any potential overtwisted disk to disjoin it from critical points of f. We know from Morse theory that all regular level sets of f are spheres transverse to X. We denote these spheres –away from two small open balls containing the critical points– by S_t, $t \in [-1, 1]$. The local model near critical points shows that the dividing set Γ_t of each S_t associated to X is connected. If we assume that ξ is overtwisted then there is a smooth isotopy φ relative to critical points and such that $\xi\varphi_1(S_0)$ has a regular closed leaf. In $[-1, 1] \times [0, 1]$ let Σ be the set of pairs (t, s) such that $\xi\varphi_s(S_t)$ has a closed leaf. By assumption, Σ intersects $[-1, 1] \times \{1\}$ but not $[-1, 1] \times \{0\}$. Let (t_*, s_*) be a point of Σ which minimizes the s coordinate in Ω. Because of the birth death lemma there is some small non-zero ε such that $\xi\varphi_{s_*}(S_{t_*+\varepsilon})$ has at least two stable closed leaves, contradicting minimality of s_*. Of course this argument is not really correct because, the projection of Σ onto $[0, 1]$ may be open and have no minimum. But it still explain how bifurcations enforced by the contact condition may be relevant. The actual proof uses also retrograde connections and information about the direction of bifurcation. This was sketched in Giroux 2000 and there are details in Massot 2014. Note that, in this situation, one can also easily prove that ξ is isotopic to the standard contact structure (without using that all tight contact structures on $\mathbb{S}^3$ are isotopic).

Normal forms on thickened tori

After the discussion of families of spheres, families of tori were used in Giroux 2000 to give normal forms for tight contact structures on torus bundles over the interval or the circle, lens spaces, and solid tori. Massot 2008b, Part 2 is a detailed introduction to these techniques. In Giroux 2000, tight contact structures on thickened tori are described using two types of building blocks: rotation sequences and orbit flips that we now briefly review.

Recall that a foliation σ on a torus T is called a *suspension* if there is a circle which transversely intersects all leaves. It then has an *asymptotic direction* $d(\sigma)$ which is a line through the origin in $H_1(T; \mathbb{R})$ spanned by limits of renormalized very long orbits of a directing vector field. For instance, if σ has a closed leaf the its homology class spans $d(\sigma)$.

Let ξ be a contact structure on $T \times [0, 1]$ and set $T_a = T \times \{a\}$. A interval $J \subset [0, 1]$ is called a *rotation sequence* for ξ if all characteristic foliations ξT_t, $t \in J$ are suspensions. We say that J is minimally twisting if the directions $d(\xi T_t)$ do not sweep out the full projective line $P(H_1(T; \mathbb{R}))$. Theorem 3.3 from ibid. guaranties that two contact structures on $T \times J$ which agree along the boundary, and have J as a minimally twisting rotation sequence, and have a non-constant asymptotic direction, are isotopic on $T \times J$ relative to boundary.

An interval $[a, b] \subset [0, 1]$ is an *orbit flip* sequence for ξ with homology class $d \in H_1(T; \mathbb{Z})$ if: ξT_a is a Morse-Smale suspension with two closed orbits whose homology

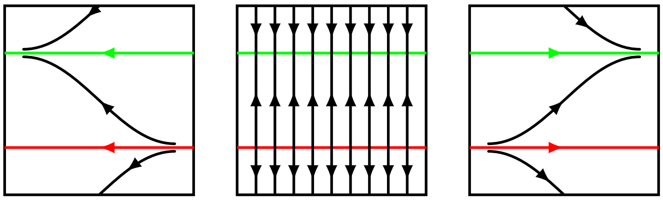


Figure 3.6.: Orbit flip. All squares have opposite edges glued to make tori. The horizontal lines are regular closed leaves in the first and last pictures while they are circles of singularities in the middle picture.

class is d ; ξT_b is a Morse-Smale suspension with two closed orbits whose homology class is $-d$; there is a multi-curve which divides all ξT_t, $t \in J$. See Figure 3.6 for an explicit model.

The strategy of ibid. is to prove, using in particular the elimination lemma and the parametric realization lemma, that any tight contact structure on a thickened torus can be decomposed into rotation sequences and orbit flips, then study how to go from one decomposition to another and, conversely, understand which sets of pieces can be stacked together to get a tight contact structure.

This strategy allows to get the list of isotopy classes of tight contact structures on this manifold, at least when both boundary components are ξ-convex and divided by only two curves. Better, Giroux was able to describe complete invariants of isotopy classes. It turns out that, besides homotopical data, only one subtle invariant was needed. The *Giroux torsion* of a contact manifold (V, ξ) is the supremum of all integers $n \geq 1$ such that there exists a contact embedding of

$$\left(T^2 \times [0,1], \ker\left(\cos(2d\pi z)dx - \sin(2d\pi z)dy\right)\right), \qquad (x, y, z) \in T^2 \times [0,1]$$

into the interior of (V, ξ) or zero if no such integer n exists. This number is invariant under isomorphisms, not only isotopies. Its definition is clearly modelled on the one of the Gromov width in symplectic geometry (see Section 2.4). Like this symplectic counterpart, the Giroux torsion is easy to bound from below but hard to bound from above, or compute exactly, without already having full classification results in terms of ad hoc invariants –like the decomposition in rotation sequences and orbit flips above. For instance ibid. proved that the torsion of ξ_d on $\mathbb{T}^3$ is $d-1$, but this needs the full classification of universally tight contact structures on thickened tori.

Bypasses

From the point of view of convex contact structures, ξ-convex surfaces correspond to regular level sets of a (local) Morse function f equipped with a contact pseudogradient. Hence the bifurcations described above should correspond to critical values. A single critical value changes the topology of level sets hence does not correspond to sweeping surfaces. However one can prove that the elementary bifurcations above correspond to pair of critical points of index 1 and 2 that are topologically in elimination position. A special case of this situation was already used in Section 3.1.3.

Yet another equivalent way of looking at those bifurcations was introduced in Honda 2000a, shortly after Giroux announced the classification results that would eventually appear in Giroux 2000. Honda defined a *bypass* as a smooth half-disk D in a contact manifold whose boundary is the union of two smooth Legendrian curves γ_1 and γ_2 from one corner to the other and such that, for some orientation, D has positive elliptic tangencies at the corners, one negative elliptic tangency on the interior of γ_1 and only positive tangencies along γ_2, see Figure 3.7.

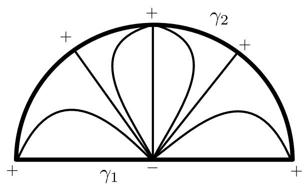


Figure 3.7.: A bypass

The name bypass comes from the fact that the contact structure rotates a full turn compared to the tangent space of the disk along the attachement arc γ_1, but does not rotate along γ_2, which is then considered to be a faster road.

A bypass D which intersects a surface S transversely along γ_1 is said to be attached to S along the *attachement arc* γ_1. Honda observed that $S \cup D$ has a regular neighborhood whose boundary consists of two ξ-convex surfaces S' and S'' isotopic to S and such that S' is contact isotopic to S while S'' has a dividing set which is obtained from a dividing set of S by an explicit surgery along the attaching γ_1, as in Figure 3.8. This surgery

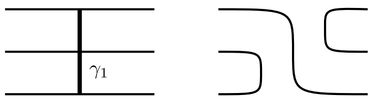


Figure 3.8.: Effect of a bypass attachement. On the left-hand side, the three horizontal lines are part of the dividing set. The right-hand side shows the new dividing set.

will be a major ingredient in Chapter 6. The main tool to detect existence of bypasses is Honda's *imbalance principle.*

Key observation 3.12. *Let $\Sigma = \mathbb{S}^1 \times [0, 1]$ be a ξ-convex annulus with Legendrian boundary inside a tight contact manifold. Assume that Σ has a dividing set Γ which has more intersection points with $\mathbb{S}^1 \times \{0\}$ than with $\mathbb{S}^1 \times \{1\}$. Then Σ is isotopic, relative to its boundary, to a surface which contains a bypass whose attaching arc is contained in $\mathbb{S}^1 \times \{0\}$.*

Indeed, the hypothesis forces Γ to contain at least one arc going from $\mathbb{S}^1 \times \{0\}$ to itself. At least one of those arcs bounds a half disk with $\mathbb{S}^1 \times \{0\}$ which contain no other component of Γ. The realization lemma then allows to isotope this half disk to a bypass.

This observation allows to decompose thickened surfaces into layers obtained by a single bypass attachement. This is especially efficient for thickened tori.

In general, existence of bypasses attached to a given Legendrian arc is difficult to check. In particular, going from γ_1 to γ_2 increases the Thurston-Bennequin invariant, hence the Bennequin inequality forbids existence of certain bypasses. The other extreme case is overtwisted contact structures, where every Legendrian arc can be connect summed with an overtwisted disk to get a bypass. This provides a new explanation of flexibility of overtwisted contact structures, see Huang 2013.

Circle bundles

Using classification results on thickened and solid tori, Giroux 2001b and Honda 2000b independently moved on to circle bundles over surfaces $\pi : V \to S$ whose topology is still dominated by tori. In view of Chapters 6 and 9, we note that, in particular, they revisited Lutz's study of $\mathbb{S}^1$-invariant contact structures on circle bundles, which we discussed in Section 2.1. Recall that Lutz proved that, up to $\mathbb{S}^1$-equivariant isotopy, those structures are characterized by the isotopy class of their dividing sets $\Gamma = \{x \in S; \pi^{-1}(x) \text{ is Legendrian}\}$. Flexibility of overtwisted contact structures shows that equivariance cannot be dropped from the classification in general.

Let ξ be such a contact structure. One can check that, for any properly embedded curve γ in S which intersects the dividing set Γ transversely (along a non-empty subset), the surface $\pi^{-1}(\gamma)$ is ξ-convex and divided by $\pi^{-1}(\gamma \cap \Gamma)$. Giroux 2001a proved that ξ is universally tight if and only if Γ has no homotopically trivial component or S is a disk or a sphere and Γ is connected (this is a generalization of Key observation 3.6). Finally, the semi-local Bennequin inequality, and Thurston characterization is isotopy classes of curves in terms of geometric intersection numbers, prove that two tight $\mathbb{S}^1$-invariant contact structures on V are isotopic (relative to ∂V) if and only if their dividing sets are isotopic (relative to ∂S). This is stated only for closed surfaces in ibid. but the proof is only easier if the boundary of S is not empty, since template matching techniques explained below are then available.

In addition to the internal analysis using sweeping surfaces which give normal forms, methods were needed to distinguish isotopy or isomorphism classes of contact structures. An idea used both in Giroux 2000, 2001b and Honda 2000a,b was called *template matching* by Honda. Suppose that V is a 3–manifold with non-empty boundary endowed with two tight contact structures ξ_0 and ξ_1 with the same germ along ∂V. In order to distinguish ξ_0 and ξ_1, it suffices to find a 3–manifold V' containing V and a contact structure ξ on $V' \setminus V$ that $\xi_0 \cup \xi$ is tight whereas $\xi_1 \cup \xi$ is overtwisted. This idea was later turned into an algebraic one using sutured Heegaard-Floer homology, see Section 4.2.3 and Chapter 6. We will also use this idea in its original topological form in Chapter 9.

Circle bundles, and more generally Seifert manifolds, have other interesting classes of contact structures. Tight contact structures are completely classified on circle bundles and there are many partial results on general Seifert manifolds. Of particular interest is the case of contact structures transverse to the fibers, especially since they have the strong property of being totally geodesic for some Riemannian metric. This aspect was studied in Massot 2008a.

3.2.4. Gluing and classifying

Classification results explained in the preceding section give building blocks (balls and thickened tori) to understand tight contact structures on 3-manifolds. It is then necessary to develop tools to guarantee that gathering several tight blocks yields a tight contact structure. Another motivation for such gluing results was to try to reprove existence results for tight contact structure that followed from Gabai's constructions of taut foliation and Eliashberg-Thurston's approximation theorem, as explained in Section 3.2.2, without going through foliations (this goal was eventually reached in Honda, Kazez, and Matić 2002).

The precursor of all gluing criteria is Colin 1997 proving that the connected sum of two tight contact manifolds is tight. Let (V_1, ξ_1) and (V_2, ξ_2) be two tight 3-manifolds and let S be a sphere separating the connected sum $V = V_1 \# V_2$. Assume for contradiction that there is an overtwisted disk D in $(V, \xi = \xi_1 \# \xi_2)$. Of course there is a smooth isotopy disjoining S from D. Colin's first observation is the following.

Key observation 3.13. *Any isotopy of embeddings of a surface S in a 3-manifold is homotopic (with fixed end points) to a concatenation of isotopies which sweep out products $S \times [0, 1]$.*

Indeed if j_t, $t \in [0, 1]$, is an isotopy of embeddings of S then each $j_t(S)$ has a tubular neighborhood which contains $j_{t'}(S)$ for t' in a neighborhood of t. By compactness one can find a finite cover of $[0, 1]$ by such neighborhoods, and use their boundary components as intermediate surfaces in the concatenation.

Hence, in order to prove tightness of connected sums of tight contact structures, it is enough to prove that, whenever we have a decomposition $V = V' \cup (S \times [0, 1]) \cup V''$ where $\partial V' = S \times \{0\}$ and $\partial V = S \times \{1\}$ are ξ-convex and both V' and $(S \times [0, 1]) \cup V''$ are tight then $V' \cup (S \times [0, 1])$ is also tight (recall that ξ-convexity can always be enforced after perturbation). Colin actually proves the stronger result that there is a contact isotopy sending $S \times \{0\}$ to $S \times \{1\}$ (using the uniqueness of tight contact structures on thickened spheres with standard boundary). The same idea can be used for disks whose boundary is Legendrian with Thurston-Bennequin invariant -1.

This idea was taken up systematically to prove tightness gluing in Honda 2000b, 2002, without necessarily having the stronger contact isotopy result at the end, but still controlling what can change in each step given by Key observation 3.13. Here it is crucial to be able to break each step into finitely many elementary steps involving only one bifurcation (or bypass attachement). This finiteness result is called Giroux's *discretization lemma* and follows from the study of generic properties of 1-parameter families of foliations in Sotomayor 1974, the bifurcation lemmas from Section 3.2.3, and careful trading of degenerate orbits against retrograde saddle connections in Giroux 2001c.

Honda calls this technique *state traversal*. It can also be used to distinguish isotopy classes of tight contact structures by proving that some ad hoc quantity is preserved under each possible elementary step. When the moving surface is not a sphere, there is no hope to prove that all bifurcations have no effect on the isotopy class of dividing set. Indeed one can always fold the surface to create pairs of dividing curves. Hence it would be useful to have general strategies to simplify a sequence of bifurcations. We will explain in Section 9.2.3 why no such strategy can exist in full generality. However, it can exist in specific examples. The most relevant one for us is *Ghiggini's torus trick*. In Ghiggini 2005b, there are two incompressible ξ-convex tori T and F intersecting transversely along a single Legendrian curve and a smooth isotopy φ. The goal is to replace φ with fixed end point by an isotopy which moves T among ξ-convex surfaces only. The trick is to apply Key observation 3.13 and the discretization lemma to F (without getting rid of bifurcations of F) and construct the new isotopy of embeddings of T along the way. This idea was generalized in Massot 2008a, and will come back in Chapter 9.

Another gluing question that is directly in line with the connected sum result is the *Legendrian surgery* question. Attaching a Weinstein 1-handle on top of a piece of symplectization replaces a neighborhood of a Legendrian knot with another one glued differently. It follows immediately from the handle picture that strong fillability is preserved by this operation. Hofer 1993 implies that *hypertightness* (existence of a Reeb vector field without contractible closed orbit) is also preserved. But this does not obviously say anything about preservation of tightness. This is a subtle question since Honda 2002 has a example of a tight manifold *with boundary* which become overtwisted after Legendrian surgery (tightness being proved by careful state traversal). But no one could embed this manifold into a closed tight contact manifold, and the question had to wait ten years before being settled, see Section 3.4.4.

In the mean time, focus shifted towards gluing along *incompressible* tori. This is the natural next step after connected sums because of structure theorems for 3-manifold (the so-called JSJ decomposition). Colin 1999a proved that *universal tightness* (i.e. tightness of the universal cover) is preserved under gluing along an incompressible pre-Lagrangian torus. All hypotheses (pre-Lagrangian, incompressible and universal tightness) are necessary in general. Note that, again, pre-Lagrangian tori play a special role.

This theorem and techniques adapted from Eliashberg and W. P. Thurston 1998 were used in Colin 2002 to construct universally tight contact structures on all *toroidal 3-manifolds*: manifolds V containing a torus whose fundamental group injects into $\pi_1(V)$. Then infinitely many of those were construct in Colin 2001b, often distinguished by their Giroux torsion in Colin 2001a. An alternative construction can be found in Honda, Kazez, and Matić 2002.

The gluing problem has later been reexamined in the context of sutured Heegaard-Floer theory, see Section 4.2.3.

3.2.5. The coarse classification theorem

The Eliashberg-Bennequin inequality on the Euler class of tight contact structures discussed in Section 3.2.1, and classification and gluing results discussed in Sections 3.2.3 and 3.2.4 led to the conjecture that, on a given closed 3-manifold, only finitely many homotopy classes of plane plane fields should contain tight contact structures, and the Giroux torsion should be responsible for all cases where there are infinitely many isomorphism classes of tight contact structures. In addition atoroidal 3-manifold should have only finitely many isotopy classes of tight contact structures. The homotopy finiteness was proved for fillable contact structure using gauge theory in Kronheimer and Mrowka 1997, see Section 4.1.1. Then Colin, Giroux, and Honda 2009 (announced in Colin, Giroux, and Honda 2003) proved the expected homotopy finiteness, the isotopy finiteness on atoroidal manifolds and the fact that for any closed 3-manifold V and any finite value n, there are finitely many isomorphism classes of tight contact structure with torsion n. However we still don't know whether every tight contact structure has finite Giroux torsion. The proofs of the main results in Colin, Giroux, and Honda 2009 are rather technical but we would like to sketch how the Giroux torsion enters the story.

The strategy of ibid. is to fix a triangulation $\mathcal{T}$ on a given closed 3-manifold V and normalize every isotopy class of tight contact structure with respect to (the same deformation of) $\mathcal{T}$. The first normalization step makes sure that each isotopy class is represented by a contact structure such that all edges of V are Legendrian, and all faces are ξ-convex. There are many possible choices here and, for each isotopy class, one selects a representative which maximizes the sum of Thurston-Bennequin invariants of all edges.

In particular each face has some dividing set and, because of uniqueness on balls and the realization lemma, each isotopy class is characterized by the combinatorics of these dividing sets.

Key observation 3.14. *The maximality enforced on Thurston-Bennequin invariants constrains the combinatorics of dividing curves. There are no dividing arc from an edge to itself (except maybe closest to vertices) and following dividing arcs around a 3-cell always lead to the same amount of spiraling.*

Stating and proving a precise version of this observation is technical, but the contradiction in case a conclusion is violated always come from finding a bypass attached to some edge, and allowing to increase the Thurston-Bennequin invariant.

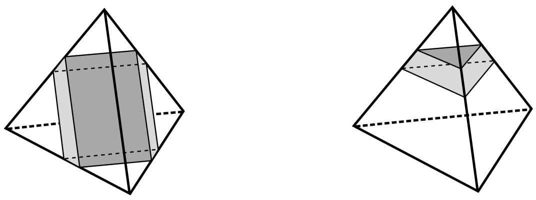


Figure 3.9.: Prisms in a triangulation

This observation, together with the realization lemma and uniqueness on balls, allows to find, for each representative ξ selected above, prisms $Y \times [0, 1]$ where Y is a triangle or a quadrilateral as in Figure 3.9 such that the contact structure is tangent to $I = [0, 1]$ and all dividing curves are contained in prisms except for a couple of them, whose number has an upper bound depending only on the triangulation. Because there are only five isotopy classes of embeddings of a prism in a tetrahedron, and only finitely many tetrahedra, we can assume when proving finiteness results that all contact structures use the same prisms. Those prisms glue together into some domain with boundary and corners fibered in intervals and the isotopy classes of contact structures tangent to this fibration are characterized by the Thurston-Bennequin invariants of these Legendrian intervals. Those invariants are bounded if the Giroux torsion is bounded otherwise one could find copies of $\mathbb{T}^2 \times I$ where the interval fibration is compatible with the fibered domain and the torsion is too large.

3.2.6. Knots in contact 3-manifolds

The proof of the main theorem of Bennequin 1983 relied on a careful study of knots in $\mathbb{R}^3$ which are transverse to the standard contact structure. It then implies constraints on *Legendrian knots*: knots that are tangent to the contact structure. So those two types of knots are sensitive to global properties of the contact structure. Their study grew as an important branch of contact topology once the topological methods of convexity theory got sufficiently developed. Early stages of these methods were used in Eliashberg and Fraser 1998 to completely understand the case of topologically trivial knots. More complete classification results from Giroux 2000; Honda 2000a could then be used to extend the class of fully understood knots, starting with Etnyre and Honda 2001. The coarse classification theorem of Colin, Giroux, and Honda 2009 also has a relative version which proves that homotopical data is enough to classify Legendrian knots in the standard $\mathbb{S}^3$ up to a finite ambiguity. Of course this finite ambiguity is the most interesting part of the story, and was investigated using both ξ-convex surface theory and Floer theoretic methods, starting with Eliashberg 1998; Chekanov 2001, and Ozsváth, Szabó, and D. P. Thurston 2008.

3.3. From convexity to open books

We take up the story of global convexity for contact structures where we left it in Section 3.1.3. The existence of an essential surface remained a rather mysterious condition (even after Giroux proved that all closed contact 3–manifolds are convex) until Torisu 2000 noticed it is linked to open book decompositions.

Let L be a fibered link with page R in a closed 3-manifold V. Let Σ be a smooth surface which is the union of two pages. In particular Σ separates V into two handlebodies H_0 and H_1. Torisu's crucial observation was the existence of an ordered Morse function f on V admitting Σ as an essential surface. Using Colin's gluing results, he also proved that the contact structure constructed using f as in Giroux 1991 is tight in restriction to both handlebodies, and characterized by this property. The characterization relied on the realization lemma and Eliashberg's uniqueness theorem on balls.

This path through classification results cannot be used in higher dimensions but it was enough to inspire the following to Giroux. Recall that any Morse function equipped with a Morse-Smale pseudogradient X can be ordered without changing X, see e.g. Laudenbach 2014.

Key observation 3.15. *Let (V, ξ) be a closed contact manifold. Assume that ξ is invariant under the flow of a vector field X which is pseudogradient for some ordered Morse function f. Let Σ be a regular level set of f separating critical points of index n and $n+1$. Then the characteristic hypersurface Σ_X is ξ-convex and divided by $K := \Sigma_X \cap \Sigma$. The part P of Σ_X below Σ is a Weinstein manifold, which is a page of an open book with binding K.*

Indeed the existence of the open book follows easily from Key observation 3.5 describing the indices of the restriction of f to Σ_X. Below (resp. above) Σ, the flow of X pushes away from (resp. towards) Σ_X at critical points, as in Figure 3.10, hence the

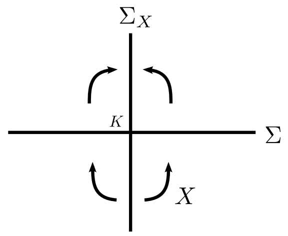


Figure 3.10.: Characteristic hypersurface Σ_X and middle level set Σ in the case of an ordered Morse function

union of level sets below and above Σ are both foliated by copies of P which form the pages of an open book. The fact that P is Weinstein was already mostly explained after Key observation 3.5.

The above observation, as well as the previous work of W. P. Thurston and Winkelnkemper 1975; Torisu 2000, all leave out the crucial question of the relation between the open book and the contact structure. In the following definitions the manifold is oriented so, together with the canonical orientation of $\mathbb{S}^1$ which coorient pages of open books, it gives an orientation to pages, and then to the binding.

Definition 3.16 (Giroux)**.** *An open book (K, θ) supports a positive contact structure ξ if:*

- *ξ induces a positive contact structure on the binding*
- *there is a contact form α for ξ such that $d\alpha$ is symplectic on all fibers, with positive orientation.*

The second condition of this definition does depend on α, and can be rephrased as saying that the Reeb field R_α is positively transverse to pages. So the structure induced on pages need some clarification. Again the cleanest statement is in terms of ideal Liouville domains. Let $F : V \to \mathbb{C}$ be a function defining (K, θ): K is the regular zero level of F and $\theta = F/|F|$. Such an F is unique up to multiplication by a smooth positive function. A compatible 1-form α being fixed, there is a non-empty convex subset of defining functions F such that $d(\alpha/|F|)$ restrict to an ideal Liouville domain structure on all pages. The induced contact structure on K is always the structure induced by ξ, it depends neither on F nor on α. This allows one to use Moser stability for ideal Liouville domains (see Courte 2015, Proposition 2.9) to prove that the ideal Liouville domain obtained on any given page is well defined up to isotopy relative to the boundary. In addition, Giroux proved that all contact structures supported by the same open book with isotopic ideal Liouville pages are isotopic. One can also construct a vector field vanishing along the binding, positively transverse to all fibers, and whose first return map is symplectic. The symplectic isotopy class of this so called monodromy map is well defined.

Conversely, starting with an ideal Liouville domain (Σ, ω) and a diffeomorphism φ supported in its interior which is symplectic, Giroux constructed a contact manifold V and an open book (K, θ) in V with page (Σ, ω) and whose monodromy is represented by φ. Again working with ideal Liouville domain avoids tweaking things near the binding.

The first lemma is that, for any Liouville form λ for (Σ, ω), one can isotope φ among compactly supported symplectomorphisms until there is some function h such that $\varphi^*\lambda - \lambda = dh$. Without loss of generality, we can assume h is positive and consider the quotient of the symplectization $\Sigma \times \mathbb{R}$ by the $\mathbb{Z}$-action generated by $(\sigma, t) \mapsto (\varphi(\sigma), t - h(t))$, which preserve the contact form associated to λ and any equation for $\partial\Sigma$ which is constant on the support of φ. The boundary of the resulting contact manifold is $\partial\Sigma \times \mathbb{S}^1$, along which the contact structure is $\xi_{\partial\Sigma} \oplus T\mathbb{S}^1$.

Here we borrow some convenient terminology from Massot, Niederkrüger, and Wendl 2013, and say that $\partial\Sigma \times \mathbb{S}^1$ is a ξ-round hypersurface. An hypersurface H in a contact manifold (V, ξ) is a *ξ-round hypersurface* modeled on some closed contact manifold (M, ξ_M) if it is transverse to ξ and admits an identification with $\mathbb{S}^1 \times M$ such that $\xi \cap TH = T\mathbb{S}^1 \oplus \xi_M$. In this definition, the word "round" is used as in "round handle". Observe that in dimension three, a ξ-round hypersurface is simply a pre-Lagrangian torus with closed characteristic leaves.

One can easily check that such an hypersurface H has a neighborhood $H \times [0, \varepsilon)$ (or $H \times (-\varepsilon, \varepsilon)$ if it is in the interior of the ambient manifold) where $\xi = \ker(\alpha_M + s dt)$ ($t \in \mathbb{S}^1$, $s \in [0, \varepsilon)$, $\xi_M = \ker \alpha_M$). When, as in the open book construction, H is a boundary component of V, we can *blow down* H to M. Let D be the disk of radius $\sqrt{\varepsilon}$ in $\mathbb{R}^2$. The map $\Psi\colon (re^{i\theta}, m) \mapsto (r^2, \theta, m)$ is a diffeomorphism from $(D \setminus \{0\}) \times M$ to $(0, \varepsilon) \times \mathbb{S}^1 \times M$ which pulls back $\alpha_M + s\,dt$ to the contact form $\alpha_M + r^2 d\theta$. Thus we can glue $D \times M$ to $V \setminus H$ to get a new contact manifold in which H has been replaced by M.

This allows one to finish the construction of the contact manifold associated to an ideal Liouville domain and a compactly supported symplectomorphism. Compared to the old fashioned construction with Liouville manifolds but without ideal Liouville domains, there is no need to extend the contact structure near the binding because the contactization of the ideal Liouville domain already involves all the needed rotation. It then suffices to blow down, without using any cut-off function.

In retrospect, one can say that W. P. Thurston and Winkelnkemper 1975 constructed contact structures compatible with open books in dimension 3, but their construction is actually not specific enough to guarantee this (they allow too much freedom in their extension near the binding).

Contact handlebodies

The contact open book picture can be seen from a slightly different perspective that is also useful, and closer to Giroux 1991.

A *contact handlebody* is a compact contact manifold (H, ξ) which admits a Morse function f (constant on ∂H) and a pseudogradient vector field X for f whose flow φ_t preserves ξ and whose limit set $\cap_{t \geq 0} \varphi_t(H)$ is a Legendrian complex L, ie a compact union of smoothly embedded isotropic submanifolds called a *Legendrian spine* of (H, ξ) (one should say a bit more about how these submanifolds fit together of course). The smooth hypersurface $R = \{X \in \xi\}$ is called a *characteristic hypersurface* of (H, ξ). As we saw, it naturally is a Weinstein manifold with Lagrangian skeleton L. The terminology comes from dimension 3 where it is customary to reserve the word handlebody for manifolds having a handle decomposition with indices at most 1. Here (H, ξ) has a decomposition into standard contact handles of index at most n in dimension $2n + 1$.

A *contact Heegaard splitting* for (V, ξ) is a triple (H, H^*, Σ) such that $V = H \cup H^*$, $\Sigma = \partial H = \partial H^*$ and $(H, \xi_{|H})$ and $(H^*, \xi_{|H^*})$ are contact handlebodies. The hypersurface Σ is then called a contact Heegaard hypersurface.

The same considerations as in the discussion of Key observation 3.15 and Figure 3.10 prove that ordered Morse function with a contact pseudogradient lead to a contact Heegaard splittings, where the Heegaard hypersurface is a regular level set separating index n and $n + 1$ critical points.

3.4. Open books in dimension 3

Uniqueness in the correspondance between contact structures and open books is better understood in dimension three.

In a 3-manifold M, let $F \subset M$ be a compact surface with boundary and $C \subset F$ a proper simple arc. We say that a compact surface $F_0 \subset M$ is obtained from F by Hopf plumbing along C if $F_0 = F \cup A$ where A is an annulus in M with the following properties: the intersection $A \cap F$ is a tubular neighborhood of C in F ; the core curve of A bounds a disk in $M \setminus F$; and the linking number of the boundary components of A is equal to ± 1. Such a plumbing is called positive or negative depending on the sign of the linking number.

According to Stallings 1978, if F is a page of an open book (K, θ) in M then any surface F_0 obtained from F by Hopf plumbing is also a page of an open book (K_0, θ_0) in M. We will say that the open book (K_0, θ_0) itself is obtained from (K, θ) by plumbing. The new monodromy is obtained by first extending the old one over $A \setminus F$ by the identity, and then composing with a positive or negative Dehn twist in A, depending on the sign of the plumbing. A stabilization of an open book (K,θ) is an open book (K_0,θ_0) that can be obtained from (K,θ) by finitely many successive Hopf plumbings.

Giroux's open book decomposition theorem in dimension 3 states that, on a closed 3-manifold M, there is a bijective correspondence between contact structures up to isotopy and open book decompositions up to isotopy and positive stabilizations. In particular any open book decomposition carries a contacts structure, unique up to isotopy. And two open books carrying isotopic contact structures become isotopic after finitely many positive Hopf plumbing of both of them. In the next two sections, we will explain some aspects of Giroux's proof, trying to highlight a few pitfalls usually not described in leisurely accounts.

3.4.1. Existence

Both the existence and uniqueness parts of the proof of the open book decomposition theorem use the contact Heegaard splitting point of view rather than directly thinking in terms of open book. Therefore we need tools to recognize contact handlebodies. Let L be a *Legendrian graph*, ie a union of Legendrian arcs and circles such that any vertex is contained in a Darboux chart $(\mathbb{R}^3, \ker dz + r^2 d\theta)$ where edges are rays in the horizontal plane $\{z = 0\}$. A *Legendrian ribbon* for L is a compact surface R with boundary which contains L in its interior, and such that one can coorient ξ and R so that ξ has no negative tangency with R, there is a flow tangent to ξR which retracts R onto L, and ξ is transverse to the boundary of R. In practice, up to shrinking the surface, it is sufficient to check that R contains L in its interior and has no negative tangency with ξ along L.

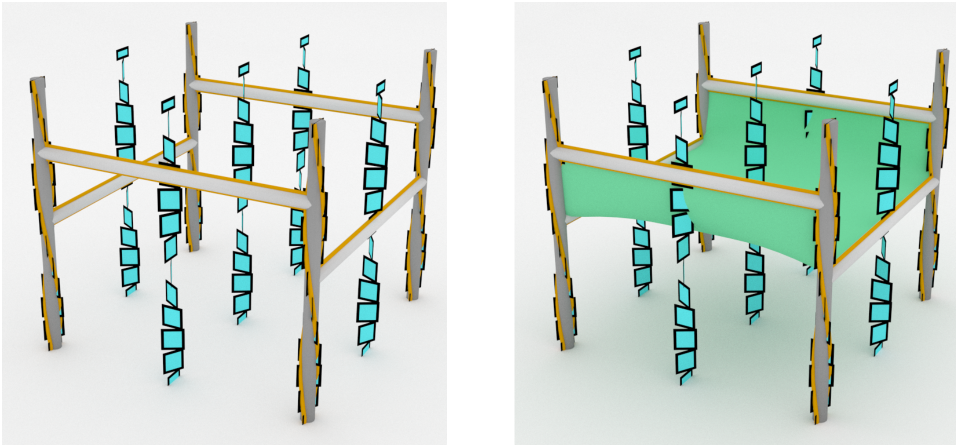
An elementary, but useful, observation is that any Legendrian graph L has a ribbon R, and any such ribbon is the characteristic surface of a contact handlebody retracting on L. This is a local way of recognizing contact handlebodies, in contrast to the following observation, essentially equivalent to the main result of Torisu 2000.

Key observation 3.17. *Let H be a handlebody, Σ its boundary and ξ a contact structure on H. The contact manifold (H,ξ) is a contact handlebody if and only if:*

- *Σ is ξ–convex and there are a dividing set Γ and a set of compression disks $D_1, \ldots,$ D_n for H, such that each ∂D_i intersects Γ exactly twice (transversely) ;*
- *the complement of the union of all compression disks is tight.*

Indeed any contact handlebody embeds into the tight $\mathbb{R}^3$ using a Weinstein neighborhood type of argument, and direct inspection gives the required compression disk. Conversely, the intersection hypothesis and the realization lemma allow to assume that all ∂D_i are Legendrian with Thurston Bennequin invariant -1, and then to normalize the contact structure along all compression disks. The complement of the union of all these disks is then a union of balls (by definition of compression disks) with controlled characteristic foliations, and we assumed they are tight so uniqueness from Eliashberg 1992 seals the characterization.

It is crucial to emphasize that the criterion is in terms of actual number of intersections, not homological number of intersections, or even number of intersections up to isotopy. This should be clear from the above sketch of proof. Overlooking this subtlety leads to very efficient “proofs” of the open book decomposition theorem.



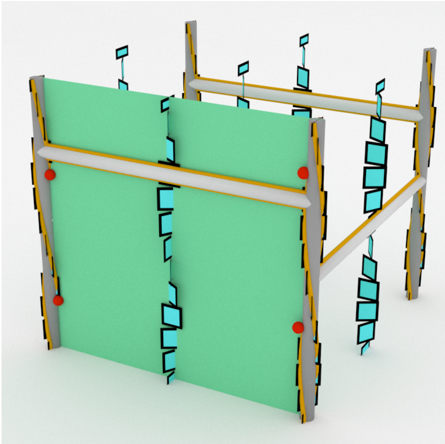
(a) A Legendrian graph whose complement is a smooth handlebody.

(b) Good compression disk.

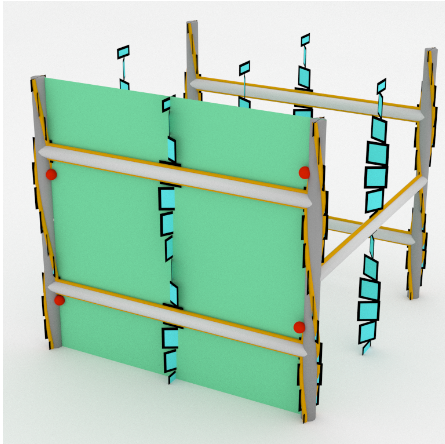
Figure 3.11.: Constructing a contact Heegaard splitting for $(\mathbb{T}^3, \xi_1)$ (I).

In principle, the existence of contact Heegaard splittings is then rather straightforward. Starting with any smooth Heegaard splitting (H_0, H_1) (given by a triangulation say) we see one handlebody as a regular neighborhood of a graph L. We first deform L until it becomes Legendrian, starting with an explicit model near vertices and then using Legendrian approximation for the interior of edges. For instance, starting with $\mathbb{T}^3$ seen as a cube with opposite faces glued by translation, we have a Heegaard splitting where one handlebody is a regular neighborhood of edges of the cube. One easily turn this graph into a Legendrian one as in Figure 3.11(a). This Legendrian L has a ribbon and a regular neighborhood H_0' which is a contact handlebody smoothly isotopic to H_0. In particular the complement H_1' of H_0 is a smooth handlebody which has a system of compression disks. Maybe after stabilizing the Legendrian L', we can assume these disks all meet a dividing set Γ of $\partial H_1'$. We want all compression disk to intersect Γ only twice, as in Figure 3.11(b), in order to apply Key observation 3.17. If there are more than two intersections, as in Figure 3.12(a), we subdivide the guilty disk into smaller one by adding more Legendrian arcs to L, as in Figure 3.12(b).

Figures 3.11 and 3.12 hide two difficulties of the general case. The minor one is that, because we know that the full $(\mathbb{T}^3, \xi_1)$ is tight, we did not worry about the tightness assumption in Key observation 3.17. This is easily arranged by starting with a graph which is the 1-skeleton of a sufficient fine triangulation so that all 3-cells are contained in Darboux charts. The serious one is we used a miracle in the subdivision of Figure 3.12(b). The miracle is not that our disk contained a convenient Legendrian arc, this actually is



(a) Bad compression disk with 4 intersection points (red balls).



(b) Disk subdivision.

Figure 3.12.: Constructing a contact Heegaard splitting for $(\mathbb{T}^3, \xi_1)$ (II).

automatic. The miracle is that the obvious ribbon for L' already contains this convenient arc, hence can be easily extended without disrupting anything. In general, the current working ribbon is isotopic to a ribbon containing the convenient arc, but this isotopy creates intersections between the boundary of the ribbon (which is the dividing set Γ) and other compression disks. This catastrophe is especially likely to happen if the convient arc has an end point which is a vertex of L'. The actual proof of existence of supporting open books first carefully cleans up neighborhoods of vertices, and then very carefully selects convenient subdivision arcs.

As discussed in Section 4.2.2, one can be tempted to rather start with a global description of the contact manifold given by a contact surgery diagram. However there does not seem to be any idea about how this could be continued to also prove the uniqueness part of the open book theorem, in contrast to the above discussion.

3.4.2. Uniqueness

The uniqueness part of the proof of the open book decomposition is a relative version of the existence part. Given two contact Heegaard splittings and associated Legendrian graphs L_1 and L_2, as in the existence proof, one rather easily constructs a third graph L_3 containing both L_1 and L_2 and whose complement is a handlebody. Then two difficulties must be addressed. First one needs to run the existence proof on L_3 with extra care to avoid disrupting L_1 or L_2 (we will ignore this problem below). Then one needs to carefully order the way new edges are added to go from L_1 or L_2 to L_3 in such a way that each move corresponds to an open book stabilization.

The first thing to explain is what is the Heegaard splitting perspective on stabilizations. A contact Heegaard splitting (H_2, H_2^*, Σ_2) is an elementary *stabilization* of another one (H_1, H_1^*, Σ_1) if there are disks D and D^* properly embedded in H_2 and H_2^* respectively and such that D and D^* intersect transversely at one point and there is a bicollar neighborhood N of D^* in H_2^* such that the boundary of $H_2^* \cup N$ is smooth and isotopic to Σ_1 through ξ-convex surfaces.

The following is the key example where stabilizations appear. Let (H_1, H_1^*, Σ_1) be a contact Heegaard splitting for (V, ξ), H_1 retracting on some Legendrian graph G_1. Let G_2 be another Legendrian graph with ribbon R_2 and let D be a disk with smooth interior and piecewise smooth boundary. Suppose that $G_1 \cap D$ is a non-empty subset of ∂D, $G_1 \subset G_2 \subset G_1 \cup D$ and $G_2 \cap D$ divides D into disks which all intersect ∂R_2 twice transversely. Then there is a contact Heegaard splitting (H_2, H_2^*, Σ_2) which is a stabilization of (H_1, H_1^*, Σ_1) and such that H_2 retracts on G_2. Note again the discussion is in terms of actual numbers of intersections between actual ribbons and disks, not in terms of homological intersection numbers or intersections after isotopy.

Given two contact Heegaard splittings, one can first stabilize them until they are associated to cell decompositions as in the existence part. In particular we have two Legendrian graphs L_1 and L_2 which are the 1-skeletons of two cell decompositions. These cell decompositions are not quite smooth at vertices (all edges come to the vertex in the same plane) but one can refine them to smooth cell decompositions and, after some small contact isotopy, assume these decompositions are in general position relative to each other. An easy variation on the uniqueness theorem from Whitehead 1940 guarantees that those decompositions can be perturbed, relative to their 1-skeleton, until they have a common refinement. Recall that a (linear) cell complex L is a subdivision of K if they coincide as subsets of $\mathbb{R}^N$ and any cell of L is contained in a cell of K. A refinement of a cell decomposition is the cell decomposition induced on a subdivision of the parametrizing linear cell complex. This common refinement obtained from Whitehead can be used as in the existence part to get a new contact Heegaard splitting. However, without further information on the refining process, there is no reason why the latter splitting should a stabilization of the original ones. The key notion here is subdivision by bissection, whose relevance to Heegaard splittings was (re-)discovered by Siebenmann 1979.

Here we will use the following ad hoc technical definition. A *special subdivision* of a cell complex K is a subdivision obtained by a finite sequence of face subdivision and 3-cell bissections. A k-cell *bissection* of a cell complex K replaces a k-cell σ by a $(k-1)$-cell F and two k-cells σ_+ and σ_- such that $\sigma_+ \cap \sigma_- = F$, $\sigma_+ \cup \sigma_- = \sigma$ and the boundary of F is a union of cells of K.

The following observation, which is the key to orderly subdivide until proving Giroux's uniqueness theorem, belongs to purely PL topology.

Key observation 3.18. *Any two cell complexes having a common refinement have a common refinement obtained by special subdivisions from both.*

Suppose K is a cell complex which is special subdivision of another one K_0. Then there is sequence of cell complexes $K_1, \cdots, K_N$ such that $K_N = K$ and, for all $i \geq 0$:

- *K_{2i+1} is obtained by subdivisions of faces of K_{2i} (or $K_{2i+1} = K_{2i}$)*

- *K_{2i+2} is obtained by bisections of 3-cells of K_{2i+1} (or $K_{2i+2} = K_{2i+1}$)*
- *every face of K_{2i+1} is a face of K.*

Indeed, the first point can be proved assuming that one cell complex is a single cell and looking at Figure 3.13 which explains the 2-dimensional case. The construction of

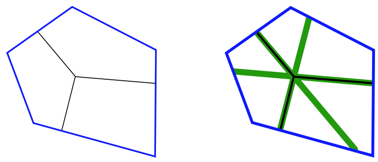


Figure 3.13.: Subdivision by bissection. On the left is a 2-cell and a complex subdividing it. On the right we have a cell complex obtained by bissections from both.

the sequence of cell complexes is done by induction on the number of steps, subdividing faces or bissecting 3-cells, that are needed to go from K_0 to K.

Using this Key observation 3.18 ensures that one can repeatedly use the key example of contact Heegaard splitting stabilization described above to finish Giroux's proof.

3.4.3. The Harer conjecture

The most direct application of the open book decomposition theorem in pure topology is the proof of Harer's conjecture announced in Giroux 2002, and detailed in Giroux and Goodman 2006. Two open books in a closed oriented three-manifold V admit isotopic stabilizations if and only if their associated oriented plane fields are homologous (ie their images in ST^*V are homologous). Besides the open book decomposition theorem (existence and uniqueness), the proof relies on the classification of overtwisted contact structures in Eliashberg 1989.

3.4.4. Tightness criterion

In principle, the open book decomposition theorem turns any question about contact 3-manifolds into a question about open books up to stabilization. In particular, questions about isomorphism classes of contact 3-manifolds are turned into questions about mapping classes of surface diffeomorphisms up to stabilization. The latter is a completely combinatorial object, and mapping class groups are much studied, but the stabilization equivalence relation is very difficult to work with. One would like to detect properties of a contact manifold by looking at any given open book decomposition. But this is not how early results looked like. The most basic ones are directly due to Giroux who proved that a contact 3-manifold is overtwisted if and only if it has some compatible open book which is a negative stabilization, and that it is Weinstein fillable if and only if it has an open book whose monodromy is a product of positive Dehn twists.

Later worked tried to improve on the above overtwisted criterion by finding less stringent condition ensuring overtwistedness. After some initial work in Goodman 2005, Honda, Kazez, and Matić 2007 proved that a contact manifold is tight if and only if all its supporting open books are *right veering*: every arc in the page is "sent to its right" by the monodromy.

The above criterion can be used to prove that a contact manifold is overtwisted but not to prove tightness since the latter would require to check *all* supporting open books. An ongoing program by Andy Wand aims to give a characterization of tightness in term of any given single open decomposition. An intermediate step in this program is to give a criterion that involves any given open book and its stabilizations, but not its destabilizations. This is explained in Wand 2015 and was enough to settle the Legendrian surgery conjecture: any contact manifold obtained by Legendrian surgery on a tight one is tight.

3.5. From open books to symplectic caps

3.5.1. Construction

In order to study contact manifolds and their various flavors of symplectic fillings, it is desirable to be able to embed any filling X into a closed manifold X'. The manifold $X' \setminus X$ is then called a symplectic cap for X (or for ∂X when this causes no ambiguity). Symplectic caps allow one to use tools and results about closed manifolds.

Lisca and Matić 1997 constructed symplectic caps for Stein fillable contact manifolds (in all dimensions). Using existence of supporting open books Etnyre and Honda 2002a constructed caps that could be glued to strong fillings of contact 3-manifolds. Still using existence of supporting open books, Eliashberg 2004 and independently Etnyre 2004a proved that any weak filling of a contact 3–manifold can be capped off.

Key observation 3.19. *Attaching 2-handles along the binding of an open book using the page framing produces a 3–manifold which is a surface bundle over the circle.*

Eliashberg proved that the smooth cobordism corresponding to this handle attachment can be endowed with a symplectic form which can be glued to any weak filling. The new 3–manifold does not have any natural contact structure in this situation and is not a convex boundary component of the cobordism (not even weakly). But it is a stable hypersurface and, using that every surface diffeomorphism of a *closed* surface is a product of left-handed Dehn twist, Eliashberg was able to cap off this cobordism by a Lefschetz fibration over the disk.

3.5.2. Planar contact structures

The construction of symplectic caps explained in the previous section opened the road to applications of symplectic topology of closed manifolds to contact manifolds. Etnyre 2004b is probably the first example of this, and highlighted the importance of *planar contact manifolds*, those contact manifolds supported by an open book whose pages embed into the plane.

Key observation 3.20. *Assume that* (V, ξ) *is a contact manifold supported by an open book with planar pages. In Eliashberg's symplectic cap, all pages can be extended to symplectic spheres with trivial normal bundle.*

Indeed Eliashberg's surgery construction turns the open book into a fibration by closed surfaces obtained by capping all binding components by disks. By definition of a supporting open book, there is a contact form α for ξ such that $d\alpha$ is positive on the interior of pages and, while α does not extend to a contact form on the surgered manifold, $d\alpha$ extends to a symplectic form positive on fibers.

As mentioned in Section 2.4, Gromov 1985 and McDuff 1990 proved that symplectic spheres with trivial normal bundle give rise to foliation by holomorphic spheres and this has strong topological consequences. Here Etnyre 2004b proved that any symplectic filling of a planar contact manifold has negative definite intersection form and connected boundary. This proves for instance that the manifold of contact elements of a closed hyperbolic surface is not planar, since McDuff 1991a proved the existence of a Liouville manifold with disconnected boundary, one of whose components is this contact elements bundle. In contrast, Etnyre 2004b proved that all overtwisted manifolds are planar. This story will continue in Section 4.3.

3.5.3. Giroux torsion and symplectic fillings

Remember from Section 2.7.2 that Eliashberg 1996 proved that, among all structures ξ_n on $\mathbb{T}^3$, only the first one admits a strong symplectic filling and this proof went through a symplectic cobordism construction. It seems that, already at that time, Giroux and Eliashberg conjectured that positive Giroux torsion always obstructs strong fillability. This conjecture received further support from classification efforts, culminating in the coarse classification theorem described in Section 3.2.5, that highlighted the role of Giroux torsion.

After several preliminary results, eg Lisca and Stipsicz 2007a, Gay 2006 managed to prove the conjecture. It used toric symplectic geometry to build cobordisms with corners that contain a nice pair of symplectic spheres and could be glued on top of any strong filling of a contact manifold with positive Giroux torsion. Then it used the symplectic cap construction of Etnyre 2004a to get a symplectic 4-manifold which is a connected sum of two manifolds, each having positive b_2^+. But this contradicts results proved in Taubes 1994 using gauge theory, as described in Section 4.1.1 below (a variation on this construction could probably get a contradiction to McDuff 1990 instead).

Besides settling an important conjecture tying the coarse classification to fillability question, this proof provided strong motivation for Wendl's cobordisms constructions described in Section 4.3 and then for Chapter 7.

4. Interactions with gauge theory and Floer homologies

4.1. Gauge theory

4.1.1. Gauge theory and symplectic geometry

Seiberg-Witten theory builds, for each smooth closed 4-manifold X, a map $\mathrm{SW}_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$ "counting" solutions of some elliptic PDE. Right from the beginning, Witten 1994 noticed that, when X is a Kähler surface, the space of solutions to this PDE is in bijective correspondance with representations of certain homology classes by sums of holomorphic curves. Taubes 1995 then proved a (much harder) version of this relation for all closed symplectic manifolds. In particular there is a canonical Spin^c structure $\mathfrak{s}_\omega$ attached to ω with $\mathrm{SW}_X(\mathfrak{s}_\omega) = \pm 1$ as well as contraints on other Spin^c structures with non-vanishing SW_X invariant.

4.1.2. Early applications to contact topology

The work of Taubes was extended to the case of symplectic 4-manifold with weakly convex boundary in Kronheimer and Mrowka 1997, where a contact structure gives a boundary condition for the Seiberg-Witten PDE. The first application of this work was finiteness of homotopy classes of plane fields containing a fillable contact structure—a precursor of the coarse classification theorem discussed in Section 3.2.5—and the corresponding result for smooth taut foliations, via the Eliashberg-Thurston theorem discussed in Section 3.2.2.

At the same time, this technology was applied to prove existence of manifolds which admit no fillable contact structure in Lisca 1998. In conjunction with progress of topological methods to prove tightness discussed in Section 3.2.4, this led to the first examples of tight non-fillable contact structures in Etnyre and Honda 2002b. In view of Chapter 7, we note right away that there is no analogue of gauge theory in higher dimensions, hence existence of tight non-fillable contact structures in higher dimension will have to rely of other techniques.

The last early application of gauge theory to contact topology was a tool developed in Lisca and Matić 1997 to distinguish isotopy classes of Stein fillable contact manifolds: if J_0 and J_1 are two Stein complex structures on the same smooth manifold X then the contact structures ξ_0 and ξ_1 induced on ∂X can be isotopic only if $c_1(J_0) = c_1(J_1)$. In order to prove this result, Lisca and Matić proved a precursor of later capping theorems: any Stein domain has a symplectic embedding into a closed projective variety. This generalizes the case of smooth affine algebraic manifolds that embed into desingularisations of their projective compactifications. The proofs uses ideas from Eliashberg and Gromov 1991 about the symplectic geometry of Stein manifolds, but also techniques from Demailly, Lempert, and Shiffman 1994 to approximate holomorphic maps by algebraic ones. This embedding result allowed Lisca and Matić to use the Seiberg-Witten equations on closed manifolds, but the same result was also proved in Kronheimer and Mrowka 1997 using their variant with contact boundary.

The Lisca-Matić rigidity of the Chern class of Stein fillings was used to prove several rigidity results, distinguishing isotopy classes of contact structures that are homotopic as plane fields. Later it was upgraded in Plamenevskaya 2004 to a proof that those contact structures are distinguished by Heegaard-Floer theory with, in dimension 4, an embedding construction using open books which was a another precursor to the capping construction of Eliashberg 2004 discussed in Section 3.5

4.2. Floer homologies

4.2.1. Flavors of Floer homology for contact manifolds

Symplectic field theory

We saw in Section 2.6.1 that Reeb fields dynamics is strongly related to Hamiltonian dynamics, therefore it was natural to try to extend Hamiltonian Floer homology to symplectizations of contact manifolds. However a new phenomenon appears. We saw in Section 2.7.2 that symplectic convexity prevents holomorphic curves to escape. But the symplectization has two ends and one of them is not convex. This difficulty was addressed in Hofer 1993.

Key observation 4.1. *For a suitable class of almost complex structures on a symplectization, the only lack of compactness arising from the non-convex end is explosion towards cylinders over homotopically trivial closed Reeb orbits.*

The immediate application of this observation was the proof of the Weinstein conjecture for overtwisted contact structures. Starting with the same family of holomorphic disks with boundary on an overtwisted disk as in Section 2.4, the only way to avoid contradiction was explosion towards closed Reeb orbits.

Even after ibid., setting up a Floer theory in symplectizations is far from obvious both from the structural point of view and because of technical issues. The structure issue is to find an algebraic way to encode the geometry of compactified moduli spaces of holomorphic curve. The technical issue is to get around the fact that those moduli spaces are not smooth objects. The structure of this theory stabilized under the name of *symplectic field theory* in Eliashberg, Givental, and Hofer 2000 (see also Cieliebak and Latschev 2009 for an alternative algebraic view of the same story). The technical issues are still the topic of very active work, see Hofer 2006; Pardon 2016, 2015. This theory has rich interactions with its symplectic counterparts, see Bourgeois and Oancea 2009.

In the mean time Hofer, Wysocki, and Zehnder 1996, 1995, 1999 refined the study of punctured holomorphic curves in symplectizations, not only pursuing Floer theoretic goals, but also more direct geometric constructions, which later resounded with contact open book decompositions. Before open books were used as a tool to study foliations and then contact structures, they appeared in dynamical systems. A surface of section for a flow on a 3 manifold is a surface whose boundary is preserved by the flow, and whose interior is transverse to it. Poincaré and Birkhoff observed that such a surface reduces the study of the continuous time dynamics of the flow to the discrete time study of the dynamics of the first return map on the surface of section. In this situation, the surface of section is the page of an open book. In Hofer, Wysocki, and Zehnder 1998, the authors proved that the Reeb flow of a strictly convex hypersurface in $\mathbb{R}^4$ admits a surface of section diffeomorphic to a disk. Pages of the corresponding open book are obtained as projections of holomorphic curves in the symplectization.

From the other direction, one can start with a compatible open book decomposition and hope to understand enough holomorphic curves to compute Floer theoretic data. This was indeed done, under some assumption, in Colin and Honda 2013.

Seiberg-Witten-Floer homology and embedded contact homology

Instanton Floer homology belongs to SU(2) gauge theory, on "Donaldson's side of gauge theory". The "Seiberg-Witten" side of this story was much longer to put on a firm basis and appeared in final form only in Kronheimer and Mrowka 2007. These *Seiberg-Witten-Floer homology* groups are defined by counting solution to some Seiberg-Witten type equations hence they can be used to guarantee existence of solutions to these equations. This was used in Taubes 2007 to prove the Weinstein conjecture in dimension three, in the spirit of Taubes's work relating Seiberg-Witten equations to holomorphic curves in dimension 4 (see Section 4.1.1). The idea is to consider a sequence of solutions to deformed Seiberg-Witten equations. The deformation involves a contact form and, under some hypotheses, the spinor component of solutions is nearly zero only on a set that closely approximates a closed Reeb orbit.

The above sketch does not directly use the algebraic structure of SWF homology groups. However, one can also recast the constructions of Kronheimer and Mrowka 1997 in this context to get invariants of isotopy classes of contact structures living in those groups.

Also, Taubes's work relating Seiberg-Witten monopoles to holomorphic curves can be adapted more fully to this context to prove that Seiberg-Witten-Floer homology is isomorphic to some homology theory counting holomorphic curves in symplectizations of contact 3–manifolds. This theory is called *embedded contact homology* and was sketched in Hutchings 2002. This name comes from its resemblance to contact homology, a part of symplectic field theory, and the fact that it mostly counts embedded holomorphic curves, but the full story is much more subtle than what this etymology suggests. The isomorphism between SWF homology and ECH was eventually fully established in a series of papers starting with Taubes 2010.

Heegaard-Floer homology

The deformation of Seiberg-Witten equations used in Taubes 2007 is not the only interesting one. Even before SWF homology was fully established, it inspired the development of *Heegaard-Floer homology*, which was meant to be some Atiyah-Floer deformation of SWF homology, but whose foundations were built in Ozsváth and Szabó 2004b without relying on any gauge theory.

Here the starting point is a *Heegaard diagram* for a 3-manifold V. Given an ordered Morse function f on V and a Morse-Smale pseudogradient X for f, the corresponding Heegaard diagram (Σ, α, β) is a surface Σ which is a level set separating critical points of indices 1 and 2, and circles $\alpha_1, \dots, \alpha_g$ (resp. $\beta_1, \dots, \beta_g$) that are the intersections of Σ and unstable manifolds of index 1 critical points (resp. stable manifolds of index 2 critical points). The surface Σ is indeed a Heegaard surface: it separates V into two handlebodies. The α circles bound disks (called compression disks) in one of them and the β circles in the other one. Hence the manifold V can be reconstructed from (Σ, α, β) by thickening $\Sigma \times [0, 1]$, attaching thickened disks to each $\alpha_i \times \{0\}$ and $\beta_i \times \{1\}$ and filling the resulting boundary by balls. The Heegaard-Floer homology group $\widehat{HF}(V)$ is (a variant of) Lagrangian Floer theory for the tori $T_\alpha = \alpha_1 \times \dots \times \alpha_g$ and $T_\beta = \beta_1 \times \dots \times \beta_g$ inside the symmetric product of g copies of Σ.

Instead of coefficients in $\mathbb{Z}$ or $\mathbb{Z}/2$, one can use coefficients in any module $\mathbb{M}$ over the group ring $\mathbb{Z}[H^1(Y, \mathbb{Z})]$. The resulting homology is called Heegaard-Floer homology with twisted coefficients in $\mathbb{M}$ and denoted by $\underline{\widehat{HF}}(V; \mathbb{M})$.

Heegaard-Floer theory is functorial with respect to cobordisms. For every pair of closed 3-manifolds Y_1 and Y_2, every 4-manifold X whose boundary is $-Y_1 \sqcup Y_2$ gives rise to a map $F_X : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$. A related construction gives an invariant of closed 4-manifolds which is conjecturally the same as the Seiberg-Witten invariant.

From the beginning, this theory was meant to be an alternative construction of Seiberg-Witten-Floer homology but, even after the latter was rigorously established in Kronheimer and Mrowka 2007, proving that they give isomorphic theory was very challenging. This was eventually proved in two independent series of papers starting with Kutluhan, Lee, and Taubes 2010 and Colin, Ghiggini, and Honda 2011. The latter crucially uses open book decompositions.

4.2.2. Contact class in Heegaard-Floer theory

Heegaard-Floer homology appeared at roughly the same time as the Giroux correspondance. In particular they were both discussed during a workshop in Oberwolfach in July 2001, where it seems that the idea of using open book decompositions to define contact invariants in $\widehat{HF}$ was born. Of course existence of these invariants was conjectured earlier, since they exist in the gauge theoretic counter part of Heegaard-Floer homology.

Ozsváth and Szabó 2005 gave two descriptions of the *contact class* $c(\xi) \in \widehat{HF}(V)/\{\pm 1\}$, in terms of knot Floer homology and using maps induced by cobordism to a fibered manifold. But its most convenient construction is due to Etnyre and Honda and is explained in Honda, Kazez, and Matić 2009a.

Key observation 4.2. *Any open book decomposition give rise to Heegaard diagrams with the following properties: the surface Σ is obtained by gluing two pages P and P' along their boundary and $\alpha_i \cap \beta_j \cap P'$ is empty if $j \neq i$ and a single point x_i if $i = j$.*

Indeed, as discussed in Section 3.4, open book decomposition give rise to Heegaard splittings. Starting from a page P of the open book, one can consider a family of disjoint arcs a_i cutting P into a disk and follow them around the open book. The span compression disks in both handlebodies whose boundary share a common arc on the page opposite P' to P. After some perturbation, one obtains a Heegaard diagram as described.

The point $x = [x_1, ..., x_g] \in T_\alpha \cap T_\beta$ is a cycle which represents the contact class. Of course one needs to prove that this class does not depend on the choice of open book and system of arcs. In particular, the uniqueness part of the open book decomposition theorem is therefore crucial.

Right from the beginning in Ozsváth and Szabó 2005, it was proved that overtwisted contact structures have vanishing contact class, because they are supported by open books that are negative stabilizations. It was also proved that Stein fillable contact structure have non-vanishing invariant. This proof used the handle decomposition discussed in Section 2.6.3. Indeed a Stein filling can be seen as a Weinstein cobordism from the standard sphere to the filled contact manifold and the associated cobordism map has to send the unknown contact invariant to the non-vanishing invariant of the standard sphere.

Compared to other Floer theories, Heegaard-Floer theory has a very topological and combinatorial flavor (according to Sarkar and J. Wang 2010, it can indeed be computed combinatorially). This is the reason why it plays nicely with open book decompositions and Weinstein handle decompositions. The non-vanishing result for Stein fillable structures was extended to weakly fillable contact structures in Ozsváth and Szabó 2004a. Here the topological description is given by Donaldson's Lefschetz fibration theorem (see Section 5.1.1) applied after capping the weak filling as explained in Section 3.5. The non-vanishing result holds only with suitably twisted coefficients, depending on the cohomology class of the restriction of the symplectic structure. For strong fillings this cohomology class vanishes, and integer coefficients are enough. This last case was also handled independently in Ghiggini 2006b.

Another topological description which interacts nicely with Heegaard-Floer theory is contact surgery diagrams. The topological effect of surgery on a knot is determined by a *framing* of the knot: a trivialization of its normal bundle. The space of framings of a cooriented knot is canonically a $\mathbb{Z}$-torsor –one can add a multiple of a meridian– and Legendrian knots have a natural choice of framing given by the contact structure. Ding and Geiges 2004a proved that every contact 3-manifold can be obtained from the standard contact structure on $\mathbb{S}^3$ by a sequence of surgeries on Legendrian knots, which are either Legendrian surgery, also called -1-contact surgery and described in Section 3.2.4, or the opposite operation called $+1$-contact surgery (here ± 1 refers the difference of framing compared to the contact one). Indeed the corresponding result in the smooth setting (without contact structure) is well known, so we have a sequence of surgery relating the given manifold to $\mathbb{S}^3$ but with unknown framings. The authors use classification results from Section 3.2.3 to turn this sequence into a longer one with framing coefficients ± 1. This allows to carry along the contact structure but the end result on $\mathbb{S}^3$ has no reason to be the standard contact structure. This is fixed by using Eliashberg's classification of contact structure on $\mathbb{S}^3$, both in the tight and overtwisted case. Admittedly, that last step is not a overly satisfying. But one can instead use the existence of a supporting open book and the fact that any surface diffeomorphism is isotopic to a product of Dehn twists to produce the required surgery diagram. The reverse strategy (going from a surgery diagram to a supporting open book) also works, see Avdek 2013 for information on both directions.

Combined with purely topological methods from Section 3.2.3, and often with contact surgery diagrams, the contact class in Heegaard-Floer turned out to be a powerful tool to distinguish tight contact structures and exhibit many tight but not fillable contact structures, e.g. in Lisca and Stipsicz 2004, 2007b; Ghiggini, Lisca, and Stipsicz 2007. It was also a crucial ingredient in Ghiggini 2005a to prove existence of strongly but not Stein fillable contact structures.

The contact class was also applied to investigate the topological content of Heegaard-Floer homology, constructing interesting elements in $\widehat{HF}$ as contact classes. Of course this strategy requires construction of interesting contact structures, and those often came from the work on Gabai constructing taut foliations, which were then turned into contact structure using the Eliashberg-Thurston approximation theorem. This theorem produces weakly fillable contact structures from taut foliations, hence it can be combined with the non-vanishing result described above. This strategy was used in Ozsváth and Szabó 2004a to prove that Heegaard-Floer homology detects the Thurston norm and the genus of knots.

A similar application of contact topology to pure topology appears in Kronheimer and Mrowka 2004 which proved the famous *property P conjecture* for knots: surgery with slope 1 on a knot K in $\mathbb{S}^3$ cannot give back $\mathbb{S}^3$ unless K is the unknot. In that case the homology theory was Floer instanton homology, but again the construction of symplectic caps for weak fillings, and the Eliashberg-Thurston perturbation theorem (using Key observation 3.8) were crucial.

Still perturbing taut foliations to contact structures and using the contact invariant, Ghiggini 2008 proved that Heegaard-Floer homology detects genus one fibered knots. A purely topological consequence was that the left-handed trefoil knot is the only knot on which surgery can produce the Poincaré sphere. Ghiggini's fiberedness criterion was quickly extended to all genus in Ni 2007

4.2.3. Sutured Heegaard-Floer homology

All those results going through Gabai's construction of taut foliations from sutured hierarchies, the Eliashberg-Thurston perturbation theorem, and contact topology, motivated the development of a direct route from sutured manifolds to Heegaard-Floer homology. This route started, after Ozsváth and Szabó 2004a but before Ghiggini 2008, with Juhász 2006 which defined *sutured Heegaard-Floer homology* SFH. The initial goals were reached in Juhász 2008, which succeeded in reproving detection of the Thurston norm and fibered knots without using contact topology.

While SFH started as an effort to reduce the need for contact topology in Heegaard-Floer theory, it actually also benefited to contact topology since it allows to work on contact manifold with ξ-convex boundary.

Gabai 1983 defined *sutured manifolds* as pairs (M, Γ) where M is a compact, oriented, not necessarily connected 3-manifold M with boundary, and Γ is an oriented embedded 1-manifold in ∂M which bounds a subsurface of ∂M. More precisely, there is an open subsurface $R_+(\Gamma) \subset \partial M$ (resp. $R_-(\Gamma)$) on which the orientation agrees with (resp. is the opposite of) the orientation on ∂M induced from M, and $\Gamma = \partial R_+(\Gamma) = \partial R_-(\Gamma)$ as oriented 1-manifolds. A sutured manifold (M, Γ) is balanced if M has no closed components, $\pi_0(\Gamma) \to \pi_0(\partial M)$ is surjective, and $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$ on the boundary of every component of M. This definition was initially introduced in order to inductively construct taut foliations, but Gabai's notion of sutures and Giroux's notion of dividing curves are clearly equivalent (it seems that Giroux was not aware of Gabai's definition in 1991, and Gabai was not aware of Lutz's work in 1983).

Sutured Floer homology is an invariant of balanced sutured manifold which is the analogue of $\widehat{HF}$. Actually one can recover $\widehat{HF}$ by deleting a ball from a closed manifold and endowing the resulting boundary sphere with a connected suture. Honda, Kazez, and Matić 2009b extended the definition of the contact class to the setup of sutured Heegaard-Floer theory. As in the closed case, one can use twisted coefficients. Whenever there is no ambiguity on the manifold M we are considering, we denote $\mathbb{Z}[H_2(M; \mathbb{Z})]$ by $\mathbb{L}$. We denote the universally twisted contact class by $\underline{c}(\xi)$, it lives in $\underline{\mathrm{SFH}}(-M, -\Gamma)/\mathbb{L}^\times$. If a contact invariant vanishes with coefficients in $\mathbb{L}$ then it vanishes for all coefficients rings.

Of course the ultimate goal of sutured invariant would be to reconstruct the contact class of closed contact manifold from the sutured contact class of pieces of the manifold. In general this is too much to hope for, there is not enough information in SFH. Even without the contact structure, reconstructing $\widehat{HF}$ from information on pieces requires the much heavier technology of bordered Heegaard-Floer theory from Lipshitz, Ozsváth, and D. P. Thurston 2011. However, Honda, Kazez, and Matić 2008 introduced gluing maps in SFH, and they were extended to twisted coefficients in Ghiggini and Honda 2008. If (M', Γ') is a sutured submanifold of (M, Γ) and ξ is a contact structure on $(M \setminus M', \Gamma \cup \Gamma')$, then there exists a linear map

$$\underline{\Phi}_\xi : \underline{\mathrm{SFH}}(-M', -\Gamma') \to \underline{\mathrm{SFH}}(-M, -\Gamma)$$

such that, for any contact structure ξ' on (M', Γ'), one has

$$\underline{c}(\xi \cup \xi') = \underline{\Phi}_\xi(\underline{c}(\xi')).$$

If every connected component of $M \setminus int(M')$ intersect ∂M then there are analogous maps over $\mathbb{Z}$ coefficients. They are denoted without underlines.

In particular, vanishing of $\underline{c}(\xi')$ implies vanishing of $\underline{c}(\xi \cup \xi')$. This allows to reprove vanishing of the contact class for overtwisted contact structures, simply by computing the sutured contact class of a neighborhood of an overtwisted disk. The same idea allowed Ghiggini, Honda, and Van Horn-Morris 2007 to prove that contact structures with positive Giroux torsion have vanishing untwisted contact class. Combined with the non-vanishing result for strongly fillable contact structures, this reproves Gay's theorem that positive Giroux torsion obstructs strong fillability. More refined information about Giroux torsion was then obtained in Ghiggini and Honda 2008 with twisted coefficients.

Given the importance of torsion in the coarse classification and as a strong filling obstruction, one could optimistically believe that it explains all cases of tight contact structure with vanishing untwisted contact class. This belief was disproved over $\mathbb{Z}/2$ coefficients in Honda, Kazez, and Matić 2008 and over $\mathbb{Z}$ coefficients in Chapter 6.

4.3. Foliations by holomorphic curves

Isolated intersections between two holomorphic curves in a 4-dimensional almost complex manifold contribute positively to their homological intersection number. This is obvious from linear algebra when we consider two immersed holomorphic curves intersecting transversely, but it also holds in general according to Gromov 1985; McDuff 1991b; Micallef and White 1995. This property is very useful in order to guarantee that spaces of holomorphic curves faithfully describe a closed symplectic manifold, and was used from the beginning in Gromov 1985; McDuff 1990.

In particular, if an embedded holomorphic curve C has topologically trivial normal bundle and belong to a 2-dimensional family, then this family foliates a neighborhood of C thanks to positivity of intersection. In order to use this observation, one needs holomorphic curves to start with. The following elementary observation provide some.

Key observation 4.3. *A smooth surface S in a symplectic manifold (X, ω) is symplectic if and only if there exists an ω-tame almost complex structure J such that S is the image of an (embedded) J-holomorphic curve.*

However, this observation is almost never useful if one cannot guarantee that this holomorphic curve (or a small deformation thereof) persists under perturbations of the almost complex structure that could be needed to enforce other useful properties.

This is where enters a second specifically 4-dimensional phenomenon: *automatic transversality.*

Key observation 4.4. *Let $(E, J) \to (\Sigma, j)$ be a complex line bundle over a Riemann surface and let D be a Cauchy-Riemann type operator on E. If $c_1(E) > -\chi(\Sigma)$, then D is automatically surjective.*

Indeed, the cokernel of D is the kernel of an adjoint operator D^* attached to a bundle E' having negative first Chern class. The local analysis of holomorphic curves ensures that elements of $\ker D^*$ behave like holomorphic sections of (integrable) complex manifolds, hence zeros of a non-zero section would contribute all contribute positively to $c_1(E)$.

We already saw in Section 3.5.2 that foliations by closed holomorphic curves can be used to study contact manifolds. But it was clear from Hofer 1993 and Hofer, Wysocki, and Zehnder 1998 that extensions of this story to punctured holomorphic curves would be desirable, especially in relation with open book decompositions.

The first result proved using open book based punctured holomorphic curves was the Weinstein conjecture for planar contact structures in Abbas, Cieliebak, and Hofer 2005. Starting with a planar open book (K, θ) and a compatible contact form α, a construction eventually published in Abbas 2011; Wendl 2010b gives an almost complex structure on the symplectization $S(\ker \alpha)$ and a foliation by J-holomorphic curves which is invariant under the action of translations and whose leaves project either onto K or onto fibers of θ (this actually holds after some isotopy of (K, θ)). In order to deduce existence of Reeb orbits for all contact form, one studies what happens to these curves when the almost complex structure changes.

In the above argument the *holomorphic open book* doesn't survive the deformation, only existence of periodic orbits does. In cases were the actual goal is to keep a foliation by holomorphic curves, we need analogues of the positivity and automatic transversality arguments. Wendl 2010a extended the automatic transversality criterion to punctured holomorphic curves. Siefring 2011 defines a suitable intersection product between punctured holomorphic curves, accounting for intersection points hidden at infinity.

Inspired by Etnyre 2004b and Abbas, Cieliebak, and Hofer 2005, and using the technology of intersection theory and automatic transversality, Wendl 2010c reproved that positive Giroux torsion obstructs strong fillability (from Gay 2006), obstructions to planarity from Etnyre 2004b and much more. The point is that foliations by holomorphic curves form actual geometric structures, that do not necessarily lead to contradictions. As seen in Gromov 1985; McDuff 1990, when they bring no contradiction, these foliation tell a lot about the geometry of their home symplectic manifold. In particular Wendl 2010c used this to prove that any minimal strong symplectic filling of a planar contact manifold has the structure of a Lefschetz fibration over the disk, which restricts to any given planar open book on the boundary. In particular such a filling is actually Weinstein. In addition, this provides one of the prime examples where open book decompositions turn a geometrical problem into a combinatorial one. Indeed Wendl's theorem proves that, in the case of planar contact structures, Weinstein fillings are in one to one correspondence with factorizations of the monodromy of any given planar open book into products of positive Dehn twists. The latter problem is purely about mapping class groups and has received quite a bit of attention since then, see e.g. Wand 2012.

Wendl 2010c was extended to weak fillings in Niederkrüger and Wendl 2011, under certain cohomological conditions that guarantee energy bounds. This paper also gave yet another proof of the Giroux torsion obstruction, and its refined version from Ghiggini, Honda, and Van Horn-Morris 2007, using holomorphic annuli with moving boundary conditions.

In the cases described above, the starting point was an already known special topological feature of the contact manifold, that gave rise to interesting holomorphic curves, and those curves in turn give topological information about symplectic fillings. The next step was to extract the minimal part required by holomorphic curves, and turn it into new definitions of topological properties. This was accomplished in Wendl 2013a which defined partially planar domains as filling obstructions. The holomorphic curves emitted by those domains were used in the context of SFT in Latschev and Wendl 2011 where they provided obstructions to strong symplectic cobordisms. Again this had been turned inside out in Wendl 2013b which took inspiration from the capping construction of Eliashberg 2004, and expected properties of holomorphic curves, to construct *weak* symplectic cobordisms based on partially planar domains. This is an example of holomorphic curve driven differential topology: holomorphic curves are not used in the construction of the symplectic structure on these cobordisms, but they strongly guide the intuition. Indeed the symplectic structure is built in such a way that it allows the holomorphic curves which obstruct the existence of a strong symplectic cobordism.

A special case of these constructions proved that any contact manifold with positive Giroux torsion is weakly symplectically cobordant to an overtwisted one, a result which was essentially proved in Gay 2006. This construction will be generalized to higher dimension in Chapter 7, with some suitable generalization of Giroux torsion.

The foliation by holomorphic curves associated to partially planar domains were also used in Wendl 2013a to compute contact classes in embedded contact homology. Those computations can be compared to the computations explained in Chapter 6 that appeared at the time on the Heegaard-Floer side.

5. Higher dimensions

5.1. Higher dimensional open books

5.1.1. Transversality and fibrations

While Gromov 1985 revolutionized symplectic topology through holomorphic maps from complex curves into symplectic manifolds, Donaldson 1996 took a somewhat dual approach and studied maps from symplectic manifolds to line bundles. An easy dimension count shows that one cannot hope for such maps to be holomorphic when the almost complex structure on the source is not integrable (on the other extreme, integrability of the complex structure obviously ensures existence of local holomorphic functions). However, Donaldson observed that holomorphicity is not strictly required to get symplectic information.

Key observation 5.1. *Let $A\colon \mathbb{C}^n \to \mathbb{C}$ be a $\mathbb{R}$-linear map with complex linear (resp. antilinear) part $A^{1,0}$ (resp. $A^{0,1}$). If $\|A^{0,1}\| < \|A^{1,0}\|$ (w.r.t the Euclidean metric on $\mathbb{C}^n$) then* $\ker A$ *is a codimension* 2 *symplectic subspace.*

Indeed one can consider the adjoint map $A^*\colon \mathbb{C} \to \mathbb{C}^n$ and compute that $\omega_0(A^*1, A^*i) = \|A^{1,0}\|^2 - \|A^{0,1}\|^2$ so the assumption implies that $\operatorname{im} A = \operatorname{Span}(A^*1, A^*i)$ is symplectic hence so is $\ker A = i(\operatorname{im} A^*)^{\omega_0}$.

So, building so-called *approximately holomorphic* sections of line bundles is enough to get codimension two symplectic submanifolds as vanishing loci. We saw in Section 2.6.2 that, in the integrable situation, existence of holomorphic sections is related to positivity of the line bundle. The model situation is on $\mathbb{C}^n$ equipped with a hermitian line bundle which is trivial but with connection 1-form $A = \frac{1}{4}\sum_{\alpha=1}^n (z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)$. This connection has curvature positive curvature ω_0 and defines the Cauchy-Riemann type operator: $\bar\partial_A = \bar\partial + A^{0,1}$ which admits the holomorphic *peak section* $f = e^{-|z|^2/4}$.

The next key observation from ibid. is that, as in Gromov 1985, non-integrability of almost complex structure mostly disappear if one zooms in sufficiently. If $\tilde{J} = \delta_k^* J$, where δ_k is the dilation $z \mapsto k^{-1/2} z$, then $|\bar\partial_{A,\tilde{J}} f| \leq C k^{-1/2} |z|^2 e^{-|z|^2/4}$.

Assume now that (X, ω) is a closed symplectic manifold and ω is rational: some integer multiple of the cohomology $[\omega/2\pi]$ admits a lift to $H^2(X;\mathbb{Z})$. Then one can consider the associated line bundle $L \to X$ and use the previous observations to construct many approximately holomorphic sections of $L^{\otimes k}$ for k large enough. The most difficult part, which does not come from non-integrability, is then to find sections s_k with the additional property that the complex linear part of the derivative is not too small along $s_k^{-1}(0)$, so that Key observation 5.1 applies. This so-called *quantitative transversality* construction relies on local results on the complexity of real algebraic sets, and a clever globalization process.

The main output of this work is thus a codimension 2 symplectic submanifold Σ whose homology class is Poincaré dual to $k[\omega/2\pi]$ for some large k. Convexity already played a role in the construction by allowing the existence of (approximately) holomorphic sections, but it also came back in the conclusion when Giroux 2002 proved that, for k large enough, the complement of Σ is Weinstein.

This decomposition gives structure to an otherwise locally homogeneous symplectic manifold, and allows to hope for inductive arguments since Σ has lower dimension, and its complement has extra structure coming from Weinstein handle decompositions. These techniques were pushed further by Donaldson and Auroux, especially in dimension 4. In particular Donaldson 1999 proved that a closed manifold with integral symplectic form can be blown up until it admits the structure of a symplectic Lefschetz fibration over $\mathbb{C}\mathrm{P}^1$, and this result was crucial in Ozsváth and Szabó 2004a, as explained in Section 4.2.1.

Those ideas were adapted to the contact setting in Ibort, Martínez Torres, and Presas 2000 to construct codimension 2 contact submanifolds. In the mean time Giroux had reformulated contact convexity in terms of symplectic open book decompositions, and the result from ibid. was quickly strengthened to existence of supporting open books in all dimensions. The setup is as follows. Let (V, ξ) be a closed contact manifold. Let $L = V \times \mathbb{C} \to V$ be the trivial hermitian complex line bundle over V equipped with the unitary connection defined by $-i\alpha$ for some contact form α on V. Let g be a Riemannian metric on V compatible with ξ. For any section s of L, denote by $\partial_\xi s$ and $\bar\partial_\xi s$ the complex linear and complex antilinear parts of $ds|_\xi$.

Key observation 5.2. *Let s_k be a sequence of sections of $L^{\otimes k}$ such that, for some constants C and η:*

- *at every point of V, $|s_k(p)| \leq C$,* $|ds_k - iks_k\alpha| \leq Ck^{1/2}$ *and* $\left|\bar\partial_\xi s_k\right| \leq C\,;$
- *at every point p where $|s_k(p)| \leq \eta$, $\left|\partial_\xi s_k(p)\right| \geq \eta k^{1/2}$.*

Then, for sufficiently large k, $K := s_k^{-1}(0)$ is a contact submanifold of (V, ξ), $\arg s_k : V \setminus K \to \mathbb{S}^1$ is fibration and $d\alpha$ is symplectic on its fibers.

Indeed those inequalities prove that, for k large enough s_k is transverse to zero, and $TK \cap \xi$ is close to a J-complex subspace, hence $d\alpha$-non degenerate, hence K is a contact submanifold. Next these estimates imply that $ds_k(R_\alpha)$ is close to iks_k, and this implies the claim about fibers.

Related ideas were used in Giroux and Pardon 2014 in order to prove existence of Lefschetz fibrations on Stein domains.

As in dimension 3, supporting open books are not unique up to isotopy and there is a standard stabilization procedure. However, one does not have a full strength uniqueness up to stabilization. Before describing this procedure, we need to recall the construction of the symplectic Dehn twist in the disk cotangent bundle $DT^*\mathbb{S}^n$. Let u be a vector tangent in $DT_q\mathbb{S}^n$ for some point q. There is a unique vector v normal to $\mathbb{S}^n$ at q, inward pointing, and such that $w = u + v$ has unit norm. One can move in $\mathbb{R}^{n+1}$ in the direction of w until one hits again $\mathbb{S}^n$ at some point q'. The projection of w onto $T_{q'}\mathbb{S}^n$ is the image of u under the positive Dehn twist, see Figure 5.1 for a picture with $n = 1$.

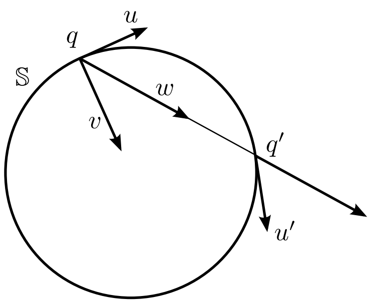


Figure 5.1.: The Dehn twist on $DT\mathbb{S}^1$

The stabilization procedure starts with a Lagrangian disk L properly embedded in the page P of a supporting open book, with Legendrian boundary Λ in the binding. One can attach a Weinstein handle to the page P along Λ to get a new page P'. The Lagrangian disk L can then be extended by the Lagrangian core of the handle to get a Lagrangian sphere L' in P'. The new monodromy is obtained from the obvious extension of the old one to P' by composition with a positive Dehn twist in a Weinstein neighborhood of L'.

If one uses instead a negative Dehn twist then the new open book is called a negative stabilization of the original one. Based on the 3-dimensional case, Giroux conjectured that contact manifolds supported by negatively stabilized open books could be the correct generalization of overtwisted contact 3-manifolds. Evidence in support of this conjecture came in Bourgeois and Koert 2010 that proved that such contact manifolds have vanishing contact homology (provided that contact homology exists). This story will continue in Section 5.2.2.

5.1.2. The existence quest

A fundamental but difficult question in contact geometry is to understand which manifolds admit contact structures. An obvious necessary condition is the existence of a *almost contact structure*: a hyperplane field equipped with a symplectic structure. It is necessary since any contact form α determines a symplectic structure $d\alpha$ on $\ker\alpha$. Key observation 2.6 ensures that a vector bundle admits a symplectic structure if and only if it admits a complex structure, hence this condition can also be used to characterize almost contact structures. These structures are also equivalent to reductions of the structure group of the tangent bundle to $U(n)$ hence it can be decided using obstruction theory and its characteristic classes. In the late sixties, general techniques explained in Gromov 1986; Eliashberg and Mishachev 2002 allowed Gromov to prove that, on open manifolds, these conditions are sufficient, and any almost contact structure is homotopic to a contact structure. Lutz 1970 and Martinet 1971 extended this result to closed 3-manifolds. Then the question stalled for a very long time.

We saw in Section 3.1.3 that, in dimension 3, contact convexity can guide the construction of a contact structure. Starting with a suitable Morse function and pseudogradient, Giroux 1991 built a contact structure by induction over handles. For each new handle, one needed to modify the contact structure provided by the induction hypothesis before being able to solve the handle extension problem. This modification was the historical motivation for the realization lemma (Key observation 3.4). In higher dimensions, one can still describe model contact handles, but the required modifications are much harder to perform, because the hypotheses from the realization lemma become genuinely symplectic and not only topological. This prevented this strategy to construct contact structures on higher dimensional manifolds.

The situation seemingly improved a lot when contact convexity was reformulated in terms of symplectic open book decompositions. Indeed it was already known, from Lawson 1978; Quinn 1979, that any $2n+1$-dimensional closed orientable manifold has an open book decomposition where pages have a handle decomposition with handles of index at most n. Giroux then proved that, under the assumption that the manifold admits a hyperplane field equipped with a complex structure, those pages can be assumed to admit an almost complex structure. These results allow one to use the topological characterization of Stein manifolds from Eliashberg 1990b to get a Stein structure on the page. Hence the comment in Giroux 2002: "Toute la difficulté serait donc vraiment de réaliser la monodromie par un difféomorphisme symplectique...".

At the same time, Frédéric Bourgeois was reading old papers constructing explicit examples of contact structures in higher dimensions, in order to find examples where he could compute contact homology using his Morse-Bott techniques from Bourgeois 2002a. Reading Lutz 1979 in this atmosphere, he realized that an open book decomposition for a contact manifold (M, ξ) could be used to construct a contact structure on $M \times \mathbb{T}^2$.

Key observation 5.3. *Let α be a contact form compatible with an open book (K, θ). Let $K \times \mathbb{D}^2$ be a tubular neighborhood of the binding K where the projection θ is the angular coordinate. For a suitable function f, the 1-form $\alpha + f(r)(cos\theta dx_1 - sin\theta dx_2)$, $x = (x_1, x_2) \in \mathbb{T}^2$, is contact on $\mathbb{T}^2 \times M$.*

This follows from a short computation in Bourgeois 2002b but was quite a surprise at that time. In addition, the contact structure obtained has quite a lot of structure. It is invariant under the obvious $\mathbb{T}^2$ action (this is what Lutz was interested in) and defines a *contact connection* on the *contact fibration* $M \times \mathbb{T}^2 \to \mathbb{T}^2$. The latter means that each fiber M_x is equipped with a contact structure ξ_x, and there is an Ehresmann connection H such that parallel transport preserve the contact structure on fibers. Such contact fibrations were systematically studied in Lerman 2004. They are easy to construct, here one can use $H = T\mathbb{T}^2$, but the special property obtained in Bourgeois's construction is that $\xi = \xi_x \oplus H$ is a contact structure on the total space.

In particular, since the fibers are codimension 2 contact submanifolds, one can perform contact branched covering, as described in Gromov 1986; Geiges 1997b, to get contact structures on $M \times \Sigma$ for any closed orientable surface Σ of positive genus.

Thanks to the existence of supporting open book for all closed contact manifolds, the above construction can be applied inductively. In particular it settled the old question of existence of contact structures on all odd dimensional tori.

Donaldson's transversality techniques and contact fibrations were used, in combination with flexibility of overtwisted 3-manifolds, to prove in Casals, Pancholi, and Presas 2015 that any almost contact structure on a closed 5-manifold is homotopic to a contact structure. However, the general existence criterion stayed out of reach until the events described in Section 5.2.2.

5.2. Overtwisted contact structures in higher dimensions

5.2.1. Plastikstufes

We saw, in Sections 2.4 and 4.2.1, that holomorphic disks with boundary on an overtwisted disk played a great role to prove non-fillability and the Weinstein conjecture for overtwisted 3-manifolds. Gromov 1985, Section 2.4.D_2' actually alludes to possible extensions of this non-fillability proof. Independently from this source, this idea was tackled in Niederkrüger 2006 which proposed a definition of overtwisted contact structures in higher dimension guided by requirements of holomorphic curves.

Key observation 5.4. *Let (V, ξ) be a contact manifold, α a contact form for ξ and J an almost complex structure on $S\xi$ which is compatible with α: if one identify $S\xi$ to $\mathbb{R} \times V$ using α then J preserves ξ and is tamed by CS_ξ, sends ∂_t to R_α and is $\mathbb{R}$-invariant. Let N be a submanifold in V. If ξ induces a foliation on N then the lift $\alpha(N)$ is totally real in $(S\xi, J)$.*

Indeed, if $\alpha(N)$ is not totally real then there exists a non-zero u in TN such that Ju is also in TN. Since $JR_\alpha = -\partial_t$, these u and Ju are necessarily in ξ so the tameness condition gives $d\alpha(u, Ju) > 0$, contradicting the Frobenius integrability criterion for $TN \cap \xi$.

More generally, one says that a submanifold N in $(V, \xi = \ker \alpha)$ is maximally foliated if the restriction of $\alpha \wedge d\alpha$ to TN vanishes and N has the maximal dimension $(\dim V + 1)/2$ allowed by this property. In this case ξ induces a singular foliation on N and the lifts considered in Key observation 5.4 is totally real outside singular points of this foliation. Suitable codimension two singularities emit holomorphic disks as in Section 2.4 and suitable codimension one singularities block them. All these properties were combined in the definition of a *plastikstufe*. A plastikstufe with singular set S in (V, ξ) is an embedding of $\mathbb{D}^2 \times S$ on which ξ prints a singular foliation which is a product of the overtwisted disk foliation on $\mathbb{D}^2$ by S. Contact manifolds containing a plastikstufe were later called PS-overtwisted.

This definition is crafted so that holomorphic disk attached to the lifted plastikstufe exist and have very controlled properties. Because the lifted manifold is totally real but not Lagrangian, one must also carefully control energy of these holomorphic curves, but this follows from geometric control coming from the maximum principle, and the fact that the symplectic structure of a (strong) symplectic filling would restrict to $d\alpha$ on V. The contradiction is then essentially the same as in dimension 3.

The proof of the Weinstein conjecture for overtwisted contact structures in Hofer 1993 was generalized to PS-overtwisted manifolds in Albers and Hofer 2009. More generally, Bourgeois and Niederkrüger 2010 announced that PS-overtwisted manifolds have vanishing SFT.

All these results provided a strong hint that the definition of PS-overtwisted contact manifold, or a variation thereof, could be the correct extension to overtwistedness to higher dimensions. However it was initially unclear whether closed PS-overtwisted manifolds existed in higher dimensions. The first example were constructed in Presas 2007 using contact fibrations.

Key observation 5.5. *Let $V \to \Sigma$ be a contact fibration with fiber (M, ξ_M) and let ξ be a contact structure on V which is a contact connection. If (M, ξ_M) is an overtwisted 3-manifold, and there is an embedded loop $\gamma \subset \Sigma$ with trivial holonomy, then (V, ξ) is PS-overtwisted.*

Indeed one can parallel transport an overtwisted disk over γ to get a plastikstufe. This construction was used as the starting point of Niederkrüger and Koert 2007 to construct PS-overtwisted contact structures on all spheres. Hence, using connected sums, any manifold carrying some contact structure carries a PS-overtwisted one.

The above parallel transport construction requires some definite amount of space around the fiber, it cannot be performed in a small standard neighborhood $(M \times \mathbb{D}^2_\varepsilon, \ker(\alpha_M + r^2 d\theta))$. This observation, together with the hunt for plastikstufes in negative stabilizations, motivated Niederkrüger and Presas 2010 to investigate large standard neighborhoods of overtwisted submanifolds. That paper proved that such neighborhood contain some special kinds of immersed plastikstufes that were already obstructing symplectic fillability (this kind of objects will be subsumed in Chapter 7). It also led Niederkrüger 2013 to conjecture that large neighborhoods of 3-dimensional overtwisted disks should be the correct definition of overtwistedness in higher dimensions.

5.2.2. Loose Legendrian submanifolds and overtwisted disks

We saw in Section 5.1.1 that the construction of contact manifolds out of an symplectic open book and the topological characterization of Weinstein manifolds reduced the question of existence of contact structures in higher dimensions to a question of isotopy of diffeomorphisms to symplectic one. One particularly optimistic version of this would be to ask whether any diffeomorphism with compact support in the interior of an ideal Liouville domain is smoothly isotopic to a symplectomorphism, through compactly supported diffeomorphisms (more realistic version would allow preliminary modifications of the open book).

Already in Eliashberg and Gromov 1991, it was noted that, thanks to some h-principles from Gromov 1986, all the rigidity of Weinstein manifolds comes from handles of maximal index. Those handles, called *critical handles* are attached to Legendrian submanifolds in the boundary of previous stages of the handle attachment process. Weinstein manifolds having a handle decomposition without critical handles are called *subcritical*. Hence it is crucial for the study of Weinstein manifolds, and possibly for the existence of contact structures, to understand the classification of Legendrian spheres up to contact isotopy. In dimension 3, Fuchs and Tabachnikov 1997 proved that any two smoothly isotopic Legendrian knots become contact isotopic after sufficiently many stabilizations, an explicit local modification which does not have much to do with stabilization of open books. In November 2010, during the EdiFest conference in Zürich, Giroux talked about the current status of the existence problem and asked Eliashberg whether this stabilization result holds in higher dimensions.

The question led to Murphy 2012 which uncovered the following phenomenon. Starting from dimension 5, there exists a class of Legendrian embeddings, called loose, which satisfy an h-principle completely analogous to the 3-dimensional overtwisted h-principle. Indeed there is a notion of formal Legendrian isotopy between Legendrian embeddings, which is analogous to a homotopy of plane fields, and such that any Legendrian embedding is formally isotopic to a loose one, and two loose Legendrian embeddings which are formally isotopic are isotopic.

We will not give the precise definition of loose Legendrian embeddings but note that it is analogous to the notion of large neighborhoods of (3-dimensional) overtwisted disks from Niederkrüger and Presas 2010 (see Section 5.2.1).

Although this flexibility theorem didn't allow to unlock Giroux's program to prove the existence of contact structure through open book, it did allow Eliashberg and Cieliebak 2012 to extend flexibility results for Weinstein manifolds from the realm of subcritical Weinstein manifolds to the much larger class of Weinstein manifolds whose critical handles are attached to loose Legendrian spheres (with some extra technical condition). Murphy et al. 2013 then brought together the theory of loose Legendrians and plastikstufes by proving that any Legendrian submanifold in the complement of a suitable plastikstufe is loose. This allowed to exhibit further hints of flexibility of PS-overtwisted manifolds.

Stimulated by all these developments, Borman, Eliashberg, and Murphy 2015 finally solved the existence problem for contact structures in higher dimensions. Based on general tools from Gromov 1986 and new specific ingredients, they proved the existence of a class of overtwisted contact manifolds in higher dimensions, satisfying a parametric h-principle.

Because of its flexibility, the class of overtwisted contact structures contain all previous attempts at defining overtwisted contact structures in higher dimensions: variations on the plastikstufe definition, large neighborhoods of overtwisted disks and negative stabilizations. Even, better Casals, Murphy, and Presas 2015 proved that almost all those definitions are actually equivalent. Thus there is now a very satisfactory understanding of overtwisted contact structures in higher dimensions.

5.3. Contact transformation groups

5.3.1. The contact mapping class group

Contact structures, like any other mathematical structure, naturally invite to study their groups of automorphisms. A *contact transformation*, or contactomorphism, of contact manifold (V, ξ) is a diffeomorphism preserving ξ. They form a subgroup $\mathcal{D}(V;\xi)$ inside the full diffeomorphism group $\mathcal{D}(V)$. Banyaga and McInerney 1995 proved that this group completely remembers the contact manifold. Every group isomorphism $\Phi\colon \mathcal{D}_o(V_1, \xi_1) \to \mathcal{D}_o(V_2, \xi_2)$ (not necessarily continuous) is induced by a contact isomorphism: there exists $\varphi\colon V_1 \to V_2$ such that $\varphi_*\xi_1 = \xi_2$ and $\Phi(g) = \varphi g \varphi^{-1}$ for every g in $\mathcal{D}_o(V_1, \xi_1)$.

Local homogeneity of contact manifolds implies that these groups are very large. They are infinite dimensional Lie groups that act transitively on k-tuples of points. While this is also true in symplectic geometry, the contact case is even more flexible since there is no volume constraint. Indeed one can prove that contact automorphisms act transitively on k-tuple of disjoint embeddings of the standard $\mathbb{R}^{2n+1}$.

One is therefore led to compare the topology of contact transformation groups $\mathcal{D}(V;\xi)$ to the topology of the full diffeomorphism group $\mathcal{D}(V)$. On closed manifolds, Gray stability strongly relates these two groups and the space of $\Xi(V)$ of contact structures on V, on which $\mathcal{D}(V)$ acts by push-forward.

Key observation 5.6. *Let (V, ξ) be a compact contact manifold. The natural map*

$$\mathcal{D}(V) \to \mathcal{D}(V) \cdot \xi, \quad \varphi \mapsto \varphi_*\xi,$$

is a locally trivial fibration whose fiber is the contact transformation group $\mathcal{D}(V;\xi) \subset \mathcal{D}(V)$.

Indeed, the Palais-Cerf fibration criterion ensures that one only needs to prove that the action of $\mathcal{D}(V)$ on $\Xi(V)$ admits local sections near any given point. These local sections are easily constructed using Moser's path method.

The first consequence of the homotopy exact sequence of this fibration is that the comparison of the contact mapping class group $\pi_0\mathcal{D}(V;\xi)$ and the smooth mapping class group $\pi_0\mathcal{D}(V)$, which is the study of the homomorphism $\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V)$, splits in two part. The image of this map is the subgroup of classes of diffeomorphisms φ such that $\varphi_*\xi$ is homotopic to ξ among contact structures (this is the usual version of Gray's theorem, and corresponds to the path lifting property of the fibration). The kernel of this map is isomorphic to the fundamental group of $\Xi(V)$ based at ξ quotiented by loops of contact structures coming from loops of diffeomorphisms.

The fundamental building block of the study of these questions on contact 3-manifolds is the contractibility of the component of the standard structure on $\Xi(\mathbb{B}^3)$ (with standard boundary condition). This was proved in Eliashberg 1992. Together with the contractibility of $\mathcal{D}(\mathbb{B}^3, \partial\mathbb{B}^3)$ proved in Hatcher 1983 and the above fibration, it proves that $\mathcal{D}(\mathbb{B}^3, \partial\mathbb{B}^3; \xi)$ is contractible.

As explained above, results on the classification of contact structures immediately imply results on the image of $\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V)$. For instance the study of incompressible pre-Lagrangian tori in Giroux 1994b led to information on the contact mapping class group of contact structures on $\mathbb{T}^3$ that were first explicitly mentioned in Eliashberg and Polterovich 1994. Giroux 2001c tackled the full study of $\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V)$ for contact element bundles, but the proofs in that paper contains a gap that will be fixed in Chapter 9, using even more technology coming from contact convexity.

In the topological study of diffeomorphism groups in higher dimensions, an important role was played by a coarser equivalence relation that isotopy. One says that two diffeomorphisms φ_- and φ_+ of some manifold V are pseudoisotopic if there is a diffeomorphism Φ of $V \times \mathbb{R}$ such that $\Phi(v,t) = (\varphi_-(v),t)$ for $t \ll 0$ and $\Phi(v,t) = (\varphi_+(v),t)$ for $t \gg 0$. Eliashberg and Cieliebak 2012, Section 14.5 introduced a symplectic version of this relation. Let $\pi\colon S\xi \to M$ be the symplectization of some contact manifold (M,ξ), equipped with its canonical Liouville form λ, its symplectic structure $\omega = d\lambda$, and its $\mathbb{R}$-action. We denote by $S_{-\infty}\xi$ its negative end and by $S_{+\infty}\xi$ its positive end. By definition, a neighborhood of $S_{\pm\infty}\xi$ is a set containing an open set which is invariant under the action of $\mathbb{R}_\pm$. A *symplectic pseudoisotopy* F of (M,ξ) is a symplectomorphism

$$F\colon (S\xi,\omega) \to (S\xi,\omega)$$

that restricts on a neighborhood of the negative end $S_{-\infty}\xi$ to the identity, and that preserves the Liouville form λ on a neighborhood of the positive end.

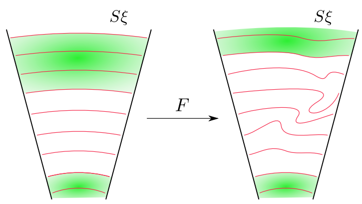


Figure 5.2.: A pseudoisotopy commutes with the $\mathbb{R}$-action on neighborhoods of $S_{\pm\infty}\xi$, but not necessarily in between.

Key observation 5.7. *A symplectic pseudoisotopy induces a contact transformation.*

Indeed, because a symplectic pseudoisotopy F preserves the canonical 1-form λ on some $\mathbb{R}_+$-invariant neighborhood U_+ of $S_{+\infty}\xi$, it also preserves the vector field X generating the $\mathbb{R}$-action, which is characterized by $d\lambda(X,\cdot) = \lambda$. Hence it commutes on U_+ with the $\mathbb{R}_+$-action and induces a contactomorphism of (M,ξ) as follows: for any x in M choose any p in U_+ above x and define $\varphi(x) = \pi(F(p))$. Such a contactomorphism is said to be *symplectically pseudoisotopic* to the identity. One can prove that this condition is implied by contact isotopy, by cutting off the lifted Hamiltonian isotopy.

Ibid. proved that, starting from dimension 5, any pseudoisotopy of a closed contact manifold is homotopic to a symplectic pseudoisotopy. In Chapter 9 we will describe, in every dimension, examples of contact transformation that are smoothly isotopic to the identity but not symplectically pseudoisotopic to the identity. In particular their contact mapping classes are in the kernel of $\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V)$.

5.3.2. Geometry of contact transformation groups

The contact mapping class group from the preceding section is, by definition, the quotient of $\mathcal{D}(V;\xi)$ by its neutral component $\mathcal{D}_o(V;\xi)$. According to Libermann 1959, the latter is the contact analogue of the group of Hamiltonian diffeomorphisms of a symplectic manifold. It is known since Hofer 1990 that an important manifestation of symplectic rigidity is the existence of an interesting bi-invariant distance of the latter group. This rich theory prompted the search for analogue distances, or other geometric structures, on $\mathcal{D}_o(V;\xi)$.

Eliashberg and Polterovich 2000 uncovered the existence of a bi-invariant cone on the universal cover of $\mathcal{D}_o(V;\xi)$. They say that a contact isotopy φ is positive if the vector field $d\varphi_t/dt$ is everywhere positively transverse to the contact structure. Hence positive isotopies are something like time-dependent Reeb flows. From any invariant cone on a group, one can try to define an invariant partial order. Here, two elements g and h of the universal cover of $\mathcal{D}_o(V;\xi)$ satisfy $g \leq h$ if $g^{-1}h$ is represented by a positive contact isotopy. In general this may fail to define an partial order for lack of antisymmetry. This failure happens exactly when there is a positive loop of contact transformations which is homotopically trivial (not necessarily among positive loops). A contact manifold is called *orderable* if there is no such failure. The first examples of orderable contact manifolds, discovered in ibid., were certain contact element bundles, certain Boothby-Wang manifolds, and the standard $\mathbb{R}P^{2n+1}$.

The study of orderability continued in Eliashberg, Kim, and Polterovich 2006 which proved surprising links between orderability and contact versions of Gromov's non-squeezing theorem. That paper also proved that the standard $\mathbb{S}^{2n+1}$ is not orderable, a surprising result that proved that the relation between orderability and rigidity is subtle.

Although the Eliashberg-Polterovich partial order is clearly a geometric structure on the universal cover of some contact transformation groups, it is not directly analogous to the Hofer distance. Sandon 2011 reproved the non-squeezing part of Eliashberg, Kim, and Polterovich 2006 using generating functions, and part of the technology developed there was later reused in Sandon 2010 to build a bi-invariant distance on some of these groups. This triggered quite a bit of activity in this direction, which is surveyed in Sandon 2015. In Chapter 11, we will see how the study of these invariant distances is linked to convexity.

5.3.3. Contact homeomorphisms

As in the symplectic case, Eliashberg proved in the early 80's that the contact transformation group is closed in the full diffeomorphism group equipped with the C^0 topology (see Müller and Spaeth 2014 for a written account of a later proof, also due to Eliashberg). In particular, one can define contact homeomorphisms of a contact manifold

(V, ξ) as the closure of $\mathcal{D}(V; \xi)$ inside Homeo(V) without getting extraneous diffeomorphisms. This is especially appealing in dimension 3 where we saw that contact topology has a very topological flavor. However it seems that no further result has been obtained in this direction, despite a lot of recent activity on the symplectic side of this story, especially Humilière, Leclercq, and Seyfaddini 2015; Buhovsky and Opshtein 2014.

Actually, even the definition of contact homeomorphism above hides considerable difficulties. It is completely unclear, even in dimension 3, whether this definition is local. If φ is a homeomorphism which is locally a C^0-limit of contact transformation then no method is known to prove or disprove that it is globally such a limit.

There is a rather different thread of ideas that starts with Gromov 1996 and leads to a (probably different) definition of contact homeomorphisms. Starting with a contact manifold (V, ξ), one can choose an auxiliary Riemannian metric g and define the *Carnot-Carathéodory distance* between two points of V as the minimal length of a Legendrian curve relating them. This distance depends on g but, provided that V is compact, its bi-Lipschitz equivalence class depends only on ξ. This is analogous to the fact that the usual Euclidean notion of bi-Lipschitz homeomorphism between compact manifolds depends only on their smooth structure.

Key observation 5.8. *A diffeomorphism of (V, ξ) is bi-Lipschitz for some Carnot-Carathéodory distance associated to ξ if and only if it preserves ξ.*

Indeed a diffeomorphism which does not preserve ξ sends some small curve tangent to ξ to a curve transverse to ξ, but these curves have different Hausdorff dimension for the induced distance. The other direction is obvious.

The above observation leads to the definition of contact homeomorphisms as those homeomorphisms which are bi-Lipschitz for some (hence every) Carnot-Carathéodory distance associated to ξ. One can also play this game with two different contact manifold and the central question becomes to decide whether the existence of a contact homeomorphism between two contact manifolds implies the existence of a smooth isomorphism. Chapter 12 will explain how contact convexity could shed light on these questions.

Part II.

Personal contributions

Contents

6. Prelagrangian tori and Heegaard-Floer homology

This chapter is a summary of Massot 2012. We saw in Part I the importance of Giroux torsion on one hand, and the contact class in Heegaard-Floer homology in the other hand. In particular Giroux torsion obstructs the existence of strong symplectic fillings, whereas the latter imply non-vanishing contact class over $\mathbb{Z}$ coefficients. It is therefore natural to investigate relations between these two invariants. Ghiggini, Honda, and Van Horn-Morris 2007 proved that, whenever the Giroux torsion is non zero, the contact invariant over $\mathbb{Z}$ coefficients vanishes. In Massot 2012, we gave a new proof of this result and, more importantly, we proved that the converse does not hold.

Theorem 6.1. *Every Seifert manifold whose base has genus at least three supports infinitely many (explicit) isotopy classes of universally tight torsion free contact structures whose Ozsváth–Szabó invariant over $\mathbb{Z}$ coefficients vanishes.*

In the above theorem, the genus hypothesis cannot be completely dropped because, for instance, on the sphere $\mathbb{S}^3$ and the torus $\mathbb{T}^3$, all torsion free contact structures have non vanishing Ozsváth–Szabó invariants. However, it may hold for genus two bases. Note that the class of Seifert manifolds is the only one where isotopy classes of contact structures are pretty well understood. So the theorem says that examples of universally tight torsion free contact structures with vanishing Ozsváth–Szabó invariant exist on all manifolds we understand, provided there is enough topology (the base should have genus at least three). In this statement, isotopy classes cannot be replaced by conjugacy classes because of the finiteness properties explained in Section 3.2.5. Along the way we proved Conjecture 7.13 of Honda, Kazez, and Matić 2008.

It is interesting to compare the above theorem (and its proof) with the results in Wendl 2013a which appeared shortly after the first version of Massot 2012. Wendl 2013a dealt with embedded contact homology, at a time when it was only conjecturally isomorphic to Heegaard–Floer theory. There he gets examples of universally tight torsion free contact structures with vanishing ECH invariants (and even some examples with vanishing twisted ECH invariants). It is intriguing to compare his list of examples with ours since, while the intersection is non empty, neither is contained in the other. Also both papers seem far from explaining clearly when Ozsváth–Szabó invariant vanish. We now have a lot of seemingly harmless contact structures with vanishing invariants but the global picture is unclear. This contrasts with the situation after Ghiggini, Honda, and Van Horn-Morris 2007 where one could have naively hoped that torsion explained all vanishings. Note however that, thanks to sutured Heegaard-Floer homology, vanishing still comes from localized parts of the manifolds: we have example of contact manifold with boundary such that any contact manifolds containing these have vanishing invariant.

Our examples also provide a corollary in the world of Legendrian knots. Ozsváth–Szabó theory provides invariants for Legendrian or transverse knots in different (related) ways, see Stipsicz and Vértesi 2009 and references therein. In the standard contact 3–spheres there are still two seemingly distinct ways to define such invariants but, in general contact manifolds, the known invariants all come from the sutured contact invariant of the complement of the knot according to the main theorem in ibid. In this paper they call strongly non loose those Legendrian knots in overtwisted contact manifolds whose complement is tight and torsion free. Corollary 1.2 of that paper states that a Legendrian knot has vanishing invariant when it is not strongly non loose. We prove that the converse does not hold.

Theorem 6.2. *There exists a specific example of overtwisted contact manifold containing a null-homologous strongly non loose Legendrian knot whose sutured invariant vanishes.*

After studying the relationship between Ozsváth–Szabó invariants and Giroux torsion, we now turn to a more specific relation between these invariants and an invariant defined only on the 3–torus. Recall from Section 3.2.1 that Giroux proved that any two incompressible pre-Lagrangian tori of a tight contact structure ξ on $\mathbb{T}^3$ are isotopic. We can then define the Giroux invariant $G(\xi) \in H_2(\mathbb{T}^3)/\pm 1$ to be the homology class of its pre-Lagrangian incompressible tori. Note that there is a *sign ambiguity* because these tori are not naturally oriented. Translated into this language, Giroux proved that two tight contact structures on $\mathbb{T}^3$ are isotopic if and only if they have the same Giroux invariant and the same Giroux torsion. This invariant is clearly $\mathcal{D}(\mathbb{T}^3)$–equivariant. Since this group acts transitively on primitive elements of $H_2(\mathbb{T}^3)$, we see that all these elements are attained by G. This also proves that all tight contact structures on $\mathbb{T}^3$ which have the same torsion are isomorphic. This classification of tight contact structures on $\mathbb{T}^3$ and the results of Section 2.7.2 show that torsion free contact structures on $\mathbb{T}^3$ are exactly the Stein fillable ones.

Theorem 6.3. *There is a unique up to sign $H_1(\mathbb{T}^3)$–equivariant isomorphism between $\widehat{HF}(\mathbb{T}^3)$ and $H^1(\mathbb{T}^3) \oplus H^2(\mathbb{T}^3)$ (on the ordinary cohomology side, H_1 sends H^1 to zero and H^2 to H^1 by slant product). Under this isomorphism, the Ozsváth–Szabó invariant of a torsion free contact structure on $\mathbb{T}^3$ is sent to the Poincaré dual of its Giroux invariant.*

Note that, on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, cohomology classes can be represented by constant differential forms and 1–dimensional homology classes by constant vector fields. The slant product of the above theorem is then identified with the interior product of vector fields with 2–forms.

The statement about torsion free contact structures is based on the interaction between the action of the mapping class group and first homology group of $\mathbb{T}^3$ on its Ozsváth–Szabó homology and ordinary cohomology. It sheds some light on the sign ambiguity of the contact invariant since the sign ambiguity of the Giroux invariant is very easy to understand.

Corollary 6.4. *There are infinitely many isomorphic contact structures whose isotopy classes are pairwise distinguished by the Ozsváth–Szabó invariant.*

Theorem 6.3 proves, via gluing, a conjecture of Honda, Kazez and Matić about the sutured invariants of $\mathbb{S}^1$–invariants contact structures on toric annuli. This conjecture is stated in Honda, Kazez, and Matić 2008, top of page 35.

Theorem 6.3 also have some consequence for the hierarchy of coefficients because $\mathbb{Z}/2$ coefficients can distinguish only finitely many isotopy classes of contact structures (since $\widehat{HF}(Y;\mathbb{Z}/2)$ is always finite).

Corollary 6.5. *There exists a manifold on which the Ozsváth–Szabó invariant over integer coefficients distinguishes infinitely many more isotopy classes of contact structures than the invariant over $\mathbb{Z}/2$ coefficients.*

In the same spirit, we prove that twisted coefficients are more powerful than $\mathbb{Z}$ coefficients even when the latter give non vanishing invariants.

Proposition 6.6. *There exist a sutured manifold with two contact structures having the same non vanishing Ozsváth–Szabó invariant over $\mathbb{Z}$ coefficients but which are distinguished by their invariants over twisted coefficients.*

6.1. Contact structures on the three torus

The proof of Theorem 6.3 relies on the action of $\mathrm{SL}_3(\mathbb{Z}) = \pi_0\mathcal{D}(\mathbb{T}^3)$ on $\widehat{HF}(\mathbb{T}^3)$. At the time of writing of Massot 2012, the status of mapping class group actions on Heegaard-Floer homology was a bit unclear and caution was needed. Fortunately the relevant setup in $\mathbb{T}^3$ was safe. Partly because of questions that I kept asking, the situation was clarified, in full generality, in Juhász and D. P. Thurston 2012.

The action of $\mathrm{SL}_3(\mathbb{Z})$ first allowed to prove the following easy lemma, which is the key algebraic trick to prove Theorem 6.3.

Lemma 6.7. *If an isomorphism $\Phi : \widehat{HF}(\mathbb{T}^3) \to H^1(\mathbb{T}^3) \oplus H^2(\mathbb{T}^3)$ is $H_1(\mathbb{T}^3)$–equivariant then it conjugates the SL_3 actions of both sides.*

Proof. In this proof we drop $\mathbb{T}^3$ from the notations. We denote by ρ the canonical action of SL_3 on H_1. Let ρ_1 and ρ_2 be two representations of SL_3 on $H^1 \oplus H^2$ which are compatible with the H_1 action, that is:

$$\forall g \in \mathrm{SL}_3, \gamma \in H_1, m \in H^1 \oplus H^2, \quad (\rho(g)\gamma)\, \rho_i(g)m = \rho_i(g)\,(\gamma m)\,.$$

We want to prove that $\rho_1 = \rho_2$ since this, applied to the standard action and to the action transported by Φ, will prove the proposition.

We first prove that, for all $g \in \mathrm{SL}_3$, $\rho_1(g)$ and $\rho_2(g)$ agree on H^2. The key property of the H_1 action is that it separates all elements of H^2: for all $m \neq m' \in H^2$, there exists γ in H_1 such that $\gamma m = 0$ and $\gamma m' \neq 0$.

Suppose by contradiction that there exists some $g \in \mathrm{SL}_3$ and $m \in H^2$ such that $\rho_1(g)m \neq \rho_2(g)m$. According to the separation property, there exists γ' in H_1 such that $\gamma'\rho_1(g)m = 0$ and $\gamma'\rho_2(g)m \neq 0$. Setting $\gamma = \rho(g)^{-1}(\gamma')$, we get $\rho(g)\gamma\rho_1(g)m = 0$ and $\rho(g)\gamma\rho_2(g)m \neq 0$, so $\rho_1(g)(\gamma m) = 0$ and $\rho_2(g)(\gamma m) \neq 0$, which is absurd since $\rho_1(g)$ and $\rho_2(g)$ are both isomorphisms.

We now prove that the representations agree on H^1. For all $m' \in H^1$, there exists $m \in H^2$ and $\gamma \in H_1$ such that $m' = \gamma m$. So for any $g \in SL_3$ and $i = 1, 2$, we get $\rho_i(g)m' = \rho_i(g)(\gamma m) = \rho(g)\gamma\rho_i(g)m$ and we know that $\rho_1(g)m = \rho_2(g)m$ thanks to the first part so $\rho_1(g)m' = \rho_2(g)m'$. □

Proof of Theorem 6.3. The existence of such an isomorphism is Proposition 8.4 of Ozsváth and Szabó 2003. The above lemma proves that, for any Φ as in the statement and any $x \in \widehat{HF}$, x and $\Phi(x)$ have the same stabilizer under the action of SL_3. The uniqueness of Φ follows since primitive elements of $H^1 \oplus H^2$ are characterized up to sign by their stabilizers. Indeed, suppose Φ_1 and Φ_2 are both isomorphisms as in the statement of the proposition. Then $\Phi_{12} := \Phi_1 \circ \Phi_2^{-1}$ is an automorphism such that, for any primitive x, there exists $\varepsilon_x \in \{\pm 1\}$ such that $\Phi_{12}(x) = \varepsilon_x x$. We now consider a $\mathbb{Z}$-basis $e_1, \dots, e_6$ of $H^1 \oplus H^2$ and compute

$$\sum \varepsilon_{\sum e_i} e_j = \varepsilon_{\sum e_i} \sum e_j = \Phi_{12}\left(\sum e_j\right) = \sum \Phi_{12}(e_j) = \sum \varepsilon_{e_j} e_j$$

so we get that all ε_{e_j} agree with $\varepsilon_{\sum e_i}$ and $\Phi_{12} = \varepsilon_{\sum e_i}\mathrm{Id}$. So Φ_1 and Φ_2 agree up to a global sign.

We now prove that the Poincaré dual of the Giroux invariant and the image of the Ozsváth–Szabó invariant coincide on torsion free contact structures. First remark that the Ozsváth–Szabó invariant belongs to $\widehat{HF}_{-1/2} \simeq H^1$ because the Hopf invariant of tight contact structures on $\mathbb{T}^3$ is $1/2$. So both invariants are primitive elements of H^1. We prove that the stabilizer of $G(\xi)$ is contained in that of $c(\xi)$ using equivariance of both invariants and the fact that G is a total invariant. For any g in SL_3 and ξ a torsion free contact structure, we have

$$\begin{aligned} gG(\xi) = G(\xi) &\iff G(g\xi) = G(\xi) \\ &\iff g\xi \sim \xi \\ &\Rightarrow c(g\xi) = c(\xi) \\ &\iff gc(\xi) = c(\xi) \end{aligned}$$

so we have the announced inclusion of stabilizers and this gives $c(\xi) = G(\xi)$. □

6.2. The contact TQFT

We now review the contact TQFT of Honda–Kazez–Matić. Let Σ be a non necessarily connected compact oriented surface with all components having non empty boundary. Let F be a finite subset of $\partial\Sigma$ whose intersection with each component of $\partial\Sigma$ is non empty and consists of an even number of points. We assume that the components of

$\partial\Sigma \setminus F$ are labelled alternatively by $+$ and $-$. This labelling will always be implicit in the notation (Σ, F). The contact TQFT associates to each (Σ, F) the graded group

$$V(\Sigma, F) = SFH(-(\Sigma \times \mathbb{S}^1), -(F \times \mathbb{S}^1))$$

(strictly speaking, one should replace F by a small translate of F along $\partial\Sigma$ in this formula).

In this construction one can use coefficients in $\mathbb{Z}/2$ or twisted coefficients (including the trivial twisting which leads to $\mathbb{Z}$ coefficients). We denote by $\underline{V}(\Sigma, F)$ the version twisted by $\mathbb{Z}[H_2(\Sigma \times \mathbb{S}^1)]$.

Proposition 6.8. *Let (Σ, F) be a surface with marked boundary points as above and $\mathbb{M}$ be any coefficient module for the sutured manifold $(\Sigma \times \mathbb{S}^1, F \times \mathbb{S}^1)$. We have, for any coherent orientations system:*

$$\underline{V}(\Sigma, F; \mathbb{M}) \simeq (\mathbb{M}_{(-1)} \oplus \mathbb{M}_{(1)})^{\otimes(\#F/2 - \chi(\Sigma))}.$$

The subscripts (-1) and (1) refer to the grading.

The analogous statement over $\mathbb{Z}$ coefficients was proved in Honda, Kazez, and Matić 2008 using product annuli decomposition and Friedl, Juhász, and Rasmussen 2011, Proposition 7.13. This technology has never been extended to twisted coefficients so, in Massot 2012, we actually drew explicit admissible sutured Heegaard diagrams with vanishing differential for these sutured manifolds.

A dividing set for (Σ, F) is a multi-curve K in Σ. The complement of a dividing set in Σ splits into two (non connected) surfaces $R_\pm$ according to the sign of their intersection with $\partial\Sigma$. The graduation of a dividing set is defined to be the difference of Euler characteristics $\chi(R_+) - \chi(R_-)$.

The following definition from Honda, Kazez, and Matić 2008 is crucial to understand contact invariants of $\mathbb{S}^1$-invariant contact structures.

Definition 6.9. *A dividing set K is said to be isolating if there is a connected component of the complement of K which does not intersect the boundary of Σ.*

To each dividing set K for (Σ, F) is associated the contact invariant of the Lutz contact structures associated to K in Section 2.1. All such contact structures are isotopic so they have the same invariant. These invariants belong to the graded part given by the graduation of K.

Theorem 6.10 (ibid.)**.** *Over $\mathbb{Z}/2$ coefficients, the following are equivalent:*

1. *$c(K) \neq 0$*
2. *$c(K)$ is primitive*
3. *K is non isolating*

Over $\mathbb{Z}$ coefficients, (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Honda, Kazez, and Matić 2008, Conjecture 7.13 stated that the assertions in this theorem are equivalent over $\mathbb{Z}$ coefficients. What remained to be proved is that isolating dividing sets have vanishing invariant. This (and more) was proved in Massot 2012 and, combined with explicit constructions and classification results, implies Theorem 6.1. We will try to give the flavor of the proof in the next section.

6.3. Vanishing results

Using the definitions and notations of the previous section, we want to discuss part of the proof of the following result.

Theorem 6.11. *If K is isolating then $c(K) = 0$ over $\mathbb{Z}$–coefficients.*

Note that the analogous statement over twisted coefficients is known to be false. For instance if we consider on $\mathbb{T}^3$ a contact structure divided by four essential circles and remove a small disk meeting one of these circles along an arc then we get an isolating dividing set on a punctured torus whose twisted invariant is sent to a non vanishing invariant since the corresponding contact structures on $\mathbb{T}^3$ are weakly fillable.

The key definition is inspired from Section 3.2.3.

Definition 6.12. *We say that dividing sets K_0, K_1 and K_2 are bypass-related if they coincide outside a disk D where they consists of the dividing sets of Figure 6.1.*

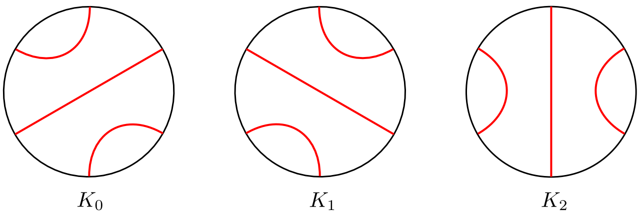


Figure 6.1.: Bypass relation

The following lemma is essentially proved, with $\mathbb{Z}$ coefficients, in Honda, Kazez, and Matić 2008 in the combination of proofs of Lemma 7.4 and Theorem 7.6. We reproduce the full proof from Massot 2012 as it illustrate how the template matching strategy, discussed in Section 3.2.3, was adapted to the algebraic setting of SFH.

Lemma 6.13. *If K_0, K_1 and K_2 are bypass-related then, for any representatives $\underline{\tilde{c}}_i \in \underline{c}(K_i)$, there exist $a, b \in \mathbb{L}^\times$ such that $\underline{\tilde{c}}_0 = a\underline{\tilde{c}}_1 + b\underline{\tilde{c}}_2$. The same holds over $\mathbb{Z}$ coefficients.*

Proof. The first part of the proof concentrate on the disk where the dividing sets differ. Let $\tilde{c}_i^D$ be representatives of the contact invariants of the three dividing sets on a disk D involved in Definition 6.12. Note that $H_2(D \times \mathbb{S}^1)$ is trivial so we now work over $\mathbb{Z}$ coefficients and suppress the underlines.

Because the c_i^D's all belong to the same rank 2 summand of $V(D, F_D)$ there are integers λ, μ and ν not all zero such that

$$\lambda \tilde{c}_0^D = \mu \tilde{c}_1^D + \nu \tilde{c}_2^D. \tag{6.1}$$

We denote by $K_\pm$ the dividing sets of Figure 6.2 and by $c_\pm$ their contact invariants.

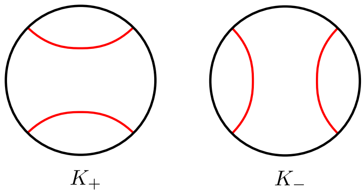


Figure 6.2.: Dividing sets used to prove Lemma 6.13

Label the points of F_D clockwise by $1, \dots, 6$ starting with the upper right point. Let Φ_j, $j = 1, 2, 3$, denote a HKM gluing map obtained by attaching a boundary parallel arc between points j and $j+1$. The gluing maps have the following effects:

$$\Phi_1 : c_0^D \mapsto c_+, \quad c_0^D \mapsto c_+, \quad c_0^D \mapsto 0 \tag{6.2}$$
$$\Phi_2 : c_0^D \mapsto 0, \quad c_0^D \mapsto c_-, \quad c_0^D \mapsto c_- \tag{6.3}$$
$$\Phi_3 : c_0^D \mapsto c_+, \quad c_0^D \mapsto 0, \quad c_0^D \mapsto c_+ \tag{6.4}$$

Using these equations and the facts that $c_\pm$ are non zero in a torsion free group (see Proposition 6.8), we get

$$(6.2) \Rightarrow \lambda = \pm\mu$$
$$(6.3) \Rightarrow \mu = \pm\nu$$
$$(6.4) \Rightarrow \lambda = \pm\nu$$

and they are all non zero so we can divide equation (6.1) by λ to get

$$\tilde{c}_0^D = \varepsilon_1 \tilde{c}_1^D + \varepsilon_2 \tilde{c}_2^D. \tag{6.5}$$

with $\varepsilon_1 = \mu/\lambda$ and $\varepsilon_2 = \nu/\lambda$.

We now return to our full dividing sets. Let D be the disk where the K_i's differ. Denote by F_D the (common) intersection of the K_i's with ∂D. Let ξ_0, ξ_1 and ξ_2 be contact structures divided by K_0, K_1 and K_2 respectively and coinciding with some ξ_b outside $D \times \mathbb{S}^1$.

Let $\underline{\Phi} : V(D, F_D) \to \underline{V}(\Sigma, F)$ be a HKM gluing map associated to ξ_b. The gluing property gives invertible elements a_i of $\mathbb{L}$ such that $\underline{\Phi}(\tilde{c}_i^D) = a_i \underline{\tilde{c}}_i$ for all i. We now apply $\underline{\Phi}$ to equation (6.5) and put $a = \varepsilon_1 a_1 a_0^{-1}$ and $b = \varepsilon_2 a_2 a_0^{-1}$ $\square$

This lemma is crucial for all computations of Massot 2012. Admittedly, most of them are rather technical. So we will only sketch how this lemma was combined with classification results on thickened and solid tori to reprove the main result of Ghiggini, Honda, and Van Horn-Morris 2007.

Proposition 6.14 (ibid.)**.** *Contact structures with positive Giroux torsion have vanishing contact invariant over $\mathbb{Z}$ coefficients.*

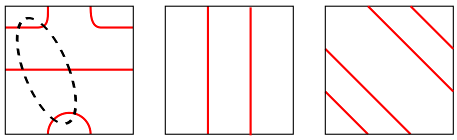


Figure 6.3.: Dividing sets for Proposition 6.14. Left and right sides of each squares should be glued to get annuli.

Proof. Let (A, F_A) be an annulus with two marked points on each boundary component and consider the dividing sets of Figure 6.3. We will denote by ξ_0, ξ_1 and ξ_2 contact structures divided by the corresponding K_i. Using the disk whose boundary is dashed, one sees that K_0 is bypass-related to K_1 and K_2. We denote $(A \times \mathbb{S}^1, F_A \times \mathbb{S}^1)$ by (N, Γ).

The study of normal forms of tight contact structures on thickened tori, discussed in Section 3.2.3, shows that there is a thickened torus N' with sutures Γ' and an explicit contact structure ξ_b on (N', Γ') such that a contact manifold has positive Giroux torsion if and only if it contains a copy of $\xi_0 \cup \xi_b$. Therefore we only need to prove that $c(\xi_0 \cup \xi_b)$ vanishes.

Let $\Phi = \Phi_{\xi_b}$ be a HKM gluing map from $\mathrm{SFH}(N, \Gamma)$ to $\mathrm{SFH}(N \cup N', \Gamma \Delta \Gamma')$. Normal form theory proves that $\xi_1 \cup \xi_b$ and $\xi_2 \cup \xi_b$ are isotopic, relative to the boundary. Using invariance under isotopy, we get $c(\xi_1 \cup \xi_b) = c(\xi_2 \cup \xi_b)$. Let $\tilde{c}_b$ be a representative of this common contact invariant. Let $\tilde{c}_1$ and $\tilde{c}_2$ be representatives of $c(K_1)$ and $c(K_2)$ such that $\tilde{c}_b = \Phi(\tilde{c}_1) = \Phi(\tilde{c}_2)$. Such representatives exist according to the gluing property. We also take any representative $\tilde{c}(K_0) \in c(K_0)$ and denote by $\tilde{c}(\xi_0 \cup \xi_b)$ its image under Φ. This image belong to $c(\xi_0 \cup \xi_b)$ according to the gluing property.

Lemma 6.13 gives $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that

$$\tilde{c}(K_0) = \varepsilon_1 \tilde{c}_1 + \varepsilon_2 \tilde{c}_2.$$

We then apply Φ to this equation to get:

$$\tilde{c}(\xi_0 \cup \xi_b) = (\varepsilon_1 + \varepsilon_2)\tilde{c}_b. \tag{6.6}$$

Let (W, ξ_W) be a standard neighborhood of a Legendrian knot (W is a solid torus). We now glue (W, ξ_W) along the boundary component of $N \cup N'$ which is in ∂N so that meridian curves have slope 0. The structure $\xi_W \cup \xi_0 \cup \xi_b$ is overtwisted whereas $\xi_W \cup \xi_1 \cup \xi_b$ (and $\xi_W \cup \xi_2 \cup \xi_b$ which is isotopic to it) is a standard neighborhood of a Legendrian curve so can be embedded into Stein fillable closed contact manifolds. Let Φ_W be a gluing map associated to ξ_W. Applying Φ_W to equation (6.6) and using the vanishing property of overtwisted contact structures, we get

$$0 = (\varepsilon_1 + \varepsilon_2)\Phi_W(\tilde{c}_b).$$

Using that $\Phi_W(\tilde{c}_b)$ is non zero and the fact that the relevant SFH group has no torsion (see Juhász 2008, Proposition 9.1) we get $\varepsilon_1 + \varepsilon_2 = 0$. Returning to Equation (6.6), we then get $c(\xi_0 \cup \xi_b) = 0$. □

6.4. Later developments and prospects

The computations from Massot 2012, including the most technical ones that are not stated above, have been re-examined from a more algebraic point of view in the series of papers Mathews 2011, 2013, 2014a,b and, over $\mathbb{Z}/2$ coefficients, from yet another point of view in Fink 2012. As already mentioned, they can also be compared with computation in embedded contact homology from Wendl 2013a. As far as geometric applications are concerned –mainly symplectic filling obstructions– most of those results should be superseded by work in progress on so-called spinal open books by Lisi, Van Horn-Morris and Wendl. So I do not intend to return to these kind of computations in Heegaard-Floer homology, although I would be mildly interested in seeing a computation of fully twisted contact classes for all tight contact structures on $\mathbb{T}^3$, including ones with positive Giroux torsion.

The main point of Massot 2012 was really to finish the exploration of interactions between Giroux torsion and Heegaard-Floer homology, and this was done. However there is still a big mystery about Giroux torsion: the finiteness conjecture. We still do not know whether all tight contact structure have finite torsion. It follows from the classification of overtwisted contact structures, or a careful study of neighborhood of an overtwisted disk, that overtwisted manifolds have infinite Giroux torsion. The coarse classification recalled in Section 3.2.5 proves that a finite value of Giroux torsion on a given manifold determines a contact structure up to isomorphism and finite ambiguity. So the finiteness conjecture is really the missing piece of the coarse classification picture. The hope to prove this conjecture is to be able to normalize a thickened torus with respect to a triangulation such as the ones appearing in the coarse classification (Section 3.2.5), and then apply classification results on thickened tori (Section 3.2.3). Here normalize is meant as in Haken's theory of normal surface but with extra information about dividing sets. This requires to prove isotopy results for ξ-convex or pre-Lagrangian tori as in Section 3.2.4, but avoiding all traps described in Section 9.2.3 below.

7. Weak and strong fillings in higher dimensions

7.1. Introduction

We saw in Part I that the study of symplectic fillings of contact 3-manifolds had a rich history, with many flavors of fillings, and interactions with the flexibility question and Reeb dynamics, especially via the concept of Giroux torsion. In contrast, nothing was known in higher dimensions until Niederkrüger 2006 proved existence of non-fillable contact structures in higher dimensions, see the discussion in in Section 5.2.1. But it said nothing about weaker versions of fillability, and interactions with flexibility and Reeb dynamics. And there were no proposed definition of Giroux torsion in higher dimensions. This chapter is a summary of Massot, Niederkrüger, and Wendl 2013 which studied all these questions. We will not give the technical details of constructions and holomorphic curves needed to prove the main results. But we will try to describe how definitions and constructions fit together.

Let us begin by recalling the phenomenon of contact structures that are tight but not (strongly) fillable. The emblematic example, in dimension 3, is the family of contact structures on $\mathbb{T}^3$ defined for $k \in \mathbb{N}$ by

$$\xi_k := \ker\left(\cos ks\ d\theta + \sin ks\ dt\right)\ ,$$

where we define $\mathbb{T}^3$ as $(\mathbb{R}/2\pi\mathbb{Z})^3$ with coordinates (s, t, θ). These contact structures are all tight due to Bennequin 1983, since they are covered by the standard contact structure on $\mathbb{R}^3$, but Eliashberg 1996 showed that only ξ_1 has a strong symplectic filling. Despite this lack of fillability, they share other important properties that are incompatible with overtwistedness. For example, they are *hypertight*, i.e. they allow Reeb vector fields without contractible closed orbits, in contrast to Hofer 1993 proving that such orbits always exist in the overtwisted case. More importantly, Giroux 1999 proved they are not flexible, meaning they are all homotopic as plane fields yet not isotopic, whereas overtwisted contact structures are maximally flexible due to Eliashberg 1989.

Although Massot, Niederkrüger, and Wendl 2013 was written before Borman, Eliashberg, and Murphy 2015, which solved the fundamental question of defining overtwisted contact structures in higher dimensions, flexibility and contractible Reeb orbits were already easy to define. Strong fillability was also defined of course. This allows us to compare the properties of the contact structures ξ_k on $\mathbb{T}^3$ discussed above with the following statement.

Theorem 7.1. *Identify the torus $\mathbb{T}^2$ with $(\mathbb{R}/2\pi\mathbb{Z})^2$ with coordinates (s,t). In any odd dimension, there is a closed manifold M carrying two contact forms α_+ and α_- such that the formula*

$$\xi_k := \ker\left(\frac{1+\cos ks}{2}\,\alpha_+ + \frac{1-\cos ks}{2}\,\alpha_- + \sin ks\; dt\right)$$

for $k \in \mathbb{N}$ defines a family of contact structures on $\mathbb{T}^2 \times M$ with the following properties:

1. *They all admit Reeb vector fields without contractible closed orbits.*
2. *They are all homotopic as almost contact structures but not contactomorphic.*
3. *$(\mathbb{T}^2 \times M, \xi_k)$ is strongly fillable only for $k = 1$.*

We recover the 3-dimensional case discussed above by taking $M = \mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ and $\alpha_\pm = \pm d\theta$ in the theorem.

As explained in Section 3.5.3, the non-fillability of the above contact structures on $\mathbb{T}^3$ was later recognized to be a consequence of the positivity of their Giroux torsion, and we'd next like to generalize this fact.

Our key insight is that, although the classification of contact structures on thickened tori uncovered Giroux torsion domains as torus bundles over the interval, and the coarse classification theorem have them appear as interval bundle over the torus, the vision relevant to fillability question is that of a circle bundle over an annulus with its ideal Liouville domain structure. Indeed, on the ideal Liouville domain

$$\Sigma = \mathbb{S}^1 \times [0,\pi], \quad \omega = \frac{1}{\sin^2 s}\, d\theta \wedge ds$$

where s is the coordinate in $[0,\pi]$ and θ the coordinate in $\mathbb{S}^1$, we can choose the Liouville form $\beta = \cot s\; d\theta$. Using the equation $f(\theta,s) = \sin s$ for the boundary, we get the contact form $f(\theta,s)\cdot(\beta + dt) = \cos s\; d\theta + \sin s\; dt$ on $\Sigma \times \mathbb{S}^1$. Thus the contactization of this ideal Liouville domain is a Giroux π-torsion domain.

Because of this observation, we decided to refer to contactizations of ideal Liouville domains as *Giroux domains*. The fact that Giroux torsion is an obstruction to strong fillability is then generalized to the following theorem.

Theorem 7.2. *If a contact manifold contains a connected codimension 0 submanifold with nonempty boundary obtained by gluing together two Giroux domains, then it is not strongly fillable.*

Observe that at least one of the Giroux domains in Theorem 7.2 must always have disconnected boundary. As discussed in Section 2.6.2, the existence of Liouville domains with disconnected boundary in dimensions four and higher is itself a nontrivial fact. Inspired by a construction from Geiges 1994, we introduce the following notion.

Definition 7.3. *A Liouville pair*[1] *on an oriented $(2n-1)$-dimensional manifold M is a pair (α_+, α_-) of contact forms such that $\pm\alpha_\pm \wedge d\alpha_\pm^{n-1} > 0$, and the 1-form*

$$\beta := e^{-s}\alpha_- + e^s\alpha_+$$

on $\mathbb{R} \times M$ satisfies $d\beta^n > 0$.

A Liouville pair allows us to construct Liouville domains with two boundary components (in fact, by attaching Stein 1-handles to these examples, one can obtain examples with any number of boundary components). These manifolds can then be used to build Giroux domains of the form $[0, \pi] \times \mathbb{S}^1 \times M$ with contact form

$$\lambda_{\mathrm{GT}} = \frac{1+\cos s}{2}\,\alpha_+ + \frac{1-\cos s}{2}\,\alpha_- + \sin s\, dt\,, \tag{7.1}$$

which can be stacked together to produce the examples described in Theorem 7.1.

In order to state an existence result for Liouville pairs, recall that a *number field of degree n* is a field that is an n-dimensional vector space over $\mathbb{Q}$. Recall also that $\mathbb{R}$ contains number fields of arbitrary degree.

Theorem 7.4. *To any number field $\Bbbk$ of degree n, one can associate canonically a $(2n-1)$-dimensional closed contact manifold $(M_\Bbbk, \xi_\Bbbk)$. If $\Bbbk$ can be embedded into $\mathbb{R}$, then $M_\Bbbk$ also admits a Liouville pair, hence $\mathbb{R} \times M_\Bbbk$ is Liouville.*

Corollary 7.5. *There exist Liouville domains with disconnected boundary in all even dimensions.*

This corollary provides a source of examples that can be plugged into Theorem 7.2 to construct non-fillable contact manifolds in all dimensions, and a special case of this leads to the examples of Theorem 7.1. The proof of Theorem 7.2 is in fact a generalization to higher dimensions of the construction mentioned in Section 4.3 to show that every contact 3-manifold with Giroux torsion is weakly symplectically cobordant to one that is overtwisted. In higher dimensions, the overtwistedness will come from a generalization of Mori 2009. Note that, already in dimension three, the cobordism argument requires the fact that overtwistedness obstructs *weak* (not only strong) fillability, a notion that has not previously been defined in any satisfactory way in higher dimensions. In dimension three of course, the subtle differences between weak and strong fillings are of interest in themselves, not only as a tool for understanding strong fillability.

As preparation for the definition of weak fillability that we will propose here, let us first have a look at the realm of (almost) complex manifolds.

Definition 7.6. *One says that a contact manifold (V, ξ) is the tamed pseudoconvex boundary of an almost complex manifold (W, J) if $V = \partial W$ and*

- *ξ is the hyperplane field $TV \cap JTV$ of J-complex tangencies,*

1. This is not what is called a Liouville pair in Weinstein 1991 but, fortunately, Eliashberg and Gromov 1991 essentially renamed those as Weinstein structures!

- *W admits a symplectic form ω taming J, and*
- *V is J-convex.*

The last point means that if we orient V as the boundary of W, then for any 1-form λ defining ξ (i.e. λ is a 1-form with $\xi = \ker \lambda$ as oriented hyperplanes), we have $d\lambda(v, Jv) > 0$ for every nonzero vector $v \in \xi$.

Note that there is no direct relation in the definition between the taming form ω and the contact structure ξ. It must also be pointed out that the existence of (W, J) is not very restrictive without the taming condition. For instance, the overtwisted contact structure on $\mathbb{S}^3$ that is homotopic to the standard contact structure can be realized as a pseudoconvex boundary of the ball for some almost complex structure, but the Eliashberg-Gromov theorem from Section 2.7.2 implies that this structure can never be tamed.

Sections 2.4, 2.6.2 and 2.7.2 explained the relevance of this definition to the study of holomorphic curves on manifolds with boundary. It also explained the definition of strong symplectic fillings and how the seemingly weaker definition of symplectic domination was irrelevant in higher dimensions.

We propose the following weak filling condition for all dimensions.

Definition 7.7. *Let ξ be a co-oriented contact structure on a manifold V. Denote by CS_ξ the canonical conformal class of symplectic structures on ξ. Let (W, ω) be a symplectic manifold with $\partial W = V$ as oriented manifolds and denote by ω_ξ the restriction of ω to ξ. We say that (W, ω) is a **weak filling** of (V, ξ) (and ω weakly dominates ξ) if ω_ξ is symplectic and $\omega_\xi + \mathrm{CS}_\xi$ is a ray of symplectic structures on ξ.*

The weak filling condition is thus equivalent to the requirement that

$$\alpha \wedge \left(d\alpha + \omega_\xi\right)^{n-1} \text{ and } \alpha \wedge \omega_\xi^{n-1}$$

should be positive volume forms for every choice of contact form α for ξ. If one fixes a contact form α, then this is equivalent to requiring $\alpha \wedge (\omega_\xi + \tau\, d\alpha)^{n-1} > 0$ for all constants $\tau \geq 0$, and it holds for instance whenever

$$\alpha \wedge d\alpha^k \wedge \omega_\xi^{n-1-k} > 0$$

for all $k \in \{0, 1, \dots, n-1\}$. In dimension three, weak domination is equivalent to domination, hence our definition of weak filling reduces to the standard one.

The first important result to state about this new definition is that it is the purely symplectic counterpart of tamed pseudoconvex boundaries.

Theorem 7.8. *A symplectic manifold (W, ω) is a weak filling of a contact manifold (V, ξ) (Definition 7.7) if and only if it admits a smooth almost complex structure J that is tamed by ω and makes (V, ξ) the tamed pseudoconvex boundary of (W, J) (Definition 7.6).*

The key to the above theorem is the following result in bilinear algebra:

Proposition 7.9. *Let E be a real vector space equipped with two symplectic forms ω_0 and ω_1. The following properties are equivalent:*

1. *the linear segment between ω_0 and ω_1 consists of symplectic forms*
2. *the ray starting at ω_0 and directed by ω_1 consists of symplectic forms*
3. *there is a complex structure J on E tamed by both ω_0 and ω_1.*

When complex structures tamed by both ω_0 and ω_1 exist, they form a contractible space. Starting from dimension 4, there are examples where such J exist, but none of them is compatible with ω_0 or ω_1.

Contrasting to the equivalence of weak fillings and tamed pseudoconvex boundaries, weak fillings are not automatically strong fillings. Indeed, weak domination of a fixed ξ is an open condition on ω, so one can easily construct weak fillings that are non-exact at the boundary by taking small perturbations of strong fillings. Less trivial examples of weak fillings non-exact at the boundary come from the generalization of the construction of weak fillings in Giroux 1994b for the tight contact structures ξ_k on $\mathbb{T}^3$. Indeed, starting from some contact manifold (V, ξ) weakly filled by some (W, ω), one can check that any contact structure on $V \times \mathbb{T}^2$ obtained through Bourgeois's construction (Key observation 5.3) is weakly filled by $(W \times \mathbb{T}^2, \omega \oplus \omega_{\mathbb{T}^2})$, where $\omega_{\mathbb{T}^2}$ is an area form on $\mathbb{T}^2$.

The next result extends the fact that weak fillability is *strictly* weaker than strong fillability beyond dimension three. Though we prove this only for dimension five, it is presumably true in all dimensions.

Theorem 7.10. *There exist 3-manifolds M with Liouville pairs (α_+, α_-) such that the contact manifolds $(\mathbb{T}^2 \times M, \xi_k)$ of Theorem 7.1 are all weakly fillable. In particular, there exist contact 5-manifolds that are weakly but not strongly fillable.*

Of course all this discussion is interesting only provided there are examples of contact manifolds that are not weakly fillable. We wrote Massot, Niederkrüger, and Wendl 2013 before the notion of overtwisted contact structures in higher dimensions stabilized in Borman, Eliashberg, and Murphy 2015; Casals, Murphy, and Presas 2015. However plastikstufes and their immersed cousins already existed. We potentially extended the class of PS-overtwisted manifolds by introducing the `bLob`, for reasons described in Section 7.2, and proved the following.

Theorem 7.11. *If (V, ξ) is a closed contact manifold that either*

(i) contains a contractible PS-overtwisted subdomain, or

(ii) is obtained as the negative stabilization of an open book,

then (V, ξ) has no (semipositive[2]) weak filling.

2. Here and elsewhere, we write the word *semipositive* in parentheses: this means that the condition is presently necessary for technical reasons, but should be removable in the future, see Section 7.7. Note that in dimensions 4 and 6, symplectic manifolds are always semipositive.

Hence any contact structure on a closed manifold V with $\dim V \geq 3$ can be modified within its homotopy class of almost contact structures to one that admits no (semipositive) weak fillings.

In the mean time, many potential definitions of overtwistedness were proved to be equivalent in Casals, Murphy, and Presas 2015. However the contractibility requirement from the above statement can be replaced by much weaker cohomological requirements, especially if one want to obstruct only strong fillings, see Section 7.3.2. Hence it is still unclear whether all flavor of PS-overtwistedness are equivalent and the precise version of the filling obstruction in Section 7.3.2 did not become obsolete. However the part concerning negative stabilization, which was based on ideas from Bourgeois and Koert 2010 but without needing conjectural aspects of SFT, did become redundant since Casals, Murphy, and Presas 2015.

We also proved that the weak filling condition is conveniently amenable to deformations near the boundary. An often used fact in dimension three, due originally to Eliashberg 1991b, is that any weak filling which is exact near the boundary can be deformed to a strong filling. This was extended in Niederkrüger and Wendl 2011 to show that every weak filling can be deformed to make the boundary a *stable hypersurface*, so that weak fillings can be studied using the machinery of SFT. Extending this idea to higher dimensions led to the notion of a *stable symplectic filling* defined in Latschev and Wendl 2011, and we proved:

Proposition 7.12. *Any weak filling can be deformed near its boundary to a stable filling. Moreover, if the symplectic form is exact near the boundary, then it can be deformed to a strong filling.*

The fact that weak fillings can be "stabilized" means that they are obstructed by the invariants defined in ibid., known as *algebraic torsion*.

Corollary 7.13. *If (V, ξ) has fully twisted algebraic torsion in the sense of ibid., then it is not weakly fillable. In particular, this is the case if (V, ξ) has vanishing contact homology with fully twisted coefficients.*

The above corollary, which was not used in Massot, Niederkrüger, and Wendl 2013, was conditional because existence of contact homology was not fully established at that time. According to Pardon 2015, Page 14, it is now fully proved.

The contact structures defined in (7.1) can be used to define a higher dimensional version of the standard 3-dimensional Lutz twist along a pre-Lagrangian torus. Notably, whenever (V, ξ) contains a hypersurface H that is isomorphic to one of the boundary components of the domain $[0, 2\pi] \times \mathbb{S}^1 \times M$ with the contact structure given by λ_{GT}, we can cut V open along H and glue in an arbitrary number of such domains to modify the contact structure on V. The contact structure obtained from this operation will never be strongly fillable, and in some cases it is not even weakly fillable:

Theorem 7.14. *By inserting contact domains of the form $([0, 2\pi k] \times \mathbb{S}^1 \times M, \ker \lambda_{\mathrm{GT}})$ for various $k \in \mathbb{N}$, one can construct closed manifolds in any dimension $2n - 1 \geq 3$*

which admit infinite families of hypertight but not weakly fillable contact structures that are homotopic as almost contact structures but not contactomorphic.

We also discussed in Massot, Niederkrüger, and Wendl 2013 a "blown down" version of the above operation, which generalizes both the classical Lutz twist along transverse knots in dimension three and a 5-dimensional version introduced in Mori 2009. This operation always produces a contact structure that is in the same homotopy class of almost contact structures, but is PS-overtwisted and thus not weakly fillable. Etnyre and Pancholi 2011 introduced a completely different generalization of the Lutz twist to higher dimensions but, in the mean time, a gap appeared in their proof.

Organization

Here is an outline of the remainder of this chapter.

In Section 7.2, we introduce the bLob as a natural generalization of the plastikstufe and briefly describe how to adapt the standard "Bishop family of holomorphic disks" argument to prove the remainder of Theorem 7.11.

The next three sections discuss the proof of Theorem 7.2, defining the first higher dimensional filling obstruction that is distinct from any notion of overtwistedness. In Section 7.3, we discuss ideal Liouville domains and Giroux domains, and state a more precise version of Theorem 7.2 that can also be applied to weak fillings. The proof requires a surgery construction explained in Section 7.4, which is inspired by the construction in Wendl 2013b of symplectic cobordisms from any contact 3-manifold with Giroux torsion to one that is overtwisted. In our case, we consider a contact manifold (V,ξ) which contains a region with nonempty boundary consisting of two Giroux domains $G_0 = \Sigma_0 \times \mathbb{S}^1$ and $G_1 = \Sigma_1 \times \mathbb{S}^1$ glued together. It turns out that one can attach along G_0 a symplectic "handle" of the form $\Sigma_0 \times \mathbb{D}^2$, the effect of which is to replace $G_0 \cup G_1$ with a region that is PS-overtwisted, thus a weak filling of (V,ξ) with suitable cohomological properties at the boundary gives rise to a larger weak filling of something PS-overtwisted and hence a contradiction. Note that since the new boundary is only weakly filled in general, the new notion of weak fillability plays a crucial role even just for proving that (V,ξ) is not *strongly* fillable.

In Section 7.6 we switch gears and address the existence of Liouville pairs in all dimensions, proving Theorem 7.4.

The main things from Massot, Niederkrüger, and Wendl 2013 that are not described here are the proof of Theorem 7.8, the part about negative stabilizations in Theorem 7.11 which became redundant since Casals, Murphy, and Presas 2015, the proof that contact structures in Theorem 1.1 are homotopic as almost contact structure (this is a rather technical computation), and the proof that they are not isotopic (this combines algebraic considerations about their fundamental groups and a contact homology computation). We also skip the variations leading to Theorem 7.14.

7.2. Bordered Legendrian open books

In this section, we will first introduce a generalization of the plastikstufe that is more natural and less restrictive than the initial version introduced in Niederkrüger 2006. Subsequently we will state that these objects, under a certain homological condition (which is trivially satisfied for the overtwisted disk), represent obstructions to weak fillability.

Definition 7.15. *Let N be a compact manifold with nonempty boundary. A relative open book on N is a pair (B, θ) where:*

- *the binding B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle;*
- *$\theta\colon N \setminus B \to \mathbb{S}^1$ is a fibration whose fibers are transverse to ∂N, and which coincides in a neighborhood $B \times \mathbb{D}^2$ of $B = B \times \{0\}$ with the normal angular coordinate.*

Definition 7.16. *Let (V, ξ) be a $(2n+1)$-dimensional contact manifold. A compact $(n+1)$-dimensional submanifold $N \hookrightarrow V$ with boundary is called a bordered Legendrian open book (abbreviated bLob), if it has a relative open book (B, θ) such that:*

(i) all fibers of θ are Legendrian;

(ii) the boundary of N is Legendrian.

A contact manifold that admits a bLob is called PS-overtwisted.

The binding B of a Legendrian open book is automatically isotropic because its tangent space is contained in the tangent space of the closure of all pages. Similarly, the fibers of θ and the boundary of N meet transversely in N, and saying that they are both Legendrian implies that the induced foliation on N is singular on B and ∂N.

A bLob is an example of a *maximally foliated* submanifold of (V, ξ), as in Section 5.2.1. Note that the definition of the bLob is topologically *much* less restrictive than the initial definition of the plastikstufe. For example, a 3-manifold admits a relative open book if and only if its boundary is a nonempty union of tori. On the other hand, a plastikstufe in dimension 5 is always diffeomorphic to a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$.

In this chapter we will discuss one setting where we can find bLobs and are unable to find plastikstufes: in Proposition 7.19, we show that bLobs always exist in certain subdomains that are naturally associated to Liouville domains with disconnected boundary, a special case of which produces the Lutz-type twist from Mori 2009.

Some bLobs also arise in relation to the results of Niederkrüger and Presas 2010, where it is shown that sufficiently large neighborhoods of overtwisted submanifolds in higher dimensional contact manifolds give a filling obstruction. In ibid. this required a rather technical argument involving holomorphic disks with an immersed boundary condition, but it can be simplified and strengthened by showing (using arguments similar to those of Proposition 7.19) that such neighborhoods always contain a bLob. Since Casals, Murphy, and Presas 2015, we also know that these neighborhood contains a plastikstufe, but it has not been exhibited explicitly, contrasting with the bLob.

Of course, finding a bLob would be useless without the following theorem.

Theorem 7.17. *If a closed contact manifold is PS-overtwisted, then it does not have any (semipositive) weak symplectic filling (W, ω) for which ω restricted to the bLob is exact.*

If $\dim V \geq 3$, then any contact structure ξ on V can be modified using Presas 2007; Niederkrüger and Koert 2007 to produce one that is PS-overtwisted. In both cases, the change produces a *small plastikstufe*: a plastikstufe contained in a ball. Hence Theorem 7.17 and the preceding section imply Theorem 7.11 stated in the introduction.

In the proof of Theorem 7.17, the general strategy is the same as in the plastikstufe case from Gromov 1985; Eliashberg 1990a; Niederkrüger 2006, but there are differences coming from two sources: the need to handle weak rather than strong fillings, and bLobs rather than plastikstufes. Working with weak fillings complicates the question of energy bounds because the integral of ω on a holomorphic curve no longer has a direct relation to the integral of $d\alpha$. This is where the homological condition comes in. Further, it is no longer obvious that we can choose our almost complex structure to be both adapted to a contact form near the binding and boundary of the bLob and tamed by ω, this is where the characterization of Theorem 7.8 is crucial. As far as the differences between the plastikstufe and the bLob are concerned, the first is the singularity along the boundary, which makes energy control easier but makes it harder to ensure that holomorphic curves cannot escape through the boundary. This difference can be handled similarly to the analogous work in Niederkrüger and Presas 2010, which dealt with the case where the fibration of the bLob becomes trivial at the boundary. The general case additionally requires somewhat technical lemmas. The second difference is of course that pages are more complicated and the interior monodromy can be anything, but this plays no role in the proof; what matters is the existence of a fibration over $\mathbb{S}^1$.

7.3. Giroux domains

7.3.1. Round hypersurfaces and blow down

Let M be a union of connected components of the boundary of a Giroux domain $\Sigma \times \mathbb{S}^1$. These components are ξ-round hypersurfaces and can thus be blown down, as explained in Section 3.3. We shall denote the resulting manifold by $(\Sigma \times \mathbb{S}^1)//M$. It inherits a natural contact structure for which each of the blown down boundary components becomes a codimension two contact submanifold.

Example 7.18. *Continuing the annulus example from Section 7.1, a Giroux π-torsion domain with one boundary component blown down is a so-called Lutz tube, i.e. the solid torus that results from performing a (half) Lutz twist along a transverse knot. With both boundary components blown down, it is the standard contact structure on $\mathbb{S}^2 \times \mathbb{S}^1$.*

In the above example, when one boundary component is blown down but not the other, the resulting domain contains an overtwisted disk. Note here how crucial it is that we use ideal Liouville domains instead of their interior, which are complete Liouville manifolds. Going all the way to infinity is needed to fully turn up to the boundary of an overtwisted disk. This motivate the definition of ideal Liouville domains independently from the Courte phenomenon. We now generalize this to higher dimensions.

Proposition 7.19. *Suppose (V, ξ) is a contact manifold containing a subdomain G with nonempty boundary, obtained from a Giroux domain by blowing down at least one boundary component. Then (V, ξ) contains a ball containing a* bLob*.*

The bLob in the above proposition comes from a Lagrangian submanifold in an ideal Liouville domain (Σ, ω). We first need a technical definition describing how these submanifolds are allowed to approach the boundary. We say that a submanifold L properly embedded inside Σ and transverse to the boundary is a *Lagrangian with cylindrical end* if: $\mathring{L}$ is Lagrangian in $\mathring{\Sigma}$, ∂L is Legendrian in $\partial\Sigma$, and there is a Liouville form β whose ω-dual vector field is tangent to L near $\partial\Sigma$.

The key lemma to prove the above proposition is then as follows. The proof relies essentially on Tischler's construction of fibration after some perturbation and Moser type arguments.

Lemma 7.20. *Let (Σ, ω) be an ideal Liouville domain. If L is a Lagrangian with cylindrical end in Σ, then $\widehat{L} := L \times \mathbb{S}^1$ inside the contactization $\Sigma \times \mathbb{S}^1$ is isotopic to a maximally foliated submanifold whose singular set is its boundary and whose foliation is otherwise defined via a fibration*

$$\vartheta \colon \widehat{L} \to \mathbb{S}^1, \quad (l, t) \mapsto F(l) + t\,,$$

for some smooth function $F \colon L \to \mathbb{S}^1$ that is constant on a neighborhood of ∂L.

Proposition 7.19 is proved using this lemma and Lagrangian constructed from trivial disks and Lagrangian surgery.

7.3.2. Obstructions to fillability

We now want to state a non-fillability result. As preparation, note that any embedding of the interior of a Giroux domain $I_\Sigma := \mathring{\Sigma} \times \mathbb{S}^1$ into a contact manifold (V, ξ) determines a distinguished subspace $H_1(\Sigma; \mathbb{R}) \otimes H_1(\mathbb{S}^1; \mathbb{R}) \subset H_2(V; \mathbb{R})$. We call its annihilator in $H^2_{\mathrm{dR}}(V)$ the space of cohomology classes *obstructed* by I_Σ, and we denote it by $\mathcal{O}(I_\Sigma)$. Classes in $\mathcal{O}(I_\Sigma)$ are exactly those whose restriction to I_Σ can be represented by closed 2-forms pulled back from the interior of Σ. If $N \subset (V, \xi)$ is any subdomain resulting from gluing together a collection of Giroux domains $I_{\Sigma_1}, \dots, I_{\Sigma_k}$ and blowing down some of their boundary components, then we define its obstructed subspace $\mathcal{O}(N) \subset H^2_{\mathrm{dR}}(V)$ to be $\mathcal{O}(I_{\Sigma_1}) \cap \dots \cap \mathcal{O}(I_{\Sigma_k})$. We will say that such a domain is *fully obstructing* if $\mathcal{O}(N) = H^2_{\mathrm{dR}}(V)$.

Example 7.21. *If Σ is homeomorphic to $[-1,1] \times M$ for some closed manifold M, and N is the result of blowing down one boundary component of the Giroux domain I_Σ, then any embedding of N is fully obstructing. Indeed, any class in $H_1(\Sigma; \mathbb{R}) \otimes H_1(\mathbb{S}^1; \mathbb{R})$ can be represented by a cycle in the $M \times \mathbb{D}^2$ part of the blown down Giroux domain and, of course, $H_1(\mathbb{S}^1; \mathbb{R})$ becomes trivial in $H_1(\mathbb{D}^2; \mathbb{R})$. For instance, a Lutz tube (see Example 7.18) in a contact 3-manifold is always fully obstructing, and the same is true for the higher dimensional generalization that we introduced in Massot, Niederkrüger, and Wendl 2013.*

The following is the refined version of Theorem 7.2 obstructing some weak fillings in addition to strong fillings.

Theorem 7.22. *Suppose (V, ξ) is a closed contact manifold containing a subdomain N with nonempty boundary, which is obtained by gluing and blowing down Giroux domains.*

- *(a) If N has at least one blown down boundary component then it contains a small bLob, hence (V, ξ) does not have any (semipositive) weak filling.*
- *(b) If N contains two Giroux domains $\Sigma^+ \times \mathbb{S}^1$ and $\Sigma^- \times \mathbb{S}^1$ glued together such that Σ^- has a boundary component not touching Σ^+, then (V, ξ) has no (semipositive) weak filling (W, ω) with $[\omega_V] \in \mathcal{O}(\Sigma^+ \times \mathbb{S}^1)$.*

In particular (V, ξ) has no (semipositive) strong filling in either case.

The first statement in this theorem follows immediately from Proposition 7.19 and Theorem 7.17. We will explain the second in Section 7.5, essentially by using the symplectic cobordism construction of the next section to reduce it to the first statement, though some care must be taken because the filling obtained by attaching our cobordism to a given semipositive filling need not always be semipositive. Ibid. also contains an alternative argument for both parts of Theorem 7.22 using J-holomorphic spheres: this requires slightly stricter homological assumptions than stated above, but has the advantage of not requiring semipositivity at all, due to the polyfold machinery developed in Hofer, Wysocki, and Zehnder 2011. See Section 7.7 for more discussion of transversality issues.

Without delving into the details, we should mention that we also expect the above filling obstruction to be detected algebraically in Symplectic Field Theory via the notion of *algebraic torsion* defined in Latschev and Wendl 2011. Recall that a contact manifold is said to be *algebraically overtwisted* if it has algebraic 0-torsion (this is equivalent to having vanishing contact homology), but there are also infinitely many "higher order" filling obstructions known as algebraic *k-torsion* for integers $k \geq 1$. It turns out that one can always choose the data on a Giroux domain $\Sigma \times \mathbb{S}^1$ so that gradient flow lines of a Morse function on Σ give rise to holomorphic curves in the symplectization of $\Sigma \times \mathbb{S}^1$, and these can be counted in SFT. The expected result is as follows:

Conjecture 7.23. *Suppose (V, ξ) contains a subdomain N as in Theorem 7.22, choose any $c \in \mathcal{O}(N)$ and consider SFT with coefficients in $\mathbb{R}[H_2(V; \mathbb{R})/\ker c]$. Then (V, ξ) has algebraic 1-torsion, and it is also algebraically overtwisted if N contains any blown down boundary components.*

7.4. Surgery along Giroux domains

In this section, we explain a surgery procedure which removes the interior of a Giroux domain from a contact manifold and blows down the resulting boundary. This surgery corresponds to a symplectic cobordism that can be glued on top of any weak filling satisfying suitable cohomological conditions, leading to a proof of Theorem 7.22.

Suppose (V,ξ) is a $(2n-1)$-dimensional contact manifold without boundary, containing a Giroux domain $G \subset V$, possibly with some boundary components blown down. Removing the interior of G, the boundary of $\overline{V \setminus G}$ is then a ξ-round hypersurface

$$\partial(\overline{V \setminus G}) = M \times \mathbb{S}^1\,,$$

where (M,ξ_M) is a (possibly disconnected) closed contact manifold. We can thus blow it down as described in Section 7.3.1, producing a new manifold

$$V' := (\overline{V \setminus G})//M$$

without boundary, which inherits a natural contact structure ξ'.

Topologically, the surgery taking (V,ξ) to (V',ξ') can be understood as a certain handle attachment. We now give a point-set description of this handle attachment which is sufficient to state the theorem below (but Massot, Niederkrüger, and Wendl 2013 does discuss a smooth model). Assume that G is obtained from the ideal Liouville domain (Σ,ω) with boundary $\partial\Sigma = M_{\mathrm{p}} \sqcup M_{\mathrm{bd}}$ by blowing down the Giroux domain $\Sigma \times \mathbb{S}^1$ at $M_{\mathrm{bd}} \times \mathbb{S}^1$ but preserving $M_{\mathrm{p}} \times \mathbb{S}^1$ as in Fig. 7.1(a) (here *bd* stands for "blown down", and *p* for "preserved"). Then topologically,

$$G = (M_{\mathrm{bd}} \times \mathbb{D}^2) \cup_{M_{\mathrm{bd}} \times \mathbb{S}^1} (\Sigma \times \mathbb{S}^1)\,.$$

Note that M_{bd} can now be regarded as a codimension 2 contact submanifold of G, namely by identifying it with $M_{\mathrm{bd}} \times \{0\}$.

Next, remove a small open collar neighborhood of M_{bd} from Σ and denote the resulting submanifold by Σ_h. We can regard $\Sigma_h \times \mathbb{S}^1$ as a subdomain of G, and consider the manifold with boundary and corners defined by

$$([0,1] \times V) \cup_{\{1\}\times(\Sigma_h \times \mathbb{S}^1)} (\Sigma_h \times \mathbb{D}^2)\,.$$

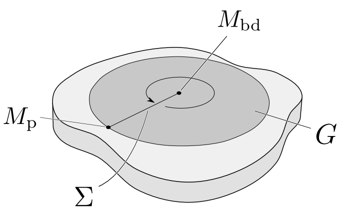
After smoothing the corners, this becomes a smooth oriented cobordism W with boundary (see Fig. 7.1(b)),

$$\partial W = -V \sqcup V' \sqcup (M_{\mathrm{bd}} \times \mathbb{S}^2)\,.$$

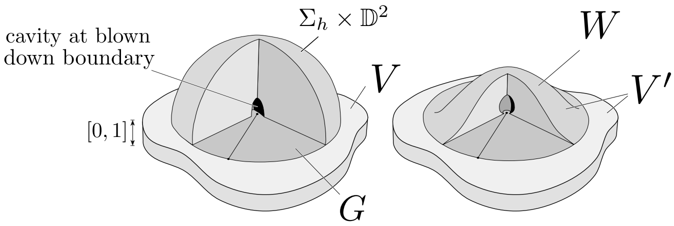
We can now state the main theorem of this section.

Theorem 7.24. *Suppose W denotes the $2n$-dimensional smooth cobordism described above, and Ω is a closed 2-form on V such that:*

- *Ω weakly dominates ξ*



(a) The domain G is obtained from the product manifold $\Sigma \times \mathbb{S}^1$ by blowing down the boundary components $M_{\mathrm{bd}} \times \mathbb{S}^1$ to M_{bd}.



(b) The cobordism is obtained by gluing $\Sigma_h \times \mathbb{D}^2$ onto G, and rounding its corners. Note that after the handle attachment the boundary of the surgered manifold consists of the contact manifold V' plus components diffeomorphic to $M_{\mathrm{bd}} \times \mathbb{S}^2$ corresponding to the blown down boundary of G.

Figure 7.1.: Giroux domain surgery

- *the cohomology class of* Ω *belongs to the obstructed subspace* $\mathcal{O}(G)$*, i.e. for every* 1*-cycle* Z *in* Σ*,*

$$\int_{Z\times\mathbb{S}^1} \Omega = 0\,.$$

Then W *admits a symplectic structure* ω *with the following properties:*

1. $\omega|_{TV} = \Omega$.
2. *The co-core* $\Sigma_h \times \{0\} \subset \Sigma_h \times \mathbb{D}^2 \subset W$ *is a symplectic submanifold weakly filling* $(\partial\Sigma_h \times \{0\}, \xi_\Sigma)$*.*
3. (V', ξ') *is a weakly filled boundary component of* (W,ω) *that is contactomorphic to the blown down manifold* $(\overline{V \setminus G})/\!/M_{\mathrm{p}}$*.*
4. *A neighborhood of* $M_{\mathrm{bd}} \times \mathbb{S}^2 \subset \partial W$ *in* (W,ω) *can be identified symplectically with*

$$\big((-\delta,0] \times M_{\mathrm{bd}} \times \mathbb{S}^2,\, \omega_0 \oplus \omega_{\mathbb{S}^2}\big)$$

for some $\delta > 0$*, where* $\omega_{\mathbb{S}^2}$ *is an area form on* $\mathbb{S}^2$ *and* ω_0 *is a symplectic form on* $(-\delta,0] \times M_{\mathrm{bd}}$ *for which the boundary* $(M_{\mathrm{bd}}, \xi_\Sigma)$ *is weakly filled. Moreover, the intersection of the co-core* $\Sigma \times \{0\}$ *with this neighborhood has the form* $(-\delta,0] \times M_{\mathrm{bd}} \times \{\mathrm{const}\}$*.*

Remark 7.25. *A pair of weak symplectic cobordisms can be smoothly glued together along a positive/negative pair of contactomorphic boundary components whenever the symplectic forms restricted to these boundary components are cohomologous. Thus the symplectic cobordism of the above theorem can be glued on top of any weak filling* (W,ω) *of* (V,ξ) *for which* $[\omega|_{TV}] \in \mathcal{O}(G)$*.*

The proof of Theorem 7.24 is somewhat technical and will not be reproduced here. It consists of the following five steps:

1. Find a standardized model with a special contact form λ for tubular neighborhoods of ∂G and the blown down components M_{bd}.
2. Construct a symplectic form on our proto-cobordism $[0,1]\times V$ that is well adjusted to both Ω and λ.
3. Carve out the interior of $\{1\} \times \Sigma \times \mathbb{S}^1$ from $[0,1] \times V$. This creates a notch with corners along its edges, and we then smoothly glue the handle $\Sigma \times \mathbb{D}^2$ into the cavity, creating a smooth manifold.
4. Study the symplectic form induced from the proto-cobordism on the glued part of the handle and extend it to the whole handle.
5. Check that the new boundary of the cobordism has the desired properties.

7.5. Giroux domains and non-fillability

We now use the cobordism of the preceding section to prove Theorem 7.22 on filling obstructions. The first few paragraphs are the expected ones where we glue the cobordism of the previous section on top of a hypothetical filling. The rest is kept there as an example of the contortions needed because of transversality issues in holomorphic curves theory. The point is that, even assuming semi-positivity of the hypothetical filling is not enough. This motivates the discussion in Section 7.7. It also explains why we stated so many geometric properties of the cobordism in Theorem 7.24.

Part (a) of the theorem follows immediately from the fact that if (V,ξ) contains a Giroux domain N that has some boundary components that are blown down and others that are not, then by Proposition 7.19 it contains a small bLob, so Theorem 7.17 implies that (V,ξ) does not admit any semipositive weak filling.

To prove part (b), suppose N has the form

$$N = (\Sigma^+ \times \mathbb{S}^1) \cup_{Y \times \mathbb{S}^1} (\Sigma^- \times \mathbb{S}^1)\,,$$

where $\Sigma^\pm$ are ideal Liouville domains with boundary $\partial\Sigma^\pm = \partial_{\mathtt{glue}}\Sigma^\pm \sqcup \partial_{\mathtt{free}}\Sigma^\pm$, $Y := \partial_{\mathtt{glue}}\Sigma^+ = \partial_{\mathtt{glue}}\Sigma^-$ carries the induced contact form α and $\partial_{\mathtt{free}}\Sigma^-$ is not empty. Arguing by contradiction, assume that (V,ξ) is weakly filled by a semipositive symplectic filling (W_0,ω) with $[\omega|_{TV}] \in \mathcal{O}(\Sigma^+)$. This establishes the cohomological condition needed by Theorem 7.24 on $\Sigma^+ \times \mathbb{S}^1$, so applying the theorem, we can enlarge (W_0,ω) by attaching $\Sigma^+ \times \mathbb{D}^2$, producing a compact symplectic manifold (W_1,ω) whose boundary (V',ξ') supports a contact structure that is weakly filled.

Since the boundary V' of the new symplectic manifold (W_1,ω) is contactomorphic to $(\overline{V \setminus (\Sigma^+ \times \mathbb{S}^1)})//Y$, we find in (V',ξ') a domain isomorphic to $(\Sigma^- \times \mathbb{S}^1)//Y$ that contains a small bLob. Unfortunately this does not directly obstruct the existence of the weak filling (W_1,ω), because even though W_0 was semipositive, W_1 might not be. We will follow the proof of Theorem 7.17, with the difference that we need to reconsider compactness to make sure that bubbling is still a "codimension 2 phenomenon".

Choose an almost complex structure J on (W_1,ω) with the following properties:

(i) J is tamed by ω and makes (V', ξ') strictly J-convex,

(ii) J is adapted to the bLob in the standard way (cf. the proof of Theorem 7.17),

(iii) for some small radius $r > 0$, $J = J_{\Sigma^+} \oplus i$ on $\Sigma^+ \times \mathbb{D}^2_r \subset W_1$, where $J_{\Sigma+}$ is a tamed almost complex structure on Σ^+ for which $\partial\Sigma^+$ is J_{Σ^+}-convex.

The third condition uses the fact from Theorem 7.24 that the co-core $\mathcal{K}' := \Sigma^+ \times \{0\}$ of the handle is a symplectic (and now also J-holomorphic) hypersurface weakly filling its boundary. The binding of the bLob lies in the boundary of the co-core $\mathcal{K}'_+$, and the normal form described in Niederkrüger 2006 is compatible with the splitting $\Sigma^+ \times \mathbb{D}^2_r$ so that (ii) and (iii) can be simultaneously achieved.

By choosing J_{Σ^+} generic, we can also assume that every somewhere injective J_{Σ^+}-holomorphic curve in Σ^+ is Fredholm regular and thus has nonnegative index. Note that any closed J-holomorphic curve in $\Sigma^+ \times \mathbb{D}^2_r$ is necessarily contained in $\Sigma^+ \times \{z\}$ for some $z \in \mathbb{D}^2_r$, and the index of this curve differs from its index as a J_{Σ^+}-holomorphic curve in Σ^+ by the Euler characteristic of its domain. This implies that every somewhere injective J-holomorphic sphere contained in $\Sigma^+ \times \mathbb{D}^2_r$ has index at least 2. Likewise, by a generic perturbation of J outside of this neighborhood we may assume all somewhere injective curves that are *not* contained entirely in $\Sigma^+ \times \mathbb{D}^2_r$ also have nonnegative index.

Now let $\mathcal{M}$ be the connected moduli space of holomorphic disks attached to the bLob that contains the standard Bishop family. We can cap off every holomorphic disk $u \in \mathcal{M}$ by attaching a smooth disk that lies in the bLob, producing a trivial homology class in $H_2(W_1)$. The cap and the co-core intersect exactly once, and it follows that u also must intersect the co-core $\mathcal{K}'_+$ exactly once, because u and $\mathcal{K}'_+$ are both J-complex.

To finish the proof, we have to study the compactness of $\mathcal{M}$ and argue that $\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of strata of codimension at least 2. A nodal disk u_∞ lying in $\overline{\mathcal{M}} \setminus \mathcal{M}$ has exactly one disk component u_0, which is injective at the boundary, and one component u_+ that intersects the co-core once; either $u_+ = u_0$ or u_+ is a holomorphic sphere. Every other non-constant connected component v is a holomorphic sphere whose homology class has vanishing intersection with the relative class $[\mathcal{K}'_+]$. So either v does not intersect the J-complex submanifold $\mathcal{K}'_+$ at all or v is completely contained in $\mathcal{K}'_+$. In either case, v is homotopic to a sphere lying in W_0: indeed, if v does not intersect the co-core, we can move it out of the handle by pushing it radially from $\Sigma^+ \times (\mathbb{D}^2 \setminus \{0\})$ into the boundary $\Sigma^+ \times \mathbb{S}^1 \subset W_0$, and if $v \subset \mathcal{K}'_+ = \Sigma_+ \times \{0\}$, then we can simply shift it to $\Sigma_+ \times \{1\} \subset W_0$. Using the fact that u_0 and u_+ are both somewhere injective, together with the semipositivity and genericity assumptions, we deduce that every connected component of u_∞ has nonnegative index, thus $\overline{\mathcal{M}} \setminus \mathcal{M}$ has codimension at least two in $\overline{\mathcal{M}}$. The rest of the proof is the same as for Theorem 7.17.

7.6. Construction of Liouville domains with disconnected boundary

7.6.1. Contact products and Liouville pairs

In this section we construct Liouville pairs on closed manifolds of every odd dimension. It is enough to prove Theorem 7.4 in the special case of a totally real number field. The general case discussed in Massot, Niederkrüger, and Wendl 2013 adds quite a bit of complications that are not needed here.

Recall that the goal is to find positive/negative pairs of contact forms (α_+, α_-) on oriented odd-dimensional manifolds M with the property that, if $s \in \mathbb{R}$ denotes the coordinate on the first factor of $\mathbb{R} \times M$,

$$\beta := e^{-s}\alpha_- + e^{s}\alpha_+$$

defines a positively oriented Liouville form on $\mathbb{R} \times M$.

Given such a pair, we define a 1-form on $\mathbb{R} \times \mathbb{S}^1 \times M$ by

$$\lambda_{\mathrm{GT}} = \frac{1+\cos s}{2}\,\alpha_+ + \frac{1-\cos s}{2}\,\alpha_- + (\sin s)\,dt, \tag{7.2}$$

and denote $\xi_{\mathrm{GT}} := \ker \lambda_{\mathrm{GT}}$.

Proposition 7.26. *The co-oriented distribution ξ_{GT} defined above is a positive contact structure on $\mathbb{R} \times \mathbb{S}^1 \times M$, which can be viewed as an infinite chain of Giroux domains $[k\pi, (k+1)\pi] \times \mathbb{S}^1 \times M = (M \times [k\pi, (k+1)\pi]) \times \mathbb{S}^1$ glued together.*

Proof. Let $\varphi\colon (0,\pi) \to \mathbb{R}$ denote the orientation reversing diffeomorphism defined by $\varphi(s) = \ln \frac{1+\cos s}{\sin s}$. This induces an orientation preserving diffeomorphism from the interior of $\Sigma := M \times [0,\pi]$ to $\mathbb{R} \times M$, so pulling back $\beta := \frac{1}{2}\left(e^{u}\alpha_+ + e^{-u}\alpha_-\right)$ gives a Liouville form which defines on Σ the structure of an ideal Liouville domain. Regarding $\partial\Sigma$ as the zero-set of the function $\sin s$ and writing $u = \varphi(s)$, the Giroux domain $\Sigma \times \mathbb{S}^1$, then inherits the contact form

$$\lambda_{\mathrm{GT}} = (\sin s) \cdot \left[dt + \frac{1}{2}\left(e^{u}\alpha_+ + e^{-u}\alpha_-\right)\right] ,$$

proving that λ_{GT} is indeed a positive contact form on $M \times [0,\pi] \times \mathbb{S}^1 = [0,\pi] \times \mathbb{S}^1 \times M$. A similar argument proves the contact condition on $[\pi, 2\pi] \times \mathbb{S}^1 \times M$, and the rest follows by periodicity. $\square$

The first example of a Liouville pair is $\pm d\theta$ on $\mathbb{S}^1$. One can construct higher dimensional examples using contact products. The contact product of (M_1, ξ_1) and (M_2, ξ_2) is defined as the product of their symplectizations $S\xi_1 \times S\xi_2$ divided by the diagonal $\mathbb{R}$-action (cf. Giroux 2010). This describes a contact manifold but, since the Liouville pair condition is really about contact forms and not only contact structures, we want a more specific construction. Suppose we have contact forms α_1 and α_2. Those give identifications between $S\xi_i$ and $\mathbb{R} \times M_i$ with fiber coordinates t_i on $\mathbb{R}$. On the product, one has the Liouville form $\lambda = e^{t_1}\alpha_1 + e^{t_2}\alpha_2$ and its dual vector field $X = \partial_{t_1} + \partial_{t_2}$. We shall say that a manifold V with a contact form λ is a *linear model* for the contact product of (M_1, α_1) and (M_2, α_2) if it is realized as a hypersurface in $S\xi_1 \times S\xi_2$ transverse to X and defined by a linear equation on t_1 and t_2. Concretely, this means $V = M_1 \times \mathbb{R} \times M_2$ is embedded into the product $(\mathbb{R} \times M_1) \times (\mathbb{R} \times M_2)$ by $\varphi(m_1, t, m_2) = (\mu t, m_1, \nu t, m_2)$ for some constants μ and ν. This gives a hypersurface positively transverse to X provided $\nu > \mu$. The contact form induced by λ on V is then $e^{\mu t}\alpha_1 + e^{\nu t}\alpha_2$. An easy computation proves the following important result (there are more general versions).

Proposition 7.27. *If M_1 is $\mathbb{R}$ or $\mathbb{S}^1$ endowed with the Liouville pair $\alpha_\pm = \pm d\theta$ and (M_2, α_2) is any manifold with a contact form, then any linear model for the contact product inherits a Liouville pair $\pm e^{\mu t} d\theta + e^{\nu t}\alpha$.*

Of course, the disadvantage of the contact product construction is that the resulting manifold is never compact, and there seems to be no general way of finding compact quotients of contact products. We shall therefore specialize further by seeking examples among Lie groups which can be seen as symplectizations of some subgroups that have co-compact lattices. (The idea to consider left-invariant contact forms on Lie groups is borrowed from Geiges 1994.)

We set

$$H_n = \left\{(t_1, \dots, t_n) \in \mathbb{R}^n \; ; \; \sum t_i = 0\right\}$$
$$G_n = H_n \ltimes \mathbb{R}^n \quad \text{where} \quad (t_1, \dots, t_n) \cdot (\theta_1, \dots, \theta_n) = (e^{t_1}\theta_1, \dots, e^{t_n}\theta_n)$$

Of course H_n is isomorphic to $\mathbb{R}^{n-1}$ but the above description is convenient in order to find a cocompact lattice in G_n.

One can show, using the above proposition or by an explicit calculation, that for a suitable choice of orientation on G_n,

$$\alpha_\pm := \pm e^{-t_1}\, d\theta_1 + e^{-t_2}\, d\theta_2 + \dots + e^{-t_n}\, d\theta_n \,. \tag{7.3}$$

is a left-invariant Liouville pair on G_n.

Proposition 7.28. *There exist lattices $\Lambda \subset H_n$ and $\Lambda' \subset \mathbb{R}^n$ such that the group action of Λ preserves Λ'. Hence, for every integer $n \geq 1$, the Liouville pair defined by (7.3) on $\mathbb{R}^{n-1} \times \mathbb{R}^n$ descends to a compact quotient which is a $\mathbb{T}^n$-bundle over $\mathbb{T}^{n-1}$.*

Proposition 7.28 is trivial when $n = 1$, and elementary when $n = 2$: for the latter case, one can choose $\Lambda \subset \mathbb{R}$ to be generated by any real number $\tau \neq 0$ such that e^τ is an eigenvalue of some matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. Then A may be viewed as the matrix of the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2 : (\theta_1, \theta_2) \mapsto (e^{-\tau}\theta_1 e^{\tau}\theta_2)$ in some other basis where it has integer coefficients. This transformation therefore preserves the lattice generated by that basis. This produces a Liouville pair on every $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with hyperbolic monodromy—these examples have appeared previously in Geiges 1995 and Mitsumatsu 1995. A hint of the general arithmetic strategy we will use below appears in this discussion, as the condition that e^τ should be an eigenvalue of some matrix in $\mathrm{SL}(2, \mathbb{Z})$ implies that e^τ belongs to a quadratic extension of the field $\mathbb{Q}$.

7.6.2. Some number theory

In this section we will need some standard notions and results from algebraic number theory, e.g. Dirichlet's Unit Theorem; a good reference for this material is Marcus 1977.

Let $\Bbbk$ be a number field, i.e. a finite degree extension of $\mathbb{Q}$, and let n denote its degree $[\Bbbk : \mathbb{Q}]$. Such a field is always isomorphic to $\mathbb{Q}[X]/(f)$ for some irreducible polynomial $f \in \mathbb{Q}[X]$ of degree n (with simple roots). Throughout this section and the next, we assume that all roots of f are real. One says that $\Bbbk$ is a totally real number field.

Each root α gives an embedding of $\Bbbk$ into $\mathbb{R}$, sending (the equivalence class of) X to α. These embeddings will be denoted by $\rho_1, \dots, \rho_n$. This method actually gives all embeddings of $\Bbbk$ into $\mathbb{C}$, and we can collect them to define an injective map

$$j\colon \Bbbk \to \mathbb{R}^n : x \mapsto (\rho_1(x), \dots, \rho_n(x)) .$$

The norm of an element of $\Bbbk$ is defined as $N(x) = \prod_i \rho_i(x)$, and the fact that f is irreducible implies that $N(x)$ vanishes only when $x = 0$. The ring of integers $\mathcal{O}_\Bbbk$ of $\Bbbk$ is by definition the set of all elements in $\Bbbk$ which are roots of monic polynomials with coefficients in $\mathbb{Z}$. These all have integer-valued norms, and an important observation is that the map j defined above sends $\mathcal{O}_\Bbbk$ to a lattice $\Lambda'_\Bbbk$ in $\mathbb{R}^n$.

Invertible elements in the ring $\mathcal{O}_\Bbbk$ are called *units* of $\Bbbk$, and they form a (multiplicative) group denoted by $\mathcal{O}_\Bbbk^\times$. They all have norm ± 1 since $N(xy) = N(x)N(y)$. We denote by $\mathcal{O}_\Bbbk^{\times,+}$ the subgroup of *positive* units: $\mathcal{O}_\Bbbk^{\times,+} = \{x \in \mathcal{O}_\Bbbk^\times \mid \rho_i(x) > 0 \text{ for all } i\}$. Dirichlet's Unit Theorem implies that $\mathcal{O}_\Bbbk^{\times,+}$ is isomorphic to $\mathbb{Z}^{n-1}$. The map j restricts to an injective group homomorphism of $\mathcal{O}_\Bbbk^{\times,+}$ into the multiplicative group $(\mathbb{R}_+^*)^n$, and since $N(\mathcal{O}_\Bbbk^{\times,+}) = \{1\}$, $\mathrm{Ln} \circ j(\mathcal{O}_\Bbbk^{\times,+})$ lands in H_n, where $\mathrm{Ln}(x_1, \dots, x_n) := (\ln x_1, \dots, \ln x_n)$. The precise formulation of Dirichlet's theorem (still in the totally real case) is that $\Lambda_\Bbbk := \mathrm{Ln} \circ j(\mathcal{O}_\Bbbk^{\times,+})$ is a lattice in H_n. Because multiplication by elements of $\mathcal{O}_\Bbbk^\times$ preserves $\mathcal{O}_\Bbbk$, this proves Proposition 7.28.

The manifold $M_\Bbbk$ of Theorem 7.4 is $G_n/(\Lambda_k \ltimes \Lambda'_\Bbbk)$. Note that the only choices we made in the construction were the ordering of the embeddings of $\Bbbk$ into $\mathbb{R}$. The manifold $M_\Bbbk$ does not depend on these choices up to diffeomorphism. Also one can easily check that G_n has a unique isotopy class of left-invariant Liouville pair.

Examples

We now discuss two examples to see all the objects discussed above appearing. The very first example of a number field is $\mathbb{Q}$ itself. In this case $n = 1$, $f = X - 1$, and j is the inclusion of $\mathbb{Q}$ in $\mathbb{R}$. The ring of integers is $\mathcal{O}_\Bbbk = \mathbb{Z}$, with $\mathcal{O}_\Bbbk^\times = \{\pm 1\}$ and $\mathcal{O}_\Bbbk^{\times,+} = \{1\}$. As expected, $M_\mathbb{Q} = \mathbb{R}/j(\mathcal{O}_\Bbbk) \cong \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$.

As a less trivial example, we consider $\Bbbk = \mathbb{Q}[\sqrt{2}]$. Here $n = 2$ and $f = X^2 - 2$ with roots $\pm\sqrt{2}$. The j map is defined by $a + bX \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$. The norm of $a + bX$ is $a^2 - 2b^2$. The integer ring $\mathcal{O}_\Bbbk$ is $\mathbb{Z} + \mathbb{Z}X$, and its image under j is the lattice

$$\{(a + b\sqrt{2}, a - b\sqrt{2}) \mid a, b \in \mathbb{Z}\} = \mathbb{Z}(1, 1) + \mathbb{Z}(\sqrt{2}, -\sqrt{2}) \subset \mathbb{R}^2 .$$

The group of units is $\mathcal{O}_{\Bbbk}^{\times} = \{\pm(1+X)^k \mid k \in \mathbb{Z}\}$, and $\mathbb{U}_{\Bbbk} = \{\pm 1\}$. Restricting to positive elements, we have $\mathcal{O}_{\Bbbk}^{\times,+} = \{(3+2X)^k \mid k \in \mathbb{Z}\}$ and $\mathbb{U}_{\Bbbk}^{+} = \{1\}$. The image of $\mathcal{O}_{\Bbbk}^{\times,+}$ in $(\mathbb{R}_+^*)^2$ is $j(\mathcal{O}_{\Bbbk}^{\times,+}) = \left\{((3+2\sqrt{2})^k, (3-2\sqrt{2})^k) \mid k \in \mathbb{Z}\right\}$, hence $\mathrm{Ln} \circ j(\mathcal{O}_{\Bbbk}^{\times,+}) = \left\{(k\ln(3+2\sqrt{2}), k\ln(3-2\sqrt{2}))\right\}$ which is indeed a lattice in $H_1 = \{(t_1,t_2) \in \mathbb{R}^2 \mid t_1 + t_2 = 0\}$ generated by $m := \big(\ln(3+2\sqrt{2}), \ln(3-2\sqrt{2})\big)$. One can check by hand that this lattice preserves $\Lambda'_{\Bbbk}$ which is generated by $(1,1)$ and $(\sqrt{2}, -\sqrt{2})$. In this basis, the matrix of m is $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, so we see that $M_{\Bbbk}$ is a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with monodromy A, which is hyperbolic. The Liouville pair we constructed yields two contact structures which rotate in opposite directions between the stable and unstable foliations of the Anosov flow defined by the monodromy (cf. Mitsumatsu 1995).

7.6.3. There is no alternative

We now briefly sketch why, as soon as one seeks Liouville pairs among left-invariant 1-forms on Lie groups, number theory is unavoidable.

First there is not much freedom in choosing Lie groups to build invariant Liouville pairs, at least in low dimension. The following result was not discussed in Massot, Niederkrüger, and Wendl 2013 so we include a proof here.

Theorem 7.29. *In dimension 5, there are only two Lie groups that have a cocompact lattice and a left invariant Liouville pair.*

One of these Lie groups is G_3 discussed above and the other is attached to the case of number fields which have one real embedding and two complex conjugate non-real embeddings in $\mathbb{C}$ (discussed in ibid. but not here).

Proof. According to Diatta and Foreman 2015, Theorem 3.1 there are exactly seven simply connected 5-dimensional Lie groups having a left invariant contact form and a cocompact lattice. We will use the notations in ibid. to name them. We want to prove that five of them do not have an invariant Liouville pair. We now list those groups with a basis $(\alpha_1, \ldots, \alpha_5)$ of their spaces of invariant 1-forms and non-vanishing exterior derivatives.

- D1 where $d\alpha_1 = \alpha_4 \wedge \alpha_2 + \alpha_5 \wedge \alpha_3$
- D2 where $d\alpha_1 = \alpha_4 \wedge \alpha_3 + \alpha_5 \wedge \alpha_2$ and $d\alpha_2 = \alpha_5 \wedge \alpha_3$
- D3 where $d\alpha_1 = \alpha_4 \wedge \alpha_3 + \alpha_5 \wedge \alpha_2$ and $d\alpha_2 = \alpha_5 \wedge \alpha_3$ and $d\alpha_3 = \alpha_5 \wedge \alpha_4$
- D5 where $d\alpha_1 = \alpha_3 \wedge \alpha_2 + \alpha_5 \wedge \alpha_4$ and $d\alpha_2 = \alpha_5 \wedge \alpha_2$ and $d\alpha_3 = \alpha_3 \wedge \alpha_5$
- D11 where $d\alpha_1 = \alpha_3 \wedge \alpha_2 + \alpha_5 \wedge \alpha_4$ and $d\alpha_2 = \alpha_3 \wedge \alpha_5$ and $d\alpha_3 = \alpha_5 \wedge \alpha_2$

In all cases, we notice that $\mathrm{vol} := \alpha_1 \wedge d\alpha_1^2$ is a volume form and:

$$\forall \gamma \in \mathrm{Span}(\alpha_2, \cdots, \alpha_5), \qquad d\gamma^2 = 0, \quad \gamma \wedge d\alpha_1^2 = 0 \quad \text{and} \quad d\alpha_1 \wedge d\gamma = 0. \tag{7.4}$$

Suppose now we have a Liouville pair $\alpha_\pm$. We write $\alpha_\pm = a_\pm \alpha_1 + \gamma_\pm$ where $\gamma_\pm$ is in $\mathrm{Span}(\alpha_2, \ldots, \alpha_5)$. Our Liouville form on $\mathbb{R} \times G$ is $\beta = e^s \alpha_+ + e^{-s}\alpha_-$ as usual.

Using Equation (7.4), one computes first that: $\alpha_\pm \wedge d\alpha_\pm^2 = a_\pm^3 \mathrm{vol}$ hence a_+ is positive and a_- is negative. Then one computes:

$$d\beta^3 = (e^s a_+ - e^{-s} a_-)(e^s a_+ + e^{-s} a_-)^2 \, ds \wedge \mathrm{vol}.$$

So the symplectic condition is violated when $s = \frac{1}{2}\ln(-a_-/a_+)$. □

Next we want to sketch why all cocompact lattices in G_n come from number theory. Suppose Γ is such a lattice. One can prove that the commutator subgroup $[\Gamma, \Gamma]$ is a lattice in $[G_n, G_n] = \{0\} \times \mathbb{R}^n$, and then $\Gamma \cap H_n$ is also a lattice. To each element $(t_1, \ldots t_n)$ of H_n we associate the diagonal matrix $\mathrm{diag}(e^{t_1}, \ldots, e^{t_n})$. Seen this way, the lattice $\Gamma \cap H_n$ is generated by diagonal matrices $A_1, \ldots, A_{n-1}$. Because the action on $\mathbb{R}^n$ of all those matrices preserves the lattice $\Gamma \cap H_n$, there is a basis where they all have integer coefficients.

Hence we see that, in order to build a lattice in G_n, we need $n-1$ matrices with integer coefficients that pairwise commute but have no other relations (they span a $\mathbb{Z}^{n-1}$ in $\mathrm{SL}_n(\mathbb{Z})$) and are diagonalizable. Building them by hand is already non-trivial for $n = 3$. One can actually prove that the algebra generated over $\mathbb{Q}$ by $\mathrm{Id}, A_1, \ldots, A_{n-1}$ is a number field with degree n. So number theory is unavoidable here. The lattice then has to be a close relative of the one we constructed above. In particular, one can replace $\mathcal{O}_\Bbbk$ by any additive subgroup M of $\Bbbk$ which is a free abelian group of rank n, and $\mathcal{O}_\Bbbk^{\times,+}$ by any of its finite index subgroups preserving M.

7.7. Later developments and prospects

The results of Massot, Niederkrüger, and Wendl 2013 are still the most advanced results on tight but not fillable contact structures in higher dimensions. The main difference with the situation at the time this paper was written is that, thanks to Borman, Eliashberg, and Murphy 2015, there is now an absolute definition of tightness (although we explained why any reasonable definition of tightness in higher dimensions was forced to make our examples tights).

Although we still don't know whether there exist contact manifolds of dimension at least 7 which are weakly but not strongly fillable, the definition of weak fillings introduced above is arguably one of the most important points in this study, both per se and as a technical tool to study other types of symplectic fillings. Bowden, Crowley, and Stipsicz 2015 constructed new examples of weak symplectic fillings.

Our construction of Liouville manifolds with disconnected boundary were also used in Bowden, Crowley, and Stipsicz 2014 in order to prove existence in all dimensions of contact manifolds which admit a Liouville filling but no Weinstein filling. Even in dimension 3, this is a relatively new result which appeared in Bowden 2012.

Very recently, my student Fabio Gironella used the notion of weak fillings and its stability under Bourgeois's construction to prove existence of closed tight but virtually overtwisted contact structures in all dimensions. He is also investigating the mysterious relation between PS-overtwisted manifolds and overtwisted manifolds.

The use of elementary number theory to produce contact and symplectic manifolds in Section 7.6 is also probably worth further investigations. For instance any number field extension correspond to some contact embedding, and it would be nice to see whether the corresponding Galois group action on contact geometry carries any interesting information.

One slightly unsatisfying aspect of all fillability obstruction results in higher dimensions is the presence of technical semi-positivity assumptions which are present only to avoid transversality issues (see also the discussion at the beginning of Section 7.5). Although this is only a minor annoyance, I think it does provide an interesting test case for any machinery designed to overcome transversality issues in holomorphic curves theory. Recall that, in the proof of the bLob (or plastikstufe) filling obstruction, we have a compactified moduli space $\mathcal{M}$ of holomorphic disks. This is a compact topological space, together with some evaluation map $\mathrm{ev}\colon \mathcal{M} \to N \setminus B$ where N is a bLob and B its binding. There is also a subset $\partial\mathcal{M}$ in $\mathcal{M}$, made of Bishop disks, which is a smooth manifold along which the linearized Cauchy-Riemann equation is surjective. This subset is sent by the evaluation map to a homologically essential submanifold in $\mathcal{M}$. All those properties hold for any almost complex structure J which is standard near the bLob. The traditional way of going on is to add technical assumptions to ensure that, for a generic perturbation of J, there is a path γ in $N \setminus B$ such that $\mathrm{ev}^{-1}(\gamma)$ is a smooth compact 1-manifold whose boundary is its intersection with $\partial\mathcal{M}$, and is connected, contradicting the classification of compact 1-manifolds. But even under these technical assumption, this is only a trick to avoid sphere bubbles. For the sake of discussion, let us make the following definition.

Definition 7.30. *A general machinery designed to overcome transversality issues in symplectic geometry is powerful enough for PS-overtwisted contact manifolds if there is a homology theory $\mathbb{H}$ for compact topological spaces and pairs of them, which reduces to singular homology for manifolds, and the machinery defines a fundamental class $[\mathcal{M}] \in \mathbb{H}(\mathcal{M}, \partial\mathcal{M})$ which is sent by the connecting map $\delta\colon \mathbb{H}(\mathcal{M}, \partial\mathcal{M}) \to H(\partial\mathcal{M})$ to the ordinary fundamental class of the topological manifold $[\partial\mathcal{M}]$.*

Lemma 7.31. *If there exists a general machinery powerful enough for PS-overtwisted contact manifolds then PS-overtwisted manifolds are non-fillable, without any technical assumption on the symplectic filling.*

Proof. We know that $\mathrm{ev}_*[\partial\mathcal{M}]$ is non-trivial in $H(N \setminus B)$ hence $[\partial M]$ cannot be the image of anything from $\mathbb{H}(\mathcal{M}, \partial\mathcal{M})$. Indeed, the commutative diagram

$$\begin{tikzcd}
\mathbb{H}(\mathcal{M}, \partial\mathcal{M}) \arrow[r, "\delta"] & H(\partial\mathcal{M}) \arrow[rr, "\iota_*"] \arrow[dr, "(\mathrm{ev}|_{\partial\mathcal{M}})_*"'] & & \mathbb{H}(\mathcal{M}) \arrow[dl, "\mathrm{ev}_*"] \\
& & H(N \setminus B) &
\end{tikzcd}$$

contradicts the fact that $(\mathrm{ev}|_{\partial\mathcal{M}})_*[\partial\mathcal{M}]$ is non-zero but is in the image of $\mathrm{ev}_* \circ \iota_* \circ \delta$ with $\iota_* \circ \delta = 0$. Note that this above proof is the dream proof that is currently beyond reach even in the semi-positive case. □

There are several transversality machineries around. The only one I have hope to understand well enough is from Pardon 2016. In particular that paper explicitly implies that it will be powerful enough for PS-overtwisted manifolds provided one can equip $(\mathcal{M}, \partial\mathcal{M})$ with an implicit atlas with boundary. I think this construction would be very interesting from a pedagogical point of view, as the minimal interesting geometric application of ibid. It would be simpler than any existing application of this technology in Pardon 2016, 2015 and I cannot think of a simpler potential application.

8. From geometry to topology

8.1. Introduction

Recall from Section 2.2 that one can easily define a natural class of Riemannian metrics compatible with a contact structure or geodesible contact structures. But the only links between their geometry and global topological properties of contact structures were established either under very restrictive symmetry conditions in Komendarczyk 2008, or in a proof where geometry and contact topology were completely uncoupled in Massot 2008a. In this chapter, which surveys Etnyre, Komendarczyk, and Massot 2012, 2016, we explain how geometry and contact topology can be more intimately linked.

A natural reference point here is the well-known sphere theorem in Riemannian geometry, proved in Berger 1960; Klingenberg 1961, which is one of the fundamental results showing how geometry can control the topology of the domain. Recall, the sphere theorem states that every simply connected n-manifold which admits $\frac{1}{4}$-pinched positive sectional curvature is homeomorphic to the n-sphere. The main result of Etnyre, Komendarczyk, and Massot 2012 is the following.

Theorem 8.1 (Contact sphere theorem)**.** *Let (M, ξ) be a closed contact 3–manifold and g a complete Riemannian metric compatible with ξ. If there is a constant $K_{max} > 0$ such that the sectional curvatures of g satisfy*

$$0 < \frac{4}{9} K_{max} < \sec(g) \leq K_{max},$$

then the universal cover of M is diffeomorphic to the 3–sphere by a diffeomorphism taking the lift of ξ to the standard contact structure on the 3–sphere.

As explained in Section 3.2.1, Eliashberg 1992 proved that ξ_{std} is the unique (up to contactomorphism) tight contact structure on S^3, and what we actually prove in the above theorem is tightness of the contact structure. So this theorem really gives Riemannian geometric conditions that imply tightness.

More generally, one might want to go beyond the tight vs overtwisted dichotomy and sort the class of tight contact structures by finding privileged subclasses. In addition to the notion of universal tightness, a classical class is that of symplectically fillable contact structures. In this chapter, we seek classes interacting nicely with curvature in Riemannian geometry. Note that such interactions will automatically be inherited by covering spaces, contrasting with fillability properties.

Our proofs of the contact sphere theorem and other global results use quantitative versions of Darboux theorem. Recall that Darboux theorem in contact geometry guaranties that each point in a contact manifold has a neighborhood which is standard, i.e.

embeds inside the standard contact structure on $\mathbb{R}^{2n+1}$. Given a contact structure ξ on a $(2n+1)$–manifold M and a Riemannian metric g we can ask for a quantitative version guaranteeing that balls up to a certain radius are standard. We define the *Darboux radius* of (ξ, g) at a point $p \in M$ as

$$\delta_p(\xi, g) = \sup\{r < \operatorname{inj}(g,p) \mid \text{the open geodesic ball } (B_p(r), \xi) \text{ at } p \text{ of radius } r \text{ is contactomorphic to an open subset in } (\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})\},$$

(where $\operatorname{inj}(g,p)$ is the injectivity radius of g at p) and the *Darboux radius of* (ξ, g) to be

$$\delta(\xi, g) = \inf_{p \in M} \delta_p(\xi, g).$$

In dimension 3, classification results from Section 3.2.1 prove that balls are standard as long as they are tight. In higher dimension this is no longer true *a priori* and we introduce the variations τ for the *tightness radius* and τ^{PS} for the *PS-tightness radius*. Results from Niederkrüger 2006; Massot, Niederkrüger, and Wendl 2013; Borman, Eliashberg, and Murphy 2015 imply that $\delta(\xi, g) \leq \tau^{\mathrm{PS}}(\xi, g) \leq \tau(\xi, g)$. In dimension 3 all three radii are equal but in higher dimension the situation is unclear, although one can say that $\delta(\xi, g) = \tau^{\mathrm{PS}}(\xi, g)$ is unlikely (see Chapter 10) and Casals, Murphy, and Presas 2015 proves a weaker version of $\tau^{\mathrm{PS}}(\xi, g) = \tau(\xi, g)$. It should be noted that both papers Etnyre, Komendarczyk, and Massot 2012, 2016 were written before the announcement of Borman, Eliashberg, and Murphy 2015 (and a fortiori before Casals, Murphy, and Presas 2015).

Of course, if we do not assume any compatibility condition between the metric and the contact structure then we cannot estimate the tightness radius. We first concentrate on what happens with the compatibility definition recalled in Section 2.2. The tightness radius is, by definition, always less than the injectivity radius but one could ask if, for compatible metrics, they always coincide. This would explain the following surprising result (which is an important ingredient of the proof of Theorem 8.1).

Theorem 8.2. *Let (M, ξ) be a contact 3–manifold and g a complete Riemannian metric that is compatible with ξ. For a fixed point $p \in M$ let $\tau_p = \tau_p(M, \xi)$ and suppose that $\tau_p < \operatorname{inj}_p(g)$. Then for all radii r with $\tau_p \leq r < \operatorname{inj}_p(g)$, the geodesic sphere $S_p(r)$ contains an overtwisted disk.*

Recall that, a priori, overtwisted disks can have a very complicated geometry and this is what makes it hard to prove tightness of contact structures. We find this theorem somewhat surprising as it says that, when a metric is compatible with a contact structure then, as soon as a geodesic ball is large enough to be overtwisted, one sees the overtwisted disk in a specific place, namely the boundary of the ball. Thus making it easy to determine when such a ball is tight (using Bennequin's theorem).

Despite this surprising result, we have numerical experiments which strongly suggest that the tightness radius can indeed be less than the injectivity radius for compatible metrics, so we search for geometrical quantities controlling the tightness radius. To this end we recall that given a Riemannian metric g on M the *convexity radius* of g is defined to be

$$\operatorname{conv}(g) = \sup\{r \mid r < \operatorname{inj}(g) \text{ and the geodesic balls of radius } r \\ \text{are weakly geodesically convex}\},$$

where $\operatorname{inj}(g)$ is the injectivity radius of (M, g). This is the convexity notion discussed at the beginning of Section 2.6.2.

Theorem 8.3. *Let (M, ξ) be a contact manifold and g a complete Riemannian metric that is compatible with ξ. Then,*

$$\tau^{PS}(M, \xi) \geq \operatorname{conv}(g). \tag{8.1}$$

In particular, if $\sec(g) \leq K$, for $K > 0$, then

$$\tau^{PS}(M, \xi) \geq \min\{\operatorname{inj}(g), \frac{\pi}{2\sqrt{K}}\}$$

and $\tau^{PS}(M, \xi) = \operatorname{inj}(g)$, if g has non-positive curvature.

We note that if M is a compact manifold then one may easily show that a lower bound for $\tau^{\mathrm{PS}}(M, \xi)$ exists. To see this note that M may be covered with Darboux balls (which are PS-tight). Then the Lebesgue number for this open cover provides the desired lower bound. Of course this bound exists for any metric and one has virtually no control over it. Theorem 8.3 shows that if the metric is compatible with the contact structure then one does not need compactness and one can estimate $\tau^{\mathrm{PS}}(M, \xi)$ below in terms of curvature and injectivity radius. In particular, our theorem shows when M is non-compact with bounded curvature and injectivity radius, the tightness radius cannot shrink to zero at infinity.

The above theorem is based on holomorphic curves in symplectizations, following Hofer 1993; Albers and Hofer 2009. The crucial point is to prevent those curves to escape the ball. In order to achieve this we need to compare Riemannian convexity and J-convexity in symplectizations of contact manifolds.

We notice that our bounds on the tightness radius are especially effective in the case of non-positive curvature.

Corollary 8.4. *Let (M, ξ) be a contact manifold and g a complete Riemannian metric compatible with ξ having non-positive sectional curvature. Then (M, ξ) and all its covering spaces are PS-tight.*

The class of compatible metrics is very natural but it is fairly restrictive. We note right away that a hyperbolic metric cannot be compatible with a contact structure on a closed manifold according to Blair 2002, p. 99. Furthermore, Blair conjectures that if a metric is compatible with a contact structure on a closed manifold and has non-positive curvature then it is flat. So the above corollary may be of very limited impact[1].

1. However it does not seem obvious that any contact structure which is compatible with a flat metric is tight, especially since there is no classification of foliations of $\mathbb{R}^3$ by lines, contrasting with the situation in $\mathbb{S}^3$.

This justifies the introduction of a more general class of metrics that can include the hyperbolic ones. Another motivation for extending the notion of compatibility comes from the theory of curl eigenfields. A curl eigenfield on a Riemannian 3–manifold is a 1–form α satisfying $*d\alpha = \theta'\alpha$ with θ' constant but $\|\alpha\|$ can vary. (We note that this equation is dual to the normal curl eigenfield equation for vector fields.) See Etnyre and Ghrist 2000 for some applications of this concept.

We say that a Riemannian metric and a contact structure ξ are *weakly compatible* if there exist a Reeb vector field for ξ which is perpendicular to ξ. This condition can be equivalently stated as there exists a contact form α such that

$$* \, d\alpha = \theta' \alpha, \tag{8.2}$$

where θ' is a positive function (which measures the rotation speed of the contact planes).

This class of metrics includes all the non-singular curl eigenfields and Beltrami fields. In addition it allows for hyperbolic metrics. It is also stable both under conformal changes and under the modifications used in Krouglov 2008, see Remark 8.6 below.

We will use several measures of how far a weakly compatible metric is from being compatible. First the rotation speed θ' and the norm ρ of the special Reeb vector field R entering in the definition are both constant in the compatible case so their gradient are such measures. We will also use the mean curvature H of the contact plane field. It vanishes for compatible metrics and its definition is recalled in Section 2.2.3. Finally we shall also use the normalized Reeb vector field $n = R/\|R\|$, which is a unit normal vector field to the contact planes. In the compatible case it is a geodesic vector field so we will consider $\nabla_n n$.

One can prove that the following two combinations of these measures give the same vector field:

$$D_g := \nabla_n n + 2Hn = (\nabla \ln \theta')^\perp - \nabla \ln \rho \tag{8.3}$$

where $v^\perp$ is the component of v perpendicular to ξ. We introduce:

$$d_g = \max_M \|D_g\|$$

Note that d_g is finite whenever M is compact and vanishes for compatible metrics (all terms in D_g vanish in this case). To extend our main theorem to weakly compatible metrics we also introduce the following notation: let $K \geq 0$ and $\sec(g) \leq \pm K$, define

$$\mathrm{ct}_K(r) = \begin{cases} \sqrt{K}\cot(\sqrt{K}r)\,, & \text{for } \sec(g) \leq K, \;\; r \leq \min\{\mathrm{inj}(g), \frac{\pi}{2\sqrt{K}}\}, \\ \frac{1}{r}, & \text{for } \sec(g) \leq 0, \\ \sqrt{K}\coth(\sqrt{K}r), & \text{for } \sec(g) \leq -K. \end{cases} \tag{8.4}$$

Here, of course, in the first case we assume $\sec(g)$ is positive somewhere and in the second case that it is 0 somewhere. Also to simplify our notations we will often write ct_K instead of ct_{-K} understanding that we mean the latter in the negative curvature setting. We may now state our result for weakly compatible metrics as follows.

Theorem 8.5. *Let (M, ξ) be a contact 3-manifold (not necessarily closed) that is weakly compatible with a Riemannian metric g. Whenever $d_g < \infty$ the tightness radius admits the following lower bound*

$$\tau(M, \xi) \geq \min \left\{ \mathrm{ct}_K^{-1} \left(d_g \right), \mathrm{inj}(g) \right\}.$$

Remark 8.6. *The above theorems can be applied only when we have control over the sectional curvature of all plane fields and not only the sectional curvature of contact planes ξ. This is natural in view of the following very slight sharpening of a result of Krouglov saying that the latter curvature is very flexible. To get this version, start with a compatible metric and observe that Krouglov's modifications do not destroy weak compatibility, although they destroy compatibility (this is another reason to use weakly compatible metrics).*

Theorem 8.7 (ibid.)**.** *Given a cooriented contact structure ξ on a closed 3–manifold M and any strictly negative function f, there is a weakly compatible metric on M such that the sectional curvatures of ξ are given by f. Moreover, if the Euler class of ξ is zero then any function f may be realized.* □

Observe that since $\mathrm{ct}_K(r) \to \infty$ as $r \to 0$ the bound in Theorem 8.5 is always nonzero. We also notice that if α is actually compatible with g then $d_g = 0$ and thus $\mathrm{ct}_K^{-1}(d_g)$ can be taken to be $+\infty$ when $K = 0$. A similar situation occurs when $\sec(g) \leq -K$, then $\mathrm{ct}_K^{-1}(r)$ is ill defined for $r \in [-\sqrt{K}, \sqrt{K}]$ and we may assume $\mathrm{ct}_K^{-1}(d_g)$ to be $+\infty$ as well. Recall that for such manifolds, the universal cover is exhausted by geodesic balls. Since an overtwisted disk has to be contained in a compact part of the universal cover, we get the following corollary.

Corollary 8.8. *Let (M, ξ) be a contact 3-manifold (not necessarily closed) that is weakly compatible with a complete Riemannian metric g of non-positive sectional curvature. If*

$$\sec(g) \leq -d_g^2 \tag{8.5}$$

at all points then the contact structure ξ is universally tight.

One remarkable property of compatible metrics is that Reeb orbits are geodesics and we use this in our study of compatible metrics. However, this is precisely what rules out closed hyperbolic manifolds in dimension 3: these manifolds cannot have any geodesic vector field Zeghib 1993. But many hyperbolic manifolds have quasi-geodesic vector fields, see e.g. Calegari 2006. These vector fields also cannot have any contractible Reeb orbits. So if a closed hyperbolic manifold has a quasi-geodesic Reeb field then the corresponding contact structure is universally tight. This observation does not explicitly use any easily defined compatibility between a metric and contact structure. However we can use an easy differential geometric criterion for quasi-geodesicity to get the following theorem which can then be compared to Theorem 8.8.

Theorem 8.9. *Let (M, ξ) be a closed contact manifold (of any dimension). Suppose M admits a metric g such that the sectional curvature of g is bounded above by some* *constant $-K < 0$, and there is a Reeb vector field R for ξ such that the normalized Reeb field $N = R/\|R\|$ satisfies*

$$\|\nabla_N N\| \leq \sqrt{K}.$$

Then (M, ξ) and all its finite covers are PS-tight.

We note that one can think of the condition $\|\nabla_N N\| \leq \sqrt{K}$ as some type of compatibility between g and ξ. We also note that, while this theorem is stronger than Theorem 8.8 when they both apply, it does require that we are working with a closed manifold. We would lastly like to point out that earlier we used n for the normalized Reeb vector field while here we used N for that purpose. We will always use n to denote a unit normal vector field to the contact planes (which, in a weakly compatible metric, the normalized Reeb vector field always is) and use N if the normalized field does not have to be normal.

Finally we return to the question of estimating the Darboux radius in higher dimensions where it probably differs from the tightness radius. The following result was proved in Etnyre, Komendarczyk, and Massot 2016 using techniques in differential geometry, briefly described in Section 8.6, without holomorphic curves.

Theorem 8.10. *Let (M^{2n+1}, ξ) be a $(2n+1)$–dimensional contact manifold and (α, g, J) be a complete compatible metric structure for ξ with rotation speed θ'. If the sectional curvature of g is contained in the interval $[-K, K]$ for some positive K then*

$$\delta(\xi, g) \geq \min\left(\frac{\operatorname{inj}(g)}{2}, \frac{1}{208n^2 \max\left(\sqrt{K}, \|[J,J]\|, \theta'\right)}\right),$$

where $[J, J]$ is the Nijenhuis torsion of the complex structure J on ξ.

8.2. Tightness radius estimates

Let (M, ξ) be a contact manifold. Let S be a regular level set of a smooth function f on M. Let W be the symplectization of (M, ξ), identified with $\mathbb{R}_+ \times M$ using some contact form α. We set $\Sigma = \mathbb{R}_+ \times S \subset W$ and $\mathcal{C} = T\Sigma \cap JT\Sigma$. Recall that the Levi form of Σ is defined on $\mathcal{C}$ by $L(u, v) = -d(df \circ J)(u, Jv)$. We want to compare pseudoconvexity of Σ, measured by its Levi form L, and Riemannian convexity of S, measured by the Hessian $\nabla^2 f$.

Proposition 8.11. *Let g be a metric compatible with the contact structure ξ on M. Then for any $v \in \mathcal{C}$ we have*

$$L(v, v) = \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv). \tag{8.6}$$

□

This formula was known before when (W, J) is Kähler, see Greene and Wu 1973, Lemma page 646. The general case is proved by computations using the special geometry of compatible metrics.

The above proposition proves that Σ is J-convex as long as S is convex (one must be careful with the extra $\mathbb{R}_+$ direction along which $\nabla^2 f$ is not positive but one still has a positive $\nabla^2 f(J\partial_t, J\partial_t)$).

This J-convexity allows to prove tightness as follows. Let $B_p(r_0)$ be the geodesic ball centered at p with radius $r_0 < \operatorname{conv}(g)$. We know that $\mathbb{R}_+ \times \partial B_p(r)$ is a strictly pseudoconvex submanifold of $\mathbb{R}_+ \times M$ for all $r \leq r_0$. Assume now for contradiction that $B_p(r)$ contains an overtwisted disk. Arguing as in Hofer 1993; Albers and Hofer 2009, one can start a family of holomorphic disks near an elliptic singularity in the overtwisted disk. Since these disk cannot touch $\mathbb{R}_+ \times \partial B_p(r_0)$ thanks to pseudoconvexity, the Gromov–Hofer compactness theorem extends to this setting and we get the existence of a closed Reeb orbit γ in $B_p(r_0)$. This orbits yields a J-holomorphic cylinder $C_\gamma = \mathbb{R}_+ \times \gamma$ in $\mathbb{R}_+ \times B_p(r_0)$. But this is a contradiction because C_γ has to touch $\mathbb{R}_+ \times \partial B_p(r)$ from the inside for some $r \leq r_0$.

The estimate in terms of curvature bounds in Theorem 8.3 comes from classical convexity estimates for the distance function, see e.g. Petersen 2006.

In the 3-dimensional case, we have the following version of Proposition 8.11 which holds for metrics that are only weakly compatible with ξ, and similarly implies Theorem 8.5 and corollary 8.8.

Proposition 8.12. *Let g be a metric weakly compatible with the contact form α on the 3–manifold M. Then for any $v \in \mathcal{C}$ we have*

$$L(v, v) = \nabla^2 f(v, v) + \nabla^2 f(Jv, Jv) - \langle \nabla f, D_g \rangle \|v\|^2$$

where $D_g = \nabla_n n + 2Hn$.

8.3. Tightness and quasi-geodesics

Here we prove Theorem 8.9, a geometric "universal tightness" criterion for contact structures using quasi-geodesics.

Suppose by contradiction (M, ξ) is overtwisted. According to Hofer 1993; Albers and Hofer 2009 there will be a closed contractible orbit in the flow of the Reeb field R, and hence in the flow of $N = R/\|R\|$. This orbit will lift to a closed orbit γ in the universal cover of M. Of course our metric, contact structure and Reeb field also lift to the universal cover and satisfy the same hypotheses (since we will work exclusively in the cover from now on, we will use the same notation for objects in the cover).

Choose any point p in the cover. There will be some r such that the embedded geodesic ball $B_p(r)$ of radius r about p will contain γ and $\partial B_p(r)$ will have a tangency with γ. Let $\mathsf{r}_p(x) = d(p, x)$ be the radial function measuring the distance from p. Convexity estimates for the distance function guarantee that

$$\sqrt{K} < \nabla^2 \mathsf{r}_p(\dot\gamma, \dot\gamma).$$

We parameterize γ such that the tangency $\partial B_p(r)$ with occurs at $\gamma(0)$. As γ lies inside $B_p(r)$ we see that

$$0 \geq \frac{\partial^2}{\partial t^2}(\mathsf{r}_p \circ \gamma)|_{t=0} = \frac{\partial}{\partial t} g(\nabla \mathsf{r}_p, \dot{\gamma}) = \nabla^2 \mathsf{r}_p(\dot{\gamma}, \dot{\gamma}) + d\mathsf{r}_p(\nabla_{\dot{\gamma}}\dot{\gamma}).$$

Thus we can compute

$$\sqrt{K} < \nabla^2 \mathsf{r}_p(\dot{\gamma}, \dot{\gamma}) \leq -g(\nabla \mathsf{r}_p, \nabla_{\dot{\gamma}}\dot{\gamma}) \leq \|\nabla_{\dot{\gamma}}\dot{\gamma}\| = \|\nabla_N N\| \leq \sqrt{K},$$

where the third inequality comes from the fact that $\nabla \mathsf{r}_p$ has unit length so $-g(\nabla \mathsf{r}_p, \nabla_{\dot{\gamma}}\dot{\gamma})$ is one component of $\nabla_{\dot{\gamma}}\dot{\gamma}$ in some orthonormal basis. Thus we arrive to the absurd consequence that $\sqrt{K} < \sqrt{K}$ and an overtwisted disk could not exist.

8.4. Overtwisted balls for compatible metrics

We now discuss the proof of Theorem 8.2 about overtwisted balls. An oriented foliation $\mathcal{F}$ on a sphere S is *simple* if it has exactly one singularity of each sign (the positive one will be called the north pole and the negative one the south pole). It is *almost horizontal* if, in addition, all its closed leaves are oriented as the boundary of the disk containing the north pole (in other words from "west" to "east"). These are (slight variations on) definitions from Eliashberg 1989.

If ξ is a contact structure on M and S is a sphere in M then we will say that ξ is simple or almost horizontal along S if $\mathcal{F} = \xi S$ has this property. The relevance of these definitions to compatible metrics comes from the following lemma which crucially use the geometry of compatible metrics.

Lemma 8.13. *Let g be a Riemannian metric compatible with the contact manifold (M, ξ). Let α be the contact form implicated in the definition of compatibility between g and ξ and R_α its Reeb vector field. Let $r < \mathrm{inj}(g)$ and S be the sphere of radius r around some point p_0. The contact structure ξ is simple along S with poles $\exp_{p_0}(\pm r R_\alpha)$.*

Proof. Let S be a sphere of radius $r < \mathrm{inj}(g)$ around p_0. Let γ be a unit speed geodesic starting at p_0 and denote $\gamma(r) \in S$ by p. Suppose p is a singularity of ξS, that is, ξ_p is tangent to S at p. By Gauss' Lemma, ξ_p is orthogonal to γ at p. As R_α is also orthogonal to ξ_p we must have $\gamma'(r) = \pm R_\alpha$. As the flow of R_α give geodesics we see that $\gamma'(t)$ is equal to $\pm R_\alpha$ for all t and hence $\gamma'(0) = \pm R_\alpha$. Thus p is $\exp_{x_0}(\pm r R_\alpha)$. □

The following proposition explores how a contact structure on a ball can be overtwisted by explaining some relations between simple, almost horizontal and tight. The first point is obvious. The second one was observed by Giroux and is used in the proof of the third one, which will be crucial for the sphere theorem.

Proposition 8.14. *Let B be a ball in a 3–dimensional contact manifold (M, ξ) which is the disjoint union of a point p and a family of spheres S_t, $t \in (0, 1]$.*

1. *If all foliations ξS_t are simple and $\xi|_B$ is tight then ξ is almost horizontal along all the S_t.*
2. *If all foliations ξS_t are almost horizontal, then $\xi|_B$ is tight.*
3. *If all foliations ξS_t are simple, and $\xi|_B$ is overtwisted then there is some radius t_1 such that all foliations ξS_t for $t \geq t_1$ have closed leaves.*

The proof of the above proposition was inspired by Giroux's proof of Bennequin's theorem. The details are a bit technical but the key idea is contained in Figure 8.1. Any closed leaf birth (resp. death) near the North pole goes West (resp East). Indeed the contact structure is positive hence a leaf which goes West becomes, for larger radii, a circle along which the characteristic foliation goes towards the North pole. The latter is repelling hence a leaf which goes West near the North pole cannot die.

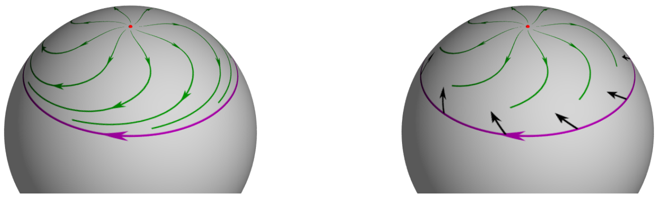


Figure 8.1.: Closed leaf near the North pole in the proof of Proposition 8.14

Proof of Theorem 8.2. Fixing $p \in M$ consider the geodesic spheres $S_p(r)$ of radius r about p and the geodesic balls $B_p(r)$ that they bound. We can use Lemma 8.13 to conclude that all the spheres $S_p(r)$, $r \leq \mathrm{inj}_p(g)$, have simple characteristic foliation. Recall that we are assuming that $\tau_p < \mathrm{inj}_p(g)$ *i.e.* $B_p(r), r < \tau_p$, is tight and $B_p(r)$, $r > \tau_p$, is overtwisted. Let

$$r' = \inf\{r \,|\, \text{such that } S_p(r) \text{ has a closed leaf in its characteristic foliation.}\}$$

Notice that $S_p(r')$ does have a closed leaf since simple foliations on spheres without closed leaves form an open set. By Proposition 8.14 the contact structure restricted to $B_p(r), r < r'$, is tight. Thus $r' = \tau_p$ and we see that $S_p(\tau_p)$ has a closed leaf in its characteristic foliation, which of course bounds an overtwisted disk. We then get overtwisted disks on spheres of higher radii using the third point of Proposition 8.14. □

8.5. The contact sphere theorem

The following proposition is a variation on a similar result used in the proof of the topological sphere theorem Klingenberg 1961, it does not involve any contact geometry. Its proof is based on comparison theorems in Riemannian geometry, especially Toponogov theorem, and Klingenberg's injectivity radius estimate. The example of lens spaces prove that the simple connectivity assumption is crucial.

Proposition 8.15. *Let M be complete simply connected Riemannian manifold whose sectional curvature satisfies $\frac{4}{9} < K \leq 1$. If p and q in M are at maximal distance, that is $d(p,q) = \operatorname{diam}(M)$, then there are radii r_p and r_q such that*

- *both closed balls $\overline{B}(p, r_p)$ and $\overline{B}(p, r_p)$ are embedded, i.e. $r_p, r_q < \operatorname{inj}(g)$,*
- *the ball $\overline{B}(q, r_q)$ is convex, i.e. $r_q < \operatorname{conv}(g)$,*
- *$M = \overline{B}(p, r_p) \cup \overline{B}(q, r_q)$, and*
- *the boundary of each ball, $\overline{B}(p, r_p)$ and $\overline{B}(p, r_p)$, is contained in the interior of the other ball.*

Proof of Theorem 8.1. We now gather the different ingredients of the contact sphere theorem, highlighting how Riemannian geometry, topological methods in contact geometry and pseudoholomorphic curves arguments interact in this proof.

Since both the hypotheses and the conclusion of the theorem are scale invariant, we can assume that the curvature is bounded above by 1 so that $\frac{4}{9} < K \leq 1$. Moreover, we can assume that M is simply connected as pulling the contact structure and metric back to the universal cover of M does not affect the curvature pinching.

Deep classical Riemannian geometry gives, through Proposition 8.15, that there are two geometrics balls B_{cvx} and B_{big} whose interior covers our manifold M and such that B_{cvx} is weakly convex.

We assume for contradiction that ξ is an overtwisted contact structure. A priori, it could be that all overtwisted disks intersect both B_{cvx} and B_{big}. However there is no loss of generality in assuming that there is one, which we denote by D, which misses the center q of B_{cvx}.

Our comparison of Riemannian and almost-complex convexity combines with pseudo-holomorphic curves arguments of Gromov and Hofer to tell us, through Theorem 8.3, that B_{cvx} is a tight ball.

We can now use either the classification of tight contact structures on balls by Eliashberg Eliashberg 1992 or, more elementarily, our description in Proposition 8.14 and its proof. Either way, we can construct a contact vector field transverse to the concentric spheres that make up $B_{\text{cvx}} \setminus \{q\}$. This contact vector fields generates a contact isotopy that will push any subset of B_{cvx} that misses q into an arbitrarily small neighborhood of ∂B_{cvx}. In particular we can push D into B_{big}.

Based on the special geometry of compatible metrics and a topological argument using Giroux's study of bifurcations for characteristic foliations, Proposition 8.14 then tells us that there is an overtwisted disk on ∂B_{big}. However, as we know $\partial B_{\text{big}} \subset B_{\text{cvx}}$, this contradicts the tightness of B_{cvx}. Hence we see that ξ must be tight.

Although this is does not completely follow from the previous discussion, the ambiant manifold is the 3-sphere as is guarantied by the classical sphere theorem. Now we get that ξ is isomorphic to the standard contact structure because all tight contact structures on the sphere are standard. This later fact is due to Eliashberg ibid. and uses purely topological method in contact geometry (see also Giroux 2000, Remark 2.18 for Giroux's alternative proof). $\square$

Remark 8.16. *As an immediate corollary of the contact sphere theorem, we obtain that any contact structure compatible with the round metric on $\mathbb{S}^3$ is isotopic to the standard one ξ_0. While this result is not obvious, it already follows from older results. Indeed, suppose that ξ is compatible with a round metric and denote by R the Reeb vector field involved. This vector field is geodesic (its orbits are great circles parametrized by arc length) and divergence free. Although there is an infinite dimensional space of geodesic vector fields on $\mathbb{S}^3$, Gluck and Gu 2001 proved that they become completely rigid if we assume in addition that they are divergence free. So ξ is actually conjugated to ξ_0 by an isometry.*

This rigidity is of a completely different nature than the statement of the sphere theorem where the isotopy with the standard structure is unrelated to any rigid structure on the sphere.

8.6. A quantitative Darboux theorem in any dimension

We now explain the steps needed to prove the Darboux radius estimate of Theorem 8.10. The statement from Section 8.1 is already a simplified weaker version of the result from Etnyre, Komendarczyk, and Massot 2016, and here we will state even simpler statements for the intermediate pieces. In the context of Theorem 8.10, we set:

$$\rho = 1/\max(\theta', \sqrt{K}, \|[J, J]\|)$$

which is the length scale appearing in the main estimate.

The goal is to embed a large geodesic ball in our contact manifold into the standard contact $\mathbb{R}^{2n+1}$. The later is the contactization of the standard Liouville structure on $\mathbb{R}^{2n}$ and we will compare it to some contactization of a natural exact symplectic manifold inside our given contact metric manifold M.

Given any point p in M and the contact hyperplane ξ_p at p, the geodesic disk $\mathbf{D}(r)$ centered at p of radius r and tangent to ξ_p is given as the image of the restriction of the exponential map to the disk of radius r in ξ_p, that is

$$\mathbf{D}(r) = \exp_p(D_\xi(r)).$$

where $D_\xi(r) = \Big(\{v \in \xi_p;\ |v| < r\}\Big)$. Denoting the Reeb flow by $\Phi(t, \mathbf{x}) : \mathbb{R} \times M \longrightarrow M$ we define the map

$$E : \mathbb{R} \times D_\xi(r) \to M : (t, v) \mapsto \Phi(t, \exp_p(v)),$$

and the R_α–invariant "cylindrical" neighborhood $\mathbf{C}(r)$ of $\mathbf{D}(r)$ to be the image of E. Of course $\mathbf{C}(r)$ is not, in general, an embedded submanifold of M, but for r small enough

$\mathcal{D}(r)$ will be an embedded disk and R_α will be transverse to $\mathbf{D}(r)$. For such an r, $\mathbf{C}(r)$ will then contain embedded neighborhoods of $\mathbf{D}(r)$, for example $E((-\epsilon,\epsilon)\times D_\xi(r))$, for sufficiently small ϵ. The Darboux radius estimate proof proceeds in the following steps.
Step I. Find an estimate on the radius r so that R_α is transverse to $\mathbf{D}(r)$.
Step II. Find an estimate on the radius r so that the pull back of the contact structure ξ to $\mathbb{R}\times D_\xi(r)$ via E embeds into the standard contact $\mathbb{R}^{2n+1}$.
Step III. Find an estimate on the size of a geodesic ball about p that embeds in M and is contained in $\mathbf{C}(r)$.

The estimate in Step I is given in the following proposition whose proof is based on Jacobi fields estimates and Gronwall type arguments.

Proposition 8.17. *Given a compatible metric structure (α, g, J) on the contact manifold (M,ξ), the disk $\mathcal{D}(r_0)$ is embedded and the Reeb vector field R_α is positively transverse to it if*

$$r_0 < r_{\pitchfork} := \min\left\{\mathrm{inj}(g), \frac{\rho}{3n^{1/4}}\right\}, \tag{8.7}$$

Moreover, if n_{D} is a unit normal vector to $\mathcal{D}(r_0)$, then along any radial geodesic $\gamma=\gamma(r)$

$$\langle R_\alpha(r), n_{\mathrm{D}}(r)\rangle \geq 1 - r/\rho - \sqrt{n}(r/\rho)^2, \tag{8.8}$$

To carry out Step II we first make an observation about contactizations of Liouville domains and exact symplectic manifolds. For the remainder of this section (W,β_0) will be a Liouville domain. Let μ denote the restriction of β_0 to ∂W. By definition μ is a contact form. The completion of W is obtained as usual by adding the cylindrical end $[1,\infty)\times\partial W$ equipped with the Liouville form $t\mu$, where t is the "radial" coordinate on $[1,\infty)$. The resulting manifold will be denoted W_∞ and we will also denote this extended 1-form by β_0. For any constant $a>1$ we set $W_a = W\cup([1,a)\times\partial W)$. We say an almost complex structure is *adapted* to β_0 if

(a) it is tamed by $d\beta_0$,

(b) it preserves the contact structure $\ker\mu$ on each $\{t\}\times\partial W$, and

(c) it sends ∂_t, point-wise, to some positive multiple of the Reeb field R_μ.

Proposition 8.18. *Suppose β_1 is a 1–form on W_T (for some $T>1$) such that $d\beta_1$ is a symplectic form on W_T and there is an almost complex structure which is adapted to β_0 and tamed by $d\beta_1$. Then, for any $T_0\in[1,T)$, the contactization of $(\overline{W}_{T_0},\beta_1)$ embeds in the contactization of (W_∞,β_0).*

The key technical step to prove the above proposition is the following interpolation lemma whose proof only needs some careful cut-off functions.

Lemma 8.19 (Interpolation lemma)**.** *Let (W,β_0) be a Liouville domain and (W_∞,β_0) its completion. Suppose β_1 is a 1–form on W_T (for some $T>1$) such that $d\beta_1$ is a symplectic form on W_T and there is an almost complex structure which is adapted to β_0 and tamed by $d\beta_1$. Then for any $T_0\in[1,T)$ there is a positive constant λ and a Liouville form $\widehat{\beta}$ on W_∞ such that* *(i)* $\hat{\beta} = \lambda\beta_0$ *outside* W_T,

(ii) $\hat{\beta} = \beta_1$ *on* W_{T_0}, *and*

(iii) $d\hat{\beta}$ *is tamed by* J.

The proposition then follows from an easy Moser type argument. In our situation, we want to apply the above proposition to the complex structure on $\mathcal{D}(r)$ obtained by pushing forward, via $\exp_p$, some complex structure on ξ_p tamed by $d\alpha_p$. Again we use Jacobi field estimates to prove the following.

Proposition 8.20. *In the above setup, the complex structure* $(\exp_p^\xi)_* J_p$ *is tamed by the restriction of* $d\alpha$ *to* $\mathcal{D}(r)$ *whenever*

$$r < \min\left(r_{\pitchfork}, \frac{\rho}{104n^2}\right).$$

The previous two propositions will guarantee that the pull back of the contact structure on $\mathbf{C}(r)$ via E will be standard, that is embed in the standard contact structure on $\mathbb{R}^{2n+1}$, thus completing Step II. So we are left to complete Step III by estimating the size of a geodesic ball that can be embedded in such a cylinder. We can make such an estimate in a more general context that does not involve anything from the special geometry of compatible metrics except that the Reeb field is geodesic. Its statement needs some reference functions sn_k, indexed by a real number k.

$$\mathrm{sn}_k(r) = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}r), & \text{if } k > 0 \\ r, & \text{if } k = 0, \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}r), & \text{if } k < 0. \end{cases}$$

Proposition 8.21. *Let* (M, g) *be a complete Riemannian manifold whose sectional curvature is bounded above by* K. *Let* X *be a unit norm geodesic vector field on* M *and* p *a point in* M. *Consider the disk*

$$\mathcal{D}(r_0) := \{\exp_p(v) : v \in X_p^\perp, \|v\| < r_0\} \quad \textit{with} \quad r_0 < \min\left(\frac{\mathrm{inj}(g)}{2}, \frac{\pi}{2\sqrt{K}}\right).$$

We denote by n *a unit vector field positively transverse to* $\mathcal{D}(r_0)$ *and assume we have the following estimate along a radial geodesic* γ

$$\langle X(\gamma(r)), n(\gamma(r))\rangle \geq 1 - P(r),$$

where $P = P(r) \geq 0$ *depends only on the distance* r *to* p *and* $P(r) \leq 1$ *on* $[0, r_0]$. *Then the cylinder* $\mathbf{C}(r_0) = \Phi((-\infty, \infty) \times \mathcal{D}(r_0))$ *given by the flow* Φ *of* X *contains a geodesic ball of radius*

$$\mathrm{sn}_K^{-1}\left((1 - P(r_0))\mathrm{sn}_K(r_0)\right)$$

about p.

8.7. Later developments and prospects

The contact sphere theorem has been slightly improved in Ge and Huang 2013 which weakened the pinching condition from 4/9 to 1/4. Their proofs relies on all steps of our proof, but adds some ideas from the theory of non-smooth convex functions and Busemann functions. Using these ideas they also prove the following nice result.

Theorem 8.22 (ibid.)**.** *Let (M, ξ) be an open contact manifold with a compatible metric g which is complete. If g has positive sectional curvature, then ξ is tight. This also holds more generally if g has nonnegative sectional curvature on M and positive sectional curvature in $M \setminus K$ for some compact subset K of M.*

Although it is nice to see new ideas about the contact sphere problem, one must emphasize that there is no reason why 1/4 should be the optimal pinching constraint. The topological 1/4-pinching sphere theorem is optimal in even dimensions since complex projective spaces have sectional curvature varying in the closed interval $[1/4, 1]$. But it is not optimal in odd dimensions. Especially in dimension 3, the first triumph of the Ricci flow was the proof, in Hamilton 1982, that positive Ricci curvature, which is a much weaker condition, is already enough to characterize $\mathbb{S}^3$ among simply connected manifolds (this was before Perelman's proof of the Poincaré conjecture of course).

This optimality question is linked to a much deeper issue. The Levi-Civita connection and its Riemann curvature tensor are not fully natural in the contact context. A more natural candidate is the Tanaka-Webster connection. This is an object coming from CR geometry but, in dimension 3, the geometry of compatible Riemannian metrics, sub-Riemannian geometry and (calibrated) CR geometry are all equivalent, this essentially follows from the fact that there is only one symplectic structure on an oriented 2-dimensional vector space up to scale. We will describe the relevant part of CR/sub-Riemannian geometry in dimension 3 only.

Let g be a sub-Riemannian structure on a cooriented contact 3-manifold (V, ξ) and J the corresponding complex structure on ξ. Let θ be the unique contact form for ξ such that

$$g = \frac{1}{2} d\theta(\cdot, J\cdot)$$

and let T be its Reeb field. We denote by π the projection onto ξ with kernel spanned by T. All this allows to define the canonical extension of g to a Riemannian metric $\hat{g}$ on V:

$$\hat{g} := g(\pi\cdot, \pi\cdot) + \theta^2.$$

Note that the $\frac{1}{2}$ in the definition of θ is there to ensure that the round metric on $\mathbb{S}^3$ with constant sectional curvature 1 is the $\hat{g}$ metric of some sub-Riemannian structure on the standard contact structure. We also define ϕ as usual by $\phi T = 0$ and $\phi_{|\xi} = J$.

Theorem 8.23 (Tanaka-Webster)**.** *On any sub-Riemannian contact 3-manifold (V, ξ, g) there is a unique connection ∇ with torsion* Tor *such that:*

- *∇ preserves ξ: for any vector field X and any Legendrian vector field Y, $\nabla_X Y$ is Legendrian*

- *the contact form θ and its derivative $d\theta$, the Reeb field T, the operator ϕ and the metric $\hat{g}$ are parallel*
- *for any Legendrian vector fields X and Y,*

$$\mathrm{Tor}(X, Y) = d\theta(X, Y)T \quad \textit{and} \quad \mathrm{Tor}(T, JX) = -J\mathrm{Tor}(T, X)$$

This connection is related to the Levi-Civita connection ∇^{LC} of $\hat{g}$ by:

$$\nabla_X Y = \nabla^{LC}_X Y - \theta(X)\phi Y - \theta(Y)\phi X - \frac{1}{2}(\phi\,\mathcal{L}_{\mathrm{T}}\phi)X + \frac{1}{2}\left(d\theta(X,Y) - g((\phi\,\mathcal{L}_{\mathrm{T}}\phi)X, Y)\right)T.$$

The torsion operator is $\tau\colon X \mapsto \mathrm{Tor}(T, X)$. It maps ξ to itself and vanishes on T. The scalar curvature of the Tanaka Webster connection is usually called the Webster curvature of (V, ξ, g), and denoted by W. These two functions W and $\|\tau\|$ locally determine (V, ξ, g) up to finite ambiguity.

Of course the contact sphere theorem (either from Etnyre, Komendarczyk, and Massot 2012 or from Ge and Huang 2013) can be rewritten in terms of those functions. But the formula relating ∇ and ∇^{LC} in Theorem 8.23 is not enticing. A much more inspiring thing to check is what those functions are for left-invariant sub-Riemannian structure on three-dimensional Lie groups. This was done (with different motivations and notations coming from sub-Riemannian geometry rather than CR geometry) in Agrachev and Barilari 2012. One immediately notes that, in this very restricted class of sub-Riemannian manifolds, SU(2) is characterized by the inequality $\|\tau\| < W$. Of course left-invariant functions on Lie groups are constant, so we do not expect to see any derivatives of W and $\|\tau\|$ here. But this inequality is a rather direct analogue of the positive Ricci assumption in Hamilton 1982. One can therefore make the following bold conjecture.

Conjecture 8.24. *Let (V, ξ, g) be a 3-dimensional complete sub-Riemannian manifold. If $\|\tau\| < W$ then the universal cover of (V, ξ) is the tight $\mathbb{S}^3$.*

Admittedly, the evidence supporting this conjecture is thin (almost all of it was discussed above). On the other hand this result would be sharp: left-invariant sub-Riemannian structures on $\mathbb{T}^3$, seen as a quotient of $\mathrm{SO}(2) \ltimes \mathbb{R}^2$, all satisfy $\|\tau\| = W$.

Another direction worth investigating relates to Theorem 8.9. The criterion given there is a local differential constraint ensuring that Reeb orbits are quasi-geodesic. This ensures tightness because all Reeb fields of overtwisted contact structure have contractible closed orbits. Although there are contact structures on the hyperbolic space satisfying this local constraint, there is no hope to apply this result explicitly to closed manifolds because the hyperbolic metric is never explicit enough. This theorem should rather be thought of as a bridge between the methods used here and the general search for quasi-geodesic Reeb vector fields on hyperbolic manifolds, hopefully making connections with Fenley and Mosher 2001; Calegari 2006; Frankel 2013, 2015.

In particular, the main result of Fenley and Mosher 2001 is that, for any closed, oriented, hyperbolic 3-manifold M, every nonzero homology class S in $H_2(M)$ and every taut, finite depth foliation $\mathcal{F}$ whose compact leaves represent S, a pseudoAnosov flow

Φ which is almost transverse to $\mathcal{F}$ is a quasi-geodesic. Almost transverse means that it becomes transverse after some sort of blow up. Note that such $\mathcal{F}$ and Φ always exist, assuming that S exists, and are constructed using a sutured hierarchy (see Section 4.2.3). Colin and Honda 2005 proved that, in the case where M has non-empty boundary which is a union of tori, it is possible to build a contact structure and a Reeb vector field transverse to a taut foliation. It would be very interesting to bridge the gap to Fenley and Mosher 2001. Note here the importance of the word "almost" in this theorem: there is no hope to get a Reeb vector field which is transverse to a close leaf. The very first case to study is when M fibers over the circle.

9. Contact mapping class groups

9.1. Introduction

This chapter surveys Giroux and Massot 2015; Massot and Niederkrüger 2016 which study contact mapping class groups.

Giroux and Massot 2015 studies contact transformations of 3-manifolds which are circle bundles equipped with contact structures tangent to the fibers. The main example of such a manifold is the unit cotangent bundle $V = ST^*S$ of a surface S, endowed with its canonical contact structure ξ: this contact manifold is also called the manifold of cooriented contact elements over S. Other examples are obtained as follows: for any positive integer d dividing $|2g-2|$ where g is the genus of S, the manifold V admits a d-fold fibered cyclic cover V_d and the pullback ξ_d of ξ on V_d is a contact structure tangent to the fibers of V_d over S. It is a nice and easy observation that all Legendrian circle bundles are of this form (see Lutz 1983, p. 179).

The main goal of Giroux and Massot 2015 is to determine the contact mapping class group of (V_d, ξ_d), namely the group $\pi_0\mathcal{D}(V_d; \xi_d)$, where $\mathcal{D}(V_d; \xi_d)$ denotes the group of contact transformations of (V_d, ξ_d) (diffeomorphisms preserving the contact structure with its coorientation). This group has an obvious homomorphism to the usual (smooth) mapping class group $\pi_0\mathcal{D}(V_d)$ (where $\mathcal{D}(V_d)$ consists of all diffeomorphisms of V_d) which has been computed in Waldhausen 1967. As explained in Section 5.3, this homomorphism is tightly related to the fundamental group of the isotopy class of ξ_d, *i.e.* the connected component of ξ_d in the space $\Xi(V_d)$ of all contact structures on V_d.

Our main result is the following theorem, in which V_d is endowed with any principal circle bundle structure inherited from one on $V = ST^*S$.

Theorem 9.1. *Let S be a closed, connected, orientable surface of genus $g \geq 1$ and d a positive integer dividing $2g-2$. Denote by $R_t : V_d \to V_d$ the action of $2\pi t \in \mathbb{R}/2\pi\mathbb{Z}$ by rotation along the fibers. Then:*

- *The fundamental group $\pi_1(\Xi(V_d), \xi_d)$ is infinite cyclic and generated by the loop $(R_t)_*\xi_d$, $t \in [0, 1/d]$.*
- *The kernel of the natural homomorphism*

$$\pi_0\mathcal{D}(V_d; \xi_d) \to \pi_0\mathcal{D}(V_d)$$

is the cyclic group $\mathbb{Z}/d\mathbb{Z}$ spanned by the contact mapping classes of the deck transformations of V_d over ST^*S.

In the torus case, $g = 1$, Geiges and Gonzalo Perez 2004 proved, also using topological methods, that the loop appearing in the previous theorem generates an infinite cyclic group in $\pi_1(\Xi, \xi_d)$ (but they did not prove that this cyclic subgroup is the full group). Bourgeois 2006 reproved this using contact homology. Then Geiges and Klukas 2014 proved the theorem when $g = 1$ and $d = 1$.

As a direct consequence of the theorem above, we obtain:

Corollary 9.2. *Let S be a closed orientable surface of genus $g \geq 2$. Then the natural homomorphism*

$$\pi_0 \mathcal{D}(S) \to \pi_0 \mathcal{D}(ST^*S; \xi)$$

induced by the differential is an isomorphism.

This corollary is stated as Theorem 1 in Giroux 2001c but the "proof" given there contains a mistake. See Section 9.2.3 for several related examples.

In the case $g = 1$, each manifold V_d is diffeomorphic to $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ fibering over $S = \mathbb{T}^2$ by the projection $(x, y, z) \mapsto (x, y)$, and its contact structure ξ_d can be defined by

$$\cos(2d\pi z)\, dx - \sin(2d\pi z)\, dy = 0, \quad x, y, z \in \mathbb{R}/\mathbb{Z}.$$

Then the results of Giroux 1994b, 1999 about pre-Lagrangian tori, discussed in Section 3.2.1 readily imply that the image of the obvious homomorphism $\pi_0 \mathcal{D}(\mathbb{T}^3, \xi_d) \to \pi_0 \mathcal{D}(\mathbb{T}^3) = \mathrm{SL}_3(\mathbb{Z})$ is the subgroup Π of transformations preserving $\mathbb{Z}^2 \times \{0\} \subset \mathbb{Z}^3$ (this was first explicitly pointed out in Eliashberg and Polterovich 1994). Therefore the induced homomorphism $\pi_0 \mathcal{D}(\mathbb{T}^3, \xi_1) \to \Pi$ is an isomorphism.

Finally, for $g = 0$, an unpublished result of Fraser 2016 shows that the contact transformation group of the standard projective 3-space (namely, the unit cotangent bundle of the 2-sphere) is connected. This completes the list of contact mapping class groups for unit cotangent bundles of closed orientable surfaces.

We now turn the higher-dimensional examples of non-trivial contact mapping classes from Massot and Niederkrüger 2016. Let $\Bbbk \subset \mathbb{R}$ be a field of real numbers such that $\dim_{\mathbb{Q}} \Bbbk$ is finite and such that $\Bbbk$ is totally real (i.e. any field embedding $\Bbbk \hookrightarrow \mathbb{C}$ is real-valued). In Section 7.6 we associated to $\Bbbk$ a compact manifold $M_{\Bbbk}$ equipped with 1-forms $\alpha_{\pm}$ such that the formula

$$\xi_n := \ker \left(\frac{1 + \cos(ns)}{2} \alpha_+ + \frac{1 - \cos(ns)}{2} \alpha_- + \sin(ns)\, dt \right)$$

for $n \geq 1$ defines a family of contact structures on $\mathbb{T}^2 \times M_{\Bbbk}$, where (s, t) are the coordinates of $\mathbb{T}^2$. For instance $M_{\mathbb{Q}} = \mathbb{S}^1$ with $\alpha_{\pm} = \pm d\theta$ so $\mathbb{T}^2 \times M_{\mathbb{Q}} = \mathbb{T}^3$ with ξ_n as above. Note that there are infinitely many such fields $\Bbbk$ for each given $\dim_{\mathbb{Q}} \Bbbk > 1$, and the corresponding $M_{\Bbbk}$ are pairwise non-homeomorphic.

Theorem 9.3. *For any totally real number field $\Bbbk$, any n greater than one and any $1 \leq m < n$, the contactomorphism*

$$\Psi_{n,m} \colon (\mathbb{T}^2 \times M_{\Bbbk}, \xi_n) \to (\mathbb{T}^2 \times M_{\Bbbk}, \xi_n),\ (s, t, \theta) \mapsto \left(s + \tfrac{2\pi m}{n}, t, \theta \right)$$

is smoothly isotopic to the identity but it is not symplectically pseudoisotopic to the identity, so in particular it is not contact isotopic to the identity. In addition, there is a contactomorphism which is conjugated to $\Psi_{n,m}$ *inside* $\mathrm{Diff}(\mathbb{T}^2 \times M_{\Bbbk})$ *but not inside* $\mathrm{Diff}(\mathbb{T}^2 \times M_{\Bbbk}, \xi_n)$*.*

The 3-dimensional results mentioned above have been obtained using Giroux's theory of ξ-convex surfaces. Such methods do not seem to be sufficiently powerful to prove the higher dimensional results and, even in dimension 3, it seems unlikely that they might yield the stronger pseudoisotopy obstruction. Instead, we will use J-holomorphic curve techniques to show that a certain pre-Lagrangian submanifold P in $\mathbb{T}^2 \times M_{\Bbbk}$ cannot be displaced from itself by any contactomorphism that is symplectically pseudoisotopic to the identity. The main theorem follows because $\Psi_{m,n}$ does displace P.

Note that a stronger non-displaceability result holds: The pre-Lagrangian P contains a Legendrian submanifold Λ which cannot be disjoined from P. This can be proved by setting up a Floer theory for Lagrangian lifts of P and Λ in the symplectization of ξ_n as was done in Eliashberg, Hofer, and Salamon 1995 (see Lemma 9.21 about why invariance under compactly supported Hamiltonian isotopies is enough). Such a strategy involves a lot more technical work than is necessary to deduce our theorem on contact transformations. An even more high-tech road would be to prove that contact transformations which are symplectically pseudoisotopic to the identity act trivially on contact homology and use it to prove Theorem 9.3. However we feel that such a monumental proof would not make sense as long as our only examples can be handled by much more elementary techniques. So we chose instead to prove the weaker non-displaceability result (which is also of independent interest and has less hypotheses). Here one can also envision variations on the argument. One referee pointed out to us that we could adapt to our setup the variation on Gromov's argument which is explained in McDuff and Salamon 2004, end of Section 9.2. This variation uses holomorphic strips instead of disks and is arguably slightly more contrived but does not set up a full Floer theory so it is also elementary in the sense of the current discussion. Note however that such a road would bypass Theorem 9.23 which has independent interest.

9.2. The 3-dimensional case

9.2.1. Spaces of surface embeddings

For any compact manifold V with (possibly empty) boundary, we denote by $\mathcal{D}(V, \partial V)$ the group of diffeomorphisms of V relative to a neighborhood of the boundary. When the boundary of V is empty, we sometimes drop ∂V from our notations.

In addition to the fibration of Key observation 5.6, we will need informations about spaces of surface embeddings, and the fibration there are involved in. Assume from now on that the contact manifold (V, ξ) has dimension 3 and let F be a compact orientable surface properly embedded in V. We denote by

- $\mathcal{P}(F, V)$ the space of proper embeddings $F \to V$ which coincide with the inclusion $\iota\colon F \to V$ near ∂F;

- $\mathcal{P}_{\mathrm{o}}(F, V) \subset \mathcal{P}(F, V)$ the connected component of the inclusion ι;
- $\mathcal{P}(F, V; \xi) \subset \mathcal{P}(F, V)$ the subspace of embeddings ψ which induce the same characteristic foliation as the inclusion, *i.e.* satisfy $\xi\, \psi(F) = \psi_*(\xi F)$;
- $\mathcal{P}_{\mathrm{o}}(F, V; \xi)$ the intersection $\mathcal{P}_{\mathrm{o}}(F, V) \cap \mathcal{P}(F, V; \xi)$.

The same standard tools as in the proof of Key observation 5.6 give the following:

Lemma 9.4. *Let (V, ξ) be a compact contact manifold of dimension 3. For every properly embedded surface $F \subset V$, the restriction map*

$$\mathcal{D}(V, \partial V; \xi) \to \mathcal{P}(F, V; \xi), \quad \phi \mapsto \phi F|,$$

is a locally trivial fibration over its image.

Remark 9.5. *The above lemma is a typical result where it is useful to work relatively to a neighborhood of the boundary and not just to the boundary itself. Indeed, any diffeomorphism relative to both ∂V and a properly embedded surface F is tangent to the identity along ∂F, and so the fibration property fails in this case. However, since the inclusion of $\mathcal{D}(V, \partial V; \xi)$ into the group of contact transformations relative to the boundary is a homotopy equivalence, this does not matter.*

The theory of ξ-convex surfaces can be used to study the homotopy type of $\mathcal{P}(F, V; \xi)$. We will need the following version of the realization lemma with parameters (see Giroux 2001c, Lemmas 6 and 7):

Proposition 9.6. *Let F be a ξ-convex surface, U a homogeneous neighborhood, and Γ the associated dividing set.*

(a) The space $\mathcal{F}(F; \Gamma)$ of singular foliations on F which are tangent to ∂F and admit Γ_U as a dividing set is an open contractible neighborhood of ξF in the space of all singular foliations on F.

(b) There exists a continuous map $\mathcal{F}(F; \Gamma) \to \mathcal{P}(F, V)$, $\sigma \mapsto \psi_\sigma$, with the following properties:

1. *$\psi_{\xi F}$ is the inclusion $F \to V$;*
2. *$\psi_\sigma(F)$ is contained in $U = F \times \mathbb{R}$ and transverse to the contact vector field ∂_t for all $\sigma \in \mathcal{F}(F; \Gamma)$;*
3. *$\xi\, \psi_\sigma(F) = \psi_\sigma(\sigma)$ for all $\sigma \in \mathcal{F}(F; \Gamma)$.*

(c) Let $\mathcal{P}(F, V; \Gamma)$ denote the space of embeddings $\psi \in \mathcal{P}(F, V)$ such that $\psi(F)$ is ξ-convex with dividing set $\psi(\Gamma)$. Then the inclusion $\mathcal{P}(F, V; \xi) \to \mathcal{P}(F, V; \Gamma)$ is a homotopy equivalence.

We will also need the following result which shows that the homotopy type of $\mathcal{D}(V, \partial V; \xi)$ is locally constant when ∂V is ξ-convex (see ibid., Proposition 8:

Proposition 9.7. *Let V be a compact 3-manifold, Δ a multi-curve on ∂V and $\Xi(V, \Delta)$ the space of contact structures ξ on V for which ∂V is ξ-convex with dividing set Δ. For $\xi \in \Xi(V, \Delta)$, the homotopy type of $\mathcal{D}(V, \partial V; \xi)$ depends only on the connected component of $\Xi(V; \Delta)$ containing ξ.*

9.2.2. Legendrian circle bundles over surfaces

In this section we consider a compact oriented surface S which is neither a sphere nor a torus. The torus case is also understood. Actually, using results of Massot 2008a, the following discussion can be carried over to orbifolds.

As in the introduction, (V_d, ξ_d) denotes the d-fold fibered cyclic cover of the unit cotangent bundle $V := V_1 = ST^*S$, equipped with the pullback of the canonical contact structure ξ of V.

The key to proving Theorem 9.1 is to go through spaces of embeddings. The fibration from Lemma 9.4 ensures that a contact transformation φ that acts non-trivially on some $\pi_0 \mathcal{P}(F, V; \xi)$ is non trivial in $\pi_0 \mathcal{D}(V; \xi)$, and that any φ which acts trivially on some $\pi_0 \mathcal{P}(F, V; \xi)$ is isotopic to some ψ that is the identity near F. In addition, spaces of embeddings conveniently allow to localize statement and, in the ξ-convex case, Proposition 9.6 allows to replace them by the more convenient $\mathcal{P}(F, V; \Gamma)$. The main technical result is the following.

Proposition 9.8. *If T is a fibered torus over a non-separating embedded circle in S then the group of deck transformations of $V_d \to V$ acts freely and transitively on $\pi_0(\mathcal{P}_{\mathrm{o}}(T, V_d; \xi_d))$. If A is a fibered annulus over a non-separating properly embedded arc in S then $\mathcal{P}_{\mathrm{o}}(A, V_d; \xi_d)$ is connected.*

Observe that the preimage $F := \pi^{-1}(\gamma)$ of any properly embedded curve γ in S is a ξ_d-convex surface in V_d. Indeed, any vector field X in S transverse to γ (and tangent to ∂S) lifts to a contact vector field $\bar{X}$ transverse to F (and tangent to ∂V_d). The dividing set of $\xi_d F$ associated with $\bar{X}$ is the set of points in F where ξ_d projects down (by the differential of π) to the line spanned by X. The first ingredient in order to prove Proposition 9.8 is the following consequence of the semi-local Bennequin inequality for tori.

Lemma 9.9. *Let F be a torus fibered over a homotopically essential circle in S and Γ a dividing set for $\xi_d F$. For any isotopy φ such that $\varphi_1(F)$ is also ξ_d-convex, the foliation $\xi_d \varphi_1(F)$ is divided by a collection of curves isotopic to the components of $\varphi_1(\Gamma)$.*

An corollary of this lemma is a constraint on the image of $\pi_0 \mathcal{D}(V; \xi) \to \pi_0 \mathcal{D}(V)$.

Corollary 9.10. *A diffeomorphism of V_d which is fibered over the identity is isotopic to a contactomorphism only if it is isotopic to the identity.*

The following lemma is useful to prove the existence of contact transformations which are smoothly but not contact isotopic to the identity:

Lemma 9.11. *Let T be a fibered torus over a homotopically non-trivial circle C in S and $i\colon T \to V_d$ the inclusion map. Let R_t be the action of $e^{2i\pi t}$ on V_d. For any non-zero integer k in $\mathbb{Z}$, the path $\gamma_k : [0,1] \to \mathcal{P}_o(T, V_d)$ defined by $\gamma_k(t) = R_{kt/d} \circ i$ is non-trivial in $\pi_1(\mathcal{P}_o(T, V_d), \mathcal{P}_o(T, V_d; \xi_d))$.*

For any integer k between 1 and $d-1$, the action of $R_{k/d}$ on $\pi_0(\mathcal{P}_o(T, V_d; \xi_d))$ is non-trivial.

Using covering spaces tricks and isotopy cut off, one can go down one more dimension and reduce Lemma 9.11 to a statement about Legendrian knots proved in Ghiggini 2006a: in $(\mathbb{T}^2 \times \mathbb{R}, \ker(\cos(2n\pi z)dx - \sin(2n\pi z)dy))$, the Legendrian circles $\{0\} \times \mathbb{S}^1 \times \{0\}$ and $\{0\} \times \mathbb{S}^1 \times \{k\}$ are not contact isotopic for $k \neq 0$. Note that this result about Legendrian knots is proved, using again a covering map trick, by reduction to the classification of $\mathbb{S}^1$-invariant contact structures! Of course the later was proved by reduction to Bennequin. This is typical of all dévissages in this section.

Alternatively, one could use the stronger result due to Eliashberg, Hofer and Salamon Eliashberg, Hofer, and Salamon 1995 saying that, in $(\mathbb{T}^3, \xi_n)$, the Legendrian circle $\{0\} \times \mathbb{S}^1 \times \{0\}$ cannot be displaced from the pre-Lagrangian torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \{0\}$ by a contact isotopy. However this result uses holomorphic curves in symplectizations so it has a different flavor from the techniques we use in this section.

The last ingredients in the proof of Proposition 9.8 are the classification of tight contact structures on thickened tori discussed in Section 3.2.3, and *Ghiggini's torus trick* described in Section 3.2.4. After proving this proposition, one can move to the actual theorem about contact transformations, whose proof we include in order to explain how pieces fit together.

Theorem 9.12. *If S is closed then the kernel of the canonical homomorphism*

$$\pi_0 \mathcal{D}(V_d, \xi_d) \to \pi_0 \mathcal{D}(V_d)$$

is the cyclic group of deck transformations of V_d over V. If V has non-empty boundary then $\pi_0 \mathcal{D}(V_d, \partial V; \xi_d) \to \pi_0 \mathcal{D}(V_d, \partial V_d)$ is injective.

Proof. We first assume V has non-empty boundary and prove that the kernel of the map $\pi_0 \mathcal{D}(V_d, \partial V_d; \xi_d) \to \pi_0 \mathcal{D}(V_d, \partial V_d)$ is trivial. The proof proceeds by induction on

$$n(S) = -2\chi(S) - \beta(S) = \beta(S) + 4g(S) - 4$$

where $\chi(S)$ and $g(S)$ are the Euler characteristic and genus of S and $\beta(S)$ is the number of connected components of ∂S. So $n(S) \geq -3$ with equality when S is a disk.

We first explain the induction step so we assume $n(S) > -3$. Let φ be a contactomorphism of V_d relative to some neighborhood U of ∂V_d and smoothly isotopic to the identity relative to U. Let a be a properly embedded non-separating arc in S and denote by A the annulus fibered over a and $i\colon A \to V_d$ the inclusion map. According to Proposition 9.8, $\mathcal{P}_o(A, V_d, \xi_d)$ is connected hence the path lifting property of the fibration $\mathcal{D}_o(V_d, \partial V_d; \xi_d) \to \mathcal{P}_o(A, V_d; \xi_d)$ from Lemma 9.4 implies that φ is contact isotopic to some φ' which is relative to A and U. Using Remark 9.5, we can assume φ' is relative to a neighborhood of $\partial V_d \cup A$ which is fibered over some neighborhood W of $a \cup \partial S$ in S. We cut S along a and round the corners inside W to get a subsurface $S' \subset S$ with $n(S') < n(S)$. By induction hypothesis applied to $\pi^{-1}(S')$, φ' is contact isotopic to identity so the induction step is completed.

The induction starts with the disk case which is already explained with all details in Giroux 2001c, Page 345. The idea is the same as for the induction step but the cutting surface in the solid torus V_d is a meridian disk. There are no such disk with Legendrian boundary in V_d but one can use the realization lemma (Proposition 9.6) to deform ξ_d near ∂V_d until such a disk exists. This does not change the homotopy type of $\mathcal{D}(V_d, \partial V_d; \xi_d)$ according to Proposition 9.7. Colin's result about embedding of disks mentioned in Section 3.2.4 then replaces Proposition 9.8 and the final isotopy is provided by Eliashberg 1992 that proves $\pi_0\mathcal{D}(B^3, \partial B^3; \xi)$ is trivial for the standard ball.

We now turn to the case where V_d is closed. We first prove that the group of deck transformations injects into $\pi_0\mathcal{D}(V_d; \xi_d)$. Let C be a non-separating circle in S and T the fibered torus over C. Denote by i the inclusion of T in V_d. Proposition 9.8 guaranties that the action of a non-trivial deck transformation f on $\pi_0(\mathcal{P}_\circ(T, V_d; \xi_d))$ is non-trivial hence f is non-trivial in $\pi_0\mathcal{D}(V_d; \xi_d)$.

We now prove surjectivity. Let φ be a contactomorphism of V_d which is smoothly isotopic to the identity. Proposition 9.8 gives a deck transformation f such that $f \circ \varphi \circ i$ is isotopic to i in $\mathcal{P}_\circ(T, V_d; \xi_d)$. As above, this implies that $f \circ \varphi$ is contact isotopic to a contactomorphism φ' which is relative to an open fibered neighborhood U of T. The circle bundle $V_d \setminus U$ has non-empty boundary hence we know that φ' is contact isotopic to identity. □

Theorem 9.12 combines with topological results from Laudenbach 1974; Hatcher 1976 describing $\pi_1(\mathcal{D}(V_d), \mathrm{Id})$ to prove the following corollary.

Corollary 9.13. *Assume that V_d has empty boundary and denote by Ξ the space $\mathcal{D}(V_d)\cdot \xi_d$ of contact structures isomorphic to ξ_d on V_d. Let R_t denote the action of $e^{2i\pi t} \in \mathbb{S}^1$ on V_d. The fundamental group $\pi_1(\Xi, \xi_d)$ is an infinite cyclic group generated by the loop $t \mapsto (R_{t/d})_*\xi_d$.*

Corollary 9.14. *The lifting map from $\pi_0\mathcal{D}(S)$ to $\pi_0\mathcal{D}(V, \xi)$ is an isomorphism.*

Proof. We denote by p the projection from V to S and by $\mathcal{D}(S, \partial S)$ the group of diffeomorphisms of S relative to a neighborhood of ∂S. In the sequence of maps:

$$\pi_0\mathcal{D}(S, \partial S) \to \pi_0\mathcal{D}(V, \partial V; \xi) \to \pi_0\mathcal{D}(V, \partial V)$$

the composite map is known to be injective (this follows from considerations of fundamental groups) so the first map is also injective. It remains to prove that it is surjective. Let φ be a contactomorphism. We want to prove that φ is contact isotopic to the lift of some diffeomorphism of S. According to Waldhausen 1967, φ is smoothly isotopic to a fibered diffeomorphism f: there exists an isotopy ψ and a diffeomorphism $\bar{f}$ in $\mathcal{D}(S, \partial S)$ such that $f = \psi_1 \circ \varphi$ and $p \circ f = \bar{f} \circ p$. We will prove that φ is contact isotopic to the lift

$D\bar{f}$. We first note that $f \circ D(\bar{f})^{-1}$ is fibered over the identity and is smoothly isotopic to a contactomorphism (through the path $t \mapsto \psi_{1-t} \circ \varphi \circ D(\bar{f})^{-1}$). Corollary 9.10 then guaranties that $f \circ D(\bar{f})^{-1}$ is smoothly isotopic to the identity. Hence φ is smoothly isotopic to $D\bar{f}$ hence contact isotopic according to Theorem 9.12. □

9.2.3. Examples of disconnected spaces of embeddings

In this section we describe examples of disconnected spaces consisting of smoothly isotopic embeddings inducing a fixed characteristic foliation. More specifically, we construct disconnected spaces of smoothly isotopic ξ-convex embeddings with a fixed dividing set, and the former spaces are deformation retracts of the latter by Proposition 9.6. Those examples should be compared to the connectedness results which were crucial in Section 9.2.2. In addition they provide counter-examples to to the incorrect statement from Giroux 2001c which was corrected in Giroux and Massot 2015, the wrong statement from Honda 2000a which was corrected in Honda 2001, and some other failed attempts at proving versions of Theorem 9.1, by other people, that never went as far as being published. All these mistakes involve over-optimistic lemmas for surface disjunction or simplifications of sequences of bifurcations whose total effect is null.

Hence we want to describe examples where we have explicit smooth isotopies among surfaces which are all ξ-convex except for a finite number of times and exhibit various behaviors for those isotopies. We also want to highlight situations where persistent intersection phenomena occur and situations where a contact isotopy exists in the ambiant manifold but not inside a smaller manifold (where a smooth isotopy still exists). For all this we need the following technical definition.

Definition 9.15. *A discretized isotopy of embeddings of an oriented surface S into a contact 3-manifold (V, ξ) is an isotopy of embeddings $j : S \times [0, 1] \to V$ such that, for some (unique) integer n:*

- *the restriction of j to $S \times [i/n, (i+1)/n]$ is an embedding for each i from 0 to $n-1$,*
- *all surfaces $j_t(S)$ are ξ-convex except when $t = i/n + 1/(2n)$ for some integer i between 0 and $n - 1$.*

Each embedding of $S \times [i/n, (i + 1)/n]$ is called a step of the discretized isotopy. It is called a forward or backward step depending on whether it is orientation preserving or reversing.

Colin's idea described in Section 3.2.4 combines with the discretization lemma to prove that any isotopy of embeddings which starts and end at ξ-convex embeddings is homotopic relative to its end-points to a discretized isotopy.

Any discretized isotopy j defines a sequence of isotopy classes of multi-curves $\Gamma_0, \dots, \Gamma_n$ such that the characteristic foliation of $j_{i/n}(S)$ is divided by $j_{i/n}(\Gamma_i)$.

Recall from Section 3.2.1 that the contact structure ξ_d on $\mathbb{T}^3$ with coordinates (x, y, z) is defined by:

$$\xi_d = \ker\left(\cos(2d\pi z)dx - \sin(2d\pi z)dy\right)$$

and they are pairwise non-isomorphic.

Proposition 9.16. *In* $(\mathbb{T}^3, \xi_d)$*, let* T *be the torus* $\{8d\, z = \cos x\}$*. Denote by* j_0 *the inclusion of* T *into* $\mathbb{T}^3$ *and by* j_1 *the embedding obtained by restriction to* T *of the rotation* $(x, y, z) \mapsto (x, y, z + 1/d)$*. Those two embeddings are smoothly isotopic and:*

- j_0 *and* j_1 *induce the same characteristic foliation on* T
- j_0 *is not isotopic to* j_1 *among* ξ_d*-convex embeddings,*
- *there is a discretized isotopy from* j_0 *to* j_1 *with only forward steps changing the direction of dividing curves,*
- *there is a discretized isotopy from* j_0 *to* j_1 *consisting of four forward steps which change the number of dividing curves without changing their direction.*

Proof. Since the rotation map is a contactomorphism, j_0 and j_1 induce the same characteristic foliation on T. Assume for contradiction that j_0 and j_1 are isotopic through ξ_d-convex surfaces. Proposition 9.6c and Lemma 9.4 then imply that there is a contact isotopy φ such that $j_1 = \varphi_1 \circ j_0$. We lift this isotopy to $\mathbb{T}^2 \times \mathbb{R}$ which covers $\mathbb{T}^3$ by $(x, y, s) \mapsto (x, y, s \mod 2\pi)$. We denote by φ' the lifted isotopy and by T' some (fixed) lift of T. We denote by τ_n the translation $(x, y, s) \mapsto (x, y, s + n)$ and by $T_{[a,b]}$ the compact manifold bounded by $\tau_a(T')$ and $\tau_b(T')$. Because T' is compact and contact isotopies can be cut-off, we can assume that φ' is compactly supported. Then there is some N such that φ_1 sends $T_{[-N,0]}$ to $T_{[-N,1]}$. In particular those submanifolds are contactomorphic. This contradicts the classification of tight contact structures on $\mathbb{T}^3$ since this contactomorphism could be used to build a contactomorphism from $(\mathbb{T}^3, \xi_{N+1})$ to $(\mathbb{T}^3, \xi_{N+2})$.

The existence of a discretized isotopy from j_0 to j_1 consisting of forward steps changing the direction of dividing curves follows from repeated uses of a small part of the classification of tight contact structures on thickened tori: if ξ is a tight contact structure on $\mathbb{T}^2 \times [0, 1]$ such that $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$ are ξ-convex with two dividing curves γ_0, γ_0' and γ_1, γ_1' respectively such γ_0 intersects γ_1 transversely at one point then ξ is isotopic relative to the boundary to a contact structure ξ' such that all tori $\mathbb{T}^2 \times \{t\}$ are ξ'-convex except $\mathbb{T}^2 \times \{\frac{1}{2}\}$.

In order to construct a discretized isotopy where the direction of dividing curves is constant, we see ξ_d as an $\mathbb{S}^1$-invariant contact structure on $\mathbb{T}^3$ with $\mathbb{S}^1$ action given by rotation in the y direction. In order to describe an $\mathbb{S}^1$-equivariant isotopy of embeddings of T, it is enough to give a isotopy of curves in $\mathbb{T}^2$. Curves corresponding to are ξ_d-convex tori are exactly those which are transverse to $\Gamma = \{x \in (\pi/d)\mathbb{Z}\}$. Figure 9.1 then finishes the proof. □

In our next example the discretized isotopy oscillates and there is persistent intersection.

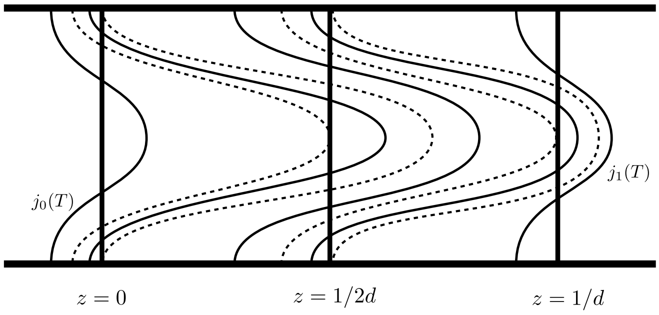


Figure 9.1.: Discretized isotopy of curves lifting to tori in $(\mathbb{T}^3, \xi_d)$. Curves lifting to non-convex tori are dashed.

Proposition 9.17. *Let V be the torus bundle over $\mathbb{S}^1$ with monodromy $B = \left(\begin{smallmatrix} 5 & 1 \\ 4 & 1 \end{smallmatrix}\right)$, ie.*

$$V = (\mathbb{T}^2 \times \mathbb{R}) / \left((Bx, t) \sim (x, t+1)\right).$$

Let T be the image of $\mathbb{T}^2 \times \{\frac{1}{2}\}$ in V and let j_0 be the inclusion map from T to V. There is a tight virtually overtwisted contact structure ξ on V and an embedding $j_1 \in \mathcal{P}_{\mathrm{o}}(T, V; \xi)$ such that:

- *j_0 is not isotopic to j_1 in $\mathcal{P}_{\mathrm{o}}(T, V; \xi)$;*
- *any $j \in \mathcal{P}_{\mathrm{o}}(T, V; \xi)$ such that $j(T)$ is disjoint from T is isotopic to j_0 in $\mathcal{P}_{\mathrm{o}}(T, V; \xi)$ (in particular $j_1(T)$ cannot be disjoined from T by contact isotopy);*
- *there is a discretized isotopy from j_0 to j_1 with one forward step and one backward step, both modifying the direction of dividing curves.*

Proof of the first point of Proposition 9.17. The proof uses the theory of normal forms for tight contact structures on V based what is explained for thickened tori in Section 3.2.3 (in particular the definition of rotation sequences and orbit flips recalled there).

Here we need two (isotopic) contact structures on $\mathbb{T}^2 \times [0, 1]$. We fix a Morse-Smale suspension σ_0 on $\mathbb{T}^2$ with two closed orbits having homology class $(1, 0)$, and we denote by σ_1 the image of σ_0 under A. We also fix a Morse-Smale suspension $\sigma_{\frac{1}{2}}$ with two closed orbits having homology class $(-1, 1)$. Let ξ be a contact structure on $\mathbb{T}^2 \times [0, 1]$ such that:

- ξ prints σ_t on $\mathbb{T}^2 \times \{t\}$ for $t \in \{0, \frac{1}{2}, 1\}$;
- $[0, 1]$ is a union of minimally twisting rotation sequences and two orbit flip sequences with homology classes $(1, 0)$ and $(-1, 1)$ respectively.

Let ξ' be a contact structure with the same properties except that orbit flip homology classes are $(1, -1)$ and $(-1, 2)$. The explicit construction of Giroux 2000, Example 3.41 guarantees that ξ and ξ' are isotopic (relative to the boundary). More specifically, it builds a contact structure printing a non-generic movie of characteristic foliations where two saddle connections happen on the same torus, and such that the movies printed by ξ and ξ' are essentially obtained by choosing the order in which these connections appear. Alternatively, one can see the isotopy between ξ and ξ' as an application of the "shuffling lemma", Honda 2000a, Lemma 4.14. We will also denote by ξ and ξ' the induced contact structures on V. And we denote by T the image in V of $\mathbb{T}^2_{1/2}$.

Let φ be a smooth isotopy of V such that $\xi' = \varphi_1^*\xi$. Assume for contradiction that $j_0 : T \hookrightarrow V$ and $j_1 = \varphi_1 \circ j_0$ are in the same component of $\mathscr{P}_o(T, V; \xi)$. Using the path lifting property for the map $\mathscr{D}_o(V; \xi) \to \mathscr{P}_o(T, V; \xi)$ guaranteed by Lemma 9.4, we get a contact isotopy θ for ξ such that $j_1 = \theta_1 \circ j_0$. Then $\psi := \theta_1^{-1} \circ \varphi_1$ is a diffeomorphism relative to T, and pulls back ξ to ξ'. Thus we can cut V along T to get a thickened torus Y, naturally identified with $\mathbb{T}^2 \times [\frac{1}{2}, 3/2]$. The diffeomorphism ψ induces a diffeomorphism of Y which is relative to the boundary, hence acts trivially on $H_1(Y)$. This is a contradiction because the restriction of ξ and ξ' to Y do not have the same relative Euler class in $H_1(Y)$. Recall that $e(Y;\xi)$ is the homology class of the vanishing locus of any generic section of ξ which spans $\xi\partial Y$ (with the correct orientation) along ∂Y. Here, contributions to this class come from orbit flips and we get $e(Y;\xi) = 2(-1,1) + 2A(1,0) = 2(0,-3)$ while $e(Y;\xi') = 2(-1,2) + 2A(1,-1) = 2(1,-7)$. Note, for sanity check, that those two classes become the same in V, since $e(Y;\xi') - e(Y;\xi) = 2(1,-4) = (\mathrm{Id} - A)(0,1)$. ☐

Finally we describe an example on a manifold with boundary with the same situation as above but things untangle inside a larger manifold.

Proposition 9.18. *Let V denote the manifold $\mathbb{T}^2 \times [0,1]$ and $V' = \mathbb{T}^2 \times [0, \frac{1}{2}]$. There is a universally tight contact structure ξ on V and two smoothly isotopic ξ-convex embeddings $j_0, j_1 : \mathbb{T}^2 \to V'$ with images T_0 and T_1 such that*

- *j_0 is isotopic to j_1 among ξ-convex embeddings in V*
- *j_0 is not isotopic to j_1 among ξ-convex embeddings in V'*
- *T_0 cannot be disjoined from T_1 by an isotopy among ξ-convex surfaces in V'*
- *there is a discretized isotopy from j_0 to j_1 in V' with one forward step and one backward step, both modifying the direction of dividing curves.*
- *there is a discretized isotopy from j_0 to j_1 in V' with one forward step and one backward step, both modifying the number of dividing curves.*

The construction is pictured in Figure 9.2. Let S be the annulus $\{1 \le |z| \le 3\} \subset \mathbb{C}$ and $S' \subset S$ the subannulus $\{1 \le |z| \le 2\}$. We fix an identification between V and $S \times \mathbb{S}^1$ which identify V' with $S' \times \mathbb{S}^1$. Let $\Gamma' = \Gamma_1' \cup \Gamma_2'$ be a disjoint union of two properly embedded arcs in S' whose end points are on the circle $\{|z| = 2\}$. Let Γ be a smooth homotopically essential circle in S such that $\Gamma \cap S' = \Gamma'$. Let ξ be a $\mathbb{S}^1$-invariant contact structure on V with dividing set Γ and denote by ξ' its restriction to V'. Let

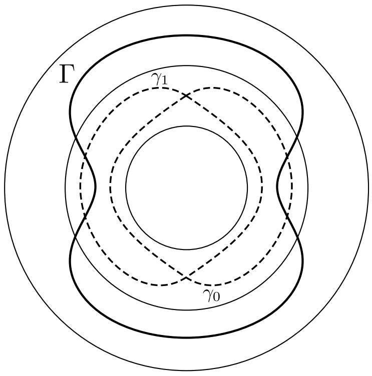


Figure 9.2.: The example of Proposition 9.18. The dividing set Γ is the thick curve, γ_0 and γ_1 are dashed.

γ_0 and γ_1 be homotopically essential circles in S' such that γ_i intersects transversely Γ'_i in two points and does not intersect the other component of Γ'. The tori we want are $T_0 = \gamma_0 \times \mathbb{S}^1$ and $T_1 = \gamma_1 \times \mathbb{S}^1$, parametrized by product maps.

The proof that they have the announced properties is based on the classification of $\mathbb{S}^1$-invariant contact structures recalled in Section 3.2.3.

9.3. Higher dimensional examples

9.3.1. From pseudoisotopic disjunctions to weakly exact Lagrangians

In this section we explain how persistence of certain Lagrangian intersections in the symplectization implies persistence of pre-Lagrangian intersections in the corresponding contact manifold. Next we explain how existence of relevant Lagrangian intersections follows from a result about weakly exact Lagrangians which will be proved in the next section.

Remember that a Lagrangian L in a symplectic manifold (W, ω) is called *weakly exact* if $\int_{\mathbb{D}^2} u^*\omega$ vanishes for every smooth map $u \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, L)$. On the pre-Lagrangian side, one can prove the following, essentially coming from Eliashberg, Hofer, and Salamon 1995:

Lemma 9.19. *Let $P \overset{\iota}{\hookrightarrow} (M, \xi)$ be a pre-Lagrangian submanifold and denote by $\mathcal{A}(P)$ the space of contact forms for ξ whose restrictions to P are closed. If P is closed then the following properties are equivalent:*

1. *there exists an α in $\mathcal{A}(P)$ such that $\int_{\mathbb{D}^2} u^*d\alpha$ vanishes for every smooth map $u \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, P)$.*
2. *for every α in $\mathcal{A}(P)$, $\int_{\mathbb{D}^2} u^*d\alpha$ vanishes for every smooth map $u \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, P)$.*
3. *there is a Lagrangian lift of P which is weakly exact*

4. all Lagrangian lifts of P are weakly exact.

A closed pre-Lagrangian submanifold with any of the above properties will be called *weakly exact*. This terminology parallels the Lagrangian case but note that there is nothing like a strongly exact closed pre-Lagrangian (see the proof below).

In the rest of this section we will explain how the following result, which is the key step to the proof of Theorem 9.3, can be translated into a statement about the non-existence of certain weakly exact Lagrangians. Remember that a contact structure is called *hypertight* if it admits a Reeb vector field without contractible closed orbits.

Theorem 9.20. *A closed weakly exact pre-Lagrangian in a closed hypertight contact manifold cannot be displaced by any contactomorphism that is symplectically pseudoisotopic to the identity.*

Note that one indeed needs the weak exactness assumption in the theorem since Darboux balls are displaceable by contact isotopy and contain plenty of closed pre-Lagrangian submanifolds.

The next lemma is a general fact about symplectic pseudoisotopy which is proved in Massot and Niederkrüger 2016, and could be useful in other contexts.

Lemma 9.21. *Let φ be a contactomorphism of (M, ξ) that is symplectically pseudoisotopic to the identity. For any compact subset K in $S\xi$, there is a compactly supported Hamiltonian isotopy Φ in $S\xi$ such that, for every $p \in K$,*

$$\pi(\Phi_1(p)) = \varphi(\pi(p)),$$

where $\pi\colon S\xi \to M$ is the canonical projection.

Next we need Gromov's trick relating Lagrangian disjunctions and Lagrangian embeddings.

Proposition 9.22 (Gromov 1985, Section 2.3.B_3')**.** *Let $(W, d\lambda)$ be an exact symplectic manifold, and let $L \subset W$ be a Lagrangian. If $\Phi_s\colon W \to W$ is a Hamiltonian isotopy, then we can find a Lagrangian immersion*

$$j\colon L \times \mathbb{S}^1 \looparrowright (W \times \mathbb{C}, d\lambda \oplus dx \wedge dy)$$

where $x + iy$ is the coordinate on $\mathbb{C}$, such that the self-intersection points of $j(L \times \mathbb{S}^1)$ are in one-to-one correspondence with the intersection points in $L \cap \Phi_1(L)$. If j is an embedding and if L is weakly exact, then j will also be weakly exact.

A detailed proof of this proposition can be found in Audin, Lalonde, and Polterovich 1994, Theorem 2.3.6. In the next section we will combine ideas from Gromov 1985, Section 2.3.B_3' with the compactness theorem in Hofer 1993 to prove the following theorem. This requires some care because the end of $S\xi \times \mathbb{C}$ is neither convex nor concave, and because neither the closed Lagrangian submanifold serving as boundary condition for an inhomogeneous Cauchy-Riemann problem, nor the perturbation term involved are in product form.

Theorem 9.23. *If (M, ξ) is a closed contact manifold that is hypertight, then $(S\xi \times \mathbb{C}, d\lambda \oplus dx \wedge dy)$ does not contain any weakly exact closed Lagrangian.*

Using all this we can prove Theorem 9.20. Suppose that P is a closed weakly exact pre-Lagrangian submanifold in a hypertight (M, ξ). Let φ be a contactomorphism symplectically isotopic to the identity and let L_P be a Lagrangian lift of P. According to Lemma 9.19, L_P is weakly exact. Assume for contradiction that $P \cap \varphi(P) = \varnothing$. Lemma 9.21 applied to $K = L_P$ and φ gives a Hamiltonian isotopy Φ in $(S\xi, d\lambda)$ which displaces L_P: $L_P \cap \Phi_1(L_P) = \varnothing$. Proposition 9.22 turns it into a weakly exact embedded Lagrangian in $(S\xi \times \mathbb{C}, d\lambda \oplus dx \wedge dy)$, which contradicts Theorem 9.23.

9.3.2. From hypertightness to absence of weakly exact Lagrangians

In this section we sketch the proof of Theorem 9.23 following the argument in Gromov 1985, Section 2.3.B_3'. The strategy is to show that there is a non-trivial holomorphic disk with boundary on any closed Lagrangian submanifold of $S\xi \times \mathbb{C}$. These disks result from bubbling of an inhomogeneous Cauchy-Riemann equation.

We fix a contact form α without contractible Reeb orbit. We identify $(S\xi, d\lambda)$ with $(\mathbb{R} \times M, d(e^t \alpha))$ using the contact form α and denote by π_ξ, $\pi_\mathbb{R}$, π_M and $\pi_\mathbb{C}$ the canonical projections of $S\xi \times \mathbb{C}$ to $S\xi$, $\mathbb{R}$, M and $\mathbb{C}$ respectively. We fix an $\mathbb{R}$-invariant almost complex structure J_α on $\mathbb{R} \times M$ which preserves ξ, is compatible with the restriction of $d\alpha$ to ξ and sends ∂_t to R_α. Let $L \subset S\xi \times \mathbb{C}$ be a closed Lagrangian, U_L a compact tubular neighborhood of L and p_0 a point in L. We assume that $\pi_\mathbb{R}(U_L)$ lies in $\{t > 1\}$ (this can be arranged by a constant rescaling of α). All these objects, including J_α, are now fixed forever. We denote by $\mathcal{B}$ the space of $W^{1,p}$-maps u from $(\mathbb{D}^2, \partial\mathbb{D}^2, 1)$ to $(S\xi \times \mathbb{C}, L, p_0)$ which are homotopic to the constant map $u_0 \colon z \mapsto p_0$.

We will consider inhomogeneous Cauchy-Riemann equations

$$\bar\partial_J u = G(u)\ ,$$

where $J = J_\alpha \oplus i$ on $S\xi \times \mathbb{C}$ and $u \in \mathcal{B}$ is the unknown. The perturbation term G is a section of the following bundle of complex-antilinear maps:

$$\overline{\mathrm{Hom}}_\mathbb{C}(T\mathbb{D}^2, T(S\xi \times \mathbb{C})) \to \mathbb{D}^2 \times (S\xi \times \mathbb{C})$$

and $G(u) \colon \mathbb{D}^2 \to \overline{\mathrm{Hom}}_\mathbb{C}(T\mathbb{D}^2, T(S\xi \times \mathbb{C}))$ denotes the restriction of G to the graph of u: $G(u)(z) = G(z, u(z))$. Let G be a family of perturbation terms G_s for $s \in [0,1]$ and set

$$\mathcal{M}(G) = \left\{ (s, u) \in [0,1] \times \mathcal{B} \;\middle|\; \bar\partial_J u = G_s(u) \right\}. \tag{9.1}$$

The spaces of perturbation terms we use are:

$$\mathcal{G}_{\varepsilon, C} = \left\{ (\mathbf{0} \oplus C s\, d\bar z) + H_s \;\middle|\; \operatorname{supp} H \subset (\varepsilon, 1-\varepsilon) \times \mathbb{D}^2 \times U_L \right\}$$

where $\mathbf{0}$ is the 0-section in $\overline{\mathrm{Hom}}_\mathbb{C}(T\mathbb{D}^2, T(S\xi))$ and Cs is in $\mathbb{R} \subset T\mathbb{C}$. The term H is a C^1 section of the bundle

$$\overline{\mathrm{Hom}}_\mathbb{C}(T\mathbb{D}^2, T(S\xi \times \mathbb{C})) \to [0,1] \times \mathbb{D}^2 \times (S\xi \times \mathbb{C})\,.$$

In this setup, the following result follows easily from standard techniques.

Proposition 9.24. *If $\varepsilon > 0$ is chosen sufficiently small and C sufficiently large, then there is some G in $\mathcal{G}_{\varepsilon,C}$ such that $\mathcal{M}(G)$ is a smooth 1-dimensional manifold whose boundary is $\{(0, u_0)\}$ where u_0 is the constant disk at p_0.*

We now choose one G given by Proposition 9.24 and keep it until the end of this section. Since $\mathcal{M}(G)$ is a 1-dimensional manifold with only one boundary point, it cannot be compact. We want to prove that, under the assumptions of hypertightness of ξ and compactness of M and L, the only source of non-compactness for $\mathcal{M}(G)$ is bubbling of holomorphic disks so that L is not weakly exact.

We first note some C^0-estimates for all solutions in $\mathcal{M}(G)$. For any $(s, (u_\xi, u_\mathbb{C}))$ in $\mathcal{M}(G)$, the component u_ξ is J_α-holomorphic outside the preimage of U_L. Hence it cannot enter any neighborhood of $S_{+\infty}\xi$ which is disjoint from the projection $\pi_\xi(U_L)$. Similarly, the component $u_\mathbb{C}$ is harmonic outside the preimage of U_L and this implies that the image of $u_\mathbb{C}$ is contained in a fixed compact subset (any disk around 0 which contains $\pi_\mathbb{C}(U_L)$ is big enough, see for example the proof of McDuff and Salamon 2004, Lemma 9.2.3). Those observations are summarized in the following lemma.

Lemma 9.25. *There is a neighborhood U_+ of $S_{+\infty}\xi$ and a compact set $K_\mathbb{C} \subset \mathbb{C}$ such that, for all (s, u) in $\mathcal{M}(G)$, $u(\mathbb{D}) \subset (S\xi \setminus U_+) \times K_\mathbb{C}$.* □

Next we need some energy bounds. In view of our later use of Hofer's energy, we will introduce the following class of symplectic forms. We consider the space of probe functions

$$\mathcal{F} := \{\psi \colon \mathbb{R} \to \mathbb{R} \mid \psi \text{ is a smooth embedding and } \psi(t) = t \text{ for } t > 1\}$$

and the associated exact symplectic forms $\omega_\psi := d(e^\psi \alpha)$ on $S\xi$.

Proposition 9.26. *There is some bound A such that $\left|\int_V u^*(\omega_\psi \oplus \omega_\mathbb{C})\right| \leq A$ for all (measurable) subsets $V \subset \mathbb{D}$, all (s, u) in $\mathcal{M}(G)$ and all ψ in $\mathcal{F}$.*

Proof. The first observation, due to Gromov, is that one can turn the inhomogeneous Cauchy-Riemann problem defining $\mathcal{M}(G)$ into an homogeneous one which allows easier energy estimates. To any u in $\mathcal{B}$ we associate its graph

$$\tilde{u} \colon \mathbb{D}^2 \to \mathbb{D}^2 \times (S\xi \times \mathbb{C}),\ z \mapsto (z, u(z))$$

and for any s in $[0, 1]$ we consider the almost complex structure J_s on $\mathbb{D}^2 \times (S\xi \times \mathbb{C})$ given by

$$J_s(\dot{z}, \dot{p}) := (i\dot{z}, J\dot{p} + 2G_s \cdot i\dot{z})$$

for every vector $\dot{z} \in T\mathbb{D}^2$ and $\dot{p} \in T(S\xi \times \mathbb{C})$. The pair (s, u) is in $\mathcal{M}(G)$ if and only if $\tilde{u}$ is a J_s-holomorphic map.

Lemma 9.27. *If $K > 0$ is large enough then $\tilde{\omega}_\psi = (K\omega_\mathbb{D}) \oplus \omega_\psi \oplus \omega_\mathbb{C}$ tames J_s for all ψ and s.* □

Note also that, because $\pi_{\mathbb{R}}(U_L) \subset [1, \infty)$, the submanifold $\mathbb{S}^1 \times L \subset \mathbb{D}^2 \times (S\xi \times \mathbb{C})$ is Lagrangian for every $\tilde{\omega}_\psi$. Stokes' formula then ensures that

$$\int_{\mathbb{D}} \tilde{u}^* \tilde{\omega}_\psi = K\pi$$

for every u in $\mathcal{B}$. If K is sufficiently large to get tameness from Lemma 9.27 and $\tilde{u}$ is J_s-holomorphic then $\tilde{u}^*\tilde{\omega}_\psi$ is non-negative on $\mathbb{D}$. Then for any $V \subset \mathbb{D}$, we have

$$\int_V u^*(\omega_\psi \oplus \omega_{\mathbb{C}}) = \int_V \tilde{u}^*\tilde{\omega}_\psi - \int_V K\omega_{\mathbb{D}}$$

with both right-hand side integrals in $[0, K\pi]$ so we can choose $A = K\pi$ to finish the proof of Proposition 9.26. □

After those preliminaries, we now consider any sequence $(s_k, u_k)_k$ which has no convergent subsequence in $\mathcal{M}(G)$. The sequence $r_k := \max_{\mathbb{D}} |du_k|$ is unbounded since otherwise the Arzelà–Ascoli theorem and elliptic regularity would provide a convergent subsequence for u_k. Let z_k be a sequence in $\mathbb{D}$ such that $r_k = |du_k(z_k)|$. After passing to a subsequence, we can assume that z_k converges to some z_∞ in $\mathbb{D}$ and r_k goes to $+\infty$. We set $\delta_k = d(z_k, \partial\mathbb{D}) = 1 - |z_k|$.

Sphere and plane bubbling

Assume for contradiction that $r_k\delta_k$ is unbounded. (this happens for instance if z_∞ lies in the interior of $\mathbb{D}$). After passing to a subsequence, we can assume that $r_k\delta_k$ is increasing and goes to infinity. We denote by $\mathbb{D}_k$ the open disk with radius $r_k\delta_k$ in $\mathbb{C}$ and consider the map $\Phi_k \colon z \mapsto z_k + z/r_k$ which, due to our choice of δ_k, sends $\mathbb{D}_k$ into $\mathbb{D}$. We set $t_k = \pi_{\mathbb{R}}(u_k(z_k))$ and

$$v_k \colon \mathbb{D}_k \to S\xi \times \mathbb{C},\ z \mapsto \tau_{-t_k} \circ u_k \circ \Phi_k\,.$$

By construction, we have $\sup |dv_k| \leq |dv_k(0)| = 1$ and $\pi_{\mathbb{R}}(v_k(0)) = 0$. The Arzelà–Ascoli theorem then proves that v_k converges uniformly on compact subsets to some $v \colon \mathbb{C} \to S\xi \times \mathbb{C}$. Since J is translation invariant, we get

$$\begin{aligned}\bar{\partial}_J v_k &= \frac{1}{r_k} d\tau_{-t_k} \circ \bar{\partial}_J u_k(\Phi_k(z)) \\ &= \frac{1}{r_k} d\tau_{-t_k} \circ G_{s_k}\left(\Phi_k(z), u_k(\Phi_k(z))\right) \to 0\end{aligned}$$

where convergence is uniform on compact sets hence, by elliptic regularity, v is genuinely J-holomorphic, and in particular the component $\pi_{\mathbb{C}} \circ v$ is a classical holomorphic function from $\mathbb{C}$ to $\mathbb{C}$, and $v_\xi := \pi_\xi \circ v$ is a J_α-holomorphic map. From Lemma 9.25 it follows that $\pi_{\mathbb{C}} \circ v$ is bounded, so that this component is in fact constant. In particular we get that, for any compact subset K in $\mathbb{C}$, we have

$$\lim_{k\to\infty} \int_{\Phi_k(K)} u_k^*\omega_{\mathbb{C}} = 0. \tag{9.2}$$

The component v_ξ by contrast cannot be constant since $|dv(0)| = 1$.

Lemma 9.28. *After passing to a subsequence of u_k, the sequence t_k diverges towards $-\infty$.*

This lemma is proved by contradiction. Assuming that t_k does not go towards $-\infty$ gives a finite area holomorphic plane v_ξ in $S\xi$. One cannot apply Gromov's removal of singularities theorem in $S\xi$ but instead we proved that either v_ξ extends to $\mathbb{C}\mathrm{P}^1$ or it is proper and both possibilities are absurd.

Remember that the Hofer energy of a J_α-holomorphic map $v\colon \Sigma \to S\xi$ in the symplectization is defined as

$$E_\alpha(v) = \sup_{\varphi \in \mathcal{F}'} \int_\Sigma v^* \omega_\varphi \,,$$

where $\mathcal{F}'$ is the space of increasing diffeomorphisms from $\mathbb{R}$ to $(-1, 0)$ and $\omega_\varphi = d(e^\varphi \alpha)$ on $S\xi$.

Lemma 9.29. *The holomorphic plane v_ξ has finite Hofer energy.*

Since v_ξ has finite Hofer energy, Hofer 1993, Theorem 31 gives a contractible T-periodic Reeb orbit γ for α and a sequence x_k such that $v_\xi(e^{2\pi(x_k+iy)})$ converges uniformly to $\gamma(Ty)$ (we cannot hope for convergence without condition on x_k because we haven't made any non-degeneracy assumption on α). This contradicts our assumption that α has no contractible closed Reeb orbit so we have proved that $r_k\delta_k$ is bounded.

Disk bubbling

Because $r_k\delta_k$ is bounded, we learn in particular that z_∞ is in $\partial\mathbb{D}$. For notational convenience only, we assume that $z_\infty = 1$. After passing to a subsequence we can assume that $r_k \geq 1$ and $r_k\delta_k$ converges to some non-negative number ν. We set

$$w_k = \left(1 - \frac{1}{r_k}\right) \frac{z_k}{|z_k|}$$

(this extra sequence of points is a minor nuisance needed because when ν is zero, the naive rescaling could lead to a constant map). We use the rescaling maps

$$\Phi_k(z) = \frac{z + w_k}{1 + \bar{w}_k z}$$

which are automorphisms of $\mathbb{D}$ sending 0 to w_k and which converge uniformly to the constant map $z \mapsto 1$ on any compact subset K of $\mathbb{D}' := \mathbb{D} \setminus \{-1\}$. Also there are positive constants $C_1(K)$ and $C_2(K)$ such that, for every z in K:

$$\frac{C_1(K)}{r_k} \leq \|d\Phi_k(z)\| \leq \frac{C_2(K)}{r_k}.$$

Our rescaled disk is then $v_k := u_k \circ \Phi_k$ which satisfies: $\|dv_k(z)\| = \|du_k(\Phi_k(z))\| \cdot \|d\Phi_k(z)\|$ for every z since $d\Phi_k(z)$ is an invertible conformal linear map. So for every compact $K \subset \mathbb{D}'$, $\|dv_k\| \leq C_2(K)$ on K. In addition each $v_k(\partial\mathbb{D}')$ is in L so $v_k(K)$ is in the $C_2(K)$-neighborhood of L for each convex compact subset K. Using an exhaustion of $\mathbb{D}'$ by such subsets we get that v_k is uniformly bounded on compact subsets of $\mathbb{D}'$. So the Arzelà-Ascoli theorem gives convergence of v_k to some $v\colon (\mathbb{D}', \partial\mathbb{D}') \to (S\xi \times \mathbb{C}, L)$ uniformly on compact subsets of $\mathbb{D}'$. Since $\|dv_k(\zeta_k)\| \geq C_1(K_0)$, we get $dv(\zeta_\infty) \neq 0$ and v is non-constant. Using that Φ_k is holomorphic we get:

$$\bar{\partial} v_k(z) = G_s\left(\Phi_k(z), v_k(z)\right) \circ d\Phi_k(z)$$

which converges to zero uniformly on compact subsets of $\mathbb{D}'$ so, by elliptic regularity, v is J-holomorphic.

The energy of v is bounded by Proposition 9.26 since:

$$\int_K v_k^*(\omega \oplus \omega_{\mathbb{C}}) = \int_{\Phi_k(K)} u_k^*(\omega \oplus \omega_{\mathbb{C}}) \leq A.$$

Again there is a version the removal of singularity theorem that allow to compactify v to a non-constant J-holomorphic disk with boundary on L. Here the key is that, although the symplectic manifold has bad infinity, L is compact. Thus as we wanted to show, L is not weakly exact.

9.3.3. Application to contact transformation

Additionally to the properties discussed in Chapter 7, the manifold $(M_{\Bbbk}, \alpha_+)$ contains a closed pre-Lagrangian submanifold P_0 such that the restriction of α_+ to P_0 is closed and $\pi_1(P_0)$ injects into $\pi_1(M_{\Bbbk})$. We use the notations of Proposition 7.28 on Page 119. Consider any of the fibers $\{\mathbf{t}\} \times (\mathbb{R}^{l+1}/\Lambda')$ in $\overline{M}/\Lambda'$ with $\mathbf{t} \in \mathbb{R}^l$. This fiber is a torus which is pre-Lagrangian in $(\overline{M}/\Lambda', \alpha_+)$, because α_+ restricts to a constant 1-form on it. Clearly this torus embeds in M under the projection $\overline{M}/\Lambda' \to M$. We choose for P_0 the image of this embedding. By construction, $\pi_1(P_0) = \Lambda'$ embeds into $\pi_1(M) = \Lambda \ltimes \Lambda'$.

We set $V_{\Bbbk} = \mathbb{T}^2 \times M_{\Bbbk}$. The pre-Lagrangian submanifold $P_0 \subset (M_{\Bbbk}, \alpha_+)$ described above extends to a pre-Lagrangian submanifold $P := \{0\} \times \mathbb{S}^1 \times P_0$ in $(V_{\Bbbk}, \xi_n)$, because the restriction of

$$\lambda_n = \frac{1 + \cos(ns)}{2}\, \alpha_+ + \frac{1 - \cos(ns)}{2}\, \alpha_- + \sin(ns)\, dt$$

to $\{0\} \times \mathbb{S}^1 \times P_0$ is the closed 1-form $\alpha_+|_{TP_0}$.

The contactomorphism $\Psi_{n,m}\colon (s, t, \theta) \mapsto (s + 2\pi m/n, t, \theta)$ obviously displaces P from itself so we only need to check that P is weakly exact and apply Theorem 9.20 to get that $\Psi_{n,m}$ is not symplectically pseudoisotopic to the identity.

We will now show $\pi_2(V_{\Bbbk}, P) = 0$, which implies that P is weakly exact. Because $\pi_1(P_0)$ embeds into $\pi_1(M_{\Bbbk})$ we get that $\pi_1(P)$ embeds into $\pi_1(V_{\Bbbk})$. The long exact sequence of the pair $(V_{\Bbbk}, P)$ contains

$$\pi_2(V_{\Bbbk}) \to \pi_2(V_{\Bbbk}, P) \to \pi_1(P) \to \pi_1(V_{\Bbbk})$$

where $\pi_2(V_{\Bbbk}) = 0$ (because by construction the universal cover of $M_{\Bbbk}$ is a Euclidean space) and $\pi_1(P) \hookrightarrow \pi_1(V_{\Bbbk})$ so $\pi_2(V_{\Bbbk}, P) = 0$.

We now prove the last part of the main theorem, about conjugations. Consider the family of contactomorphisms

$$\Phi_\tau \colon (\mathbb{T}^2 \times M_{\Bbbk}, \xi_n) \to (\mathbb{T}^2 \times M_{\Bbbk}, \xi_n),\ (s, t; x) \mapsto (s, t + \tfrac{2\pi m \tau}{n}; x)$$

for $\tau \in [0, 1]$.

The contactomorphism Φ_1 is conjugated to $\Psi_{n,m}$ by the diffeomorphism

$$A \colon (V_{\Bbbk}, \xi_n) \to (V_{\Bbbk}, \xi_n),\ (s, t; x) \mapsto (t, -s; x)$$

which satisfies $A\Phi_1 A^{-1} = \Psi_{n,m}$. But A cannot be replaced by a contactomorphism since Φ_1 is, by construction, contact isotopic to the identity.

9.4. Later developments and prospects

Both papers Giroux and Massot 2015 and Massot and Niederkrüger 2016 are rather recent and there is not much new in these directions, except for the appearance of Lanzat and Zapolsky 2015 which constructs new examples of non-trivial contact mapping classes in higher dimensions. However these examples do not live on closed contact manifolds but on contactizations $W \times \mathbb{S}^1$ of Liouville manifolds. The fact that they are non-trivial improves corresponding results about the Liouville base in Khovanov and Seidel 2002. It would be interesting to know whether this result survives compactification of the ambient manifold, say by blow down as in Section 7.3.1 (this operation turns $W \times \mathbb{S}^1$ into the closed manifold supported by the open book with page W and monodromy identity).

In the three dimensional case, what is missing is a general result analogous to the coarse classification of tight contact structures (see Section 3.2.5). There is no general criterion guaranteeing that $\ker(\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V))$ is finite, or even finitely generated. As proved in Ding and Geiges 2010, this kernel is isomorphic to $\mathbb{Z}$ if $V = \mathbb{S}^1 \times \mathbb{S}^2$ and ξ is tight. Since $\mathcal{D}(\mathbb{S}^2 \times \mathbb{S}^1)$ has a lot of topology (although it has finitely many connected components), it may be an exceptional case from the contact point of view too. From the coarse classification theorem and results in this chapter, it is natural to restrict attention to irreducible atoroidal manifolds. In that case it follows from Perelman's proof of the geometrization conjecture that V is hyperbolic. Mostow rigidity then proves that $\pi_0\mathcal{D}(V)$ is finite. Better, Gabai 2001 proved that each connected component is contractible. So the kernel of $\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V)$ is isomorphic to $\pi_1\,\Xi(V;\xi)$. One could hope to study this fundamental group by techniques extending the ones used in the coarse classification theorem. Another possibility is to try and mimic the methods used to get generating sets for surface mapping class groups. The strategy there is to find a complex onto which the mapping class group acts nicely, typically a variation on the complex of curves. One could hope to use convexity here but I was not able to implement this strategy up to now.

In higher dimensions, computing $\ker(\pi_0\mathcal{D}(V;\xi) \to \pi_0\mathcal{D}(V))$ seems rather hopeless since there is no single example of a computed $\pi_0\,\Xi(V)$. However one can still hope for some weak version of our results about contact element bundles. It could be that, for almost every closed manifold M, every contact diffeomorphism φ of ST^*M is contact isotopic to the lift of a diffeomorphism f of M. A more realistic goal could be to prove that φ is homotopic to such a lift. My hope is to use sheaf theoretical methods to prove that. Indeed the contact graph in $ST^*(M \times M)$ of the lift of any f is the microsupport of the constant sheaf on the graph of f. If one can prove that the graph of every φ is a microsupport then it would certainly have topological consequences. Here the inspiration is Guillermou 2012 which proved that every closed exact Lagrangian in the cotangent bundle of a closed manifold is the microsupport of some nice complex of sheaves, and used it to reprove that it is homotopy equivalent to the base.

In a different direction, it is interesting to study the case of overtwisted contact 3-manifolds. Because of the parametric version of the classification of overtwisted contact structures from Eliashberg 1989, this is essentially equivalent to studying the space of overtwisted disks inside a given overtwisted manifold. This is a question that I asked my student Fabio Gironella to think about, but I quickly found out that Thomas Vogel was preparing a paper on the same examples, hence I redirected Fabio towards something else.

Part III.

Ongoing work

Contents

10. Exotic tight contact structures on $\mathbb{R}^{2n-1}$

We saw in Section 3.2.1 that Eliashberg 1992 proved that all tight contact structures on the closed ball $\mathbb{B}^3$ which coincide with the standard one along the boundary are isotopic to it, relative to the boundary. Together with genericity of ξ-convex surfaces and the realization lemma (or by direct use of the elimination lemma), this can be used to prove that $\mathbb{R}^3$ has a unique isomorphism class of tight contact structures.

In higher dimension this strategy fails at all stages. First we do not have genericity of ξ-convex surfaces (or the analogue of the elimination lemma). And there are exotic contact structures on closed balls. Indeed Eliashberg 1991b; Geiges 1997a; Ustilovsky 1999; Ding and Geiges 2004b proved that spheres have strongly fillable contact structures that are homotopic to the standard one but not isomorphic. Removing a standard open Darboux ball to such a sphere produces an exotic closed ball. Note that the latter argument does not tell anything about the complement of an open ball or a point. At this stage we still don't know whether $\mathbb{R}^{2n-1}$ admits an exotic tight contact structure when n is larger than two.

Many exotic contact structures on $\mathbb{S}^{2n-1}$ are actually Stein fillable, they arise as link of isolated complex singularities. The corresponding Stein filling has some non-trivial topology, and this was used to produce exoticness. Indeed the Eliashberg-Floer-McDuff theorem from McDuff 1991a guarantees that all Stein fillings of the standard contact structure on $\mathbb{S}^{2n-1}$ are diffeomorphic to balls.

Joint work in progress with Klaus Niederkrüger aim to prove the following variation of this result, which would imply the existence of exotic tight contact structures on $\mathbb{R}^{2n-1}$.

Conjecture 10.1. *Let ξ be a contact structure on $\mathbb{S}^{2n-1}$ having some Stein filling W. If the complement of a point in $(\mathbb{S}^{2n-1}, \xi)$ is contactomorphic to the standard contact structure on $\mathbb{R}^{2n-1}$ then W is diffeomorphic to a ball.*

The following definition appeared in Niederkrüger and Rechtman 2011 generalizing the proof the Weinstein conjecture for tight 3-manifolds having non-zero π_2 in Hofer 1993.

Definition 10.2. *Let (M, ξ) be a $(2n-1)$-dimensional contact manifold. An n-dimensional submanifold $N \hookrightarrow M$ is called a **Legendrian open book (Lob)**, if ξ induces a singular foliation $\mathcal{F}$ on N that is diffeomorphic to an open book decomposition, i.e., the singular set $\operatorname{sing}(\mathcal{F}) = \{p \in N \mid T_pN \subset \xi_p\}$ is the binding of an open book on N, and each regular leaf of the foliation corresponds to a page of the open book.*

Legendrian open books can be used to understand properties of symplectic fillings of a contact manifold (M, ξ) by studying the holomorphic disks whose boundaries lie in the Lob. Near the binding the situation is the same as for bLobs, and one has a source of J-holomorphic disks.

Proposition 10.3. *Let* $(\mathbb{S}^{2n-1}, \xi_0)$ *be the standard contact sphere, and let* $\mathbb{S}^{n-2}_{\mathrm{sing}}$ *be the isotropic submanifold*

$$\mathbb{S}^{n-2}_{\mathrm{sing}} := \left\{(0, i\mathbf{y}) \in \mathbb{C} \times \mathbb{C}^{n-1} \;\middle|\; \mathbf{y} \in \mathbb{R}^{n-1}, \|\mathbf{y}\| = 1\right\}.$$

The complement $\mathbb{S}^{2n-1} \setminus \mathbb{S}^{n-2}_{\mathrm{sing}}$ *is foliated by a family of* n*-dimensional spheres* $L_{\mathbf{b}}$ *that is parametrized by* $\mathbf{b}$ *in the interior of* $\mathbb{D}^{n-1}$*. Each of these spheres* $L_{\mathbf{b}}$ *is a* Lob *with Legendrian open book* $(B_{\mathbf{b}}, \vartheta_{\mathbf{b}})$*, where the binding* $B_{\mathbf{b}}$ *is the subset where* $\{r = 0\}$*, and the fibration is* $\vartheta_{\mathbf{b}} \colon L_{\mathbf{b}} \setminus B_{\mathbf{b}} \to \mathbb{S}^1, (re^{i\varphi}, \mathbf{x}) \mapsto e^{i\varphi}$*.*

In order to prove Conjecture 10.1, we assume the existence of an isomorphism between the standard $\mathbb{R}^{2n-1}$ and the complement of a point in the boundary of some Stein manifold W. Transporting the family of Lobs from Proposition 10.3 to $\mathbb{R}^{2n-1}$ and then ∂W, we get the same picture in ∂W except that everything become wild near one point.

Our work in progress aims to prove that holomorphic curves can still be used in this situation. In particular the total lack of energy bound is made up for by the assumption that the filling W is Stein. For instance, the proof in a tamer context could use the following steps to analyse compactness of some family of curves. First use a bubbling off analysis to build a bounded holomorphic plane $v \colon \mathbb{C} \to W^+$ with finite area, where W^+ is a completion of W. Next use Gromov's removal of singularities to extend v to a holomorphic sphere $\mathbb{CP}^1 \to W^+$ and get a contradiction to some assumption on W, typically symplectic asphericity. In our case, the contradiction comes from the fact that W^+ embeds into some $\mathbb{C}^N$, and Liouville's theorem forbidding the existence of a non-constant bounded holomorphic map from $\mathbb{C}$ to $\mathbb{C}^N$, without any energy bound assumption.

This argument, and others similar in spirit, are efficient but they require to keep the almost complex structure integrable (at least near the boundary). In particular we cannot modify the complex structure to build the Bishop family of holomorphic disks near the binding of each Lob and guarantee that no other disk approach the binding. The existence part is more or less already proved in the complex analysis literature, although we have a more symplectic friendly proof. But the uniqueness is still in progress.

11. Open books and invariant norms

Section 5.3 reviewed what is known about invariant norms on contact transformation groups. A conjugation invariant norm on a group G is a function $\nu : G \to [0, \infty)$ satisfying the following properties:

1. $\nu(\mathrm{Id}) = 0$ and $\nu(g) > 0$ for all $g \neq \mathrm{Id}$.
2. $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$.
3. $\nu(g^{-1}) = \nu(g)$ for all $g \in G$.
4. $\nu(h^{-1}gh) = \nu(g)$ for all $g, h \in G$.

Bi-invariant distances are another point of view on the same objets. Such a distance d defines a norm $\nu = d(\cdot, \mathrm{Id})$ and, starting from a norm ν, one gets a distance $d(f, g) = \nu(fg^{-1})$.

Inspired by the Hofer distance in symplectic geometry, there have been several recent works on invariant norms on groups of contact transformations isotopic to the identity or its universal cover, see Sandon 2010; Fraser, Polterovich, and Rosen 2012; Colin and Sandon 2015; Borman and Zapolsky 2015 and the survey Sandon 2015.

A conjugation invariant norm is a purely algebraic object (e.g. $\nu(\mathrm{Id}) = 0$ and $\nu(g) = 1$ for all other g is very much invariant). The topology given by such a norm can be very different from the smooth one (again this happens in the symplectic case with Hofer and Viterbo's distances).

In the contact case, norms are discrete except maybe on $\pi_1(G)$. The usual way of defining a discrete invariant is through generating sets. To any generating set S of a group G we can associate the norm

$$\nu_S(g) = \min\{k \in \mathbb{N}\,;\, g \in S^k\}.$$

Non-degeneracy and triangle inequality are obvious. Invariance by inversion and conjugation hold whenever S is invariant by these operations. In that case we will say that S is an invariant generating set.

Recall a group is called *perfect* if any element is a product of commutators. In this case the set of commutators is an invariant generating set. The resulting norm is called the *commutator length*. A group is *uniformly perfect* is this norm is bounded. If furthermore G is a simple group then every non-trivial element g gives a generating set $\{g, g^{-1}\}$ and a corresponding norm ν_g. If ν_g is bounded for some g then any invariant norm ν on G is bounded, by $\max(\nu_g)\nu(g)$. A group is *uniformly simple* if all these ν_g norms are bounded (by the same number).

As an example, we discuss a famous subgroup of contact transformations of the standard contact $\mathbb{S}^3$.

Example 11.1 (The rotation group $\mathrm{SO}_3(\mathbb{R})$)**.** *Every element is the product of two reflections hence the product of two conjugates of any given rotation with angle π . One then check that, for every non-trivial rotation R, there is a product of two conjugate of R which has angle π. Hence every element of $\mathrm{SO}_3(\mathbb{R})$ is a product of four conjugates of R. Let A be the rotation with angle $\pi/2$ around the z-axis and let B be the half-turn around the x-axis. Then $[A, B]$ is a half-turn (around the z-axis). Hence every element of $\mathrm{SO}_3(\mathbb{R})$ is the product of at most two commutators. Note every element is conjugate to its inverse in this group.*

The only general result known about contact transformation groups is in Rybicki 2010 which proves that, for every contact manifold (V, ξ), $G = \mathcal{D}(V; \xi)$ and its universal cover are simple. In particular these groups are perfect, a crucial information for us.

The most basic observation about the algebraic structure of transformation groups is that two transformations with disjoint support commute. A refined version of this observation involve commutators. It goes back at least as far as Thurston's work on diffeomorphism groups.

Observation 11.2. *Let a, b and g be three transformations such that g displaces the support of a from the support of b. Then the commutator $[a, b]$ is a product of four conjugates of g or g^{-1}.*

Proof. Indeed, if we set $c = g^{-1}ag$ then b and c have disjoint support hence $cb = bc$ and:

$$\begin{aligned}
aba^{-1}b^{-1} &= gcg^{-1}bgc^{-1}g^{-1}b^{-1} \\
&= gcg^{-1}c^{-1}cbgc^{-1}b^{-1}bg^{-1}b^{-1} \\
&= g(cg^{-1}c^{-1})\left((bc)g(bc)^{-1}\right)(bg^{-1}b^{-1}).
\end{aligned}$$ □

Hence commutators of transformations with displaceable support can be replaced by conjugates of elements depending only on the support. The next step in this line is Burago, Ivanov, and Polterovich 2008, Theorem 2.2(i) which guarantees that, if F is a transformation and U is a subset whose iterated images under F are pairwise disjoint ($F^i(U) \cap F^j(U)$ is empty for all $i \neq j$ in $\mathbb{Z}$) then any product of commutators of elements supported in U can be rewritten as a product of only two such commutators.

This motivates the following definition, originally introduced in the smooth setting in ibid., and later adapted to the contact setting in Fraser, Polterovich, and Rosen 2012.

Definition 11.3 (Burago, Ivanov, and Polterovich 2008; Fraser, Polterovich, and Rosen 2012)**.** *An open contact manifold (V, ξ) is called contact portable if there exists a compact set $V_0 \subset V$ and a contact isotopy $\{P_t\}$ of V, $t \geq 0$, $P_0 = \mathrm{Id}$ such that following hold:*

- *The set V_0 is an attractor of $\{P_t\}$, i.e. for every compact set $K \subset V$ there exists some $t > 0$ such that $P_t(K) \subset V_0$.*

- *There exists a contactomorphism θ of V displacing V_0 (θ may have non-compact support).*

In this situation, V_0 is called a core of V.

The algebraic ideas mentioned above prove the following key statement.

Proposition 11.4 (Burago, Ivanov, and Polterovich 2008; Fraser, Polterovich, and Rosen 2012)**.** *If (V,ξ) is a portable contact manifold then any contact transformation with compact support in V is a product of two commutators. If θ is a displacing contactomorphism as in Definition 11.3, then each of these commutators is a product of four conjugates of θ or θ^{-1}.*

It is an easy observation that interiors of contact handlebodies are contact portable, with their Legendrian spine as attractors. Actually I do not know any other example (the only example in Fraser, Polterovich, and Rosen 2012 is the standard $\mathbb{R}^{2n+1}$).

The next crucial fact is that the open book decomposition theorem decomposes any closed contact manifold into a union of two contact handlebodies (see Section 3.3). Hence it is useful to understand when contact transformations can be decomposed with respect to such a contact Heegaard splitting.

Proposition 11.5. *Let (V,ξ) be a closed orientable contact manifold and let G be the connected component of the identity in the contact transformation group $\mathcal{D}(V,\xi)$. Assume that ξ is supported by an open book whose pages are subcritical Weinstein manifolds. Then Every element of G or its universal cover is the composition of two elements with compact support in the interior of contact handlebodies.*

Corollary 11.6. *If (V,ξ) is supported by an open book with subcritical pages then every norm on G or its universal cover is bounded.*

The above proposition is rather simple to prove. Let $V = H_0 \cup H_1$ be a contact Heegaard splitting coming from an open book with subcritical pages. There are Legendrian spines L_0 and L_1 for H_0 and H_1 that do not have any Legendrian pieces: they are union of isotropic submanifolds of lower dimensions. The key point is that, in a generic contact isotopy φ, there will not be any intersection between L_0 and some $\varphi_t(L_1)$.

The above assumption on open book is of course rather strong and such manifold definitely have a flexible flavor, although they can be Stein fillable, e.g. if the monodromy is trivial. In a more general situation, one can imagine the following scenario. During the isotopy φ, there are finitely many collisions between $\varphi_t(L_1)$ and L_0 but one can prevent them by composing φ_t with an isotopy having support in a ball which slows down φ_t to avoid collision. This is the strategy used in Burago, Ivanov, and Polterovich 2008; Tsuboi 2008 to prove that invariant norms on diffeomorphism groups are bounded.

In the contact case we know this strategy can not succeed in all situations since there are examples of unbounded norms on the universal cover of G. What happens is that the slow down move needs some definite amount of room. Something like a loose chart for the Legendrian part of spines is required. Note however that the complement of a Legendrian spine of one handlebody in a contact Heegaard splitting is tight (it embeds into the fillable manifold constructed from the same page with trivial monodromy). It seems very likely that the Legendrian spine of an open book with flexible page has the right kind of loose charts (by definition, critical handles are attached to loose Legendrians, and the Liouville flow provide room in the relevant extra dimension). However the class of contact structures supported by open books with flexible pages is rather mysterious. It definitely has some flexible flavor but does not seem to have any direct relation to overtwistedness.

We encode this discussion into a technical definition and a conjecture.

Definition 11.7. *An element of G or its universal cover has a good decomposition if it can be written the composition of two elements with compact support in the interior of contact handlebodies and an element with compact support in a Darboux ball.*

Conjecture 11.8. *If (V, ξ) is overtwisted or supported by an open book with flexible pages then every element of G or its universal cover has a good decomposition.*

It follows from the algebraic discussion above that this conjecture (or its proved version for subcritical open books) would imply that all invariant norm on the corresponding group are bounded. But the proof actually give explicit bounds in many cases. For instance, let φ be a strictly monotone isotopy. It means that the vector field $d\varphi_t/dt$ is never in ξ (φ could be a Reeb flow for instance). Then there is a positive ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0]$, every element of G can be written as a product of at most 20 conjugates of φ_ε and φ_ε^{-1}. This essentially follows from the fact that φ_ε displaces isotropic submanifolds.

On generic Legendrian isotopies

A technical ingredient in order to tackle Conjecture 11.8 is to prove generic properties of the trace $\varphi(L \times [0,1])$ where $\varphi \colon V \times [0,1] \to V$ is a contact isotopy and L is a Legendrian submanifold. We know since Whitney 1943, 1944 what kind of singularities to expect if we perturb φ among all smooth maps: φ will become an immersion outside finitely many Whitney umbrellas, and we also know things about double and triple points. But we need to stay among contact isotopies. I wasn't able to find this statement in the literature so I proved it.

A smooth map from $L \times I$ to (V, ξ) is a trace of Legendrian isotopy if and only if its 1-jet extension lands in:

$$\Lambda_*(L \times I, V; \xi) = \left\{((l,t), v, A) \in J^1(L \times I, V)\, ;\, A \colon TL \times \{0\} \overset{\text{Lag}}{\hookrightarrow} \xi_v \right\}$$

This space is an open subset of the smooth codimension $n(n+1)/2$ submanifold $\Lambda(L \times I, V; \xi) \subset J^1(L \times I, V)$ defined by asking that $A(TL \times \{0\})$ is isotropic in ξ_v without requiring any rank condition.

Singularities of maps from $L \times I$ to V correspond in $J^1(L \times I, V)$ to the stratified submanifold $\Sigma(L \times I, V)$ of 1-jets $((l,t), v, A)$ where A does not have full rank. This $\Sigma(L \times I, V)$ has codimension $n+1$ in $J^1(L \times I, V)$.

In a Darboux chart, both Λ_* and Σ become real algebraic sets. There are not transverse to each other so studying their intersection requires some care. But I proved that this intersection is of codimension $n+1$ in the smooth variety Λ_*. This is in accordance with the hope that non-immersed points are isolated in $L \times I$. But it is not enough since we want to perturb while staying in Λ_* so we cannot direct apply the usual "Thom transversality in jet spaces" theorem.

Instead one can use a front projection and the associated lift map. The 1-jet of a lifted front can be written in terms of the 2-jet of the front and I proved that this jet lift is an algebraic submersion. Hence the preimage of $\Lambda_* \cap \Sigma$ has the expected codimension and Thom transversality applies.

12. Towards contact homeomorphisms

As explained in Section 5.3, one can hope to define contact homeomorphisms, at least in dimension 3. In this chapter we explain how convexity could help to prove foundational results in this area.

12.1. Carnot-Carathéodory bilipschitz homeomorphisms

The Heisenberg group $\mathbb{H}_n$ in dimension $2n+1$ is constructed from the $2n$-dimensional symplectic vector space (E, ω_0) (unique up to isomorphism) as $\mathbb{H}_n = E \times \mathbb{R}$ equipped with the following semi-direct product group law:

$$(v, z)(v', z') = (v + v', z + z' + \omega_0(v, v')/2).$$

Let β_0 be the canonical primitive of ω_0 (i.e. $(\beta_0)_v = \iota_v \omega_0$). Then $\beta_0 + dz$ is a left invariant contact form on $\mathbb{H}_n$. Any choice of Euclidean structure on E compatible with ω_0 gives rise to a left-invariant Riemannian metric on $\mathbb{H}_n$. The corresponding sub-Riemannian metric for $(\mathbb{H}_n, \xi_0 = \ker(\beta_0 + dz))$ is independent of the choice of Euclidean structure up to homothety. By a slight abuse of notation we will call it "the natural sub-Riemannian structure on $\mathbb{H}_n$".

It is also convenient to consider the "norm"

$$N(v, z) = \left(\|v\|^4 + |z|^2\right)^{1/4}$$

which defines a left invariant distance which is equivalent to the sub-Riemannian ones.

The next crucial geometric objects are dilatations

$$\delta_\lambda(v, z) = (\lambda v, \lambda^2 z).$$

For each pair (x, y) we have $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$ hence δ_λ is bilipschitz with constant $\mathrm{BiLip}(\delta_\lambda) = \lambda$. Also $N(\delta_\lambda(x)) = \lambda N(x)$.

Examples of bilipschitz homeomorphisms

We need some examples of bilipschitz homeomorphisms in dimension 3. Balogh, Hoefer-Isenegger, and Tyson 2006 proves that a Lipschitz map f of $\mathbb{R}^2$ such that there is a constant λ such that a.e. $J(f) = \lambda$ lifts to a unique Lipschitz map in the Heisenberg group. For instance one could lift $f(x, y) = (x + g(y), y)$ where g is any Lipschitz map from $\mathbb{R}$ to itself. As a very simple example we can lift $g_t(y) = -t|y|$ to get the bilipschitz flow

$$\varphi_t(x, y, z) = \left(x - t|y|, y, z - \frac{t}{2}y|y|\right).$$

This map is not differentiable in the ∂_y direction along the plane $y = 0$ but one can explicitly check that $\varphi_t^*\alpha = \alpha$ everywhere else. Of course on can use more complicated maps, say $g(y) = y^2 \sin(1/y))$ for instance.

One can also lift maps like $f(r, \varphi) = (r, \varphi + \theta(r^2))$ in polar coordinates (r, φ). One can compute the matrix of Df in the orthonormal moving frame $(\partial_r, (1/r)\partial_\varphi)$:

$$Df(r, \varphi) = \begin{pmatrix} 1 & 0 \\ 2r^2\theta' & 1 \end{pmatrix}$$

and we get that f is bilipschitz if $\theta' = 1/r^2$ for instance because both Df and Df^{-1} are bounded. Note that f sends any line through the origin to a spiral crossing each coordinate axis infinitely many times. Those curves lift to Legendrian with this behavior.

The examples above are not truly 3-dimensional so they can be somewhat misleading. For instance they do not create Legendrian curves that have intersections with arbitrarily small Reeb translates. An easy way to build some is to craft a bilipschitz map which commutes with one dilatation. Denote by $B(O, r)$ the ball of radius r for the norm N. Any smooth contactomorphism u_1 supported in $B(O, 2) \setminus B(O, 1)$, can be extended by conjugation by δ_2:

$$\forall n \in \mathbb{Z}; \quad u_{B(O,2^{n+1})\setminus B(O,2^n)} = \delta_2^n \circ u_1 \circ \delta_2^{-n}.$$

The crucial point is that $\mathrm{BiLip}(u) = \mathrm{BiLip}(u_1)$. In particular the extension by $u(O) = O$ is (globally) biLipschitz everywhere. If u_1 sends the Legendrian line $L = \{x = z = 0\}$ to a Legendrian which has a Reeb chord then $u(L)$ will have shorter and shorter Reeb chords accumulating on 0.

The Pansu-Rademacher theorem

The main result of Pansu 1989 is a Carnot-Carathéodory version of Rademacher's theorem asserting that Lipschitz maps are almost everywhere differentiable. This requires a notion of differentiability adapted to the sub-Riemannian setting.

Definition 12.1. *A map h from $\mathbb{H}_n$ to itself with $h(0) = 0$ is Pansu-differentiable (or P-differentiable) at 0 if*

$$h_t(x) = \delta_{1/t} \circ h \circ \delta_t(x)$$

converges uniformly on compact subsets when t goes to zero.

One can use this definition to define P-differentiability at any point of any homeomorphism between contact manifolds. Pansu's theorem is then that CC-lipschitz maps are almost everywhere P-differentiable. If the map is bilipschitz, and one uses a local

Darboux chart, one has the extra conclusion that, almost everywhere, the limit map from the definition is a group automorphism of $\mathbb{H}_n$.

This is clearly a strong result but one must keep in mind that the Euclidean version does not prevent the existence of exotic smooth structures that are bilipschitz homeomorphic to standard ones, see Sullivan 1979.

12.2. An anisotropic contact manifold

We want to know whether CC-bilipschitz maps remember contact topology. Hence the most pressing question is whether a neighborhood of an overtwisted disk can be CC-bilipschitz embedded into $\mathbb{H}_3$. In this section we explain how the results discussed in Section 3.2.1 could help answering this question, using characteristics of invariant contact structures on $\mathbb{T}^2 \times \mathbb{R}$ that could be seen from a metric perspective.

In this section, $V = T^2 \times \mathbb{R}$ with coordinates (x, y, t) and $\xi = \ker(\cos(ky)dx - \sin(ky)dt)$ for some fixed integer k. We denote by $H \in H_1(V, \mathbb{Z})$ the homology class of $\{y = t = 0\}$.

Proposition 12.2. *The contactomorphism group* $\mathrm{Diff}(V, \xi)$ *has the following rigidity properties:*

- *Let* $\iota : T^2 \times [-\varepsilon, \varepsilon] \to V$ *be a homologically essential contact embedding. Then* $\iota_* H = H$*.*
- *A diffeomorphism* ψ *of* V *is isotopic to a element of* $\mathrm{Diff}(V, \xi)$ *if and only if* $\psi_* H = H$*.*
- *If* $\rho : \mathbb{S}^1 \hookrightarrow \mathrm{Diff}(V, \xi)$ *is a 1-parameter subgroup smoothly conjugated to a rotation in* T^2 *then for any* $v \in V$*,* $[\rho(\mathbb{S}^1)v] = H$*.*
- *For any positive integers* n *and* m*, one has the self covering map* $\pi_{n,m} : V \to V$ *defined by* $\pi_{n,m}(x, y, t) = (nx, my, t)$*. The contact structure* $\pi_{n,m}^* \xi$ *is isotopic to* ξ *if and only if* $m = 1$*.*

Also, H is the only primitive homology class in $H_1(V, \mathbb{Z})$ which is not the direction of a foliation by Legendrian circles.

Conjecture 12.3. *Let* $\iota : T^2 \times [-\varepsilon, \varepsilon] \to V$ *be a homologically essential locally bilipschitz embedding. Then* $\iota_* H = H$*.*

Conjecture 12.4. *Let* $\rho : \mathbb{S}^1 \hookrightarrow \mathrm{Diff}(V, \xi)$ *be the rotation group* $\rho(\theta) = (x, y, t) \mapsto (x + \theta, y, t))$*. Let* h *be a homeomorphism of* V*. If* $h \circ \rho(\theta) \circ h^{-1}$ *is a group of locally bilipschitz homeomorphisms then* $h_* H = H$*.*

We now link this manifold with the tight/overtwisted dichotomy. First it is pretty clear that the smooth version of Conjecture 12.3 implies Bennequin but the same is not so clear for Conjecture 12.4. This could mean the later is more promising.

In Conjecture 12.4 we already have a $\mathbb{S}^1$ but the exotic one does not necessarily commutes with it. Also note that a contact manifold can have a free $\mathbb{S}^1 \times \mathbb{S}^1$ action: think of a neighborhood of a prelagrangian torus.

Proposition 12.5. *If Conjecture 12.3 or Conjecture 12.4 is true then there is no bilipschitz embedding of an overtwisted disk into a tight contact manifold.*

The proof of the above proposition will use Bennequin's theorem and classification results by Eliashberg and Giroux.

Proof. Assume for contradiction there is a tight (M, ξ) with a bilipschitz image $\varphi(B)$ of a neighborhood B of overtwisted disk D. After shrinking slightly B, we can assume $\varphi(B)$ is contained in a smooth ball B_1 in M. Genericity of ξ-convex surfaces, the realization lemma and Eliashberg's uniqueness ensures that (B_1, ξ) embeds into the standard $\mathbb{S}^3$. In addition, the neighborhood B contains an unknotted Lutz tube $(D^2 \times \mathbb{S}^1, \xi_L = \ker(cos(\pi r)dz + r\sin(\pi r)d\theta))$. The boundary of this tube is prelagrangian with meridian direction. It can be perturbed to a torus T whose characteristic foliation is a Morse-Smale suspension with two meridian closed leaves. Let U be a homogeneous neighborhood of T and $\varphi : U \to \mathbb{S}^3$ the restriction of the given bilipschitz embedding. Let ψ be a diffeomorphism approximating φ in C^0-topology. Using genericity of ξ-convex surfaces, we can further assume that $T' = \psi(T)$ is ξ-convex. Because of Bennequin's theorem, the dividing set of T' is not meridian. So φ^{-1} already gives a contradiction to Conjecture 12.3.

We denote by U' a homogeneous neighborhood of T' inside $\varphi(U)$. We set $U_1 = \varphi^{-1}(U')$. The restriction ξ_1 of ξ_L to U_1 is tight since ξ_L is tight on U. The semi-local Bennequin inequality and the classification of tight structures on thickened tori implies the existence of free smooth $\mathbb{S}^1$ action with meridian direction on (U_1, ξ_1). Hence we get a free bilipschitz $\mathbb{S}^1$ action on U' which contradicts Conjecture 12.4. □

12.3. Handle straightening approach

In order to smooth contact homeomorphisms, we can try to use a contact handle decomposition given by the open book theorem and smooth one handle at a time.

Conjecture 12.6 (0-handle straightening)**.** *We set*

$$H_0 = \{x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3 \text{ and } \xi_0 = \ker(dz - ydx + xdy).$$

For any CC-bilipschitz embedding $\varphi : (H_0, \xi_0) \to (\mathbb{R}^3, \xi_0)$, there is a CC-bilipschitz embedding $\psi : (H_0, \xi_0) \to (\mathbb{R}^3, \xi_0)$ which coincides with φ near ∂H_0 and is smooth in a neighborhood of the origin.

I expect that the above conjecture will follow from Pansu's theorem explained in Section 12.1, compare the Euclidean case in Kirby 1966, Theorem 6. The crucial step would then be:

Conjecture 12.7 (1-handle straightening)**.** *We set*

$$H_1 = \{x^2 + z^2 \leq 1; y^2 \leq 1\} \subset \mathbb{R}^3 \text{ and } \xi_1 = \ker(dz + ydx + 2xdy).$$

For any CC-bilipschitz embedding $\varphi : (H_1, \xi_1) \to (\mathbb{R}^3, \xi_1)$ *which is smooth near* $H_1 \cap \{y = \pm 1\}$*, there is a CC-bilipschitz embedding* $\psi : (H_1, \xi_1) \to (\mathbb{R}^3, \xi_1)$ *which coincides with* φ *near* ∂H_1 *and is smooth in a neighborhood of* $H_1 \cap \{x = z = 0\}$*.*

For completeness we state what would be the 2-handle straightening conjecture, but we will explain below that it can be bypassed.

Conjecture 12.8 (2-handle straightening)**.** *We set*

$$H_2 = \{x^2 + z^2 \leq 1; y^2 \leq 1\} \subset \mathbb{R}^3 \text{ and } \xi_2 = \ker(dz + ydx + 2xdy).$$

For any CC-bilipschitz embedding $\varphi : (H_2, \xi_2) \to (\mathbb{R}^3, \xi_2)$ *which is smooth near* $H_2 \cap \{x^2 + z^2 = 1\}$*, there is a CC-bilipschitz embedding* $\psi : (H_2, \xi_2) \to (\mathbb{R}^3, \xi_2)$ *which coincides with* φ *near* ∂H_2 *and is smooth in a neighborhood of* $H_2 \cap \{y = 0\}$*.*

Eliashberg's uniqueness on balls combines with Moise-Munkres smoothing of 3-dimensional homeomorphisms (the combination of Moise 1952; Munkres 1960) to give the 3-handle case.

Proposition 12.9 (3-handle straightening)**.** *We set*

$$H_3 = \{x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3 \text{ and } \xi_3 = \ker(dz - ydx + xdy).$$

For any CC-bilipschitz embedding $\varphi : (H_3, \xi_3) \to (\mathbb{R}^3, \xi_3)$ *which is smooth near* ∂H_3*, there is a CC-bilipschitz embedding* $\psi : (H_3, \xi_3) \to (\mathbb{R}^3, \xi_3)$ *which coincides with* φ *near* ∂H_3 *and is smooth everywhere.*

Giroux's theorem about existence of contact handle decompositions (which is equivalent to existence of supporting open books) guarantees that the handle straightening conjectures are sufficient to smooth any CC-bilipschitz homeomorphism between closed contact 3-manifolds. Actually, using the characterization of contact handlebodies, one can get away with 0-handle straightening and a weak version of 1-handle straightening. Below we give a couple of possible weakening of this conjecture, there are many more.

Conjecture 12.10 (weak 1-handle straightening)**.** *We set*

$$H_1 = \{x^2 + z^2 \leq 1; y^2 \leq 1\} \subset \mathbb{R}^3 \text{ and } \xi_1 = \ker(dz + ydx + 2xdy).$$

For any CC-bilipschitz embedding $\varphi : (H_1, \xi_1) \to (\mathbb{R}^3, \xi_1)$ *which is smooth near* $H_1 \cap \{y = \pm 1\}$*, there is a topological embedding* $\psi : H_1 \to \mathbb{R}^3$ *which coincides with* φ *near* ∂H_1 *and is smooth in* $H_1 \cap \{x^2 + z^2 \leq \varepsilon\}$ *for some positive* ε *and such that the arc* $\gamma = H_1 \cap \{x^2 + z^2 = 0\}$ *is sent by* ψ *to a* ξ_1*-Legendrian curve whose* ξ_1*-framing is homotopic rel end-points to the one defined by* $\psi_* \xi_1$*.*

Alternatively, one can ask that ψ *is smooth in* $H_1 \cap \{x^2 + z^2 \leq \varepsilon\}$ *for some positive* ε *and such that the annulus* $A = H_1 \cap \{x^2 + z^2 = \varepsilon/2\}$ *is sent by* ψ *to a* ξ_1*-convex annulus divided by a pair of arcs isotopic rel end-points to* $\psi(A \cap \{z = 0\})$*.*

Proposition 12.11. *The 0-handle straightening conjecture and any version of the 1-handle straightening conjecture imply that any CC-bilipschitz homeomorphism between closed contact manifolds can be smoothed to a C^0-close contact diffeomorphism (maybe not isotopic to the original homeomorphism through CC-bilipschitz maps though).*

Sketch of proof. This follows from the existence of arbitrarily fine contact handle decompositions and the characterization of contact handlebodies in Key observation 3.17. In order to get C^0-close approximation one must follows the proof of the characterization in order to see that it gives a weak 2-handle straightening and then use 3-handle straightening. □

12.4. From quadri-lipschitz structures to homotopy classes

The conjectures from the preceding section are probably quite hard. One can instead try to prove weaker results. For instance one could hope that a CC-bilipschitz map has to preserve the homotopy class of (unoriented) plane field. Since we know that any homeomorphism in dimension 3 is isotopic to a diffeomorphism unique up to smooth isotopy, a cheap way of making sense of the previous sentence is to asked whether a (non-contact) smoothing of a CC-bilipschitz homeomorphism has to preserve the homotopy class of plane field. One can even put stronger hypothesis and assume that our homeomorphism is quadri-lipschitz: it is bilipschitz both in the Carnot-Carathéodory sense and in the Euclidean sense.

Here we will sketch how a theory inspired by the microbundle theory from Milnor 1964 could prove the above conjecture.

Definition 12.12. *A bundle family over a topological space X is a triple (H, p, s) where $H = (H_\lambda)_{\lambda \in (0,\infty]}$ is a family of topological spaces with $H_\lambda \subset H_{\lambda'}$ when $\lambda \leq \lambda'$, $p : H_\infty \to X$ is called the projection map, $s : X \to \cap H_\lambda$ is called the zero section and $p \circ s = \mathrm{Id}$.*

Two such objects (H, p, s) and (H', p', s') over X are called equivalent if there exist neighborhoods V and V' of $s(X)$ and $s'(X)$, a homeomorphism $\varphi : V \to V'$ and a positive constant C such that the diagram

$$\begin{tikzcd} & V \arrow[dd, "\varphi"] \arrow[dr, "p"] & \\ X \arrow[ur, "s"] \arrow[dr, "s'"'] & & X \\ & V' \arrow[ur, "p'"'] & \end{tikzcd}$$

commutes and:

$$V' \cap H'_{\lambda/C} \subset \varphi(V \cap H_\lambda) \subset V' \cap H'_{C\lambda}.$$

One has an obvious definition of pull-back of a bundle family by a continuous map.

Example 12.13. *To any pair of metrics* d_1 *and* d_2 *on* X *(defining the given topology on* X*), we associate the bundle family defined by:*

$$H_\lambda = \{(x, y) \in X;\ d_1(x, y) \leq \lambda\sqrt{d_2(x, y)}\}$$

with $p(x, y) = x$ *and* $s(x) = (x, x)$*.*

The equivalence class of the bundle family in the above example depends only on the bilipschitz equivalence classes of d_1 and d_2. In particular, a map which is quadri-Lipschitz from (X, d_1, d_2) to (X', d_1', d_2') pulls back H' to a bundle family equivalent to H.

Definition 12.14. *A diffuse contact structure on* X *is a bundle family over* X *which is locally equivalent to one associated to* $(\mathbb{R}^{2n+1}, \mathrm{d}_{\mathrm{CC}}, \mathrm{d}_{\mathrm{euc}})$ *where* d_{CC} *is the Carnot-Caratheodory distance and* $\mathrm{d}_{\mathrm{euc}}$ *is the Euclidean one.*

Example 12.15. *In particular, any contact structure on a closed* $(2n+1)$*-dimensional contact manifold has a canonical equivalence class of diffuse contact structures. Indeed one can choose some sub-Riemannian and Riemannian metrics and the bundle family associated to corresponding distances will be locally equivalent to the model one.*

Definition 12.16. *A microplane field on a topological space* X *is a tuple* (E, F, π, σ) *where* E *and* $F \subset E$ *are topological spaces, the projection* π *is continuous map from* E *to* X*, the zero section* σ *is a continuous map from* X *to* E *such that* $\pi \circ \sigma = \mathrm{Id}$*. Furthermore we require that, for each* x *in* X*, there are neighborhoods* U *of* x *and* V *of* $\sigma(x)$ *with* $\sigma(U) \subset V$ *and* $\pi(V) \subset U$ *and a homeomorphism* φ *from* V *to* $U \times \mathbb{R}^n$ *which sends* $F \cap V$ *to* $\mathbb{R}^{n-1} \times \{0\}$ *and makes the following diagram commute:*

$$\begin{tikzcd}
& V \arrow[dr, "\pi"] \arrow[dd, "\varphi"] & \\
U \arrow[ur, "\sigma"] \arrow[dr, "\mathrm{Id}\times 0"'] & & U \\
& U \times \mathbb{R}^n \arrow[ur, "pr_1"'] &
\end{tikzcd}$$

Two microplane fields (E, F, π, σ) and (E', F', π', σ') over X are equivalent if there are neighborhoods V and V' of $\sigma(X)$ and $\sigma'(X)$ and a homeomorphism φ from $(V, F \cap V)$ to $(V', F' \cap V')$ which is compatible with sections and projections.

Example 12.17. *Let* ξ *be a plane field on a closed smooth manifold* X*. For any connection* ∇ *on* X *there is a neighborhood* W *of the zero section in* TX *such that the exponential map of* ∇ *is a diffeomorphism from* W *to a neighborhood of the diagonal in* $X \times X$*. We can then set* $E = \exp(W)$*,* $F = \exp(\xi \cap W)$*,* $\pi(x, y) = x$ *and* $\sigma(x) = (x, x)$*. The tuple* $\mu_\xi := (E, F, \pi, \sigma)$ *is microplane field on* X *whose equivalence class is independent of the choices of* ∇ *and* W*. We will say that* μ_ξ *is obtained by exponentiating* ξ*.*

Definition 12.18. *A microplane field* (E, F, π, σ) *is compatible with a diffuse contact structure* (H, p, s) *if, for every* λ *smaller then some* λ_0*, there exist nested neighborhoods* V_λ *of* $\sigma(X)$ *and* W_λ *of* $s(X)$ *and a homeomorphism* $\varphi : V_{\lambda_0} \to W_{\lambda_0}$ *compatible with projections and sections and such that*

$$\varphi(V_\lambda \cap F) \subset W_\lambda \cap H_\lambda.$$

Note that the compatibility condition defined above depends only on the equivalence classes of microplane field and diffuse contact structure.

Lemma 12.19. *Let* ξ *be a contact structure on a smooth manifold* X *and* $H\xi$ *a diffuse contact structure on* X *obtained from* ξ *as in Example 12.15. Then any microplane field obtained by exponentiating* ξ *as in Example 12.17 is compatible with* $H\xi$*.*

Conjecture 12.20. *Any two microplane fields compatible with a given diffuse contact structure are equivalent.*

Conjecture 12.21. *There is a classifying space* $\mathrm{BMP}(n)$ *and a universal microplane field* $\mathrm{EMP}(n) \to \mathrm{BMP}(n)$ *such that any microplane field over* X *is equivalent to* $f^* \mathrm{EMP}(n)$ *for some* $f : X \to \mathrm{BMP}(n)$ *. In addition there is a map from* $\mathrm{BMP}(n)$ *to* $\mathrm{BTop}(n)$ *such that, if a microplane field is contained in the tangent microbundle of a topological manifold* X *then there is a commutative diagram*

$$\begin{tikzcd} & \mathrm{BMP}(n) \arrow[d] \\ X \arrow[ur] \arrow[r] & \mathrm{BTop}(n) \end{tikzcd}$$

producing the relevant equivalence classes of objects.

Conjecture 12.22. *There are fibrations* $\mathrm{B}\Xi(n) \to \mathrm{BMP}(n)$ *such that*

$$\begin{tikzcd} \mathrm{B}\Xi(n) \arrow[r, "\pi"] \arrow[d] & \mathrm{BMP}(n) \arrow[d] \\ \mathrm{BO}(n) \arrow[r] & \mathrm{BTop}(n) \end{tikzcd}$$

is commutative up to homotopy.

If μ_ξ *is a microplane field obtained by exponentiating a plane field* ξ *then the corresponding classifying maps satisfies:* $B\mu_\xi \sim \pi \circ B\xi$ *among lifts of* $X \to \mathrm{BTop}(n)$*.*

Conjecture 12.23. *If* $n = 3$*, both horizontal maps in Conjecture 12.22 are homotopy equivalences in a compatible way.*

We can now put all pieces together assuming the above conjectures.

Let $H\xi$ and $H\xi'$ be diffuse contact structures associated to contact structures ξ and ξ' as in Example 12.15. Let μ_ξ and $\mu_{\xi'}$ be microplane fields obtained by exponentiating

ξ and ξ'. Lemma 12.19 ensures that $H\xi$ and $H\xi'$ are compatible with μ_ξ and $\mu_{\xi'}$ respectively.

Assume that φ is a quadri-Lipschitz homeomorphism from (X, ξ) to (X, ξ'). By construction, $H\xi$ and $\varphi^*H\xi'$ are equivalent. Conjecture 12.20 ensures that $\varphi^*\mu_{\xi'}$ is equivalent to μ_ξ. By Conjecture 12.21 it means the classifying maps $B\varphi^*\mu_{\xi'} = B\mu_{\xi'} \circ \varphi$ and $B\mu_\xi$ are homotopic. Since μ_ξ and $\mu_{\xi'}$ are obtained by exponentiation, Conjecture 12.22 gives $\pi \circ B\xi' \circ \varphi \sim \pi \circ B\xi$ where π is the fibration from $\mathrm{B}\Xi(2)$ to $\mathrm{BMP}(2)$. By Conjecture 12.23 so we get that $B\xi' \circ \varphi \sim B\xi$ (among lifts of BTX). Since φ is isotopic to some diffeomorphism ψ, we have $B\xi' \circ \psi \sim B\xi$ (among lifts of BTX) and $\psi_*\xi$ is homotopic to ξ' as unoriented plane fields.

12.5. Limits in C^0-topology

A less radical approach to contact homeomorphisms and topological contact manifolds is offered by the Eliashberg-Gromov theorem guaranteeing that a diffeomorphism which is the C^0-limit of a sequence of contact diffeomorphisms is contact, see Section 5.3.

Before trying to work with this definition, a test case is the following conjecture whose topological version (without contact structure) follows from Cerf 1961, 1968.

Conjecture 12.24 (Local connectedness)**.** *Let (M, ξ) be a closed contact 3–manifold. The contactomorphism group $\mathcal{D}(M; \xi)$ equipped with the C^0 topology is locally arcwise connected.*

Again the hope is to use contact handle decompositions to prove the conjecture. For instance, the next lemma follows from Eliashberg's uniqueness of tight contact structures on balls.

Lemma 12.25. *Let B^+ be a Darboux ball inside (M, ξ) and $B \Subset B^+$ a smaller Darboux ball. For every φ in $\mathcal{D}(M; \xi)$ such that $\varphi(B) \subset B^+$, there exists a ψ in $\mathcal{D}_c(B^+; \xi)$ such that $\psi \circ \varphi$ is the identity on a neighborhood of B.*

Similar lemmas for higher index contact handles follow from the semi-local Bennequin inequality and Colin's disk theorem (see Section 3.2.4). A version of these lemmas where the C^0 norm of ψ is controlled would readily imply Conjecture 12.24. More specifically, the following would be enough for 0-handles and gives the general flavor:

Conjecture 12.26. *Let ξ be the standard contact structure on $V = \mathbb{S}^2 \times \mathbb{R}$ and let $j_0 : \mathbb{S}^2 \to V$ be the inclusion $x \mapsto (x, 0)$. For every positive ε and every $j : \mathbb{S}^2 \to V$ which induces the same characteristic foliation as j_0 on $\mathbb{S}^2$, and is ε-close to j_0 in C^0 distance, there is a compactly supported contact transformation φ such that $\varphi \circ j = j_0$ and φ is 100ε-close to Id in C^0 distance.*

Note that the PL analogue of this conjecture occupies roughly 15 pages in Moise 1952, which Cerf 1961 refers to in the smooth case: “La démonstration de ce résultat est assez longue, et comme elle suit pas à pas celle du lemme 4 de Moise 1952, nous ne la donnerons pas ici.”.

In the contact setting, even the case of $j : x \mapsto (x, \varepsilon)$ is non-obvious since no map $(x,t) \mapsto (x, \rho(t))$ will be contact and solve the problem.

Assuming that one can prove Conjecture 12.24, the next step (which shouldn't be much harder) would be to prove that the definition of contact homeomorphisms in terms of C^0 limits is local:

Conjecture 12.27 (From local to global)**.** *Let (M, ξ) be a closed contact 3–manifold. Any homeomorphism of M which is locally a C^0-limit of smooth contact embeddings is globally a C^0-limit of contact diffeomorphisms.*

Once contact homeomorphisms are well understood (whatever the definition), the next step is to use them to define C^0 contact manifolds as topological manifolds equipped with an atlas whose transition functions are contact homeomorphisms. Again, in dimension 3, we expect to see nothing new here, confirming that contact structures on 3-manifolds are very much topological objects. The following is one way this could be stated precisely.

Conjecture 12.28 (Smoothing)**.** *Let M be a closed topological manifold and $\boldsymbol{\mathcal{A}}$ an atlas for M whose transition maps are C^0-limits of smooth contact embeddings of $(\mathbb{R}^3, \xi_0)$.*

Let $\mathrm{Homeo}(M; \boldsymbol{\mathcal{A}})$ be the space of homeomorphisms φ such that, for every x in M, there are charts $\chi_1 : U_1 \hookrightarrow \mathbb{R}^3$, $\chi_2 : U_2 \hookrightarrow \mathbb{R}^3$, and an open set $U \subset U_1$ such that x is in U, $\varphi(U) \subset U_2$, and $\chi_2 \circ \varphi \circ \chi_1^{-1} : \chi_1(U) \to \chi_2 \circ \varphi(U)$ is a C^0-limit of smooth contact embeddings.

Then $\mathrm{Homeo}(M; \boldsymbol{\mathcal{A}})$ is a locally connected subgroup of $\mathrm{Homeo}(M)$, and there is a smooth structure Σ on M and a contact structure ξ on M_Σ such that $\mathcal{D}(M_\Sigma; \xi)$ is dense in $\mathrm{Homeo}(M; \boldsymbol{\mathcal{A}})$.

The pair (Σ, ξ) is unique up to isotopy: if Σ' and ξ' also lead to a dense $\mathcal{D}(M_{\Sigma'}; \xi')$ in $\mathrm{Homeo}(M; \boldsymbol{\mathcal{A}})$ then there is a homeomorphism ψ topologically isotopic to Id_M which is a diffeomorphism from M_Σ to $M_{\Sigma'}$ and $\psi_* \xi = \xi'$.

Autors index

Subject index

Personal publications

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