Metric problems concerning nilpotent groups

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November 13th, 2010
We present four open problems about maps between nilpotent groups.

1. (Continuation of Tessera’s talk). Heisenberg groups do not biLipschitz embed in $\ell_2$. But snowflaked versions do. Want bounds on distortions, dimensions of such embeddings.

2. Gromov’s Hölder homeomorphism problem and a variant.
Heisenberg group $\text{Heis}^3$ is the 3-dimensional Lie group with Lie algebra spanned by $\xi$, $\eta$ and $\zeta$ with $[\xi, \eta] = \zeta$. The left-invariant vector fields $\xi$ and $\eta$ span a plane field $H$, Carnot-Carathéodory distance $d_{cc}(x, x')$ is the inf of length of curves tangent to $H$ joining $x$ to $x'$. Dilation $\delta_t$ is an automorphism induced by $\delta_t(\xi) = t\xi$, $\delta_t(\eta) = t\eta$, $\delta_t(\zeta) = t^2\zeta$. It multiplies Carnot distances by $t$.

Finiteness of Carnot distance follows from picture:
Translation and dilation invariance implies

1. \( d_{cc}(x, x \exp(t^2 \zeta)) = td_{cc}(1, \exp(\zeta)) = \text{const. } t. \)
2. \( \text{volume} B(x, r) = r^4 \text{volume} B(x, 1) = \text{const. } r^4, \) thus Hausdorff dimension is 4.
3. The same number of balls of radius \( r/2 \) suffice to cover every ball \( B(x, r). \)

Heisenberg group in its Carnot-Carathéodory metric gives a sharp approximation of the word metric on the integral Heisenberg group

\[
\text{Heis}_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbb{Z} \right\}.
\]

In general, if \( \Gamma \) is a finitely generated nilpotent group, given a finite generating system, the word metric space \((\Gamma, d_w)\) admits an asymptotic cone, which is a Carnot group. A Carnot group is a nilpotent Lie group equipped with a Carnot-Carathéodory metric homogeneous under dilations.
Definition

The doubling dimension \( \dim(X) \) of a (doubling) metric space \( X \) is the least \( d \) such that for all \( R \), every \( R \)-ball can be covered by \( 2^d \frac{R}{2} \)-balls.

Example

1. \( \mathbb{R}^n \), \( \text{Heis}^n \) (\( n \) odd) have doubling dimension linear in \( n \).
2. The Internet network equipped with its latency metric is believed to have low dimension.

Theorem

(Assouad 1983). For every \( \epsilon \in (0, 1) \) and \( d > 0 \), there exist \( D(d, \epsilon) \) and \( N(d, \epsilon) \) such that for every \( d \)-dimensional metric space \( X \), the snowflaked metric \( (X, d_X^{1-\epsilon}) \) embeds in \( \ell_2^N \) with distortion \( \leq D \).

So snowflaked Heisenberg group does biLipschitz embed in \( \ell_2 \).

Question

Give sharp bounds on \( D \) and \( N \).
Theorem

(Gupta, Krauthgamer, Lee 2003; Lee, Mendel, Naor 2004). In Assouad’s theorem, one can take $N = O\left(\frac{d \log d}{\epsilon}\right)$ and $D = O\left(\frac{d}{\sqrt{\epsilon}}\right)$.

Unclear whether dimension bound is sharp or not.

Question

What is the minimal dimension $N(\epsilon)$ of a Euclidean space in which $(\text{Heis}^3, d_{cc}^{1-\epsilon})$ admits a biLipschitz embedding?

Remark

$N(\epsilon) > 4$.

Indeed, the Hausdorff dimension of $(\text{Heis}^3, d_{cc}^{1-\epsilon})$ is $\frac{4}{1-\epsilon} > 4$. 
Theorem

(Gupta, Krauthgamer, Lee 2003; Lee, Mendel, Naor 2004). In Assouad's theorem, one can take $N = O\left(\frac{d \log d}{\epsilon}\right)$ and $D = O\left(\frac{d}{\sqrt{\epsilon}}\right)$. For fixed $d$, the distortion bound is sharp (Lee, Mendel, Naor 2004): $(1 - \epsilon)$-snowflaked Laakso spaces require distortion $\Omega\left(\frac{1}{\sqrt{\epsilon}}\right)$ when embedded in $\ell_2$.

Dependance on $d$? Heisenberg groups do not help. Indeed, (Lee, Naor 2006): $(\text{Heis}^n, d^{1-\epsilon}_{cc})$ embed in $\ell_2$ with distortion $O\left(\frac{1}{\sqrt{\epsilon}}\right)$ independent on $n$.

Question

What is the minimal distortion of a biLipschitz embedding of $(\text{Heis}^3, d^{1-\epsilon}_{cc})$ in $\ell_2$?
Laakso graphs
Proof (Lee, Mendel, Naor 2004) that $1 - \epsilon$-snowflaked Laakso graph $(G_j, d_j^{1-\epsilon})$ requires distortion $D_j \geq \Omega\left(\frac{1}{\sqrt{\epsilon}}\right)$ when embedded in $\ell_2$.

By induction on $j$. Show that $D_j^2 \geq 4^{-\epsilon} D_{j-1}^2 + \frac{1}{4}$.

In rescaled Laakso graph, $uavb$ is a unit square with a diagonal of length 2. When mapped to a quadrilateral $u'a'v'b'$ in Euclidean space in a distance nondecreasing manner, parallelogram inequality

$$|u' - v'|^2 + |a' - b'|^2 \leq |u' - a'|^2 + |a' - v'|^2 + |v' - b'|^2 + |b' - u'|^2$$

implies

$$4 \frac{|u' - v'|^2}{|u - v|^2} + \frac{|a' - b'|^2}{|a - b|^2} \leq \frac{|u' - a'|^2}{|u - a|^2} + \frac{|a' - v'|^2}{|a - v|^2} + \frac{|v' - b'|^2}{|v - b|^2} + \frac{|b' - u'|^2}{|b - u|^2},$$

or

$$4^{1-\epsilon} \frac{|u' - v'|^2}{|u - v|^{2(1-\epsilon)}} + \frac{|a' - b'|^2}{|a - b|^{2(1-\epsilon)}} \leq \frac{|u' - a'|^2}{|u - a|^{2(1-\epsilon)}} + \frac{|a' - v'|^2}{|a - v|^{2(1-\epsilon)}} + \frac{|v' - b'|^2}{|v - b|^{2(1-\epsilon)}} + \frac{|b' - u'|^2}{|b - u|^{2(1-\epsilon)}}.$$
Question

(Gromov 1993). Let $G$ be a Carnot group of dimension $n$. For which $\alpha \in (0, 1)$ does there exist locally a homeomorphism $\mathbb{R}^n \to G$ which is $C^{\alpha}$-Hölder continuous?

Definition

Let $X$, $Y$ be metric spaces. Let $\text{Holder}(X, Y) = \sup\{\alpha \in (0, 1) \mid \exists \text{ locally a } C^{\alpha}\text{-Hölder continuous homeomorphism } X \to Y \text{ whose inverse is Lipschitz}\}$.

Example

If $G$ is a $r$-step Carnot group, the exponential map $g = \text{Lie}(G) \to G$ is locally $C^{1/r}$-Hölder continuous and its inverse is Lipschitz. Thus $\text{Holder}(\mathbb{R}^n, G) \geq 1/r$.

Proposition

Let $G$ have dimension $n$ and Hausdorff dimension $Q$. Then $\text{Holder}(\mathbb{R}^n, G) \leq \frac{n}{Q}$.
Proposition

Let $G$ have dimension $n$ and Hausdorff dimension $Q$. Then $\text{Holder}(\mathbb{R}^n, G) \leq \frac{n-1}{Q-1}$.

Proof. Use the Varopoulos (1985) isoperimetric inequality for piecewise smooth domains $D \subset M$,

$$\text{vol}(D)^{Q-1/Q} \leq \text{const.} \, \mathcal{H}^{Q-1}(\partial D).$$

It follows that the boundary of any non smooth domain $\Omega$ has Hausdorff dimension at least $Q - 1$. Indeed, cover $\partial \Omega$ with balls $B_j$ and apply (*) to $\Omega \cup \bigcup B_j$. This gives a lower bound on $\mathcal{H}^{Q-1}(\partial (\bigcup B_j)) \leq \sum \mathcal{H}^{Q-1}(\partial B_j) \leq \text{const.} \sum \text{diameter}(B_j)^{Q-1}$. 
(Gromov 1993). Let $n = 2m + 1$, let Heis$^n$ denote $n$-dimensional Heisenberg group. Let $V \subset \text{Heis}^n$ be a subset of topological dimension $m + 1$. Then the Hausdorff dimension of $V$ is at least $m + 2$. It follows that $\text{Holder}(\mathbb{R}^n, \text{Heis}^n) \leq \frac{m+1}{m+2}$.

**Proof.** According to topological dimension theory (Alexandrov), there exists an $m$-dimensional polyhedron $P$ and a continuous map $f : P \to \text{Heis}^n$ such that every map sufficiently $C^0$-close to $f$ hits $V$.

Gromov approximates $f$ with piecewise horizontal maps which sweep an open set $U$. This gives rise to a local projection $p : U \to \mathbb{R}^{m+1}$ such that for every ball $B$, the tube $p^{-1}(p(B))$ has volume $\leq \text{const. diameter}(B)^{m+2}$.

Cover $V$ with balls $B_j$. The corresponding tubes $T_j = p^{-1}(p(B_j))$ cover $U$. Then the volume of $U$ is less than $\sum \text{diameter}(B_j)^{m+2}$, which shows that $\text{dim}_{\text{Hau}}(V) \geq m + 2$.

**Theorem**

(Gromov 1993). Let $G$ be a generic Carnot group of dimension $n$, Hausdorff dimension $Q$, with an $h$-dimensional distribution. Let $k \leq h$ be such that $h - k \geq (n - h)k$. Then $\text{Holder}(\mathbb{R}^n, G) \leq \frac{n-k}{Q-k}$. 

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**Facts about Heisenberg group**

**Snowflake embeddings**

**Hölder homeomorphism problem**

**Isoperimetric inequality**

**Horizontal submanifolds**

**Conformal Hölder homeomorphism problem**

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**P. Pansu**

**Metric problems concerning nilpotent groups**
Curvature pinching

Definition

Let $M$ be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say $M$ is $\delta$-pinched if sectional curvature ranges between $-1$ and $\delta$. Define the optimal pinching $\delta(M)$ of $M$ as the least $\delta \geq -1$ such that $M$ is quasiisometric to a $\delta$-pinched simply connected Riemannian manifold.

Example

Rank one symmetric spaces of noncompact type are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$. Real hyperbolic space has sectional curvature $-1$. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched.

Question

What is the optimal pinching of $H^m_{\mathbb{C}}$?
Definition

Say two geodesic rays in a Riemannian manifold are asymptotic if their Hausdorff distance is finite. The visual boundary of a negatively curved manifold is the set of asymptoticity classes of geodesic rays.

Facts.

- The visual boundary, seen from a point \( o \), is a sphere (use polar coordinates).
- It carries a visual metric \( d_o \).
- Different visual metrics \( d_o \) and \( d_{o'} \) are equivalent.
- Quasiisometries between negatively curved Riemannian manifolds induce quasisymmetric maps between ideal boundaries.
Example

If $M$ is a rank one symmetric space, the visual metrics on its ideal boundary are locally equivalent to Carnot-Carathéodory metrics on Carnot groups.

Proposition

Let $M$ be a simply connected $\delta$-pinched Riemannian manifold. Equip the ideal boundary $\partial M$ of $M$ with a visual metric. The natural homeomorphism $S^{n-1} \to \partial M$ is $C^\alpha$ with $\alpha = \sqrt{-\delta}$, and its inverse is Lipschitz. Therefore $\text{Holder}(\mathbb{R}^{n-1}, \partial M) \geq \sqrt{-\delta}$.

Indeed, geodesics from a unit ball to a point come together exponentially fast, with exponents ranging from $\sqrt{-\delta}$ to 1 (Rauch comparison theorem, 1950's).

Question

Let $G$ be a Carnot group. Let $\alpha > 1/2$. Does there exist quasisymmetrically equivalent metrics on $G$ which locally admit $C^\alpha$ homeomorphisms from Euclidean space? With Lipschitz inverses?

If no, then optimal pinching of $H^m_C$ is $-\frac{1}{4}$.
**Definition**

Let $f : X \to Y$ be a homeomorphism. The conformal Hölder exponent $\text{CHolder}(f)$ of $f$ is the supremum of $\alpha$’s such that for all $\ell > 0$, there exists $L > 0$ such that for all $x, x', x''$ in $X$,

$$d(f(x), f(x'')) \leq \ell d(f(x), f(x')) \Rightarrow d(x, x'') \leq L d(x, x')^\alpha.$$ 

Let $\text{CHolder}(X, Y)$ denote the supremum of $\alpha$’s such that there locally exist homeomorphisms $X \to Y$ with conformal Hölder exponents $\geq \alpha$.

**Lemma**

1. If $f : X \to Y$ is $C^\alpha$ and $f^{-1}$ is $C^\beta$, then $\text{CHolder}(f) \geq \alpha \beta$. In particular, $\text{Holder}(X, Y) \leq \text{CHolder}(X, Y)$.

2. Let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms. Assume that $g$ is quasisymmetric. Then $\text{CHolder}(g \circ f) = \text{CHolder}(f)$. 

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**Definition**

Let $X$ be a compact metric space. The conformal dimension of $X$ is the infimum of Hausdorff dimensions of metric spaces quasisymmetrically equivalent to $X$.

**Example**

(Pansu 1990). Carnot groups have conformal dimension equal to their Hausdorff dimension.

**Corollary**

Let $G$ be a Carnot group of dimension $n$ and Hausdorff dimension $Q$. Then $\text{CHolder}(\mathbb{R}^n, G) \leq \frac{n}{Q}$.

**Theorem**

(Pansu 2009). The optimal pinching of $H^2_C$ is $-\frac{1}{4}$.

Gives some hope for $\text{CHolder}(\mathbb{R}^3, \text{Heis}^3) = \frac{1}{2}$ and therefore $\text{Holder}(\mathbb{R}^3, \text{Heis}^3) = \frac{1}{2}$. 

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Metric problems concerning nilpotent groups
Metric geometry, algorithms and groups

Paris, January 10th - April 8th, 2011

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Organizers: Guy Kindler (Jerusalem), James Lee (U. Washington), Claire Mathieu (Brown), Ryan O’Donnell (Carnegie Mellon), Pierre Pansu (Paris-Sud/ENS), Nicolas Schabanel (LIAFA-CNRS), Lior Silberman (Vancouver)

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January 17-21: embeddings, algorithms, complexity
March 21-25: expanders, derandomization

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