Natural fibrations in contact topology

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Abstract

This expository note is an expanded version of the first Section of [GM15] about fibrations in contact topology.

1 The Cerf-Palais fibration criterion

The following basic criterion will be convenient to prove that maps are fibrations by constructing local sections of group actions. It appeared independently in [Cer61, Lemma 2 of Section 0.4.4 p. 240] and [Pal60, Theorem A], see also [Cer68, Lemma 1 in the appendix].

Lemma 1.1 (Cerf-Palais fibration criterion)**.** *Suppose G is a topological group acting continuously on two topological spaces E and B. A continuous map $p : E \to B$ is a fibration as soon as it is G-equivariant and, for every x in B, the map $g \mapsto gx$ has a continuous section near x: there is an open set U containing x and a map $s\colon (U, x) \to (G, e)$ such that $s(x')x = x'$ for all x' in U.*

Proof. A local trivialization is given by $(x', v) \mapsto s(x')v$ from $U \times p^{-1}(x)$ to $p^{-1}(U)$. □

Remark 1.2. In the preceding lemma, asking that $s(x) = e$ does not cost anything. If s is a any section then one can replace it by $\bar{s}\colon x' \mapsto s(x')s(x)^{-1}$. All sections we will construct automatically enjoy this property and we won't comment further on it.

2 Contact structures and contact transformations

For any topological group G, we denote by G_o the neutral component of G. If G acts on a space X and x is in X then the stabilizer of x is denoted by G_x.

Until the end of this section, we fix a compact oriented manifold V and denote by $\mathcal{D}$ the group of diffeomorphism of V which restrict to the identity on a neighborhood of the (maybe empty) boundary of V, endowed with the compact-open topology. This group is homotopically equivalent to the group of diffeomorphisms relative to the boundary and will be technically more convenient.

We denote by $\mathcal{G}$ the group of isotopies of V. Its elements are paths $(\varphi_t)_{t\in[0,1]}$ in $\mathcal{D}$ starting at Id. Those paths are required to be smooth in the sense that the associated map of $V \times [0,1]$ defined by $(x,t) \mapsto (\varphi_t(x), t)$ is a diffeomorphism[1]. The space $\mathcal{G}$ will be endowed with the topology and group structure induced by $\mathrm{Diff}(V \times [0,1])$. In particular $(\varphi\varphi')_t = \varphi_t \circ \varphi'_t$.

We fix a reference positive cooriented contact structure ξ_0 on V. We denote by Ξ the space of positive cooriented contact structures on V which coincide with ξ_0 near the boundary of V. It is equipped with the quotient topology coming from the space of 1-forms on V.

Let F be a compact orientable surface properly embedded in V. The characteristic foliation $\xi_0 F$ printed by ξ_0 on F is the singular foliation tangent to $\xi_0 \cap TF$. More precisely, a (cooriented) singular foliation is an equivalence class of 1-forms modulo multiplication by positive functions. The foliation $\xi_0 F$ is the equivalence class of the restriction to F of any contact form for ξ_0.

The stability of contact structures [Gra59] give rise to several natural fibrations in contact geometry. Beware that none of those fibrations are surjective in general. The stability statement precisely give local sections of group actions needed to apply Lemma 1.1.

Lemma 2.1 (Stability lemma)**.** *The push forward action of $\mathcal{G}$ on Ξ has local sections. More precisely:*

1. *For any ξ in Ξ, there is a neighborhood U_Ξ of ξ and a map $s\colon U_\Xi \to \mathcal{G}$ such that $s(\xi) = e$ the identity isotopy and*

$$\forall \xi' \in U_\Xi, \quad s_1(\xi')_* \xi = \xi'.$$

2. *If F is a surface properly embedded in V and ι the inclusion map, then one can construct s such that, in addition:*

$$\xi' F = \xi F \implies \forall t \in [0,1],\ s_t(\xi')_{|F} = \iota.$$

Proof. Using an auxiliary Riemannian metric, one can smoothly assign a contact form α_ξ to any ξ in Ξ. Because the contact condition is open in the space of 1-forms, any ξ has a neighborhood U_Ξ in Ξ made of contact structures ξ' such that, for any $t \in [0,1]$, $\alpha_t = (1-t)\alpha_\xi + t\alpha_{\xi'}$ is a contact form. The proof of Gray's theorem using Moser's method then gives the desired local section $U_\Xi \to \mathcal{G}$. More explicitly, $s(\xi')$ is the flow of the unique time-dependant vector field X_t such that

- $X_t \in \ker \alpha_t$
- $\iota_{X_t}(d\alpha_t)_{| \ker \alpha_t} = -\dot{\alpha}_{| \ker \alpha_t}$

If a surface F is fixed, the construction can be modified as follows. First one picks up contact forms such that $\iota^*\alpha_{\xi'} = \iota^*\alpha_\xi$ whenever $\xi' F = \xi F$ and ξ' is sufficiently close to ξ. Then we fix a tubular neighborhood $F \times \mathbb{R}$ and a cut-off function $\rho : \mathbb{R} \to [0,1]$ which equals one near the origin and vanishes outside some compact set. We denote by s the real coordinate in this neighborhood.

[1] Of course $V \times [0,1]$ has corners if V has non-empty boundary, see [Cer61] for a description of the topology of manifolds with corners and their diffeomorphisms groups.

Given a contact form $\alpha_{\xi'}$ we construct α_t as above and we set $H = -s\rho(s)\dot{\alpha}(\partial_s)$. Since all forms α_t are contact, there is a unique time dependant vector field $Y_t \in \ker \alpha_t$ such that

$$\iota_{Y_t}(d\alpha_t)_{|\ker \alpha_t} = -(dH + \dot{\alpha})_{|\ker \alpha_t}.$$

We then set $X_t = HR_t + Y_t$ where R_t is the Reeb vector field of α_t. This vector field vanishes as soon as $\dot{\alpha}$ does so it vanishes near ∂V. It generates an isotopy pushing ξ to ξ'. In addition, H vanishes along F and, for all ξ' such that $\xi' F = \xi F$, $dH + \dot{\alpha}$ also vanishes because $\iota^* \dot{\alpha} = 0$. So X_t vanishes along F and $s_t(\xi')_{|F} = \iota$. □

In view of the fibration criterion of Lemma 1.1, an immediate corollary of the preceding lemma is the following fibration result.

Corollary 2.2. *The map $\mathcal{D} \to \Xi$ sending ϕ to $\phi_* \xi_0$ is a fibration.*

The usual statement of Gray stability (with any number of parameters) follows from the this fibration statement together with the homotopy lifting property of fibrations. A priori the fibration statement is much stronger but we saw that its proof costs exactly the same modulo the elementary fibration criterion.

Moving our attention from contact structures to contact transformations, we can also use the preceding lemma to prove that $(\mathcal{D}, \mathcal{D}_{\xi_0})$ is a good pair of topological spaces.

Corollary 2.3. *There is a neighborhood $U_\mathcal{D}$ of the group of contactomorphisms $\mathcal{D}_{\xi_0}$ inside $\mathcal{D}$ and a deformation retraction $r\colon U_\mathcal{D} \to \mathcal{D}_{\xi_0}$.*

Given a surface F properly embedded in V, one can construct r such that, in addition, if $\varphi \in U_\mathcal{D}$ satisfies $(\varphi^* \xi_0)F = \xi_0 F$ then, for all $t \in [0,1]$, $r_t(\varphi)_{|F} = \varphi_{|F}$.

Proof. Let U_Ξ be a neighborhood of ξ_0 on which the stability lemma constructs a map s to $\mathcal{G}$. Define $U_\mathcal{D}$ as the inverse image of U_Ξ under the pull-back map $\varphi \mapsto \varphi^* \xi_0$. For any φ in $U_\mathcal{D}$, we set $r_t(\varphi) = \varphi \circ s_t(\varphi^* \xi_0)$. In particular $r_0(\varphi) = \varphi$ and $r_1(\varphi)_* \xi_0 = \varphi_* s_1(\varphi^* \xi_0)_* \xi_0 = \varphi_* \varphi^* \xi_0 = \xi_0$. The property relative to surfaces comes directly from the corresponding property of s. □

3 Surfaces in contact manifolds

Suppose F is a surface properly embedded in V. We denote by $\mathcal{P}(F)$ the space of proper embeddings of F in V which coincide with the inclusion near ∂F. Note that elements of $\mathcal{P}(F)$ are therefore *parametrized* copies of F in V. We denote by $\mathcal{P}_0(F)$ the connected component of the inclusion in $\mathcal{P}(F)$.

There is a natural map $\mathcal{D} \to \mathcal{P}(F)$ given by composing the inclusion with any diffeomorphism. Palais [Pal60] and Cerf in a more general setup [Cer61] proved independently that this map is a fibration. They used the fibration criterion with the isotopy group $\mathcal{G}$. For the sake of completeness, we sketch the construction of local sections of the $\mathcal{G}$-action. Fix some metric on V and a reference embedding $j_0 \in \mathcal{P}(F)$. If j is close enough to j_0 then, for all x in F, there is a unique minimizing geodesic from $j_0(x)$ to $j(x)$. Hence there is a unique section $X(j)$ of $TV_{|j_0(F)}$ such that $j(x) = \exp_{j_0(x)}(X(j))$. Using a tubular neighborhood of $j_0(F)$ one can continuously extend $X(j)$ to a smooth vector field on V still small enough to exponentiate to a diffeomorphism (recall that $\mathcal{D}$ is open in the space of smooth maps from V to V). The isotopy pushing j_0 on j is then $t \mapsto \exp(tX(j))$.

Returning to contact geometry, we denote by $\mathcal{P}(F, \xi_0)$ the subset of embeddings j endowing F with the same characteristic foliation as inclusion: $\xi_0 j(F) = j_* \xi_0 F$. We set $\mathcal{P}_\circ(F, \xi_0) = \mathcal{P}_\circ(F) \cap \mathcal{P}(F, \xi_0)$. Combining Palais-Cerf sections with the retraction of Corollary 2.3, we get a new fibration.

Corollary 3.1. *If F is a surface properly embedded in V then the map $\mathcal{D}_{\xi_0} \to \mathcal{P}(F, \xi_0)$ composing the inclusion of F with a contactomorphism relative to a neighborhood of ∂V is a fibration.*

Proof. We will use the fibration criterion with $G = \mathcal{D}_{\xi_0}$ acting by left composition on both spaces. So we fix j_0 in $\mathcal{P}(F, \xi_0)$ and seek a local section of the map from $\mathcal{D}_{\xi_0}$ to $\mathcal{P}(F, \xi_0)$ defined by $\varphi \mapsto \varphi \circ j_0$. Let s be a section of the action of $\mathcal{D}$ on $\mathcal{P}(F)$ near j_0. Let $r \colon U_{\mathcal{D}} \to \mathcal{D}_{\xi_0}$ be a retraction given by Corollary 2.3 with respect to the surface $j_0(F)$. We set $U_P = s^{-1}(U_{\mathcal{D}}) \cap \mathcal{P}(F, \xi_0)$ and define $\bar{s} \colon U_P \to \mathcal{D}_{\xi_0}$ as $r \circ s$.

For any $j \in U_P$, we have $s(j) \circ j_0 = j$ and, because both j_0 and j endow F with the same characteristic foliation as the inclusion, ξ_0 and $s(j)^* \xi_0$ print the same characteristic foliation on $j_0(F)$. Hence $r(s(j))_{|j_0(F)} = s(j)_{|j_0(F)}$. So $r(s(j)) \circ j_0 = j$ and the map $\bar{s} = r \circ s$ is a section of $\mathcal{D}_{\xi_0} \to \mathcal{P}(F, \xi_0)$ on U_P. □

Remark 3.2. The above corollary is one place where it's more convenient to work relative to a neighborhood of the boundary and not only relative to the boundary. Indeed any diffeomorphism which is relative to ∂V and to some properly embedded surface F is tangent to identity along ∂F so we wouldn't have a fibration in this setting.

However, this technical point has no impact on our study of the topology of contactomorphism groups because the inclusion of the space of contactomorphisms relative to a neighborhood of ∂V into the space of those relative to ∂V is a homotopy equivalence. So we are free to restrict our attention to the most convenient class.

The path lifting property of the fibration of Corollary 3.1 allows to convert isotopies of embeddings with "constant" characteristic foliations into contact isotopies. However, it is rather hard to control directly the characteristic foliation. In addition, the map $p_{\mathscr{F}} \colon \mathcal{P}(F) \to \mathscr{F}(F)$ which sends an embedding j to the foliation $j^{-1}(\xi_0 j(F))$ on F is not a fibration. Although it has local sections defined near $\xi_0 F$ (which send $\xi_0 F$ to the inclusion), one cannot hope to lift long paths in general. For instance, in the standard sphere $\mathbb{S}^3$, let F be an unknotted prelagrangian torus. The space of linear foliations on a torus is connected yet the Bennequin inequality forbids the existence of embeddings realizing linear foliations whose direction is meridian viewed from either one of the solid tori bounded by F.

We will now explain how the theory of ξ_0-convex surfaces introduced in [Gir91] still gives a way to understand and control characteristic foliations. From now on, we assume that F has empty or Legendrian boundary. We say that F is ξ_0-convex whenever there is a contact vector field X which is transverse to

F and tangent to ∂V. In that case there is a unique isotopy class of properly embedded one-dimensional submanifolds Γ which *divide* $\xi_0 F$. A foliation σ is divided by Γ if it is transverse to Γ and, on each connected component of $F \setminus \Gamma$, there is a vector field directing σ which either dilates an area form and points outward along Γ or contracts an area form and points inward. To any contact vector field X as above, on can associate the curve Γ_X where X belongs to ξ_0 along F. The space of curves dividing a given foliation is contractible (if non-empty). Among those, the curves coming from a contact vector field as X are exactly those intersecting ∂F at the points where ξ_0 is tangent to ∂V.

Proposition 3.3 ([Gir91, Gir01])**.** *Let F be a ξ_0-convex surface and X a contact vector field transverse to F and tangent to ∂V. Let U be the image of F under the flow of X. Let Γ_X be the dividing set defined by X on F and denote by $\mathscr{F}(F, \Gamma_X)$ the space of foliations on F which are divided by Γ_X and tangent to ∂F. Denote by $\mathcal{P}(F, \Gamma_X)$ the preimage of $\mathscr{F}(F, \Gamma_X)$ under $p_{\mathscr{F}}$. One has the following properties:*

1. *$\mathscr{F}(F, \Gamma_X)$ is an open contractible neighborhood of $\xi_0 F$ in $\mathscr{F}(F)$*
2. *There is section $\sigma \mapsto \psi_\sigma$ of $p_{\mathscr{F}}$ defined on $\mathscr{F}(F, \Gamma_X)$ with the additional properties:*
   - *(a) $\psi_{\xi_0 F}$ is the inclusion of F into V;*
   - *(b) for any σ, the surface $\psi_\sigma(F)$ is contained in U and transverse to X.*
3. *The inclusion of $\mathcal{P}(F, \xi_0)$ into $\mathcal{P}(F, \Gamma_X)$ is a homotopy equivalence.*

The second point of the above proposition is a version of the so-called realization lemma for ξ_0-convex surfaces (sometimes called the flexibility lemma). This version supports any number of parameters.

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