Quasiconformal mappings and manifolds of negative curvature

Pierre Panu
Centre de Mathématiques
Ecole Polytechnique
F-91128 Palaiseau Cedex
(U.A. 169 du CNRS)

In 1972, H. Grötzsch [1] observed that the classical Liouville and Picard theorems—non-existence of entire functions which are bounded or, more generally, exist only at two points—extended to a class of non-holomorphic maps. A holomorphic bijection between plane domains is conformal with respect to Euclidean metric; its differential at each point is a similarity, i.e., an isometry times a homothety. H. Grötzsch considered maps whose differential sits at a bounded distance from similarities. If, at some point \( x \), the differential takes a circle to an ellipse with axes \( a \) and \( b \), he defined the distortion \( Q(x) \) as the least number such that
\[
1/b \leq a/b \leq 2.
\]
He furthermore allowed a discrete set of singular points where the map is a ramified covering. He showed that the Picard theorem extends to the class of maps defined on the whole plane which have bounded distortion.

Nowadays, these maps are called quasiregular maps. A smooth diffeomorphism with bounded distortion is called quasiconformal, and the word "quasiregular" includes maps which are not 1-1 and whose differential may vanish.

It is clear that one can define quasiconformality, at least for smooth diffeomorphisms between Riemannian manifolds in any dimension. This notion plays a crucial role in G.D. Mostow's rigidity theorem for compact Riemannian manifolds of constant sectional curvature -1 and dimension \( \geq 3 \). This theorem states that two such manifolds, if diffeomorphic, have to be isometric. The argument involves quasiconformal mappings in the following way. A diffeomorphism \( \phi \) between two quotients \( D/R \) and \( D'/R' \) of the unit disk \( D \) in \( \mathbb{R}^2 \) lifts to a quasiconformal mapping \( \hat{\phi} \) of \( D \). \( \hat{\phi} \) extends by continuity to a quasiconformal homeomorphism \( \hat{f} \) of the unit sphere \( S^2 \). It satisfies
\[
f \circ g = g \circ (f \circ g), \quad g \in G
\]
where \( G \) denotes the isomorphism induced by \( \hat{f} \) on the fundamental groups \( R \) and \( R' \). Equation (1), together with some regularity of \( f \), implies that \( f \) is in fact conformal. The corresponding hyperbolic isometry of \( D \) descends to an isometry between the quotients.

In this talk, we investigate whether the first steps of the argument carry over to general simply connected manifolds of negative curvature. In this case, there is a natural notion of "ideal boundary" [7], thus two questions arise:

1. do quasiconformal mappings extend to this boundary?
2. is the extension quasiconformal in some sense?

In section 1, we discuss a generalization of Schwarz' Lemma which shows that a quasiconformal mapping \( f \) between manifolds of bounded negative curvature is a quasiconformality, i.e., satisfies metric inequalities.

\[
C + d(x,y)/L \leq d(f(x),f(y)) \leq Ld(x,y) + C
\]
where \( C \) and \( L \) are large constants. Such a map extends to ideal boundaries. Our proof relies on conformal distances constructed by means of capacities. Section 2 contains capacity estimates which lead to a nice application: locally symmetric spaces do not admit very pinched metrics. The precise definition of quasiregular maps is delayed until section 3, where the conformal structure on the boundary of homogeneous Riemannian manifolds of negative curvature is described. In sections 4 and 5, we organise material found in several papers by Pekka Tutka and Dennis Sullivan. In particular, we observe that one can speak of conformal and quasiconformal mappings on the boundary of a manifold of negative curvature as soon as there is a cocompact isometry group.

I learned most of the material described here in conversations with Mikhail Gromov. I hope that, in the present paper, it is apparent how much I owe him. Section 2 was completed in Japan during the Symposium. It is a pleasure to thank the Taniuchi Foundation for its generous support.

1. Schwarz Lemma and conformal distances

The Schwarz Lemma claims that, if a holomorphic function \( f \) on the unit disk fixes the origin, i.e., \( f(0) = 0 \), and if \( |f(z)| \leq 1 \) for all \( z \) with \( |z| = 1 \), then \( |f'(0)| \leq 1 \). The normalisation \( f(0) = 0 \) can be avoided by expressing the result in terms of the hyperbolic metric \( d \) of the unit disk.

1. Schwarz Lemma. Let the unit disk be equipped with its hyperbolic metric \( d \) of constant curvature -1. Then any holomorphic map of the disk to itself is distance decreasing, i.e.,
\[
d(f(x), f(y)) \leq d(x, y)
\]
In this form, the lemma belongs to Riemannian geometry. This is even more clear with L. Ahlfors' extension of Schwarz' Lemma to surfaces of variable curvature.

2. Theorem [12]. Let \( D \) be a surface endowed with a Riemannian metric of curvature \( \leq -1 \). Any holomorphic function on the disk with values in \( S \) is distance decreasing.

It is apparent in L. Ahlfors' paper [12] that Schwarz' Lemma only depends on the isoperimetric behaviour of the target surface \( S \). Let us define the isoperimetric profile \( I(v) \) of a Riemannian manifold \( M \) as follows. For a real number \( v \), volume(\( M \)), let \( I(v) \) be the infimum of the volumes of the hypersurfaces in \( M \) which bound a compact domain of volume \( v \). If \( I(v) \) is finite, \( I(v) = \inf \{ \text{vol}(D) : \text{compact, vol}(D) = v \} \).

Remember that the Classical Isoperimetric Inequality states that the isoperimetric profile of Euclidean space \( \mathbb{R}^n \) has the form
\[
I(v) = (4\pi)^{n/2} v^{1/n} / \Gamma(1 + n/2)
\]
for a sharp constant \( c_n \).

3. Theorem [M. Gromov] (10). Compare I.S. Rosenthal [493]. Let \( N \) be a \( k \)-dimensional Riemannian manifold whose isoperimetric profile \( I(v) \) satisfies the following two inequalities:

\( I(v) \leq c_N(v(1 + I(v)) \),

\( c_N \) for small \( v \), \( N \) is at least as good as Euclidean space, i.e., if one writes
then the integral
\[ \int_{VC} (\nu - v \cdot \nu) \, dv \]
should be finite.

Let \( M \) be a Riemannian manifold with sectional curvature \( K = -1 \).
Then any conformal (resp. quasiregular) immersion of \( M \) to \( N \) is
Lipschitz (resp. Hölder) continuous with constants which depend only on
\( K \), the function \( \nu \) and the deviation from conformality.

This theorem is sharp under the condition that a sharp
isoperimetric inequality is used. For example, in order to conclude
from it that isometries are the only conformal self maps of a rank one
symmetric space \( M \) (which is in fact known, see [25]), one needs to,
assuming density in \( M \) of a given volume, balls have minimum boundary
volume. This is known only when the sectional curvature is constant.

4 Corollary.- Let \( M \), \( N \) be complete simply connected Riemannian
manifolds with bounded negative sectional curvature, i.e.,
\[ K \leq b > 0 \]
Then any quasiconformal diffeomorphism \( f \) of \( M \) onto \( N \) is a
bijection, i.e., satisfies inequalities of the form
\[ -C + G(\alpha, \beta) \leq f(\alpha, \beta) - G(\alpha, \beta) + C \]
for some constants \( C \) and \( L \) which depend only on \( K \), \( b \) and
the deviation from conformality.

Proof.- It is known that the assumption on \( M \) implies a linear
isoperimetric inequality (see [16] chap. 6), thus condition (ii) is
satisfied. Condition (i) follows from a very general principle: as far
as small volumes are concerned, all Riemannian manifolds behave almost
like Euclidean space, see [24]. Finally, the Hölder condition for \( f \) and
\( f^{-1} \), as well as any relation of the type
\[ d(f(\alpha, \beta), G(\alpha, \beta)) \leq |f(\alpha, \beta) - G(\alpha, \beta)| \]
for some homeomorphic \( G \) of \( M \), implies that \( f \) is a quasiconformity.

Let us now return to H. Bröcker's original motivation for the study of
quasiregular mappings. Schwarz's lemma implies Lipschitz's theorem as
follows. A bounded entire function \( f \) is interpreted as a quasiregular
map \( f \) from \( C \) to the disk. Let us composer \( f \) with homothetic of
\( C \), and restrict then to the unit disk in \( C \). With respect to
hyperbolic metrics, all these maps should be uniformly Hölder, thus
equicontinuous. Considering larger and larger homotheties at a point \( z \)
shows that \( f(z) \) vanishes, thus \( f \) is constant. The same argument
gives Picard's theorem, once one observes that the plane minus two
points admits a conformal complete metric with bounded negative
curvature, the quasi-hyperbolic metric which will be defined in \( \S 10 \).

However, the isoperimetric method does not provide any extension of
Picard's theorem to dimensions \( \geq 3 \). The following theorem by B. McAneney
requires some more value distribution theory.

5 Theorem ([9]).- For \( n \geq 3 \), there is a lower bound
\( K_0(n) > 0 \) for the deviation from conformality of quasiregular maps on \( \mathbb{R}^n \)
that at least \( \alpha \) values.

It is striking that there exist such maps in dimension \( 3 \) ([22]). The
preceding theorem has a quantitative version ([21]): A quasiregular
map on the disk which takes enough points \( A_1, \ldots, A_m \) is Lipschitz (as
far as large distances are concerned) with respect to a complete
conformal metric on the complement of these points. This metric is not
the quasi-hyperbolic metric.

Question.- What are the invariance properties of this metric?

The Schwarz lemma, as a tool to prove nonexistence of holomorphic
maps from \( \mathbb{C} \) to certain complex manifolds, has had many developments
(see [25]). The idea is to construct functorial pseudo-distances on
complete manifolds which are distance deforming increasing.
Such a distance has to be identically zero for \( \mathbb{C} \) thus no holomorphic
maps can exist from \( \mathbb{C} \) to a manifold for which the pseudo-distance is a
distance ("hyperbolic" complex manifolds). There are analogues in
projective (\( \mathbb{P} \)) and affine geometry (\( \mathbb{A} \)).

In conformal geometry, functorial pseudo-distances can be constructed
using capacities. Conformal capacity has been used in the plane for a
long time. Its introduction in higher dimensions is due to Ch. Loewner
([34]).

6 Definition.- A condensor in a manifold is a triple \((C, B_1, B_2)\) where
\( C \) is open, \( B_1 \) and \( B_2 \), the "plates", are closed and contained in
the closure of \( C \). We shall admit \( B_2 = \emptyset \) when \( C \) is unbounded. Assume
that the manifold is Riemannian. The conformal capacity of a condensor
is the infimum of the volumes of the conformal metrics on \( C \) for which
the plates stay at distance \( 1 \). i.e.,
\[ \inf \{ \text{vol}(\mathcal{M}(\mathcal{B}, \mathcal{B})) \} \]

7 Definition.- Let \( M \) be Riemannian manifold. We define two
conformal pseudo-distances \( a \) and \( b \) as follows. For two points \( x \) and \( y \),
\[ a(x, y) = \inf \{ \text{cap}(\mathcal{M}(B, B)) : B \subset \mathcal{M}(x, y) \}
\]
where \( B \) is compact, connected and contains \( x \) and \( y \).

Both these quantities enter as tools in a number of papers (for
example, see the papers of M. Vuorinen [48] and H. Tanaka [54]).
They have been studied for their own sake by J. Ferrand [31] and

8 Properties.

(i) \( a(x, y) \) is a conformal invariant.
(ii) a conformal homeomorphism is bi-Lipschitz with respect to \( a \)
and \( b \).
(iii) a quasiregular map is Lipschitz with respect to \( b \).

Example.- For two point homogeneous spaces, the conformal pseudo-
distances and the symmetrical Riemannian metric \( d \) are functionally
independent, i.e., \( d = \text{vol} \), but the function \( \text{vol} \) is more or
less unknown. In the constant curvature case, F.W. Buzing [19] has shown
that, in the definition of \( d \) or \( \text{vol} \), the minimizing geodesic is
the one whose geodesic segments (known as the Teichmüller
geodesics) defines \( d \). This is unknown for other rank one (non
compact) symmetric spaces. This yields inequalities of the form
\[ a(x, y) \leq (A + B) \cdot \text{vol}(\mathcal{M}(x, y)) \]
for some constants \( A, B > 0 \) depending only on the dimension, see
[35]. A similar inequality is obtained for \( b(x, y) \) thanks to the identity
\[ d(x, y) = 2^{\frac{n}{2}} \cdot \text{vol}(\mathcal{M}(x, y)) \]
for some constants \( N > 0 \).

In dimension 2, more is known. Indeed, the Schwarz-Christoffel
formula for the conformal mapping of the half-plane onto the
Teichmüller condensor leads to a representation of \( d \) by means of an
elliptic integral, see [30].

It is obviously important to know for which manifolds the functions
\( a(x, y) \) and \( b(x, y) \) are distance-like, i.e., are positive. This has been
studied by M. Vuorinen in the case of Euclidean domains. Things are
10 Definition (12).- Let $\Sigma$ be an open, connected subset of Euclidean space. The quasi-hyperbolic metric space $(\Sigma, \rho_\Sigma)$, corresponding to Euclidean distance, is obtained from the Euclidean metric by a conformal factor equal to the inverse of the Euclidean distance to the boundary $\partial \Sigma$. The distance $\rho_\Sigma(x, y)$ is defined by $\rho_\Sigma(x, y) = \log(1 + |x - y|/d(x, y)/d(0, 0))$.

11 Theorem (60).- For a domain $\Omega$ in $\mathbb{R}^n$, the pseudo-distances $\rho_{\pm, k}$ and $\rho_{\pm, m}$ are linked by the following inequalities:

$$\rho_{\pm, k} \leq \rho_{\pm, m} \leq k \rho_{\pm, k}$$

if $\rho_{\pm, k}$ is connected, $\rho_{\pm, m} \leq C \rho_{\pm, k}$.

where $A, B, C, D, E$ are constants, $\rho_{\pm, k}$ and $\rho_{\pm, m}$ are homeomorphisms of $\Omega$, depending only on the dimension $n$.

These inequalities show that, as the Euclidean domains the conformal invariants $\rho_{\pm, k}$ and $\rho_{\pm, m}$ are connected. This is not that clear for general Riemannian manifolds, again, some geometric inequalities give a useful criterion concerning $\rho_{\pm, m}$.

12 Theorem (43).- Assume that the $n$-dimensional complete Riemannian manifold $\mathcal{M}$ satisfies a strictly stronger isoperimetric inequality than Euclidean $n$-space, i.e., condition (ii) in Theorem 3. Then the conformal pseudo-distance $\rho_{\pm, m}$ is a distance.

If furthermore $\mathcal{M}$ has bounded geometry, i.e., bounded sectional curvature and a positive injectivity radius, and if for example one has an isoperimetric inequality of type

$$\text{vol}(\mathcal{M}) \leq C \text{vol}(\mathcal{B})$$

then, for all $x, y \in \mathcal{M}$,

$$\rho_{\pm, m}(x, y) \geq d(x, y)^{1/n}$$

for some constants $A$ and $B$.

13 As a consequence, all simply connected manifolds of bounded negative curvature, all non almost abelian solvable Lie groups (N. Varopoulos [64] and [65]) have a conformal distance. They do not admit maps from Euclidean space or, more generally, from any manifold with vanishing $\rho_{\pm, m}$. Any quasi-conformal mapping between two such manifolds is a quasi-isometry, as defined in the introduction.

14 For manifolds of bounded negative curvature, Theorem 12 is reasonably sharp. Indeed, the reverse inequality

$$\rho_{\pm, m}(x, y) \leq d(x, y) + B$$

holds. To check this, one merely needs compute the $L^p$ norm of the gradient for some function of the distance to the geodesic segment through $x$ and $y$.

On the other hand, let $\mathcal{M}$ be the 3-dimensional Heisenberg group, i.e., the simply connected nilpotent nonabelian group in dimension 3. Then an isoperimetric inequality holds with exponent $\tau = 1/4$ (43), thus Theorem 12 yields $\rho_{\pm, m} \leq 16 d^{1/4}$. One easily sees that, conversely,

$$\liminf \rho_{\pm, m}(x, y)/d(x, y) = 1$$

Indeed, given a left-invariant metric $\rho$, split $\rho$ orthogonally as $\rho = \rho_h + \rho_g$ where $h$ is the direction of the center $H$ and $\rho$ its orthogonal complement. Define a group automorphism $\Delta_\tau$ by

$$\Delta_\tau = \tau t_{id} + t_{id}$$

so that $\Delta_\tau$ is a homothety by a factor $t$ of $\rho$ onto the metric $\rho_h$.

Then $d(\Delta_\tau x, \Delta_\tau y) = t d(x, y)$, whereas

$$d(\rho_h(x, y)) = t^2 d^2(x, y) \leq t \cdot \rho_{\pm, m}(x, y)$$

When $t$ goes to infinity, the metrics $\rho_h$ converge to a Carnot metric $d_\rho$ (see section 2) and the Hausdorff dimension $\dim H$ is equal to 4. The capacity $\Cap_h = \Cap^2$, with respect to $\rho_h$, which vanishes, since $H$ has dimension 3 with respect to $d_\rho$.

Question. Determine the asymptotics of $\rho_{\pm, m}$ on nilpotent groups.

To show that the conformal invariant $\rho_{\pm, m}$ is non trivial, one merely needs produce a condenser with finite capacity.

15 Proposition. (See section 2) Let $\mathcal{M}$ be a non-dimensional simply connected Riemannian manifold with pinched negative curvature.

$$d(x, y) = \exp(d(0, x) + d(0, y))$$

Its conformal pseudo-distance $\rho_{\pm, m}$ is

$$\rho_{\pm, m} = 1 + d(x, y)^{1/n}$$

On the other hand, it is very likely that the invariant $\rho_{\pm, m}$ vanishes for a smaller pinching, in particular for quaternionic hyperbolic spaces (compare Corollary 21 of section 2). This would show that no general inequality holds between $\rho_{\pm, m}$ and $\rho_{\pm, k}$.

16 In [35], chapter 17, J. Väisälä introduces properties $P_1$ for a domain $\Omega$ in $\mathbb{R}^n$. $\Omega$ has property $P_1$ at a boundary point $b$ if, for all connected subsets $E$ and $F$ of $\Omega$ containing $b$ in their closure, the capacity $\Cap_{\Omega}(E, F)$ is finite. He then shows that, if $\Omega$ has property $P_1$ at $b$ and if $\Omega'$ is an other domain which is finitely connected at each boundary point, then every quasi-conformal mapping of $\Gamma$ onto $\Omega'$ extends continuously to the boundary. A Riemannian manifold has vanishing invariant $\rho_{\pm, m}$ (the class $\mathcal{M}_\omega$ of [35]) if and only if it has property $P_1$ at $b$.

It would be interesting to have natural examples of manifolds with this property.

One can find other kinds of conformally invariant distances in the literature. The Yamabe conjecture - in a conformal class of Riemannian metrics, find one, often unique, with constant scalar curvature - produces a conformally invariant Riemannian metric. It has been studied exactly for this purpose by Ch. Loewner and L. Nirenberg [35] on Euclidean domains. Another method consists in starting with some Riemannian metric and normalizing it by a clever conformal factor, a suitable power of the length $|M|$ of the Weil tensor, see [42]. Both these tricks have pseudo-conformal analogues, see [65] and [68]. These metrics behave badly under quasi-conformal mappings. Indeed, there are quasi-conformal mappings which are not locally Lipschitz, and thus, not Lipschitz under any Riemannian metric.

To study a conformally flat manifold $\mathcal{M}$, one may be tempted to imitate the construction of the Kobayashi holomorphic or projective distances, i.e., define

$$d(z, w) = \inf \{ t \in \mathbb{R} : z, w \text{ in the disk } D \leq R^2, \text{ and there exists an}\}$$

$$\text{conformal embedding } z, w \text{ in } D \leq M \text{ such that}\}$$

$$f(z, w) = (z, w)$$

and make a distance out of the function $f$. For Euclidean domains, inequalities as in Theorem 11 hold, but it is unclear for which conformally flat manifolds this distance vanishes.

2. Capacity estimates

In this section, we prove two inequalities concerning conformal capacities.
not vanish. The proof applies to capacities with arbitrary exponents, which is also of interest. (Remember that capacity originates from electrostatic capacity in physics, which, in dimension 3, has exponent 2, see (47)).

17 Definition.- Let \((C_{u/v}, B)\) be a condenser in a Riemannian manifold. Let \( p \) be a positive real number. The \( p \)-capacity is

\[
\inf \left\{ h^p \mid u \text{ smooth on } C, \text{ extends continuously to } \mathcal{C}, \text{ values } 0 \text{ on } B_0 \text{ and } 1 \text{ on } B_1 \right\}
\]

Conformal capacity is obtained for \( p = 1 \) equal to the dimension (see (101) for the equivalence with definition 6).

Initiating J. Ferrand's definition of conformal distance \( e \), we introduce a notion of critical exponent for a simply connected Riemannian manifold of non positive sectional curvature.

18 Definition.- Let \( M \) be an open Riemannian manifold. Its critical exponent \( p(M) \) is the least exponent \( q \) such that there exist condensers with connected and unbounded plates which have a finite \( q \)-capacity.

If furthermore \( M \) has non positive sectional curvature, it has an Eberlein-O'Neill boundary \( \partial M \). One can consider condensers of type \((M, X, Y)\) where \( X \) and \( Y \) are points at infinity. We define the modified exponent \( p(M) \) as the infimum of exponents \( q \) such that there exist points \( x, y \in \partial M \) with \( h^p(M, x, y) < \infty \).

It is likely that \( p(M) = p_M \). On the other hand, \( p(M) > dim M \) implies that the conformal distance \( e \) does not vanish. Thus Proposition 15 follows from the following Lemma.

19 Lemma.- Let \( M \) be simply connected and have bounded negative curvature \( -\frac{1}{2} < h \leq 0 \). Then both \( p(M) \) and \( r(M) \) are constant on rays through \( m \). Indeed, they are constant on rays through \( m \). We claim that \( h \) is integrable and \( e^{hM} \) is constant on rays through \( m \). Thus, \( p(M) = p_M \).

The exponent \( q > n-1 \) implies that \( \int_0^\infty e^{-eh} h^{q-n} h(r) dr < \infty \) is finite if \( q < n-1 \).

In fact, if we denote the volume entropy by

\[
h_{vol}(M) = \limsup_{r \to \infty} \frac{\log \text{vol}(B_r)}{r},
\]

we have proven the following inequality

\[
x^q \leq b^q N \Rightarrow p(M) \leq h_{vol}(M) / b.
\]

For example, volume entropy for a rank one symmetric space with sectional curvature normalized by \(-4 \leq K \leq -1\) and dimension \( n \) is equal to \( n+k-2 \), where \( k = 2 \) for rank \( 1 \) hyperbolic spaces, \( k = 4 \) for quaternionic hyperbolic spaces, \( k = 8 \) for Cayley hyperbolic planes. Thus these spaces have \( p \leq k + 2 \). This inequality is sharp.

20 Lemma.- Let \( M \) be a rank one symmetric space with sectional curvature normalized by \(-4 \leq K \leq -1\). For each \( n-1 < k < n+k-2 \), there exists a positive constant \( c_{k,n} \) such that, if \( h \) denotes a homothetic attached to a point at infinity \( x \) and \( B \) is any closed subset of \( M \), then

\[
\text{Cap}_{h}(M, B) \leq c_{k,n} \text{length}(h(B)).
\]

21 Corollary.- For such a symmetric space, \( p = n + k - 2 \). Indeed, for \( q < n+k-2 \), \( x, y \in \partial M \), \( u \) smooth on \( M \) with \( u(x) = 1, u(y) = 0 \) and \( h \) a homothetic attached to \( x \) or \( y \), one has

\[
2 \int d\mu \leq 2 \text{length}(h(B)) + \text{length}(h(B)) = +\infty.
\]

22 Corollary.- Let \( M \) be a compact quotient of a rank one symmetric space of dimension \( n \leq 2k \). Then \( M \) does not admit any metric with pinching better than \((n-1)/(n-2k-2)) \).

The sharp result \( 1/(n-1) \) is the best possible pinching known in dimension 4 (H. M. Ville (67)).

The invariants \( p \) and \( q \) can be defined in a combinatorial way for nets. It is likely that they are quasi-isometry invariants (see the work of M. Kanai (24) for 2-capacities). If this is true, then the conclusion of Corollary 22 extends to all compact manifolds whose fundamental group is isomorphic to a cocompact subgroup of U(n), Sp(n, C) or SO(n).

Question (Gromov).- Compare \( p(M) \) with the exponents for which \( L^n \) or \( L^n - \text{cohomology} \) of \( M \) in degree 1 vanishes.

Proof of Lemma 20.- Let us foliate the symmetric space by parallel horospheres \( N \) centered at \( x \) - the level of the horofunction \( h \). Let \( u \) be a function on \( M \) which takes value 1 on \( B \) and extends by continuity to value 0 at \( x \). By the coarse formula, it suffices to uniformly estimate

\[
\int \left| du \right|^{n}.
\]

For the horospheres which hit \( B \) - a set of measure \( \text{length}(h(B)) \) - the function \( u \) takes the value one on \( N \), thus the integral is greater than some capacity \( \text{Cap}(N, \text{point}, x) \). This capacity does not depend on the particular point, since \( N \) is homogeneous. It does not depend on the particular horosphere, since they are all pairwise isometric. It is zero for two reasons.

1) Since the exponent \( q > n-1 \), the Sobolev embedding of \( M \) into \( C^{n-1} \) allows one to replace the point by a ball of finite size as a plate of the condenser.

2) The horosphere \( N \) is isometric to a nilpotent lie group with left-invariant metric, whose isoperimetric profile satisfies \( \text{vol}(B) \) where \( p = n + k - 2 \) (M. Varopoulos (60)), thus Theorem 12 applies.

For \( M \) a symmetric space and \( x, y \) points at infinity, we show that \( \text{Cap}_{h}(M, x, y) = +\infty \) for all \( q < n+k-2 \). Question.- What happens when \( q = 2 \) is the critical exponent?

3. Regularity properties of quasiconformal mappings.

Early, it has turned out to be necessary to consider quasiregular mappings which are not of class \( \mathcal{C}^1 \). In Teichmüller's theory (see (55), (51), (51)) one obtains as solutions of a variational problem mappings which are smooth except at a finite number of points. Furthermore, in the deformation theory of Riemann surfaces, there definitely occur quasiconformal mappings which are nowhere smooth, as we shall see below. We give two equivalent definitions of quasiregular maps. A quasiconformal mapping in a punctured domain is

\[
\text{Cap}_{h}(M, k, B) \leq c_{k,n} \text{length}(h(B)).
\]
23 Analytic definition. (C.B. Morrey [37]) A continuous map $f$ between Riemannian manifolds of dimension $n \geq 2$ is $K$-quasiregular if it admits a differential $df$ in the sense of distributions which is a locally $L^1$-integrable function and satisfies
$$|df| \leq K |f|.$$ 

The number $K$ is only one of the various ways to measure the deviation from conformality, i.e., the distance between the differentials $df$ and the shifts of the eigenvalues $\lambda^i, \ldots, \lambda^n$ of the endomorphism $\text{deg} \cdot df$, one has
$$\text{deg} \cdot df = K(\lambda^n)/n.$$ 

An equally satisfactory quantity is
$$Q = n/\lambda^n,$$
which satisfies
$$\log Q \leq \log K \cdot (n-1) \log Q.$$

For a linear map $A$ between Euclidean spaces, the number $Q$ has a metric interpretation: Given a ball $B$, its image is pinched between two balls $B(S)$ and $B(B) - i e.,$
$$B(S) \leq B < B(2S)$$
such that $Q = B/2S$.

More generally, if $f$ is a continuous, discrete, open map between Riemannian manifolds, $x$ is a point and $r$ is a real number, one can define the ratio $Q(x,r) = s/\delta$ where $s$ is the minimum radius of a ball centered at $f(x)$ which contains $B(x,r)$, and $\delta$ is the maximum radius of a ball centered at $f(x)$ which is contained in $B(x,r)$.

24 Metric Definition. (N. N. Lavrentiev [29]) A continuous map between Riemannian manifolds is quasiregular if it is orientation preserving, open, discrete, and if
$$Q(x) = \limsup_{r \to 0} Q(f(x), r).$$

is bounded.

There is a third characterization of quasiconformality by means of capacities, [46] (23, [36]. The fact that, in dimensions $\geq 2$, these definitions coincide is a sequence of theorems by I. N. Vekua [45], J. A. Jenkins [22], J. H. Behling [22], J. Väisälä [14], D. Martin - B. Rickman - J. Väisälä [22]. This is the conclusion of long-standing efforts to determine to which class of regular quasiregular maps exactly belong. This regularity is expressed by the following properties.

25 Properties. In dimensions $\geq 2$, quasiregular mappings are absolutely continuous on lines, i.e., in a coordinate patch, a quasiregular map is absolutely continuous on almost every line. As a consequence, they need Lebesgue null sets to null sets.

Quasiregular mappings have a differential almost everywhere, which is $L^n$ integrable. These properties have turned out to be essential in G.D. Mostow's rigidity theorems for compact manifolds of constant sectional curvature.

26 Theorem [39]. If two compact Riemannian manifolds of dimension $\geq 2$, with constant sectional curvature $-1$, are diffeomorphic, then they are isometric.

27 Here is a sketch of the proof. A diffeomorphism between two such manifolds lifts to a quasiconformal mapping $f$ of the universal covers, i.e., the unit disk in $\mathbb{H}$. Let us denote by $f$ and $f'$ the fundamental groups of the manifolds. They act conformally on the disk. The diffeomorphism induces an isomorphism $\iota : f \equiv f'$ and, for $g \in \Gamma$, one has $f \cdot g = f(g) \cdot f$. The quasiconformal mapping $f$ extends to the unit sphere (Property 21 of [36]) and the extension, still denoted by $f$, is a quasiconformal homeomorphism of the $(n-1)$-sphere (Schwarz' reflection principle). We now show, following P. Tukia [40], that $f$ is a conformal mapping of the sphere. This is due to the fact that the action of $f$ on the sphere is highly transitive, and normalizes little regularity of $f$.

Still, it fails when $n = 2$. Choose the upper half-space model, and normalize $f$ so that $f(0) = 0$ and $f(1) = \infty$. Consider the line $\gamma$ through $0$ and $f(1)$. Since $h^+ \cdot f$ is compact, there exist elements $\alpha_0 \in \Gamma$ such that $h^{-1} \cdot f \cdot h = \alpha_0 \cdot f$ and $\alpha_0 \cdot f$ is bounded in $0$. Then one can write $(\gamma_0) \cdot f = \gamma_0 \cdot f + \gamma_0 \cdot f$. The conjugacy condition now reads
$$h^{-1} \cdot f \cdot h = \alpha_0 \cdot f = \alpha_0 \cdot f.$$

Choose subsequences such that $\alpha_0 \cdot f$ and $\gamma_0 \cdot f$ converge, and if $f$ is differentiable at $0$, then in the limit $k + f \cdot \gamma = h(0)$. From there on, it is easy to show that $f$ is conformal.

28 In [40, 41], G.D. Mostow generalized the rigidity theorems to all locally symmetric spaces without $2$-dimensional factors. The argument in the rank one case also relies on the theory of quasiconformal mapping, but in a slightly extended context. Indeed, the first steps are the same. A symmetric space of rank one is a simply connected Riemannian manifold with negative sectional curvature. As such, it is an "ideal boundary", defined by means of asymptotic geodesics [7]. The lift of a diffeomorphism - in fact, of any homotopy equivalence - is a quasiconformity (as defined in the introduction). It extends to a homeomorphism of the ideal boundary (a fact which can be traced back to M. Morse [30]). This extension is not quasiconformal with respect to any Riemannian metric on $\mathbb{E}$, in general, it fails to do so.

29 Let us define a family of distances on $\mathbb{E}$ adapted to the situation. Fix a point $p \in \mathbb{E}$. There is a unique Euclidean metric $g_*$ on the subbundle $V$ which is invariant under the isometries fixing $p$, and it allows one to define the length of curves tangent to $V$ and, we set, for two points $p, q$ in $\mathbb{E}$,
$$d(p, q) = \inf \text{length}_{c} \in C \text{ joins } p \text{ to } q \text{ in the boundary, } c$$

This number is finite since the distribution $V$ is non integrable, and the distance $d$ changes conformally, i.e., a small $d$-ball is very close to a $d$-ball. Thus we defined a conformal structure (in a generalized sense) on the boundary $\mathbb{E}$.

Now the boundary extension of a quasiconformity of $\mathbb{E}$ is a quasiconformal mapping with respect to any of the metrics $d$. Here we take the metric definition for quasiconformal maps, which is meaningful for arbitrary metric spaces. The class of maps obtained coincides with G.D. Mostow's "quasiconformal mappings over a division algebra" [40]. These maps are absolutely continuous on a suitably class of lines" [41] and almost everywhere differentiable [44] in a sense which we explore in Tukia's argument, as well as G.D. Mostow's, to extend the rigidity theorem in rank one.

Let $M$ be a rank one symmetric space with isometry group $\mathbb{G}$. To a choice of a point $x$ in $M$ and a boundary point $p \in \mathbb{E}$, there
In the constant curvature case, $N$ is abelian and $A$ consists of homotheties. In the other cases, $N$ is two-step nilpotent, its Lie algebra splits as
\[ n = v + l(n), \]
and the element $\delta$ of $A$ acts on $n$ by multiplication by $t$ on $v$ and $\delta^2$ on $l(n).$ Thus the ideal boundary of a rank one symmetric space identifies with a nilpotent Lie group. The results of absolute continuity and differentiability of quasi-conformal mappings will in fact apply to the whole class of Carnot groups, which we define now.

**Definition.**- A Carnot group is a simply connected nilpotent Lie group whose Lie algebra $n$ splits as $n = V_1 \oplus \ldots \oplus V_k$, where $[V_i, V_j] = V_{i+j}$.

A Carnot group $N$ admits a one-parameter group of homotheties $\delta_t \in \text{Aut}(N)$, $\delta_t$ is multiplication by $t$ on $V_i$.

By a norm $\| \cdot \|$ we mean a left-invariant distance on $N$ which is homogeneous of degree one under the group of homotheties. Particular norms are the Carnot metrics, given by a Banach space structure on $V_i$, one can define the length of curves in $N$ which are tangent to the left-invariant subbundle of $TN$ generated by $V_i$. One defines quasi-conformal mappings using the metric definition. The class obtained does not depend on the particular choice of norm.

A continuous map $f$ between Carnot groups $N$ and $N'$ equipped with homotheties $(\delta_t)$ and $(\delta'_t)$ is said to be $\delta$-differentiable at $x$ if the limit
\[ D(f)(x) := \lim_{t \to 0} \frac{f(\delta_t(x)) - f(x)}{t} \]
exists for all $x$.

A line is an orbit of a left-invariant vector field which is tangent to $V_i$.

For a smooth function $u$ on $N$, let $\nabla u(x) := \sup_{\gamma(x) = x} \text{Inf}_{\gamma} (Du_{\gamma(t)})$ be the infimum of the integrals (with respect to Haar measure)

\[ \int |Du| \]

over all smooth functions $u$ on $N$ which tend to $0$ on $B_r$ and to $1$ on $B_r$. Conformal capacity is obtained for $p = 1$ equal to the group's Hausdorff dimension $p = \dim V_i$.

**Theorem (441).**- A quasi-conformal homeomorphism between open subsets of Carnot groups admits almost everywhere a $\delta$-differential which is a group isomorphism intertwining the two one-parameter groups of homotheties.

It is absolutely continuous on almost every line (414) and, as a consequence, it send null-sets to null-sets.

1-quasi-conformal mappings preserve conformal capacities, and $K$-quasi-conformal mappings multiply then at most by $K$ (for a suitable measurement $K$ of the deviation from conformality).

In other words, a big part of the analytic theory of quasi-conformal mappings in Euclidean space can be carried out on Carnot groups. However, one needs further information - still unknown - on the function $c(r)$ to conclude that quasi-conformal mappings are Holder continuous. It is also unclear whether the condenser whose plates are two arbitrary curves has a non-zero capacity.

32. A new feature of the nilpotent theory is that, in general, there are no quasi-conformal mappings at all. The reason is that there is too little choice for differentials. Indeed, these should live in the group $\text{Aut}(N)$ of automorphisms of $N$ which commute with the homotheties. In the abelian case, this is the whole linear group, and every smooth differentiable is locally quasi-conformal. The Iwasawa component of $N(n,1)$ is the Heisenberg group. The group $\text{Aut}(N)$ consists of homotheties times symplectic $2n-2$ by $2n-2$ matrices a smooth differentiable is locally quasi-conformal if and only if it is a contact transformation, i.e., it preserves the plane distribution $V (223).$ This still produces an infinite dimensional group of quasi-conformal mappings. In contrast, when $N$ is the Iwasawa component of $Sp(n,1)$, $n \geq 2$, the group $\text{Aut}(N)$ consists of homotheties and a compact group $Sp(n-1)Sp(1)$. In this case, any quasi-conformal mapping is $1$-quasi-conformal.

33. Corollary (441).- The elements of $Sp(n,1)$ are the only global quasi-conformal self-maps of the boundary of quaternionic hyperbolic $n$-space.

In the same vein, if $\text{Aut}(N)$ consists of homotheties only - case which definitely occurs, see (441) - then any (even local) quasi-conformal mapping of $N$ is the restriction of a translation or homothety.

It would be interesting to have a local version of the preceding corollary. This amounts to prove that local quasi-conformal mappings are smooth. Proofs of this fact in the Euclidean case are due to I. M. Remshunajk and F.W. Gehring (111). They rely on non-linear elliptic regularity theory. In the nilpotent case, the corresponding equations are hyperbolic and the necessary regularity is not available.

34. There is much room left for further generalizations, since the metric definition for quasi-conformal maps can be taken using arbitrary metric spaces.

1) The solvable Lie groups which admit left-invariant metrics with strictly negative curvature have been classified by E. Heintze (221). They are of the form $AN$ where $N$ is a nilpotent Lie group and $A$ a one-parameter group of contracting automorphisms. The data of $A$ together with a norm is a very natural generalization of a Banach space with its homotheties. One can speak of differentiability and of quasi-conformal mappings, a class which will not depend on the particular norm. A new feature is that these groups are not length spaces (cf [143]), i.e., the distance between two points is not the length of any curve joining them. In fact, a group $N$ admits a length norm if and only if it is a Carnot group.

2) A question of Gromov: as far as we know, nothing is known about quasi-conformal mappings between separable Hilbert spaces. What about Liouville theorem? Notice that traditional methods use integration, and thus do not extend to infinite dimensions.

3) Lift the assumption of discreteness and you can speak of quasi-conformal maps between manifolds of different dimensions. Is Liouville theorem still true?
4. Quasiconformal groups

35 Definition (13) - A quasiconformal group on a metric space is a group of uniformly quasiconformal homeomorphisms.

36 We are concerned with the following question, originally due to F.W. Gehring and R.F. Paley: when is a quasiconformal group of the standard sphere (resp., ideal boundary of a manifold of negative curvature) quasiconformally conjugate to a group of conformal transformations?

A general method to address this question is due to D. Sullivan (53). He observes that every quasiconformal group leaves invariant at least one measurable conformal structure. Indeed, at each point, the space of conformal structures on the tangent space identifies with \( B(n) \otimes D(n) \), which admits an invariant metric with non-positive sectional curvature. The set of pull-backs of the standard structure by the elements of the group is bounded, thus one can attach to it a unique point, the center of the smallest ball which contains it, for example (see (59)). These data form a measurable conformal structure, almost everywhere invariant under the group. Notice that this argument carries over to Carnot groups, since all what is needed is the a.e. differentiability of quasiconformal mappings. In this case, by a measurable conformal structure, we mean the data at each point of a Euclidean metric on the left-invariant subbundle generated by \( V \). The metric should depend measurably on the point.

37 In dimension 2, the sphere has only one conformal structure, by Riemann's mapping theorem. (The extension to measurable conformal structures is due to C.B. Morrey (37)). Thus the invariant conformal structure is quasiconformally conjugate to the standard one, and we are done (37). As a consequence, all quasiconformal groups in dimension 2 are known.

38 The argument fails in higher dimensions, and in fact, P. Tukia has constructed domains in \( \mathbb{R}^3 \) which made a transitive, connected quasiconformal group which is not isomorphic to any subgroup of \( \text{SO}(n,1) \), which, in turn, needs extra assumptions. A typical example is a space that is expanding at the point \( x \). Assume that the invariant conformal structure is equipped with a standard structure at \( x \). Using iterates of \( g \), one sees that some neighborhood of \( x \) can be mapped conformally to smaller and smaller neighborhoods of \( x \), which are very close to a standard disk. One concludes that the group is conjugate to a standard disk. In fact, as proved by P. Tukia, this argument works under no extra assumption on the invariant conformal structure.

39 Theorem (59). - Let \( g \) be a measurable conformal structure on the sphere. If its conformal group is isomorphic to a group of quasiconformal homeomorphisms of the disk, then \( g \) is the standard conformal structure.

Clearly, this applies to Carnot groups too. D. Sullivan has a different result: let \( g \) be invariant under a discrete subgroup \( \Gamma \) of \( \text{SO}(n,1) \). Then \( g \) is standard under a weaker assumption on \( \Gamma \): it approaches almost every point horizontally (53). The assumption in P. Tukia's theorem is comical approach a.e.

Since the connected subgroups of \( \text{SO}(n,1), \text{Un}(n), \ldots \) are known, this leads to a method to decide where two homogeneous Riemannian manifolds are quasiconformally equivalent. For the case of Euclidean domains, see the forthcoming work by F.W. Gehring and G. Martin.

40 Corollary. - Let \( N \) be a Carnot group with homothety group \( \cdot A \), let \( M \) denote the group \( AN \) endowed with a left-invariant Riemannian metric of negative sectional curvature. Let \( N \) be a simply connected Riemannian manifold with negative sectional curvature and cocompact isometry group. Assume that \( M \) and \( M' \) are quasiconformally equivalent. Then \( \text{Isom}(M) \) and \( \text{Isom}(M') \) are cocompact subgroups in a common topological group.

Notice that, if \( \text{Isom}(M') \) is discrete, we may conclude that both \( \text{Isom}(M) \) and \( \text{Isom}(M') \) are subgroups of a simple Lie group \( \text{Un}(n,1), U(n,1), \ldots \)

5. Global characterization of quasiconformal mappings

41 I want to emphasize the fact that the quasiconformality of a manifold \( M \) can be checked from its behaviour under conjugacy with conformal mappings of \( M \). This applies only when the conformal group of \( M \) is large enough. Therefore, in the sequel, \( B \) denotes either the boundary of a rank one symmetric space (i.e., a sphere with an exctic conformal structure) or a Carnot group \( N \). We denote by \( B \) its conformal group (i.e., a simple group \( N \)). If \( \text{Isom}(M) \) is cocompact in the symmetrized case, the group \( B \) is a compact quotient of \( \text{Isom}(M) \). Let us begin with a consequence of the preceding discussion.

42 Corollary (see (62) for an elementary proof in the case of Euclidean space). - A quasiconformal group on \( \mathbb{R}^n \) consists only of 1-quasiconformal mappings (conformal mappings in the symmetric case).

One of the applications of the methods of section 1, especially Theorem 3, is to equicontinuity properties of "normalized" quasiconformal mappings (see also (63), chap. 20). Given balls \( D_1, D_2 \subset \mathbb{R}^n \), and a point \( x \in D_1 \), we say that a homeomorphism \( f \) of the sphere \( S \) is normalized if \( f(x) = x \) and \( D_1 \subset f(D_2) \subset D_2 \), then the normalized quasiconformal mappings of \( S \) with a given distortion are equicontinuous. Any quasiconformal mapping can be normalized - depending only on its distortion - by multiplying it with suitable elements of the conformal group. Thus one can state: if \( D \) denotes the set of quasiconformal mappings on \( S \) with a fixed distortion less than \( K \), then \( D \) is a compact subset of the homeomorphic group of the sphere, a kind of converse is true.

43 Proposition. - Let \( f \) be a cocompact subgroup of the conformal group \( G \). A homeomorphism \( f \) of \( S \) is quasiconformal if and only if \( f \) is composed with a compact subgroup of the homeomorphic group of the sphere.

44 Corollary. - Let \( f \) be a cocompact group \( G \). A homeomorphism \( f \) is 1-quasiconformal (conformal in the symmetric case) if and only if \( f \) is conjugate to a quasiconformal subgroup of \( G \).

45 Remark. - Since the boundary of a simply connected manifold of negative curvature can be reconstructed functorially from any discrete cocompact group of isometries, this corollary shows that the conformal group \( G \) can be recovered from any discrete cocompact subgroup. By the way, this is Mostow's rigidity theorem: any isomorphism between
lattices extends to an isomorphism between the Lie groups. This leads us to a notion of conformal mappings between the boundaries of universal covers of arbitrary compact manifolds with negative curvature.

46 Definition.- Let $N$, $M$ be simply connected manifolds with negative sectional curvature. Let $F = F_1$ be cocompact groups of isometries. A homeomorphism $f : M \to M'$ is said to be conformal if the multiplicative set generated by $f$ and $f'$ is contained in $\Gamma_{\text{fix}}(B')$ where $B' \subset \text{Hom}_G(M)$ is compact. The map $f$ is called quasi-conformal if $f \subset F' \subset F$ where $B' \subset \text{Hom}_G(M)$ is compact.

47 Theorem (P. Tukia [613].) With the above definition, a quasi-conformal mapping extends to a quasiconformity of $M$ onto $M'$, well-defined modulo maps with bounded displacement. Conversely, a quasiconformity extends to a quasi-conformal mapping.

48 Question.- Does the conformal group of $M$ preserve some conformal class of distances? In case the group $F$ is smooth with respect to some differentiable structure on $M$, is there any relation between the quasi-conformal group attached to the smooth structure and the quasiconformal group attached to the smooth structure? The case of symmetric spaces already shows that there is no inclusion.

49 Invariants of patterns of points. It is easy to construct conformal invariants of a finite number of points on the standard sphere if given $k$ distinct points, take the various simplices they generate in hyperbolic space, choose a combination of volumes, mutual angles and distances. I claim that any such invariant is quasi-invariant under quasiconformal mappings of the sphere. This just comes from compactness modulo conformal normalization.

Conversely, quasi-conformal mappings of the standard sphere can be defined using the conformal invariant of $k$ points $a, b, c, \ldots$, which tends to infinity when exactly two points become close to each other. Indeed, taking a homeomorphism $f$, we use a conformal mapping so that $f$ fixes three given points $a, b, c$, then the range of variation of $f$ is controlled by the $k$-points invariant $\alpha(a, b, c, \ldots)$ and a modulus of continuity at $d$ is given by the invariant $\alpha(d, a, b, \ldots)$. Thus the set of normalized homeomorphisms which almost preserve $(a, b, c, \ldots)$ is compact (uniforimly), and the conformal group of homeomorphisms of $S^k$ strictly containing $\operatorname{SO}(n)$ cannot preserve such an invariant. Does this imply some dynamical property of the action of $S^k$ on quadruples of distinct points?

There are two famous examples of conformal invariants of several points, and the cross-ratio on the sphere. There, the invariant has values in the two-sphere itself. Second, the volume of the hyperbolic simplex generated by $k$ points on the $n$-sphere, which an...ertes a key role in the proof of Mostow's rigidity theorem [613]. Usually, $k$ is 4, but it is known that $k = 2$ posses.

50 One easy wonder whether the conformal mappings on the boundary of a manifold of negative curvature as defined in § 46 preserve some kind of rigidity. One can characterize the class of $C^\infty$ functions on $M$ whose derivative is $L^1$-integrable, since they have adequate compactness properties, see [612], but it is unclear whether one can reconstruct the whole Royden algebra of continuous functions with $L^1$-integrable derivative, together with its norm $\|f\|_{L^1}$.

It is known (see [612], [633]) that this algebra completely determines the conformal structure.

References

[22] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[23] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[27] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[29] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[31] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[33] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[34] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[37] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[38] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
[40] I. S. GALE, Invariants Mathématiques, chap. 6, notes de cours rédigées par J. Lafontaine.
503) B. RICKMAN, On the number of omitted values of entire quasiregular mappings, J. d'Analyse Math. 32, 100-117 (1978).
512) P. TUKIA, On quasiconformal groups, Preprint Helsinki (1982).