Two remarks on the support genus question

Patrick Massot

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Abstract

I first recall the mostly undocumented link between convex contact structures and open book decompositions. Then I use this to explain why, for any surface of genus $g \geq 1$, the canonical contact structure on the bundle of cooriented lines ("unit cotangent bundle") has support genus one. Then I explain why I think the canonical contact structure on a circle bundle with Euler number -1 over a genus g surface is a better candidate for support genus g.

1 Convex contact structures and open books

Recall from [EG91] that a contact structure ξ is called convex if it is invariant under the flow of a gradient X of a Morse function f. We will call (f, X) a ξ-convex Morse function for brevity. Giroux proved that a ξ-convex Morse function can always be ordered without loosing the relation to the contact structure. Also recall from [Gir91] that the characteristic hypersurface of a contact vector field X is the set Σ_X where X belongs to the contact structure. If X has non-degenerate singularities, e.g. it is a gradient of a Morse function, then Σ_X is a smooth hypersurface tangent to X [Gir91, Proposition 2.5 and Exemple 2.6].

If (f, X) is a ξ-convex Morse function, the critical points of f are exactly critical points of $f_{|\Sigma_X}$ and an index i critical point of f gives a critical point of $f_{|\Sigma_X}$ whose index is i if $i \leq n$ or $i - 1$ otherwise [Gir91, Proposition 4.5].

Proposition 1 (Giroux)**.** *Let ξ be a coorientable contact structure on a closed $2n+1$–manifold M. Suppose (f, X) is an ordered ξ-convex function. Let Σ be a regular level set of f above critical values of index n and below critical values of index $n+1$. Then Σ is transverse to Σ_X and their intersection K is the binding of an open book on M. This binding cuts Σ and Σ_X into four pages of the open book, see Figure 1. This open book supports ξ (maybe only after a perturbation near the binding).*

Conversely, any supporting open book comes from this construction.

2 Open books on bundles of contact elements

2.1 General discussion

Let B be a closed $(n + 1)$-manifold. Let V be the bundle of cooriented hyperplanes tangent to B and ξ its canonical contact structure. We now follow

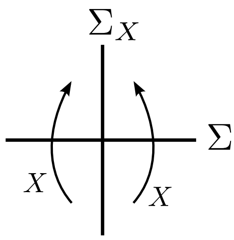

Figure 1: The four pages picture

[Gir91, Exemple 4.8] to construct ξ-convex functions on V (Giroux works on non-cooriented hyperplanes bundles but this is the same up to a two-fold cover).

Any diffeomorphism φ of B lifts to a contactomorphism $\hat{\varphi}$ of V defined by $\hat{\varphi}(x, H) = (\varphi(x), D_x\varphi(H))$. Any vector field X on B lifts to a contact vector field on V. Indeed, if φ_t is the flow of X then one can define $\hat{X} = \frac{d}{dt}_{|t=0}\hat{\varphi}_t$.

Suppose now that $f_0 : B \to \mathbb{R}$ is a Morse function on B. One can choose a gradient vector field X for f_0 such that, for any critical point p of f_0, the eigenvalues of D_pX are real and simple. Let $\hat{X}$ be the lifted contact vector field. We want to prove that it is a pseudo-gradient for some Morse function f on V.

First note that $\hat{X}$ projects to X so singularities of X are above critical points of f_0. Let p be such a critical point. Since the flow $\hat{\varphi}_t$ of $\hat{X}$ projects to the flow φ_t of X and $\varphi_t(p) = p$, we get that $\hat{X}$ is vertical above p, ie belongs to $\mathrm{Ver}_p V := \ker \pi_*$ where π is the projection from V to B. The fact that eigenvalues of X are real and simple prove that the restriction of $\hat{X}$ to the sphere V_p is the gradient of some self-indexed Morse function $g_p : V_p \to \mathbb{R}$ having exactly $2n + 2$ critical points corresponding to eigendirections of D_pX. The function we want on V is then $f = f_0 + \sum_{p \in \mathrm{Crit}(f)} \chi_p g_p$ where χ_p is a cut-off function. This is all explained in [Gir91, Exemple 4.8]. We now want to check explicitely that everything works, compute the indices of critical points and explain why f can be made self-indexed for free if f_0 is and $\dim V = 3$.

Around each $p \in \mathrm{Crit}_i(f_0)$, we choose a Morse chart where

$$f_0(x) = i - \sum_{k=0}^{i-1} x_k^2 + \sum_{l=i}^{n} x_l^2.$$

Suppose we have a sequence of real eigenvalues $\lambda_0 < \cdots < \lambda_{i-1} < 0 < \lambda_i < \cdots < \lambda_n$ and set

$$X = \sum_{q=0}^{n} \lambda_q x_q \frac{\partial}{\partial x_q}.$$

The vector field X is a gradient for f whose linearisation at p has eigenvalues λ_q. In this chart, the bundle V becomes trivial with fiber $\mathbb{R}^{n+1}/\mathbb{R}_{>0}$ where $\mathbb{R}^{n+1}$ with coordinate $(y_0, \ldots, y_n)$ is the dual of the base $\mathbb{R}^{n+1}$. The flow of X is $\varphi_t : (x_q) \mapsto (e^{\lambda_q t} x_q)$. The lifts to V maps (y_q) to $(e^{-\lambda_q t} y_q)$. So $\hat{X}$ is the sum of X and the projection X' to $\mathbb{R}^{n+1}/\mathbb{R}_{>0}$ of

$$\tilde{X}' := -\sum_{q=0}^{n} \lambda_q y_q \frac{\partial}{\partial y_q}.$$

Note that X' is indeed invariant under homothety $y \mapsto \mu y$, $\mu \in \mathbb{R}_{>0}$. We can now define $\tilde{g}$ on $\mathbb{R}^{n+1}$ which will project to g (we forget the subscript p since p is fixed in this discussion).

$$\tilde{g} = -\frac{1}{\|y\|^2} \sum_q \lambda_q y_q^2$$

We need to check that X' is a pseudo-gradient for g. We consider the unit sphere $S = \{\sum y_q^2 = 1\}$, the orthogonal projection $\pi : T\mathbb{R}^{n+1}_{|S} \to TS$ and the restriction g_S of $\tilde{g}$ to S. We can now compute

$$\begin{aligned} dg(X') &= dg_S(\pi(\tilde{X}')) \\ &= -2 \sum_q \lambda_q y_q^2 \left(\sum_r (\lambda_r y_r^2) - \lambda_q \right) \\ &= -2 \sum_{q,r} \lambda_q y_q^2 y_r^2 (\lambda_r - \lambda_q) \\ &= -2 \sum_{q<r} (\lambda_q - \lambda_r) y_q^2 y_r^2 (\lambda_r - \lambda_q) \\ &= 2 \sum_{q<r} y_q^2 y_r^2 (\lambda_r - \lambda_q)^2 \end{aligned}$$

Now we can use that $\lambda_r \neq \lambda_q$ when $r \neq q$ to see that the above is positive unless exactly one y_q is non-zero (remember $y = 0$ is not considered here). It's easy to see that critical points of g are indeed the projections of the points of $v_q^\pm := (0, \dots, 0, \pm 1, 0, \dots, 0)$ (intersections of S with coordinate axes) and they are non-degenerate. In order to understand the index of $v_q^\pm$, we remark that $T_{v_q^\pm} S$ is spanned by lines coming from planes in $\mathbb{R}^{n+1}$ where all coordinates are zero except y_q and one other y_r. In this direction, $v_q^\pm$ is attractive (resp. repulsive) if and only if $-\lambda_q > -\lambda_r$ (resp. $-\lambda_q < -\lambda_r$). So the number of repulsive directions is $\#\{r\ ; \lambda_r < \lambda_q\} = q$. So the index of $v_q^\pm$ is $n - q$ (remember the index is the dimension of the unstable manifold of the *descending* gradient so here we count attractive directions).

We now come to the cut-off functions. Suppose our Morse chart has radius $2\sqrt{\varepsilon}$. Let ρ be a cut-off function on $\mathbb{R}_+$ with value 1 on a neighborhood of the origin, support in the interior of $[0, 4\varepsilon]$ and derivative $\rho'(t) \geq -1/\varepsilon$. We will use the cut-off function $\chi(x) = \rho(\|x\|^2)$. The requiered Morse function on V is defined near p by

$$f(x, y) = f_0(x) + \eta\chi(x)g(x).$$

By construction, we have $2(n+1)$-critical points in the fiber over p. We need to check that χ does not introduce any extra critical point. The danger comes from

$$\frac{\partial f}{\partial x_k} = -2x_k \left(1 + \eta\rho'(\|x\|^2)g(y)\right)$$

(where $k < i$ as above) so that this derivative could accidentaly vanish outside p. But

$$1 + \eta\rho'(\|x\|^2)g(y) \geq 1 - \frac{\eta}{\varepsilon} \max g$$

so we can choose η small enough to avoid this problem. Note however that this trick would prevent us from getting a self-indexed function in high dimensions without first tweaking f_0. In dimension 3, a miracle will help anyway.

2.2 Hyperbolic surfaces

We now restrict the general discussion to the base where the base B is a genus g surface. Let f_0 be a self-indexed Morse on B having $c_i(f_0)$ critical points of index i, $i \in \{0, 1, 2\}$. The lifted function f described above has $c_i(f)$ critical points of index i where

$$
\begin{aligned}
c_0(f) &= 2c_0(f_0) & c_1(f) &= 2c_0(f_0) + 2c_1(f_0) \\
c_2(f) &= 2c_1(f_0) + 2c_2(f_0) & c_3(f) &= 2c_2(f_0)
\end{aligned}
$$

The restriction f_Σ of f to $\Sigma_{\hat{X}}$ has

$$
\begin{aligned}
c_0(f_\Sigma) &= c_0(f) = 2c_0(f_0) \\
c_1(f_\Sigma) &= c_1(f) + c_2(f) = 2c_0(f_0) + 4c_1(f_0) + 2c_2(f_0) \\
c_2(f_\Sigma) &= c_3(f_0) = 2c_2(f_0)
\end{aligned}
$$

So the Euler characteristic of $\Sigma_{\hat{X}}$ is $\chi(\Sigma_{\hat{X}}) = -4c_1(f_0)$. In addition the Morse function f is ordered if η is chosen small enough in absolute value. Indeed, critical points of index 0 will have slightly negative values, index 1 will have values either slightly positive or slightly less than 1 (depending whether they live above index 0 or 1 critical points of f_0), index 2 will have values either slightly more than 1 or slightly less than 2 and index 3 will have values slightly more than 2. So $\Sigma_{\hat{X}}$ is the double of a page P whose boundary K is $\Sigma_{\hat{X}} \cap f^{-1}(1)$. This boundary is the binding of open book supporting the canonical contact structure on V so we now want to understand its number of connected component. This will give the genus of P since we know $\chi(P) = \chi(\Sigma_{\hat{X}})/2$.

Note that $\Sigma_{\hat{X}}$ is the conormal of X_0, the set of cooriented lines containing X_0. We now check that the binding K is a trivial 2-fold cover of the critical level $f_0^{-1}(1)$. This critical level is a union of smooth circles $C_1, \dots, C_n$ intersecting transversely at critical points. Away from critical points, $f^{-1}(1)$ is the inverse image of f_0^{-1} and $\Sigma_{\hat{X}}$ corresponds to lines containing X_0, which does not vanish, hence we have two points in each fiber over a non-critical point of f_0^{-1}. In addition, the projection of $f^{-1}(1)$ clearly doesn't intersect Morse charts of critical points of index 0 or 2 of f_0. So we only need to understand what happens above a Morse chart centered around a critical point of index 1 for f_0. Say we have chosen eigenvalues $\lambda_0 = -1$ and $\lambda_1 = 1$. So

$$f(x, y) = 1 - x_0^2 + x_1^2 + \eta\chi(x)(y_0^2 - y_1^2).$$

We can parametrize the fiber by an angle θ so that $y_0 = \cos(\theta)$ and $y_1 = \sin(\theta)$.

$$
\begin{aligned}
f(x, y) &= 1 - x_0^2 + x_1^2 + \eta\chi(x)\cos(2\theta) \\
\hat{X} &= -x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} - \sin(2\theta)\frac{\partial}{\partial \theta} \\
\xi &= \ker\big(\cos(\theta)dx_0 - \sin(\theta)dx_1\big).
\end{aligned}
$$

One can see in this explicit model that

$$K = \begin{cases} x_0^2 - x_1^2 = 0 \\ \cos(2\theta) = 0 \end{cases}$$

which glues smoothly with the previous description. The 2-fold cover is trivial because one can use the coorientation of a circle C_m to understand points of K living over it. Hence we get $2n$ binding components where n is the number of circles we have in $f_0^{-1}(1)$. So $\chi(P) = 2 - 2g(P) - 2n$ and the genus of the page we constructed is:

$$g(P) = 1 + c_1(f_0) - n. \tag{1}$$

It's time for a concrete example which, if I understand correctly, comes from [Bir17], see the discussion in [Deh12] to see how many people discussed this example. Embed B as usual in $\mathbb{R}^3$ so that it intersects the y-axis in $2g + 2$ points, see Figure 2. The intersection with the plane $\{x = 0\}$ gives you $g + 1$ vertical circles. The intersection with the plane $\{z = 0\}$ gives you $g+1$ horizontal circles, one of them being much larger than the other. The complement of those

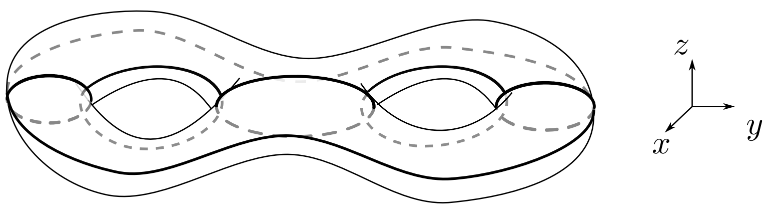

Figure 2: Morse function for genus 1 open books

$2(g + 1)$ circles has four connected components parametrized by the signs of x and z. There is a self-indexed Morse function f_0 on B whose critical level $f_0^{-1}(1)$ is the union of those $2(g + 1)$ circles. It has

$$c_0(f_0) = 2, \quad c_1(f_0) = 2(g + 1), \quad c_2(f_0) = 2.$$

Hence we get an open book supporting the canonical contact structure on the bundle of cooriented contact elements with $4(g + 1)$ binding components and Equation (1) proves that is has genus 1.

3 A candidate for high support genus

Let $\pi : V \to B$ be a circle bundle with Euler number -1 over a surface of genus g. Let ξ be a contact structure on V in the canonical isotopy class. It means ξ has a Reeb field R which generates a free circle action on V.

Since the Euler number is -1, there is a section of π over the complement of a single point k_0 whose closure has boundary $K_0 := \pi^{-1}(k_0)$. The $\mathbb{S}^1$-action on this section gives a supporting open book (K_0, θ_0). The monodromy is a right-handed Dehn twist τ along the boundary. Note that I used the general procedure to find supporting open books for Boothby-Wang contact structures: $\{k_0\}$ is the relevant Donaldson hypersurface in B and τ is the relevant fibered Dehn twist (sorry for being pedantic).

It is very easy to understand all open books compatible with R. Indeed the binding K is a collection of Reeb orbits hence a collection of fibers. Pages are transverse to R hence π restricts to pages as a covering map onto the complement in B of $\pi(K)$. Using multiplicativity of the Euler characteristic, it is very easy to see that our favorite open book (K_0, θ_0) minimizes genus among open books compatible with R.

Now what about open books compatible with ξ? My remark is that, the appendix of [BDT06] proves that the genus $2g$ Heegaard splitting associated to (K_0, θ_0) minimizes genus among all Heegaard splitting. Better, any genus $2g$ Heegaard splitting for V is isotopic to that one.

Does it help? Well, it certainly proves that (K_0, θ_0) is unstabilized. But you guys proved that it's not enough. Of course we also know that, if there is a better supporting open book then its Heegaard splitting could be stabilized as a Heegaard splitting. We also know from $\mathbb{S}^3$ that you can have non-isotopic contact Heegaard splittings which are isotopic as Heegaard splittings. So I really don't know but I would find it surprising if (K_0, θ_0) is not genus minimizing. Any idea?

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