Université Paris-Sud • M2 CS & EDP • M2 AAG
Introduction to the spectral theory (2014–2015)
Webpage: `http://www.math.u-psud.fr/~pankrash/2014spec/`
Intermediate Exam

The solutions must be written either in English or in French. The result of any question (even if you have no solution) can be used to study the subsequent questions of the same exercise.

Exercise 1
We are going to work in the Hilbert space $\mathcal{H} = L^2(0,1)$.

1. Consider the following sesqulinear form
$$t(u,v) = \int_0^1 \overline{u'(t)}v'(t)dt, \quad D(t) = \big\{u \in H^1(0,1) : u(0) = 0\big\}.$$
   (a) Show that the form t is closed and describe the operator T generated by the form t.
   (b) Calculate the eigenvalues and the eigenfunctions of T.
   (c) Calculate the spectrum of T.

2. Consider the following operator B in $\mathcal{H}$:
$$(Bf)(x) = \int_0^x f(t)\,dt.$$
   (a) Calculate the adjoint B^* of B.
   (b) Show that the operator $K := BB^*$ is Hilbert-Schmidt and calculate its Hilbert-Schmidt norm.
   (c) For $f \in C_c^\infty(0,1)$, calculate $(Kf)''$, $(Kf)(0)$ and $(Kf)'(1)$, then establish a link between the operator K and the operator T from question (1).
   (d) Describe the point spectrum, the discrete spectrum, the essential spectrum of K.

3. Use the preceding questions to compute the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

Exercise 2

1. Let T be a self-adjoint semibounded from below operator in a Hilbert space $\mathcal{H}$. Assume that the essential spectrum of T is non-empty and denote
$$\Sigma := \inf \operatorname{spec}_{\mathrm{ess}} T.$$
Furthermore, assume that there exist N linearly independent vectors $f_1, \dots, f_N$ in $D(T)$ such that all the eigenvalues of the $N \times N$ matrix
$$\Big(\langle f_j, (T - \Sigma) f_k \rangle\Big)_{j,k=1}^{N}$$
are strictly negative. Show that the operator T has at least N eigenvalues in $(-\infty, \Sigma)$.

2. We are now working in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Consider the following self-adjoint operator:
$$T = \frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1, \quad D(T) = H^4(\mathbb{R}).$$
   (a) Calculate the spectrum of T.

   (b) Let $V \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be real-valued. Show that the operator
$$S := T + V, \quad D(S) = H^4(\mathbb{R}),$$
   is self-adjoint and calculate its essential spectrum.

   (c) Let $\mathcal{F}$ be the Fourier transform in $L^2(\mathbb{R})$ and $\widehat{V} := \mathcal{F}V$. Give an explicit expression for the operator $\widehat{S} := \mathcal{F}S\mathcal{F}^*$ and describe its domain.

   (d) Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ and $q \in \mathbb{R}$ consider the following functions $\varphi_{q,\varepsilon}$,
$$\varphi_{q,\varepsilon}(p) = \frac{1}{\varepsilon}\,\varphi\Big(\frac{p-q}{\varepsilon}\Big).$$
   Show that these functions belong to $D(\widehat{S})$ and that
$$\lim_{\varepsilon \to 0+} \big\langle \varphi_{q,\varepsilon}, \widehat{S}\varphi_{r,\varepsilon} \big\rangle = \widehat{V}(q - r) \quad \text{for } q, r = \pm 1.$$

   (e) Assume that $\widehat{V}(0) < 0$ and $\big|\widehat{V}(2)\big| < \big|\widehat{V}(0)\big|$. Show that the operator S has at least two negative eigenvalues.