RANDOM DATA CAUCHY THEORY FOR SUPERCRITICAL WAVE EQUATIONS II : A GLOBAL EXISTENCE RESULT

by

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Abstract. — We prove that the subquartic wave equation on the three dimensional ball Θ, with Dirichlet boundary conditions admits global strong solutions for a large set of random supercritical initial data in $\cap_{s<1/2}(H^s(\Theta) \times H^{s-1}(\Theta))$. We obtain this result as a consequence of a general random data Cauchy theory for supercritical wave equations developed in our previous work [**6**] and invariant measure considerations, inspired by earlier works by Bourgain [**2, 3**] on the non linear Schrödinger equation, which allow us to obtain also precise large time dynamical informations on our solutions.

1. Introduction

In our previous work [**6**], we developed a general theory for constructing local strong solutions to nonlinear wave equations, posed on compact riemannian manifolds with supercritical random initial data. The goal of this article is to show that in a *very particular case* we can combine this local theory with some invariant measure arguments (see the work by Bourgain [**2, 3**] and the authors [**10, 11, 5**]) to obtain *global* solutions. Namely, we shall consider the nonlinear wave equation with Dirichlet boundary condition posed on Θ, the unit ball of $\mathbb{R}^3$,

$$(1.1) \qquad (\partial_t^2 - \mathbf{\Delta})w + |w|^\alpha w = 0, \quad (w, \partial_t w)|_{t=0} = (f_1, f_2), \quad w\,|_{\mathbb{R}_t \times \partial\Theta} = 0, \qquad \alpha > 0$$

with radial *real valued* initial data (f_1, f_2). Our aim is to give a proof of the following result.

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Theorem 1. — *Suppose that $\alpha < 3$. Let us fix a real number p such that* $\max(4, 2\alpha) < p < 6$. *Let $((h_n(\omega), l_n(\omega))_{n=1}^{\infty}$ be a sequence of independent standard real Gaussian random variables on a probability space $(\Omega, \mathcal{A}, p)$. Consider (1.1) with initial data*

$$f_1^{\omega}(r) = \sum_{n=1}^{\infty} \frac{h_n(\omega)}{n\pi} e_n(r), \quad f_2^{\omega}(r) = \sum_{n=1}^{\infty} l_n(\omega) e_n(r)\,,$$

where $(e_n(r))_{n=1}^{\infty}$ is the orthonormal basis consisting in radial eigenfunctions of the Laplace operator with Dirichlet boundary conditions, associated to eigenvalues $(\pi n)^2$.

Then for every $s < 1/2$, almost surely in $\omega \in \Omega$, the problem (1.1) has a unique global solution

$$u^{\omega} \in C(\mathbb{R}, H^s(\Theta)) \cap L^p_{loc}(\mathbb{R}_t; L^p(\Theta))\,.$$

Furthermore, the solution is a perturbation of the linear *solution*

$$u^{\omega}(t) = U(t)(f_1^{\omega}, f_2^{\omega}) + v^{\omega}(t) = \cos(t\sqrt{-\mathbf{\Delta}}) f_1^{\omega} + \frac{\sin(t\sqrt{-\mathbf{\Delta}})}{\sqrt{-\mathbf{\Delta}}} f_2^{\omega} + v^{\omega}(t),$$

where $v^{\omega} \in C(\mathbb{R}, H^{\sigma}(\Theta))$ for some $\sigma > 1/2$. Moreover

$$\|u^{\omega}(t)\|_{H^s(\Theta)} \leq C(\omega, s) \log(2 + |t|)^{\frac{1}{2}}\,.$$

Remark 1.1. — *Notice that, having normalized the Lebesgue measure*

$$dL = \frac{1}{4\pi} r^2 d\theta dr,$$

the eigenfunctions, e_n, have the following explicit form:

$$e_n(r) = \sqrt{2}\frac{\sin(n\pi r)}{r} \tag{1.2}$$

Remark 1.2. — We have (see [**5**, Lemma 3.2]) that almost surely

$$(f_1^{\omega}, f_2^{\omega}) \in \bigcap_{s<1/2} (H^s(\Theta) \times H^{s-1}(\Theta))$$

but the probability of the event $\{(f_1^{\omega}, f_2^{\omega}) \in H^{\frac{1}{2}}(\Theta) \times H^{-\frac{1}{2}}(\Theta)\}$ is zero. Thus, the randomization process has no smoothing property in the scale of (L^2-based) Sobolev spaces, and in the above statement we obtain global solutions for data which are not in $H^{1/2}(\Theta) \times H^{-1/2}(\Theta)$. Observe that the equation (1.1) is $H^{3/2-2/\alpha} \times H^{1/2-2/\alpha}$ critical. As a consequence, for $2 < \alpha < 3$, we obtain global existence of strong solutions for a supercritical model, a result which seems to be completely out of reach of the present deterministic methods, even for the local existence theory.

Remark 1.3. — The initial data we consider can be arbitrarily large in $L^2(\Theta)$. Consequently, our result is *not* a "small data result".

The rest of the paper is organized as follows. In the next section, we analyze the Hamiltonian structure of the equation and we introduce the suitable finite dimensional model which approximate (1.1). In Section 3, we establish the key probabilistic estimate concerning the L^p space-time norms of the free evolution. In Section 4, we recall the deterministic Strichartz estimates for the free evolution. Section 5 is devoted to local well-posedness results. In Section 6, we perform the globalization argument. The analysis of Section 6 has much in common with some arguments already appeared in [**10, 11, 5**].

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2. Reduction of the problem and approximating ODE

For $\sigma \in \mathbb{R}$, we define (see [**5**] for more details) $H^\sigma(\Theta)$ as

$$H^\sigma(\Theta) \equiv \Big\{ u = \sum_{n=1}^{\infty} c_n e_n,\, c_n \in \mathbb{C} \,:\, \|u\|^2_{H^\sigma(\Theta)} = \sum_{n=1}^{\infty} (n\pi)^{2\sigma} |c_n|^2 < \infty \Big\}$$

Remark (see [**8**]) that for $-1/2 < \sigma < 1/2$ these spaces coincide with the classical Sobolev spaces (of radial functions) and are independent of the choice of boundary conditions we made. Following [**5**], we make some algebraic manipulations on (1.1) allowing to write it as a first order equation in t. Set $u \equiv w + i\sqrt{-\mathbf{\Delta}}^{-1}\partial_t w$. Then we have that u solves the equation

$$(i\partial_t - \sqrt{-\mathbf{\Delta}})u - \sqrt{-\mathbf{\Delta}}^{-1}\big(|\mathrm{Re}(u)|^\alpha \mathrm{Re}(u)\big) = 0, \quad u|_{t=0} = u_0, \quad u|_{\mathbb{R}\times\partial\Theta} = 0, \tag{2.1}$$

where $u_0 = f_1 + i\sqrt{-\mathbf{\Delta}}^{-1} f_2$. Equation (2.1) is (formally) an Hamiltonian equation on $L^2(\Theta)$ with Hamiltonian,

$$\frac{1}{2}\|\sqrt{-\mathbf{\Delta}}(u)\|^2_{L^2(\Theta)} + \frac{1}{\alpha+2}\|\mathrm{Re}(u)\|^{\alpha+2}_{L^{\alpha+2}(\Theta)}$$

which is (formally) conserved by the flow of (2.1).

In order to prove Theorem 1, we will need to study (2.1) with initial data given by

$$u_0(r,\omega) = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{n\pi} e_n(r),$$

where $g_n(\omega) = h_n(\omega) + i l_n(\omega)$ are independent normalized complex Gaussian random variables.

For $N \geq 1$, we denote by E_N the N dimensional vector space on $\mathbb{C}$ spanned by $(e_n)_{n=1}^N$. Fix $\chi \in C_0^\infty(-1,1)$ equal to 1 on $(-1/2,1/2)$. Let us define $S_N = \chi(\frac{-\mathbf{\Delta}}{\pi^2 N^2})$. This operator sends L^2 to E_N and satisfies

$$S_N\Big(\sum_{n=1}^{\infty} c_n e_n\Big) = \sum_{n=1}^{\infty} \chi(\frac{n^2}{N^2}) c_n e_n\,.$$

Let us observe that the map S_N we use in this paper is slightly different than the one involved in [**5**]. The reason is that we will use L^p, $p \neq 2$ mapping properties of S_N which do not hold for the map used in [**5**]. More precisely, we have the following statement.

Lemma 2.1. — *Let us fix $p \in [1, \infty]$. There exists $C > 0$ such that for every integer $N \geq 1$, $\|S_N\|_{L^p \to L^p} \leq C$. Moreover, for every $f \in L^p$, $S_N(f) \to f$ as $N \to \infty$ in L^p.*

The proof of Lemma 2.1 is essentially contained in [**4**], where the case of manifolds without boundary is considered. The general case is more involved technically and requires a precise description of the operator S_N (see for example [**9**, section 4.3]). Notice however that the radial assumption we made here would allow a rather direct proof.

We shall approximate the solutions of (2.1) by the solutions of the ODE

$$(2.2) \quad (i\partial_t - \sqrt{-\boldsymbol{\Delta}})u - S_N\left(\sqrt{-\boldsymbol{\Delta}}^{-1}\big(|S_N(\mathrm{Re}(u))|^{\alpha} S_N(\mathrm{Re}(u))\big)\right) = 0,\ u|_{t=0} = u_0 \in E_N\ .$$

Let us define the measure μ_N on E_N as the image measure under the map from $(\Omega, \mathcal{A}, p)$ to E_N (equipped with the Borel sigma-algebra) defined by

$$(2.3) \qquad \omega \longmapsto \sum_{n=1}^{N} \frac{h_n(\omega) + i l_n(\omega)}{n\pi} e_n\,,$$

where $h_n(\omega), l_n(\omega)$, $n = 1, \cdots N$ is a sequence of independent standard real gaussian random variables ($h_n, l_n \in \mathcal{N}(0,1)$). We next define the measure ρ_N as the image measure by the map (2.3) of the measure

$$\exp\Big(-\frac{1}{\alpha+2}\|S_N(\sum_{n=1}^{N}\frac{h_n(\omega)}{n\pi}e_n)\|_{L^{\alpha+2}(\Theta)}^{\alpha+2}\Big)dp(\omega)\,.$$

It turns out that ρ_N is invariant under the flow of (2.2).

Proposition 2.2. — *For every $u_0 \in E_N$ the flow of (2.2) is defined globally in time. Moreover the measure ρ_N is invariant under this flow.*

Proof. — The proof of this result is essentially (in a slightly different context) in [**5**]. For the sake of completeness, we recall briefly the proof. The local existence and uniqueness for the ODE (2.2) follows from the Cauchy-Lipschitz theorem. We can extend globally in time the solutions of (2.2) thanks to the energy conservation law associated to (2.2).

Indeed if we multiply (2.2) by $\mathbf{\Delta} u - S_N(S_N(|\mathrm{Re}(u))|^\alpha S_N(\mathrm{Re}(u)))$ (which is an element of E_N) and integrate over Θ, we get that the solutions of (2.2) satisfy

$$\frac{d}{dt}\Big[\frac{1}{2}\|\sqrt{-\mathbf{\Delta}}(u)\|^2_{L^2(\Theta)}+\frac{1}{\alpha+2}\|S_N(\mathrm{Re}(u))\|^{\alpha+2}_{L^{\alpha+2}(\Theta)}\Big]=0\,.$$

Thus we have a control uniform with respect to time and therefore the solutions of (2.2) are defined globally in time. Let us now turn to the proof of the measure invariance. Let us decompose the solution of (2.2) as

$$u(t)=\sum_{n=1}^{N}\big(a_n(t)+ib_n(t)\big)e_n,\quad a_n(t),b_n(t)\in\mathbb{R}\,.$$

Then, if we set

$$H(a_1,\dots,a_N,b_1,\dots,b_N)\equiv\frac{1}{2}\sum_{n=1}^{N}(\pi n)^2(a_n^2+b_n^2)+\frac{1}{\alpha+2}\int_\Theta\big|S_N(\sum_{n=1}^{N}a_ne_n)\big|^{\alpha+2}$$

the problem (2.2) may be rewritten in the coordinates a_n, b_n as

$$\dot{a}_n=(\pi n)^{-1}\frac{\partial H}{\partial b_n},\quad \dot{b}_n=-(\pi n)^{-1}\frac{\partial H}{\partial a_n},\quad n=1,\dots,N \tag{2.4}$$

(e_n are real valued). Let us first observe that thanks to the structure of (2.4) the quantity $H(a_1,\dots,a_N,b_1,\dots,b_N)$ is conserved under the flow of (2.4). Therefore we may apply Liouville's theorem for divergence free vector fields to obtain that the measure

$$\prod_{n=1}^{N}(\pi n)^2da_ndb_n$$

is conserved by the flow of (2.4). Since $H(a_1,\dots,a_N,b_1,\dots,b_N)$ is also conserved under the flow of (2.4) we obtain that the measure

$$\frac{1}{(2\pi)^N}\exp\big(-H(a_1,\dots,a_N,b_1,\dots,b_N)\big)\prod_{n=1}^{N}(\pi n)^2da_ndb_n$$
$$=\exp\Big(-\frac{1}{(\alpha+2)}\int_\Theta\big|S_N(\sum_{n=1}^{N}a_ne_n)\big|^{\alpha+2}\Big)\prod_{n=1}^{N}\sqrt{\frac{\pi}{2}}ne^{-(\pi n)^2(a_n^2/2)}da_n\sqrt{\frac{\pi}{2}}ne^{-(\pi n)^2(b_n^2/2)}db_n$$

is also conserved by the flow of (2.4) which, coming back to E_N, completes the proof of Proposition 2.2. □

Let us fix from now on in the rest of this paper a number $s < 1/2$. Let us define the measure μ on $H^s(\Theta)$ as the image measure under the map from $(\Omega, \mathcal{A}, p)$ to $H^s(\Theta)$ equipped with the Borel sigma algebra, defined by

$$\omega \longmapsto \sum_{n=1}^{\infty} \frac{h_n(\omega) + il_n(\omega)}{n\pi} e_n \,, \tag{2.5}$$

where $((h_n, l_n))_{n=1}^{\infty}$ is a sequence of independent standard real Gaussian random variables.

Using [**1**, Theorem 4], we have that for $\alpha < 4$ the quantity

$$\| \sum_{n=1}^{\infty} \frac{h_n(\omega) + il_n(\omega)}{n\pi} e_n \|_{L^{\alpha+2}(\Theta)}$$

is finite almost surely. Therefore, we can define a nontrivial measure ρ as the image measure on $H^s(\Theta)$ by the map (2.5) of the measure

$$\exp\Big(- \frac{1}{\alpha+2} \| \sum_{n=1}^{\infty} \frac{h_n(\omega)}{n\pi} e_n) \|_{L^{\alpha+2}(\Theta)}^{\alpha+2} \Big) dp(\omega) \,.$$

Observe that if a Borel set $A \subset H^s(\Theta)$ is of full ρ measure then A is also of full μ measure. Therefore, we need to solve (2.1) globally in time for u_0 in a set of full ρ measure.

We next turn to the limits of the measures ρ_N. We have the following statement.

Lemma 2.3. — *Set*

$$f(u) = \exp\Big(- \frac{1}{\alpha+2} \|u\|_{L^{\alpha+2}(\Theta)}^{\alpha+2} \Big), \quad f_N(u) = \exp\Big(- \frac{1}{\alpha+2} \|S_N(u)\|_{L^{\alpha+2}(\Theta)}^{\alpha+2} \Big) \,. \tag{2.6}$$

Then

$$\lim_{N\to\infty} \int_{H^s(\Theta)} |f_N(u) - f(u)| d\mu(u) = 0 \,.$$

In particular,

$$\lim_{N\to\infty} \rho_N(E_N) = \rho\big(H^s(\Theta)\big). \tag{2.7}$$

Proof. — The argument is very close to the proof of [**11**, Lemma 3.7] and therefore we will only sketch it. Thanks to the analysis of [**1**] (see also [**11**, Lemma 2.3]), we have that $\|S_N(u)\|_{L^{\alpha+2}(\Theta)}^{\alpha+2}$ converges, as $N \to \infty$ to $\|u\|_{L^{\alpha+2}(\Theta)}^{\alpha+2}$ in $L^1(d\mu)$. Therefore $f_N(u)$ converges in measure, as $N \to \infty$ to $f(u)$ with respect to the measure $d\mu$. For $\varepsilon > 0$, we consider the set

$$A_{N,\varepsilon} \equiv \big(u \in H^s(\Theta) \,:\, |f_N(u) - f(u)| \leq \varepsilon\big).$$

Using the Cauchy-Schwarz inequality, we get

$$\int_{A_{N,\varepsilon}^c} |f_N(u) - f(u)| d\mu(u) \leq \|f_N - f\|_{L^2(d\mu)} [\mu(A_{N,\varepsilon}^c)]^{\frac{1}{2}} \leq 2[\mu(A_{N,\varepsilon}^c)]^{\frac{1}{2}} \,.$$

On the other hand

$$\int_{A_{N,\varepsilon}} |f_N(u) - f(u)| d\mu(u) \leq \varepsilon\,.$$

Finally, the convergence in measure of f_N to f implies that for a fixed ε,

$$\lim_{N\to\infty} \mu(A^c_{N,\varepsilon}) = 0.$$

This completes the proof of Lemma 2.3. □

3. Gaussian estimates

Let us recall the following standard Gaussian estimate (see e.g. [**10, 11, 5**]).

Lemma 3.1. — *Let c be a positive constant satisfying $c < \pi/2$. Denote by $B(0,\lambda)_s$ the open ball of center 0 and radius λ in $H^s(\Theta)$. Then there exists $C_s > 0$ such that for every N, λ,*

$$\rho_N(B(0,\lambda)^c_s \cap E_N) \leq \mu_N(B(0,\lambda)^c \cap E_N) \leq C_s e^{-c\lambda^2}\,.$$

Let $S(t) = e^{-it\sqrt{-\mathbf{\Delta}}}$ denote the free evolution operator. Let us observe that for every $t \in \mathbb{R}$, $S(t+2) = S(t)$. The following large deviation estimate will play a crucial role in our analysis.

Proposition 3.2. — *For any $2 \leq p < 6$, there exists $C, c > 0$ such that for any $N, \lambda > 0$,*

$$\mu_N(\{f \in E_N \,:\, \|S(t)f\|_{L^p((0,2)\times\Theta)} > \lambda\}) \leq Ce^{-c\lambda^2}$$

Proof. — We need to show that there exists $C, c > 0$ such that for any $N, \lambda > 0$,

$$p(\{\omega \in \Omega \,:\, \|S(t)f^\omega_N\|_{L^p((0,2)\times\Theta)} > \lambda\}) \leq Ce^{-c\lambda^2}, \tag{3.1}$$

where

$$f^\omega_N = \sum_{n=1}^{N} \frac{g_n(\omega)}{\pi n} e_n\,.$$

Observe that in order to prove (3.1) it suffices to establish the bound

$$\exists\, C > 0, \quad \forall\, q \geq p, \quad \|S(t)f^\omega_N\|_{L^q(\Omega;L^p((0,2)\times\Theta))} \leq C\sqrt{q}. \tag{3.2}$$

Indeed, if we suppose that (3.2) holds true then by the Bienaymé-Tchebichev inequality, we have

$$p(\{\omega \in \Omega \,:\, \|S(t)f^\omega_N\|_{L^p((0,2)\times\Theta)} > \lambda\}) \leq \lambda^{-q}\|S(t)f^\omega_N\|^q_{L^q(\Omega;L^p((0,2)\times\Theta))} \leq C\Big(\frac{\sqrt{q}}{\lambda}\Big)^q$$

and (3.1) follows by taking $q = \lambda^2/2$.

Let us now turn to the proof of (3.2). Recall the general Gaussian bound (see e.g. [**1, 6**])

$$\exists\, C>0,\quad \forall\, q\geq 2,\quad \forall\,(c_n)_{n=1}^{\infty}\in l^2,\quad \big\|\sum_{n=1}^{\infty}c_n g_n(\omega)\big\|_{L^q(\Omega)}\leq C\sqrt{q}\big(\sum_{n=1}^{\infty}|c_n|^2\big)^{\frac12} \tag{3.3}$$

(observe that (3.3) also follows from [**6**, Lemma 3.1]).

For $q\geq p$, using the Minkowski inequality, we can write

$$\|S(t)f_N^{\omega}\|_{L^q(\Omega;L^p((0,2)\times\Theta))}\leq \|S(t)f_N^{\omega}\|_{L^p((0,2)\times\Theta;L^q(\Omega))}\,. \tag{3.4}$$

For fixed t, r, using the Gaussian bound (3.3), we get

$$\|S(t)f_N^{\omega}\|_{L^q(\Omega)}\leq C\sqrt{q}\Big(\sum_n\Big|e^{-it\pi n}\frac{1}{\pi n}e_n(r)\Big|^2\Big)^{1/2}.$$

Therefore, using that $p\geq 2$,

$$\begin{aligned}\|S(t)f_N^{\omega}\|_{L^q(\Omega;L^p((0,2)\times\Theta))} &\leq C\sqrt{q}\Big\|\Big(\sum_n\Big|e^{-it\pi n}\frac{e_n(r)}{n}\Big|^2\Big)^{1/2}\Big\|_{L^p((0,2)\times\Theta)}\\ &\leq C\sqrt{q}\Big\|\sum_n\Big|\frac{e_n}{n}\Big|^2\Big\|_{L^{p/2}(\Theta)}^{1/2}\\ &\leq C\sqrt{q}\Big(\sum_n\Big\|\Big|\frac{e_n}{n}\Big|^2\Big\|_{L^{p/2}(\Theta)}\Big)^{1/2}\\ &\leq C\sqrt{q}\Big(\sum_n\Big\|\frac{e_n}{n}\Big\|_{L^p(\Theta)}^2\Big)^{1/2}.\end{aligned}$$

Next (see [**1**, Lemma 2.5], or the explicit form of e_n (1.2)), we use the estimate

$$\forall\, p\geq 2,\quad \exists\, C>0,\quad \forall\, n\geq 1,\quad \|e_n\|_{L^p(\Theta)}\leq\begin{cases}Cn^{1-\frac{3}{p}}, & 3<p\\ C(\log(n))^{\frac13}, & p=3\\ C, & 2\leq p<3.\end{cases}$$

This gives ($p<6$)

$$\|S(t)f_N^{\omega}\|_{L^q(\Omega;L^p((0,2)\times\Theta))}\leq C\sqrt{q}\,\Big(\sum_n\frac{1}{\min(n^2\log^{-2/3}(n),n^{6/p})}\Big)^{1/2}\leq C\sqrt{q}$$

which completes the proof of (3.2). This ends the proof of Proposition 3.2. □

4. Deterministic Strichartz estimates

In this section we recall the Strichartz estimates for the free evolution (see [**5**, Section 4] and [**6**, Section 2] for the proofs).

Definition 4.1. — *For $T > 0$ and $0 < \sigma < 1$, we define the spaces*

$$X_T^\sigma = C^0([-T,T]; H^\sigma(\Theta)) \cap L^p((-T,T); L^q(\Theta)), (p = \frac{2}{\sigma}, q = \frac{2}{1-\sigma}) \tag{4.1}$$

and its "dual" space

$$Y_T^\sigma = L^1([-T,T]; H^{-\sigma}(\Theta)) + L^{p'}((-T,T); L^{q'}(\Theta)), (p = \frac{2}{\sigma}, q = \frac{2}{1-\sigma}) \tag{4.2}$$

equipped with their natural norms ((p', q') being the conjugate couple of (p, q)).

Remark that a simple interpolation argument gives the following statement.

Lemma 4.2. — *Assume that $0 \le \sigma < 1$ and*

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \sigma, \quad \frac{2}{\sigma} \le p \le +\infty\,.$$

Then

$$\|f\|_{L^p([0,T];L^q(\Theta))} \le C\|f\|_{X_T^\sigma}, \qquad \|f\|_{Y_T^\sigma} \le C\|f\|_{L^{p'}([0,T];L^{q'}(\Theta))} \tag{4.3}$$

Recall that $S(t) = e^{-it\sqrt{-\boldsymbol{\Delta}}}$. We next state several Strichartz inequalities for $S(t)$. We refer to [**5**, Section 4] for the proof.

Proposition 4.3. — *For every $0 < \sigma < \sigma_1 < 1$, there exists $C > 0$ such that for every $T \in]0, 2]$, every $f \in H^\sigma(\Theta)$, $g \in Y_T^{1-\sigma}$, $h \in Y_T^{1-\sigma_1}$,*

$$\|S(t)(f)\|_{X_T^\sigma} \le C\|f\|_{H^\sigma(\Theta)}, \tag{4.4}$$

$$\|\int_0^t S(t-\tau)\sqrt{-\boldsymbol{\Delta}}^{-1}(g)(\tau)d\tau\|_{X_T^\sigma} \le C\|g\|_{Y_T^{1-\sigma}} \tag{4.5}$$

$$\|(1-S_N)\int_0^t \sqrt{-\boldsymbol{\Delta}}^{-1} S(t-\tau)(h)(\tau)d\tau\|_{X_T^\sigma} \le CN^{\sigma-\sigma_1}\|h\|_{Y_T^{1-\sigma_1}}\,. \tag{4.6}$$

Remark 4.4. — The map S_N involved in (4.6) is slightly different than the corresponding one involved in [**5**]. However the proof of [**5**] still works since we have that $(1 - S_N)$ is bounded from H^{σ_1} to H^σ with norm $\le CN^{\sigma-\sigma_1}$.

We shall also make use of the next Strichartz estimate.

Proposition 4.5. — *Let p be such that $4 < p < 6$. Define σ by $\sigma = \frac{3}{2} - \frac{4}{p}$. Then there exist $C > 0$ such that for every $T \in]0, 2]$, every $f \in H^\sigma(\Theta)$ one has*

$$\|S(t)(f)\|_{L^p([-T,T]\times\Theta)} \leq C\|f\|_{H^\sigma(\Theta)}\,.$$

Proof. — Let q be such that $1/p + 1/q = 1/2$. Then the Sobolev inequality and the endpoint of (4.4) yield

$$\begin{aligned}\|S(t)(f)\|_{L^p([-T,T]\times\Theta)} &\leq C\|(1-\mathbf{\Delta})^{\frac{1}{2}(\frac{3}{2}-\frac{6}{p})}S(t)(f)\|_{L^p([-T,T];L^q(\Theta))} \\ &\leq C\|(1-\mathbf{\Delta})^{\frac{1}{2}(\frac{3}{2}-\frac{6}{p})}(f)\|_{H^{\frac{2}{p}}(\Theta)} = C\|f\|_{H^\sigma(\Theta)}\,.\end{aligned}$$

This completes the proof of Proposition 4.5. □

5. Local well-posedness

The problem (2.1) is reduced to the integral equation

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}\big(|\mathrm{Re}(u(\tau))|^\alpha \mathrm{Re}(u(\tau))\big)d\tau\,. \tag{5.1}$$

The next statement provides bounds on the right hand-side of (5.1).

Proposition 5.1. — *For a given positive number $\alpha < 3$ we choose a real number p such that* $\max(4, 2\alpha) < p < 6$. *Then we fix a real number σ by $\sigma = \frac{3}{2} - \frac{4}{p}$. Set $F(u) = |Re(u)|^\alpha u$. Then there exist $C > 0$, $\delta > 0$ such that for every $T \in]0, 2]$, every $u_1, u_2 \in X_T^\sigma$, every $v_1, v_2 \in L^p((-T,T)\times\Theta)$ (radial with respect to the second variable) every $u_0 \in H^\sigma(\Theta)$,*

$$\big\|S(t)u_0\big\|_{X_T^\sigma} \leq C\|u_0\|_{H^\sigma(\Theta)}\,, \tag{5.2}$$

$$\Big\|\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}F(u_1(\tau)+v_1(\tau))d\tau\Big\|_{X_T^\sigma} \leq CT^\delta\big(\|u_1\|_{X_T^\sigma}^{\alpha+1} + \|v_1\|_{L_T^pL^p}^{\alpha+1}\big), \tag{5.3}$$

where $L_T^pL^p$ denotes the norm in $L^p((-T,T)\times\Theta)$. Moreover

$$\begin{aligned}\Big\|(1-S_N)\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}F(u_1(\tau)+v_1(\tau))d\tau\Big\|_{X_T^\sigma} \\ \leq CT^\delta N^{-\delta}\big(\|u_1\|_{X_T^\sigma}^{\alpha+1} + \|v_1\|_{L_T^pL^p}^{\alpha+1}\big)\,,\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\Big\|\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}\Big(F(u_1(\tau)+v_1(\tau)) - F(u_2(\tau)+v_2(\tau))\Big)d\tau\Big\|_{X_T^\sigma} \\ &\leq CT^\delta\Big(\|u_1\|_{X_T^\sigma}^\alpha + \|u_2\|_{X_T^\sigma}^\alpha + \|v_1\|_{L_T^pL^p}^\alpha + \|v_2\|_{L_T^pL^p}^\alpha\Big)\Big(\|u_1-u_2\|_{X_T^\sigma} + \|v_1-v_2\|_{L_T^pL^p}\Big)\end{aligned} \tag{5.5}$$

and

$$\Big\| \int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1} S_N\Big(F(u_1(\tau)+v_1(\tau)) - F(u_2(\tau)+v_2(\tau))\Big) d\tau\Big\|_{X_T^\sigma} \tag{5.6}$$
$$\leq CT^\delta\Big(\|u_1\|^\alpha_{X_T^\sigma} + \|u_2\|^\alpha_{X_T^\sigma} + \|v_1\|^\alpha_{L^p_T L^p} + \|v_2\|^\alpha_{L^p_T L^p}\Big)\Big(\|u_1-u_2\|_{X_T^\sigma} + \|v_1-v_2\|_{L^p_T L^p}\Big)$$

Proof. — Let us first observe that thanks to the assumption $p>4$, we have that $\sigma>1/2$ and thus $p>2/\sigma$. Estimate (5.2) follows from Proposition 4.3. Let us next show (5.4). Using (4.6) and Lemma 4.2 the left hand side of (5.4) is bounded by

$$CN^{\sigma-\sigma_1}\Big(\||\mathrm{Re}(u_1)|^\alpha \mathrm{Re}(u_1)\|_{L^{p'}((-T,T);L^{q'}(\Theta))} + \||\mathrm{Re}(v_1)|^\alpha \mathrm{Re}(v_1)\|_{L^{p'}((-T,T);L^{q'}(\Theta))}\Big) \tag{5.7}$$

where σ_1 (close to σ) is such that $\sigma<\sigma_1<1$ and will be fixed later and (p',q') are such that

$$\frac{1}{p'}+\frac{3}{q'}=\frac{5}{2}+(1-\sigma_1)\,.$$

We take $p'=q'$ and thus $\frac{1}{p'}=\frac{1}{q'}=\frac{7}{8}-\frac{\sigma_1}{4}$. Therefore we can evaluate (5.7) by

$$CN^{\sigma-\sigma_1}\Big(\|u_1\|^{\alpha+1}_{L_T^{(\alpha+1)p'}L^{(\alpha+1)p'}} + \|v_1\|^{\alpha+1}_{L_T^{(\alpha+1)p'}L^{(\alpha+1)p'}}\Big)\,.$$

Thanks to Lemma 4.2 and the Hölder inequality, the proof of (5.4) will be completed if we can provide that $(\alpha+1)p'<p$, i.e.

$$\frac{\alpha+1}{p}<\frac{1}{p'}=\frac{7}{8}-\frac{\sigma_1}{4}\,. \tag{5.8}$$

Let us choose σ_1 as $\sigma_1=\sigma+\varepsilon$, where $\varepsilon>0$ is to be specified. Thus

$$\frac{7}{8}-\frac{\sigma_1}{4}=\frac{1}{2}+\frac{1}{p}-\frac{\varepsilon}{4}\,.$$

Hence (5.8) can be assured if we can choose $\varepsilon>0$ such that

$$\frac{\alpha+1}{p}<\frac{1}{2}+\frac{1}{p}-\frac{\varepsilon}{4},$$

i.e. $\frac{\varepsilon}{4}<\frac{1}{2}-\frac{\alpha}{p}$. Thanks to the assumption $p>2\alpha$, we have that $\frac{1}{2}-\frac{\alpha}{p}>0$ and thus a proper choice of $\varepsilon>0$ is indeed possible. This completes the proof of (5.4). The proof of (5.3) is the same as the proof of (5.4) by choosing $\sigma_1=\sigma$. The proofs of (5.5) and (5.6) are very similar to that of (5.4) by invoking the inequality

$$\exists\, C>0,\quad \forall\,(x,y)\in\mathbb{R}^2,\quad ||x|^\alpha x-|y|^\alpha y|\leq C|x-y|(|x|^\alpha+|y|^\alpha)\,.$$

This completes the proof of Proposition 5.1. □

As a consequence of Proposition 5.1, we infer the following well-posedness results for (2.1).

Proposition 5.2. — *For a given positive number $\alpha < 3$ we choose a real number p such that $\max(4, 2\alpha) < p < 6$. Then we fix a real number σ by $\sigma = \frac{3}{2} - \frac{4}{p}$. There exist $C > 0$, $c \in]0, 1]$, $\gamma > 0$ such that for every $A > 0$ if we set $T = c(1 + A)^{-\gamma}$ then for every radially symmetric u_0 satisfying $\|S(t)u_0\|_{L^p((0,2)\times\Theta)} \leq A$ there exists a unique solution u of (2.1) such that $u(t) = S(t)u_0 + v(t)$ with $v \in X_T^\sigma$. Moreover $\|v\|_{X_T^\sigma} \leq CA$. In particular, since $S(t)$ is 2 periodic and thanks to the Strichartz estimate of Proposition 4.5,*

$$\sup_{t\in[-T,T]} \|S(\tau)u(t)\|_{L^p(\tau\in(0,2);L^p(\Theta))} \leq CA\,.$$

In addition, if $u_0 \in H^s(\Theta)$ (and thus $s < \sigma$) then

$$\|u(t)\|_{H^s(\Theta)} \leq \|S(t)u_0\|_{H^s(\Theta)} + \|v(t)\|_{H^s(\Theta)} \leq \|u_0\|_{H^s(\Theta)} + CA\,.$$

Remark 5.3. — Thanks to Proposition 3.2 the data

$$f^\omega = \sum_{n=1}^{\infty} \frac{g_n(\omega)}{\pi n} e_n$$

satisfies the assumption $\|S(t)f^\omega\|_{L^p((0,2)\times\Theta)} < \infty$, almost surely in ω. Therefore, despite the fact that f^ω is essentially in $H^{1/2}$ and not more regular, and thus supercritical for (2.1) for $2 < \alpha < 3$, Proposition 5.2 establishes a local well-posedness theory for data f^ω almost surely in ω. We refer to Part I (cf. [**6**]) for a general local well-posedness theory for the cubic wave equation posed on a compact manifold with random initial data.

Proof of Proposition 5.2.. — If we write $u(t) = S(t)u_0 + v(t)$ then $v(0) = 0$ and v solves the problem

$$(i\partial_t - \sqrt{-\mathbf{\Delta}})v - \sqrt{-\mathbf{\Delta}}^{-1}\big(|\mathrm{Re}(S(t)u_0 + v)|^\alpha \mathrm{Re}(S(t)u_0 + v)\big) = 0$$

with corresponding integral equation

$$v(t) = -i\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}\big(|\mathrm{Re}(S(\tau)u_0 + v(\tau))|^\alpha \mathrm{Re}(S(\tau)u_0 + v(\tau))\big)d\tau \equiv K_{u_0}(v)\,.$$

Using (5.3) and (5.5) of Proposition 5.1, we infer that for u_0 such that for $T \in]0, 2]$, $\|S(t)u_0\|_{L^p((0,2)\times\Theta)} \leq A$,

$$\|K_{u_0}(v)\|_{X_T^\sigma} \leq CT^\delta A^{\alpha+1} + CT^\delta \|v\|_{X_T^\sigma}^{\alpha+1}$$

and

$$\|K_{u_0}(v_1) - K_{u_0}(v_2)\|_{X_T^\sigma} \leq CT^\delta \|v_1 - v_2\|_{X_T^\sigma}(A^\alpha + \|v_1\|_{X_T^\sigma}^\alpha + \|v_2\|_{X_T^\sigma}^\alpha)\,.$$

Proposition 5.2 follows by applying the contraction mapping principle to the nonlinear map K_{u_0} on the ball of radius A of X_T^σ (centered at the origin) with $T = c(1 + A)^{-\gamma}$ for a suitable choice of $c \ll 1$ and $\gamma \gg 1$. □

Thanks to (5.6) and the fact that S_N is (uniformly with respect to N) bounded on X_T^σ (see Lemma 2.1), we can apply the argument of the proof of Proposition 5.2 to obtain a well-posedness in the context of (2.2) with bounds independent of N.

Proposition 5.4. — *For a given positive number $\alpha < 3$ we choose a real number p such that* $\max(4, 2\alpha) < p < 6$. *Then we fix a real number σ by $\sigma = \frac{3}{2} - \frac{4}{p}$. There exist $C > 0$, $c \in]0,1]$, $\gamma > 0$ such that for every $A > 0$ if we set $T = c(1+A)^{-\gamma}$ then, for every $N \geq 1$ and for every $u_{0,N} \in E_N$ satisfying $\|S(t)u_{0,N}\|_{L^p((0,2)\times\Theta)} \leq A$ the unique solution u_N of (2.2) given by Proposition 2.2 satisfies*

$$u_N(t) = S(t)u_{0,N} + v_N(t)$$

with

$$\|v_N\|_{X_T^\sigma} \leq CA$$

$$\sup_{t\in[-T,T]} \|S(\tau)u_N(t)\|_{L^p(\tau\in(0,2);L^p(\Theta))} \leq CA\,,$$

$$\|u_N(t)|_{H^s(\Theta)} \leq \|S(t)u_{0,N}\|_{H^s(\Theta)} + \|v_N(t)\|_{H^s(\Theta)} \leq \|u_{0,N}\|_{H^s(\Theta)} + CA\,.$$

6. Global existence for (2.1) on a set of full ρ measure

Let us denote by $\Phi_N(t) : E_N \to E_N$, $t \in \mathbb{R}$ the flow of (2.2) defined in Proposition 2.2. In the next proposition, we obtain a crucial long time bound for the solutions of (2.2) (a similar argument was already performed in [**10, 11, 5**]).

Proposition 6.1. — *Let us fix p such that $2\alpha < p < 6$. Then for every integer $i \geq 1$, every integer $N \geq 1$, there exists a ρ_N measurable set $\Sigma_N^i \subset E_N$ such that $\rho_N(E_N \backslash \Sigma_N^i) \leq 2^{-i}$ and there exists a constant C such that for every $i, N \in \mathbb{N}$, every $u_0 \in \Sigma_N^i$, every $t \in \mathbb{R}$,*

$$\|S(\tau)(\Phi_N(t)(u_0))\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|\Phi_N(t)(u_0)\|_{H^s(\Theta)} \leq C(i + \log(1+|t|))^{\frac{1}{2}}\,. \tag{6.1}$$

Proof. — For i, j integers ≥ 1, we set

$$B_N^{i,j}(D) \equiv \big\{u \in E_N \,:\, \|S(t)u\|_{L^p((0,2)\times\Theta)} + \|u\|_{H^s(\Theta)} \leq D(i+j)^{\frac{1}{2}}\big\},$$

where the number $D \gg 1$ (independent of i, j, N) will be fixed later. Thanks to Lemma 3.1, Proposition 3.2, we have that

$$\rho_N(E_N \backslash B_N^{i,j}(D)) \leq Ce^{-cD^2(i+j)}\,. \tag{6.2}$$

Thanks to Proposition 5.4, there exist $c > 0$, $C > 0$, $\gamma > 0$ only depending on α such that if we set $\tau \equiv cD^{-\gamma}(i+j)^{-\gamma/2}$ then for every $t \in [-\tau, \tau]$,

$$\Phi_N(t)\big(B_N^{i,j}(D)\big) \subset \Big\{u \in E_N \,:\, \|S(t)u\|_{L^p((0,2)\times\Theta)} + \|u\|_{H^s(\Theta)} \leq C\,D(i+j)^{\frac{1}{2}}\big\}\,. \tag{6.3}$$

Next, we set

$$\Sigma_N^{i,j}(D) \equiv \bigcap_{k=-[2^j/\tau]}^{[2^j/\tau]} \Phi_N(-k\tau)(B_N^{i,j}(D)),$$

where $[2^j/\tau]$ stands for the integer part of $2^j/\tau$. Using the invariance of the measure ρ_N by the flow Φ_N (Proposition 2.2), we can write

$$\rho_N(E_N \backslash \Sigma_N^{i,j}(D)) \leq (2[2^j/\tau]+1)\rho_N(E_N\backslash B_N^{i,j}(D)) \leq C2^j D^\gamma (i+j)^{\gamma/2}\rho_N(E_N\backslash B_N^{i,j}(D)).$$

Using (6.2), we now deduce

$$\rho_N(E_N\backslash\Sigma_N^{i,j}(D)) \leq C2^j D^\gamma (i+j)^{\gamma/2} e^{-cD^2(i+j)} \leq 2^{-(i+j)}, \tag{6.4}$$

provided $D \gg 1$, independently of i, j, N. Thanks to (6.3), we obtain that for $u_0 \in \Sigma_N^{i,j}(D)$, the solution of (2.2) with data u_0 satisfies

$$\big\|S(\tau)\big(\Phi_N(t)(u_0)\big)\big\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|\Phi_N(t)(u_0)\|_{H^s(\Theta)} \leq CD(i+j)^{\frac{1}{2}}, \quad |t| \leq 2^j. \tag{6.5}$$

Indeed, for $|t| \leq 2^j$, we may find an integer $k \in [-[2^j/\tau],[2^j/\tau]]$ and $\tau_1 \in [-\tau,\tau]$ so that $t = k\tau + \tau_1$ and thus $u(t) = \Phi_N(\tau_1)\big(\Phi_N(k\tau)(u_0)\big)$. Since $u_0 \in \Sigma_N^{i,j}(D)$ implies that $\Phi_N(k\tau)(u_0) \in B_N^{i,j}(D)$, we may apply (6.3) and arrive at (6.5). Next, we set

$$\Sigma_N^i = \bigcap_{j=1}^{\infty} \Sigma_N^{i,j}(D).$$

Thanks to (6.4), $\rho_N(E_N\backslash\Sigma_N^i) \leq 2^{-i}$. In addition, using (6.5), we get that there exists C such that for every i, every N, every $u_0 \in \Sigma_N^i$, every $t \in \mathbb{R}$,

$$\big\|S(\tau)\big(\Phi_N(t)(u_0)\big)\big\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|\Phi_N(t)(u_0)\|_{H^s(\Theta)} \leq C(i+\log(1+|t|))^{\frac{1}{2}}. \tag{6.6}$$

Indeed for $t \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $2^{j-1} \leq 1+|t| \leq 2^j$ and we apply (6.5) with this j. This completes the proof of Proposition 6.1. □

For integers $i \geq 1$ and $N \geq 1$, we define the cylindrical sets

$$\tilde{\Sigma}_N^i \equiv \big\{u \in H^s(\Theta) \,:\, S_N(u) \in \Sigma_N^i\big\}.$$

Next, for an integer $i \geq 1$, we set

$$\Sigma^i \equiv \limsup_{N\to\infty} \tilde{\Sigma}_N^i \equiv \bigcap_{N=1}^{\infty} \bigcup_{N_1=N}^{\infty} \tilde{\Sigma}_{N_1}^i,$$

we get

$$\rho(\limsup_{N\to\infty} \tilde{\Sigma}_N^i) = \lim_{N\to\infty} \rho(\bigcup_{N_1=N}^{\infty} \tilde{\Sigma}_{N_1}^i) \geq \limsup_{N\to\infty} \rho(\tilde{\Sigma}_N^i). \tag{6.7}$$

We have that

$$\rho(\tilde{\Sigma}^i_N) = \int_{\tilde{\Sigma}^i_N} f(u)d\mu(u)$$

and

$$\rho_N(\Sigma^i_N) = \int_{\Sigma^i_N} f_N(u)d\mu_N(u) = \int_{\tilde{\Sigma}^i_N} f_N(u)d\mu(u)$$

where f and f_N are defined by (2.6). Therefore, thanks to Lemma 2.3, we get

$$\lim_{N\to\infty} \big((\rho(\tilde{\Sigma}^i_N) - \rho_N(\Sigma^i_N)\big) = 0\,.$$

Therefore, using Proposition 6.1 and (2.7), we obtain

$$\limsup_{N\to\infty} \rho(\tilde{\Sigma}^i_N) = \limsup_{N\to\infty} \rho_N(\Sigma^i_N) \geq \limsup_{N\to\infty} \big(\rho_N(E_N) - 2^{-i}\big) = \rho\big(H^s(\Theta)\big) - 2^{-i}. \tag{6.8}$$

Collecting (6.7) and (6.8), we arrive at

$$\rho(\Sigma^i) \geq \rho\big(H^s(\Theta)\big) - 2^{-i}.$$

Now, we set

$$\Sigma \equiv \bigcup_{i\geq 1} \Sigma^i\,.$$

Thus Σ is of full ρ measure. It turns out that one has global existence for $u_0 \in \Sigma$.

Proposition 6.2. — *Choose a real number p such that $\max(4, 2\alpha) < p < 6$ and then a real number σ by $\sigma = \frac{3}{2} - \frac{4}{p}$ (so that we are in the scope of the applicability of Propositions 5.2, 5.4, 6.1). Let us fix $i \in \mathbb{N}$. Then for every $u_0 \in \Sigma^i$, the local solution u of (2.1) given by Proposition 5.2 is globally defined. In addition there exists $C > 0$ such that for every $u_0 \in \Sigma^i$,*

$$\forall t \in \mathbb{R}, \quad \|u(t)\|_{H^s(\Theta)} + \|S(\tau)(u(t))\|_{L^p(\tau\in(0,2);L^p(\Theta))} \leq C(i + \log(1+|t|))^{\frac{1}{2}}\,. \tag{6.9}$$

Proof. — Let $u_0 \in \Sigma^i$. Then there exists $N_k \to \infty$ such that $S_{N_k}(u_0) \in \Sigma^i_{N_k}$. Set $u_{0,k} \equiv S_{N_k}(u_0)$. Then $u_{0,k}$ is a sequence such that

$$\lim_{k\to\infty} \|u_0 - u_{0,k}\|_{H^s(\Theta)} = 0.$$

Furthermore, thanks to (6.1), we have

$$\|S(t)u_{0,k}\|_{L^p((0,2)\times\Theta)} \leq Ci.$$

After possibly extracting a subsequence, we have that $S(t)u_{0,k}$ converges in L^p for the weak topology to a function $g \in L^p((0,2)\times\Theta)$. But, as $S(t)u_{0,k}$ converges in $\mathcal{D}'$ to $S(t)u_0$, we deduce that $S(t)u_0 = g \in L^p((0,2)\times\Theta)$. But, thanks to Lemma 2.1, the family $(S_{N_k})_{k=1}^{\infty}$ is uniformly bounded on $L^p((0,2)\times\Theta)$, and

$$\forall\, g \in L^p((0,2)\times\Theta), \quad \lim_{k\to+\infty} \|g - S_{N_k} g\|_{L^p((0,2)\times\Theta)} = 0.$$

Indeed, it is true if $g \in C_0^\infty((0,2)\times\Theta)$ and follows for general g by density. As a consequence, we deduce

$$\lim_{k\to\infty} \|S(t)(u_0 - u_{0,k})\|_{L^p((0,2)\times\Theta)} + \|u_0 - u_{0,k}\|_{H^s(\Theta)} = 0. \tag{6.10}$$

Let us fix $T > 0$. Our aim is to extend the solution of (2.1) given by Proposition 5.2 to the interval $[-T,T]$. Using Proposition 6.1, we have that there exists a constant C such that for every $k \in \mathbb{N}$, every $t \in \mathbb{R}$,

$$\|S(\tau)(\Phi_{N_k}(t)(u_{0,k}))\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|\Phi_{N_k}(t)(u_{0,k})\|_{H^s(\Theta)} \leq C(i + \log(1+|t|))^{\frac{1}{2}}. \tag{6.11}$$

To prove (6.9), we are going to pass to the limit $k \to +\infty$ in (6.11). If we set $u_{N_k}(t) \equiv \Phi_{N_k}(t)(u_{0,k})$ and $\Lambda \equiv C(i + \log(1+T))^{\frac{1}{2}}$, we have the bound

$$\|S(\tau)(u_{N_k}(t))\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|u_{N_k}(t)\|_{H^s(\Theta)} \leq \Lambda, \quad \forall\, |t| \leq T,\ \forall\, k \in \mathbb{N}. \tag{6.12}$$

In particular

$$\|S(\tau)(u_0)\|_{L^p(\tau\in(0,2);L^p(\Theta))} + \|u_0\|_{H^s(\Theta)} \leq \Lambda \tag{6.13}$$

(apply (6.12) with $t = 0$ and let $k \to \infty$). Let $\tau > 0$ be the local existence time for (2.1), provided by Proposition 5.2 for $A = \Lambda$. Recall that we can assume $\tau = c(1+\Lambda)^{-\gamma}$ for some $c > 0$, $\gamma > 0$ depending only on p. We can also assume that $T > \tau$. Denote by $u(t)$ the solution of (2.1) with data u_0 on the time interval $[-\tau,\tau]$. Define v by $u(t) = S(t)(u_0) + v(t)$. Thanks to (6.13) and Proposition 5.2, we have that

$$\|v\|_{X^\sigma_\tau} + \|u(t)\|_{H^s(\Theta)} + \|S(\tau_1)(u(t))\|_{L^p(\tau_1\in(0,2);L^p(\Theta))} \leq C\Lambda, \quad t \in [-\tau,\tau], \tag{6.14}$$

where C depends only on p. Next we define $v_{N_k}(t)$ by $u_{N_k}(t) = S(t)(u_{0,k}) + v_{N_k}(t)$. Thanks to (6.12) and Proposition 5.4, we have that

$$\|v_{N_k}\|_{X^\sigma_\tau} + \|u_{N_k}(t)\|_{H^s(\Theta)} + \|S(\tau_1)(u_{N_k}(t))\|_{L^p(\tau_1\in(0,2);L^p(\Theta))} \leq C\Lambda, \quad t \in [-\tau,\tau]. \tag{6.15}$$

We have that $w_{N_k} \equiv v - v_{N_k}$ solves the equation

$$(i\partial_t - \sqrt{-\mathbf{\Delta}})w_{N_k} = \sqrt{-\mathbf{\Delta}}^{-1}\Big(F(u) - S_{N_k}(F(S_{N_k}(u_{N_k})))\Big), \quad w_{N_k}|_{t=0} = 0, \tag{6.16}$$

where $F(u) = |\mathrm{Re}(u)|^\alpha \mathrm{Re}(u)$. Next, we write

$$F(u) - S_{N_k}(F(S_{N_k}(u_{N_k}))) = S_{N_k}\big(F(u) - F(S_{N_k}(u_{N_k}))\big) + (1 - S_{N_k})F(u).$$

Therefore

$$(6.17)\quad w_{N_k}(t) = -i\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1} S_{N_k}\big(F(u(\tau)) - F(S_{N_k}(u_{N_k})(\tau))\big)d\tau$$
$$- i\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}(1-S_{N_k})F(u(\tau))d\tau.$$

Using Proposition 5.1, we obtain that there exist $C>0$ and $\theta,\delta>0$ depending only on p such that one has the bound

$$\|(1-S_{N_k})\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}F(u(\tau))d\tau\|_{X^\sigma_\tau} \leq C\tau^\theta N_k^{-\delta}\big(\|S(t)(u_0)\|^{\alpha+1}_{L^p((-\tau,\tau)\times\Theta)} + \|v\|^{\alpha+1}_{X^\sigma_\tau}\big).$$

Another use of Proposition 5.1 yields

$$\|\int_0^t S(t-\tau)\sqrt{-\mathbf{\Delta}}^{-1}S_{N_k}\big(F(u(\tau)) - F(S_{N_k}(u_{N_k})(\tau))\big)d\tau\|_{X^\sigma_\tau}$$
$$\leq C\tau^\theta\big(\|S(t)(u_0 - S_{N_k}(u_{0,k}))\|_{L^p((-\tau,\tau)\times\Theta)} + \|v - S_{N_k}(v_{N_k})\|_{X^\sigma_\tau}\big)\times$$
$$\big(\|S(t)(u_0)\|^\alpha_{L^p((-\tau,\tau)\times\Theta)} + \|S(t)(u_{0,k})\|^\alpha_{L^p((-\tau,\tau)\times\Theta)} + \|v_{N_k}\|^\alpha_{X^\sigma_\tau} + \|v\|^\alpha_{X^\sigma_\tau}\big).$$

Collecting the last two bounds (6.12), (6.13), (6.14), (6.15), coming back to (6.17) yields

$$\|w_{N_k}\|_{X^\sigma_\tau} \leq C\tau^\theta(1+\Lambda)^\alpha\|w_{N_k}\|_{X^\sigma_\tau} + o(1)_{k\to+\infty}\,.$$

Recall that $\tau = c(1+\Lambda)^{-\gamma}$, where $c>0$ and $\gamma>0$ are depending only on p. In the last estimate the constants C and θ also depend only on p. Therefore, if we assume that $\gamma > \alpha/\theta$ then the restriction on γ remains to depend only on p. Similarly, if we assume that c is so small that $C\tau^\theta(1+\Lambda)^\alpha \leq Cc^\theta(1+\Lambda)^{-\gamma\theta}(1+\Lambda)^\alpha \leq Cc^\theta < 1/2$ then the smallness restriction on c remains to depend only on p. Therefore, we have that after possibly slightly modifying the values of c and γ (keeping c and γ independent of N_k) in the definition of τ that

$$(6.18)\qquad \lim_{k\to\infty}\|w_{N_k}\|_{X^\sigma_\tau} = 0,$$

where $\tau = c(1+\Lambda)^{-\gamma}$ and the constants c and γ depend only on p. Hence

$$(6.19)\qquad \lim_{k\to\infty}\|u - u_{N_k} - S(t)(u_0 - u_{0,k})\|_{X^\sigma_\tau} = 0\,.$$

Coming back to (6.10), we obtain that

$$(6.20)\qquad \lim_{k\to\infty}\|u(\tau) - u_{N_k}(\tau)\|_{H^s(\Theta)} = 0\,.$$

Moreover combining (6.19) with (6.10) and the Strichartz inequality of Proposition 4.5 yields

$$(6.21)\qquad \lim_{k\to\infty}\|S(\tau_1)(u(\tau) - u_{N_k}(\tau))\|_{L^p(\tau_1\in(0,2);L^p(\Theta))} = 0.$$

As a consequence of (6.20), (6.21) and (6.12), we infer that

$$\|u(\tau)\|_{H^s(\Theta)} + \|S(\tau_1)(u(\tau))\|_{L^p(\tau_1\in(0,2);L^p(\Theta))} \leq \Lambda\,. \tag{6.22}$$

Thanks to (6.20), (6.21) and (6.22) we can repeat the argument on $(\tau, 2\tau)$, $(2\tau, 3\tau)$, ...$([\frac{T}{\tau}]\tau, ([\frac{T}{\tau}]+1)\tau)$ (and similarly for negative times), giving existence up to the time T (which was an arbitrary number) and (6.9). This completes the proof of Proposition 6.2. □

Therefore we solved globally the problem (2.1) on a set of full ρ measure. This completes the proof of Theorem 1.

Remark 6.3. — *It is likely that as in* [**5**]*, where the easier sub-critical problem is studied, we may further push the analysis in order to prove that the measure ρ is indeed invariant under the flow of (2.1) established by Theorem 1. We decided not to pursue this issue here since our main concern in the present paper is to establish random data Cauchy theory for supercritical problems. We refer to* [**2, 3, 7, 10, 11, 12**] *for results concerning the existence of invariant Gibbs measures in the closely related context of the Nonlinear Schrödinger equation.*

References

[1] A. Ayache, N. Tzvetkov, *L^p properties of Gaussian random series*, to appear in Trans. AMS.

[2] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. 166 (1994) 1-26.

[3] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996) 421-445.

[4] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the non linear Schrödinger equation on compact manifolds*, Amer. J. of Math., 126 (2004) 569-605.

[5] N. Burq, N. Tzvetkov, *Invariant measure for the three dimensional nonlinear wave equation*, Preprint 2007, *http://arxiv.org/abs/0707.1445*.

[6] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations I : local existence theory*, Preprint 2007, *http://arxiv.org/abs/0707.1447*.

[7] S. Kuksin, A. Shirikyan, *Randomly forced CGL equation : stationary measures and the inviscid limit*, J. Phys A 37 (2004) 1-18.

[8] J.L. Lions, Magenes *Problèmes aux limites non homogènes et applications*, Dunod, Paris 1968.

[9] J. Szeftel, *Propagation et réflexion des singularités pour l'équation de Schrödinger non linéaire*, Ann. Inst. Fourier, 55 (2005) 573–671.

[10] N. Tzvetkov, *Invariant measures for the Nonlinear Schrödinger equation on the disc*, Dynamics of PDE 3 (2006) 111-160.

[11] N. Tzvetkov, *Invariant measures for the defocusing NLS*, Preprint 2007, to appear in Ann. Inst. Fourier.

[12] P. Zhidkov, *KdV and nonlinear Schrödinger equations : Qualitative theory*, Lecture Notes in Mathematics 1756, Springer 2001.

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