Université Paris-Sud • M2 CS & EDP
Introduction to the spectral theory (2013–2014)
Webpage: `http://www.math.u-psud.fr/~pankrash/2013spec/`
Final Examination

The solutions must be written either in English or in French. The result of any question (even if you have no solution) can be used to study the subsequent questions of the same exercise.

Exercise 1. Let $\mathcal{H}$ be a separable Hilbert space, A be a self-adjoint operator if $\mathcal{H}$ such that $A \geq 0$ and $\ker A = \{0\}$. We say that a function $u : \mathbb{R} \to \mathcal{H}$ is a solution of the wave equation

$$u''(t) + Au(t) = 0. \tag{1}$$

if $u \in C^2(\mathbb{R}, \mathcal{H})$, and the inclusion $u(t) \in D(A)$ and the identity (1) are valid for any $t \in \mathbb{R}$.
For $t \in \mathbb{R}$ we define $C_t, S_t : \mathbb{R} \to \mathbb{R}$ by

$$C_t(x) = \cos(t\sqrt{x}) \text{ and } \quad S_t(x) = \frac{\sin(t\sqrt{x})}{\sqrt{x}} \text{ for } x > 0, \quad C_t(x) = S_t(x) = 0 \text{ for } x \leq 0.$$

Let us fix $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$ and consider the functions $\varphi, \psi : \mathbb{R} \to \mathcal{H}$ defined by

$$\varphi(t) = C_t(A)u_0, \quad \psi(t) = S_t(A)u_1.$$

(1) Show that $\varphi(t)$ and $\psi(t)$ belong to $D(A)$ for any $t \in \mathbb{R}$.
(2) Show that $\varphi \in C^1(\mathbb{R}, \mathcal{H})$ and that $\varphi'(t) = -AS_t(A)u_0$ for any $t \in \mathbb{R}$.
(3) Show that $\psi \in C^1(\mathbb{R}, \mathcal{H})$ and that $\psi'(t) = C_t(A)u_1$ for any $t \in \mathbb{R}$.
(4) Show that both φ and ψ are solutions of (1).

Now we would like to show that $u(t) = \varphi(t) + \psi(t)$ is the unique solution to (1) satisfying the initial conditions $u(0) = u_0$ and $u'(0) = u_1$. Let w be another solution satisfying the same initial conditions. Set $v(t) := u(t) - w(t)$.

(5) Show the equality

$$\frac{d}{dt}\big\langle v(t), Av(t)\big\rangle = \big\langle v'(t), Av(t)\big\rangle + \big\langle Av(t), v'(t)\big\rangle.$$

Remark: one may use the direct definition of the derivative.
(6) Show that the value $E(t) = \big\langle v'(t), v'(t)\big\rangle + \big\langle v(t), Av(t)\big\rangle$ is independent of t.
(7) Show that $v(t) = 0$ for all $t \in \mathbb{R}$.

Exercise 2. Let $\alpha \in \mathbb{R}$. Denote by T_α the operator in $L^2(\mathbb{R})$ associated with the sesquilinear form

$$t_\alpha(u, v) = \big\langle u', v'\big\rangle_{L^2(\mathbb{R})} + \alpha\overline{u(0)}v(0), \quad D(t_\alpha) = H^1(\mathbb{R}).$$

(1) Describe the domain and the action of the operator T_α and show that it is self-adjoint.
(2) Let S be the restriction of T_α to $D(S) = D(T_\alpha) \cap D(T_0)$. Show that $S \subset T_0$.
(3) Let $z \in \mathbb{C} \setminus \mathbb{R}$. Calculate $\dim \operatorname{ran}(S - z)^\perp$.
(4) For the same z, show that $(T_0 - z)^{-1} - (T_\alpha - z)^{-1}$ is a finite-dimensional operator.
(5) Calculate the essential spectrum of T_α.
(6) Calculate the discrete spectrum of T_α.
(7) Find the *sharp* constant $C_\alpha \in \mathbb{R}$ such that for any $u \in H^1(\mathbb{R})$ there holds

$$\int_{\mathbb{R}} \big|u'(x)\big|^2 dx + \alpha\big|u(0)\big|^2 \geq C_\alpha \int_{\mathbb{R}} \big|u(x)\big|^2 dx.$$

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary. The following facts concerning the Sobolev spaces may be used without proofs:

- if $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in H^2(\Omega)$
- the expression $\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$ defines a norm on $H^2(\Omega)$, and this norm is equivalent to the usual H^2-norm.
- for any $a > 0$ one can find $b > 0$ such that $\|j_0 u\|^2_{L^2(\partial\Omega,ds)} \le a\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^d)} + b\|u\|^2_{L^2(\Omega)}$ for all $u \in H^1(\Omega)$; here and below $j_0 u$ means the trace of u on $\partial\Omega$ and ds means the $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$.
- $H^1_0(\Omega) = \big\{u \in H^1(\Omega) : j_0 u = 0\big\}$.

Let $\alpha \in \mathbb{R}$, $0 < m < M < +\infty$. Pick a measurable function $\gamma : \partial\Omega \to [m, M]$. In $L^2(\Omega)$ consider the operator T_α defined through the sesqulinear form

$$t_\alpha(u,u) = \langle \nabla u, \nabla v\rangle_{L^2(\Omega,\mathbb{C}^d)} + \alpha\,\langle j_0 u, \gamma j_0 v\rangle_{L^2(\partial\Omega,ds)}, \quad D(t_\alpha) = H^1(\Omega).$$

(1) Show that T_α is self-adjoint, semibounded from below and with a compact resolvent.

(2) Show that $D(T_\alpha) \subset H^2(\Omega)$ and that $T_\alpha u = -\Delta u$ for any $u \in D(T_\alpha)$.

Denote by $E(\alpha)$ the lowest eigenvalue of T_α. In addition, let Λ denote the lowest Dirichlet eigenvalue of Ω.

(3) Show that $\lim_{\alpha\to-\infty} E(\alpha) = -\infty$.

(4) Show that $\alpha \mapsto E(\alpha)$ is a non-decreasing function.

(5) Show that $\alpha \mapsto E(\alpha)$ is a continuous function.

(6) Show that $E(\alpha) \le \Lambda$ for any α and deduce the existence of the finite limit $\lim_{\alpha\to+\infty} E(\alpha) =: \mu$.

Now we would like to show that $\mu = \Lambda$. For any α, let u_α be an L^2-normalized eigenfunction of T_α for the eigenvalue $E(\alpha)$.

(7) Show that there exists a sequence (α_n) with $\lim \alpha_n = +\infty$ such that u_{α_n} converge to some $u_\infty \in L^2(\Omega)$ in the L^2-norm.

(8) Show that Δu_{α_n} converge to some $v \in L^2(\Omega)$ in the L^2-norm.

(9) Deduce that $u_\infty \in H^1_0(\Omega)$ and that $t_\alpha(u_\infty, u_\infty) \le \mu\|u_\infty\|^2_{L^2(\Omega)}$.

(10) Show that $\mu = \Lambda$.

Exercise 4. Let $\alpha \in \mathbb{R}$ be fixed. In $\mathcal{H} := L^2(\mathbb{R}^2, \mathbb{C}^2)$ consider the operator T_0 given by

$$T_0 = \begin{pmatrix} -\dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2} & \alpha\Big(i\dfrac{\partial}{\partial x_1} - \dfrac{\partial}{\partial x_2}\Big) \\ \alpha\Big(i\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}\Big) & -\dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2} \end{pmatrix}$$

(1) Show that T_0 is self-adjoint on the domain $D(T_0) = H^2(\mathbb{R}^2, \mathbb{C}^2)$.

(2) Calculate the spectrum of T_0. Does T_0 have any eigenvalue?

Introduce the function $v : \mathbb{R}^2 \to \mathbb{R}$, $v(x) = \dfrac{\big(\ln|x|\big)^2}{1+|x|}$. For $\lambda \in \mathbb{R}$, consider the operator T_λ given by

$$T_\lambda = T_0 + \lambda v \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

(3) Show that T_λ is self-adjoint on $D(T_\lambda) = H^2(\mathbb{R}^2, \mathbb{C}^2)$.

(4) Describe the essential spectrum of T_λ.