Negative curvature pinching

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November 7th, 2013

Definition

Let M be a Riemannian manifold. Let $0 < \delta \leq 1$. Say M is δ-pinched *if sectional curvature ranges between δ and 1. Define the* optimal pinching $\delta(M)$ *of M as the largest $\delta \leq 1$ such that M is diffeomorphic to a δ-pinched Riemannian manifold.*

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Projective spaces over the complexes, quaternions and octonions have $\delta(M) \geq \frac{1}{4}$.

Indeed, in their canonical (Fubini-Study) metric, lines are totally geodesic of curvature 1 and real projective subspaces are totally geodesic of curvature $\frac{1}{4}$. Other sectional curvatures lie in between.

Theorem (Berger, Klingenberg 1959)

Let M be a $\frac{1}{4}$-th pinched even dimensional simply connected Riemannian manifold. Then

- *either M is homeomorphic to a sphere,*
- *or M is isometric to a projective space.*

This implies that the optimal pinching for projective spaces equals $\frac{1}{4}$.
Proof: cover M with two geodesic balls, use angle comparison theorems.

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Proof: cover M with two geodesic balls, use angle comparison theorems.

Recent improvement:

Theorem (Brendle, Schoen 2007)

Let M be a $\frac{1}{4}$-pinched Riemannian manifold. Then

- *either M is diffeomorphic to a quotient of the sphere,*
- *or M is isometric to a projective space.*

Proof : "$M \times \mathbb{R}$ has nonnegative isotropic curvature" is preserved by the Ricci flow. And this follows from $\frac{1}{4}$-pinching.

Definition

*Let M be a compact Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-*pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching *of M as the least $\delta \geq -1$ such that M is diffeomorphic to a δ-pinched Riemannian manifold.*

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Examples

Each projective space has a dual hyperbolic space.

Complex hyperbolic space $H^m_{\mathbb{C}}$ is a metric on the ball in $\mathbb{C}^m$ which is invariant under all holomorphic automorphisms.
Spheres in $H^m_{\mathbb{C}}$ are homogeneous under conjugates of $U(m)$. Horospheres are homogeneous under Heisenberg group $Heis^{m-1}$.

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All have compact quotients M which have $\delta(M) \leq -\frac{1}{4}$.

Theorem (Many people)

Let N be a compact quotient of $H_{\mathbb{C}}^m$, $H_{\mathbb{H}}^m$ ($m \geq 2$) or $H_{\mathbb{O}}^2$. If a metric on N is $-\frac{1}{4}$-pinched, then it lifts to a symmetric metric.

This is due to

- M. Ville, 1984 for $H_{\mathbb{C}}^2$ (pointwise estimate on the characteristic class $\chi - 3\sigma$),
- L. Hernández-Lamoneda, 1991, and independently S.T. Yau and F. Zheng, 1991 for $H_{\mathbb{C}}^m$ (harmonic maps),
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Harmonic map approach based on vanishing theorem: if M is compact Kähler (resp. quaternionic, octonionic Kähler), N is $-\frac{1}{4}$-pinched,

$$f : M \to N \text{ harmonic} \Rightarrow f \text{ pluriharmonic}.$$

(non linear Hodge theory).

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Let M be a complete simply connected Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is quasi-isometric to a δ-pinched complete simply connected Riemannian manifold.*

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M is isometric to a left-invariant metric on a Lie group of the form $\mathbb{R} \ltimes \mathbb{R}^3$.

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Let M be a Riemannian manifold. Let $p > 1$. L^p-cohomology *of M is the cohomology of the complex of L^p-differential forms on M whose exterior differentials are L^p as well,*

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For (uniformly) contractible spaces, L^p-cohomology is quasi-isometry invariant. Wedge product $\alpha, \beta \mapsto \alpha \wedge \beta$ induces cup-product $[\alpha] \smile [\beta] : H^{k,p} \times H^{k',p} \to H^{k+k',p/2}$ in a quasi-isometry invariant manner.

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$T^{1,p}$ *is non zero and thus infinite dimensional*. Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in L^p for all $p > 1$ but it is not the differential of a function in L^p.

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which is Sobolev space $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

Proposition

Let M be a simply connected negatively curved Riemannian manifold. Functions u on M whose differential belongs to L^p have boundary values u_∞ on the visual boundary. The cohomology class $[du] \in H^{1,p}(M)$ vanishes if and only if u_∞ is constant.

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This suggests

Definition

(Bourdon-Pajot 2004). For a negatively curved manifold M, define the Royden algebra *$\mathcal{R}_p(M)$ as the space of L^∞ functions u on M such that $du \in L^p$, modulo $L^p \cap L^\infty$ functions.*

Then $\mathcal{R}_p(M)$ identifies with an algebra of functions on the visual boundary of M. If M is homogeneous, $\mathcal{R}_p(M)$ is a (possibly anisotropic) Besov space.

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Use the radial vectorfield $\xi = \frac{\partial}{\partial r}$ in polar coordinates and its flow ϕ_t, whose derivative is controlled by sectional curvature.

Use Poincaré's homotopy formula :
For α a closed 2-form in L^q,

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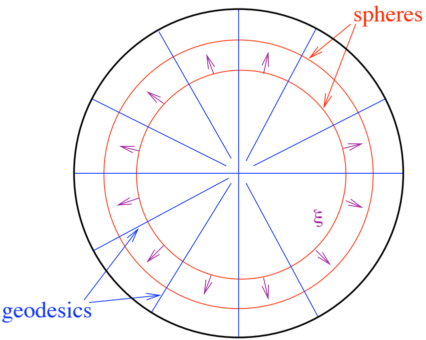

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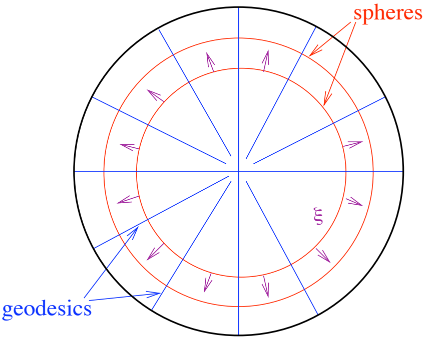

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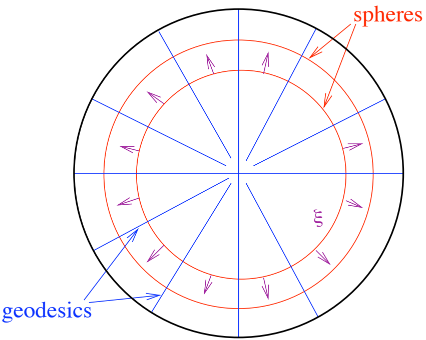

Step 2. *Boundary value determines cohomology class.*

Theorem

If $\dim(M) = 4$, M is δ-pinched and $q < 1 + 2\sqrt{-\delta}$, then a boundary value operator is defined, it injects $H^{2,q}$ into closed forms on the boundary. In particular, $T^{2,q} = 0$.

δ-pinched means sectional curvature $\in [-1, \delta]$.

Back to the left-invariant metric on the Lie group $\mathbb{R} \ltimes \mathbb{R}^3$ where $\mathbb{R}$ acts on $\mathbb{R}^3$ by

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

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Indeed, one constructs explicit nonzero classes in $T^{2,q}(M)$ for $2 < q < 4$. Unfortunately, this does not work with $H^2_{\mathbb{C}}$.

Theorem

$T^{2,q}(H^2_{\mathbb{C}}) = 0$ for $2 < q < 4$.

Recall that $H^2_{\mathbb{C}}$ can be viewed as $\mathbb{R}^4$ with metric $dt^2 + e^t dx^2 + e^t dy^2 + e^{2t}(dz - xdy)^2$.

Here is a strategy for proving that the optimal pinching of $H^2_{\mathbb{C}}$ is equal to $-\frac{1}{4}$.

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Scheme of proof

- *Recall Royden algebras $\mathcal{R}_p(M)$, $p > 1$, are quasi-isometry invariants.*
- *Given $u \in \mathcal{R}_p$, define a vectorsubspace $\mathcal{S}_p(u) \subset \mathcal{R}_p$, in a quasi-isometry invariant manner.*
- *If M is δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for every u, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p$.*
- *If $M = H^2_{\mathbb{C}}$, for all $p \in (4, 8)$, there exists (locally) $u \in \mathcal{R}_p$ such that $\mathcal{S}_p(u)$ is not a subalgebra of $\mathcal{R}_p$.*

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Definition

Let M be a simply connected negatively curved manifold, let $p>4$, let $u\in\mathcal{R}_p(M)$. Define

$$\mathcal{S}_p(u)=\{v\in\mathcal{R}_p(M)\,|\,[dv]\smile[du]=0\in H^{2,p/2}(M)\}.$$

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Conjecture

If M is 4-dimensional, δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for every u, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p(M)$.

Naive attempt. Let $v,\ v' \in \mathcal{S}_p(u)$. Then $[dv] \smile [du]$ vanishes if and only if its boundary value $dv_\infty \wedge du_\infty = 0$ a.e. Then $v'_\infty dv_\infty \wedge du_\infty + v_\infty dv'_\infty \wedge du_\infty = 0$ a.e., showing that $[d(vv')] \smile [du] = 0$, i.e. $vv' \in \mathcal{S}_p(u)$.

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Why it fails. a.e. no. In distributional sense. Multiplying distributions is delicate.

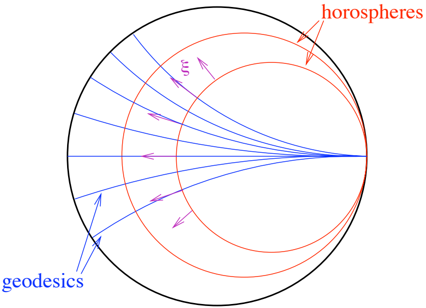
Now we compute $H^{2,q}(H^2_{\mathbb{C}})$ for $2 < q = p/2 < 4$.

Step 1. *Switch point of view. Use horospherical coordinates. View $H^2_{\mathbb{C}}$ as a product $\mathbb{H}^1 \times \mathbb{R}$. Prove a Künneth type theorem.*

For $q \notin \{4/3, 2, 4\}$, differential forms α on $H^2_{\mathbb{C}}$ split into components α_+ and α_+ which are contracted (resp. expanded) by ϕ_t. Then

$$h_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds - \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator h on L^q. $P = 1 - dh - hd$ retracts the L^q de Rham complex onto a complex $\mathcal{B}$ of differential forms on $\mathbb{H}^1$ with missing components and weakly regular coefficients.



Step 2. *If $2 < q < 4$, this complex is nonzero in degrees 1 and 2.*
$\mathcal{B}^1$ consists of 1-forms which are multiples of the left-invariant contact form τ on $\mathbb{H}^1$.

Step 3. *If $2 < q < 4$, vanishing of degree 2 cohomology classes is characterized by a differential equation.*
If $\alpha \in \mathcal{B}^2$ is a 2-form, then $\alpha \in d\mathcal{B}^1$ if and only if α satisfies the linear differential equation

$$\alpha = d(\frac{\tau \wedge \alpha}{\tau \wedge d\tau}\tau).$$

If $dv \wedge du$ is a solution, $d(v^2) \wedge du$ is not a solution, unless dv is proportional to du.

Failure of the subalgebra theorem for $H^2_{\mathbb{C}}$.
In coordinates (x, y, z) on $\mathbb{H}^1$, one can take (locally) $u = y$ and $v = x$. Then $dv \wedge du = -d\tau$ belongs to $d\mathcal{B}^1$, whereas $d(v^2) \wedge du$ does not. So for $4 < p = 2q < 8$, $\mathcal{S}_p(u)$ is not (locally) a subalgebra of $\mathcal{R}_p(H^2_{\mathbb{C}})$.

Other rank one symmetric spaces.
The comparison theorem should work for all of them: in the definition of $\mathcal{S}_\kappa$, replace du by a cohomology class κ of degree 1, resp. 3 resp. 7. Steps 1 and 2 of the L^q computation in degree 2 resp. 4 resp. 8 are unchanged. It turns out that for all spaces but $H^2_{\mathbb{C}}$, the differential equation of Step 3 is a consequence of $d\alpha = 0$, so $\mathcal{S}_\kappa$ is an algebra in these cases.