STRICHARTZ ESTIMATES FOR WATER WAVES

by

T. Alazard, N. Burq & C. Zuily

Abstract. — In this paper we investigate the dispersive properties of the solutions of the two dimensional water-waves system with surface tension. First we prove Strichartz type estimates with loss of derivatives at the same low level of regularity we were able to construct the solutions in [**2**]. On the other hand, for smoother initial data, we prove that the solutions enjoy the optimal Strichartz estimates (i.e, without loss of regularity compared to the system linearized at $(\eta = 0, \psi = 0)$).

Résumé. — Nous nous intéressons dans cet article aux propriétés dispersives du système des ondes de surface en dimension 2, avec tension de surface. Nous démontrons tout d'abord des estimées de Strichartz, avec pertes de dérivées, au niveau de régularité où nous avons construit des solutions dans [**2**]. Ensuite, pour des données initiales plus régulières, nous démontrons les estimées de Strichartz optimales (i.e. sans perte de régularité par rapport à celles du système linéarisé en $(\eta = 0, \psi = 0)$)

1. Introduction

In a time-dependent domain $\Omega_t \subset \mathbf{R}^2$ which is located between a free hypersurface Σ_t and a fixed known bottom Γ, consider a potential flow $v = \nabla_{x,y}\phi$, with

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega_t, \quad \partial_n\phi = 0 \quad \text{on } \Gamma.$$

The surface-tension water-waves problem is given by two equations: a kinematic condition (which states that the free surface moves with the fluid), and a dynamic condition (that expresses a balance of forces across the free surface). The system reads

$$\begin{cases} \partial_t\eta = \partial_y\phi - \nabla\eta\cdot\nabla\phi & \text{on } \Sigma_t = \{y = \eta(t,x)\}, \\ \partial_t\phi + \dfrac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + g\eta = H(\eta) & \text{on } \Sigma_t, \end{cases} \tag{1.1}$$

Support by the French Agence Nationale de la Recherche, project EDP Dispersives, référence ANR-07-BLAN-0250, is acknowledged.

where $\nabla = \partial_x$, $g > 0$ is the acceleration of gravity and

$$H(\eta) = \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right).$$

is the mean curvature of the free surface.

1.1. Assumptions. — We work in a fluid domain such that there is uniformly a minimum depth of water, more precisely we assume that for each time t one has

$$\Omega_t = \Omega_{1,t} \cap \Omega_2$$

where $\Omega_{1,t}$ is the half space located below the free surface Σ_t,

$$\Omega_{1,t} = \{\, (x,y) \in \mathbf{R}\times\mathbf{R} \,:\, y < \eta(t,x) \,\}$$

for some unknown function η and Ω_2 contains a fixed strip around Σ_t, that means that there exists $h > 0$ such that,

$$\{(x,y) \in \mathbf{R}\times\mathbf{R} \,:\, \eta(t,x) - h \le y \le \eta(t,x)\} \subset \Omega_2, \tag{1.2}$$

for all $t \in [0,T]$. We shall also assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

We emphasize that no regularity assumption is made on the bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We consider both cases of infinite depth and bounded depth bottoms (and all cases in-between). Finally, we could consider the cases where the free surface is a graph over a given smooth hypersurface and the bottom is time dependent.

1.2. Main results. — Following Zakharov we reduce the system to a system on the free surface. If $\psi = \psi(t,x) \in \mathbf{R}$ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then $\phi(t,x,y)$ is the unique variational solution of

$$\Delta\phi = 0 \quad \text{in } \Omega_t, \qquad \phi(t,x,\eta(t,x)) = \psi(t,x). \tag{1.3}$$

The Dirichlet-Neumann operator is then defined by

$$(G(\eta)\psi)(t,x) = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)} = \partial_y\phi - \nabla\eta\cdot\nabla\phi\Big|_{y=\eta(t,x)}.$$

(we refer to Section 2 in [**2**] for a precise construction).

Then (η,ϕ) is solution of the water-waves system (1.1) if and only if (η,ψ) solves the system

$$\begin{cases} \partial_t\eta - G(\eta)\psi = 0, \\ \partial_t\psi + g\eta - H(\eta) + \dfrac{1}{2}|\partial_x\psi|^2 - \dfrac{1}{2}\dfrac{\big(\partial_x\eta\cdot\partial_x\psi + G(\eta)\psi\big)^2}{1+|\partial_x\eta|^2} = 0. \end{cases} \tag{1.4}$$

Concerning the Cauchy theory for the water waves with surface tension, there are many results starting from the pionneering work of K. Beyer and M. Günther [**11**]. See D. M. Ambrose and N. Masmoudi [**7**], B. Schweiser [**25**], T. Iguchi [**20**], D. Coutand and S. Shkoller

[**18**], J. Shatah and C. Zeng [**26**], M. Ming and Z. Zhang [**23**], F. Rousset and N. Tzvetkov [**24**]. In [**2**], we established new local well posedness results for the system (1.4) under sharp (as long as no dispersive effects are taken into account) regularity assumptions on the initial data. We refer to the introduction of [**2**] for references and a short historical survey of the background of these problems.

The purpose of this work is precisely, in the case $d = 1$, to investigate the dispersive properties of these solutions. Our results are twofold: first we prove Strichartz type estimates with loss of derivatives at the very same level of regularity we were able to construct the solutions in [**2**]. On the other hand, for smoother initial data, we prove that the solutions enjoy the optimal Strichartz estimates (i.e, without loss of regularity compared to the system linearized at $(\eta = 0, \psi = 0)$).

Define the usual Besov space,

$$u \in B^{\sigma}_{\infty,2}(\mathbf{R}) \Longleftrightarrow \sum_{j\in\mathbf{N}} 2^{2j\sigma}\|\Delta_j(u)\|^2_{L^\infty(\mathbf{R})} < +\infty,$$

where $u = \sum_j \Delta_j(u)$ is the standard Littelwood-Paley decomposition of u. Notice that if $\sigma \notin \mathbf{N}$, we have (with continuous injection)

$$B^{\sigma}_{\infty,2}(\mathbf{R}) \subset W^{\sigma,\infty}(\mathbf{R}),$$

where $W^{\sigma,\infty}(\mathbf{R})$ is the usual Hölder C^σ space (which, if $\sigma \notin \mathbb{N}$, is characterized by the fact that $\big(2^{j\sigma}\|\Delta_j(u)\|_{L^\infty(\mathbf{R})}\big)_{j\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, see for example [**16**, Proposition 2.3.1].

Our main results are the following.

Theorem 1.1. — *Let $s > 5/2$ and $T > 0$. Consider a solution (η, ψ) of* (1.4) *on the time interval $I = [0, T]$ such that Ω_t satisfies* (1.2) *for $t \in I$. If*

$$(\eta, \psi) \in C^0\big(I, H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})\big),$$

then

$$(\eta, \psi) \in L^4\big(I, B^{s+\frac{1}{4}}_{\infty,2}(\mathbf{R}) \times B^{s-\frac{1}{4}}_{\infty,2}(\mathbf{R})\big).$$

Theorem 1.2. — *Let $s > 11/2$, $T > 0$. Consider a solution (η, ψ) of* (1.4) *on the time interval $I = [0, T]$ such that Ω_t satisfies* (1.2) *for $t \in I$. If*

$$(\eta, \psi) \in C^0\big(I, H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})\big),$$

then

$$(\eta, \psi) \in L^4\big(I, B^{s+\frac{3}{8}}_{\infty,2}(\mathbf{R}) \times B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R})\big).$$

Remark 1.3. — (i) Theorem 1.1 was obtained recently under the assumption $s \geq 15$ by Christianson-Hur-Staffilani [**17**] .

(ii) Let $s > 5/2$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})$ satisfying $\mathrm{dist}(\Sigma_0, \Gamma) \geq c > 0$, we proved in [**2**] that there exist $T > 0$ and a solution $(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})\big)$ satisfying $\mathrm{dist}(\Sigma_t, \Gamma) \geq c > 0$.

(iii) The gain of regularity exhibited in Theorem 1.2 is optimal as can be seen at the level of the linearized system around the trivial solution $(\eta, \psi) = (0, 0)$ which reads (when $g = 0$),

$$\partial_t \eta - |D_x| \psi = 0, \quad \partial_t \psi - \Delta \eta = 0.$$

Indeed $u = |D_x|^{\frac{1}{2}} \eta + i\psi$ is a solution of the equation $i\partial_t u - |D_x|^{\frac{3}{2}} u = 0$, for which one can prove the optimal estimate

$$\left\| \exp(-it\,|D_x|^{\frac{3}{2}}) u_0 \right\|_{L^4\left(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R})\right)} \leq C \|u_0\|_{H^s(\mathbf{R})},$$

which gives the desired regularity on (η, ψ).

(iv) It is most likely that Theorem 1.1 remains valid when $\mathbf{R}$ is replaced by the one dimensional torus $\mathbf{T}$. Indeed, our proof relies on a semi-classical parametrix (on time intervals taylored to the frequency) which exhibits finite speed of propagation and which can consequently be easily localized in space.

(v) Notice that the dispersive estimates proved in this paper can be combined with our previous work to improve the regularity threshold obtained in [**2**] and give local well posednesss for initial data below the $s = 2 + \frac{1}{2}$ threshold. This will be the matter of a forthcoming paper (including the 3-d water-waves system) [**4**].

(vi) Notice finally that dispersive properties of the operator linearized at $(\eta = 0, \psi = 0)$ were used recently by Wu [**32, 33**] and Germain-Masmoudi-Shatah [**19**] to prove global existence results for gravity waves.

1.3. Strategy of the proofs. — Following the approach in Alazard-Métivier [**1**], after suitable paralinearizations, we have shown in [**2**] that the water waves system can be arranged into an explicit paradifferential symmetric equation of Schrödinger type, and we deduced the smoothing effect for the 2-d surface tension water waves. Here, we will also take benefit of this paralinearization reduction, and this reduced system will be our starting point. The guiding line for the rest of our proof is very classical: construction of a parametrix to prove dispersion ($L^1 - L^\infty$ estimates), and then TT^* argument.

There are two main difficulties in the analysis of this equation. First the coefficients of the operator are time dependent and consequently we cannot get rid of the lower order terms by simple conjugation arguments (see Burq-Planchon [**15**]). Second the coefficients enjoy poor regularity, and finally, whereas the principal part in the operator is of order 3/2, the subprincipal part in the operator is of order 1 which gives only a 1/2 difference compared to the usual 1 difference encountered for magnetic Schrödinger operators. As will be shown in our analysis, the presence of such subprincipal parts will produce non trivial oscillations which here have to be taken into account in the analysis.

The first common step for both theorems is to perform several reductions for the paradifferential equation. The first one is to use Alinhac's para-composition theory [**5**] (see also Burq-Planchon [**15**] where a similar idea was used) to reduce the matters to the study of a Schrödinger type operator with constant coefficients principal part. This is particular to space dimension 1 and reflects the fact that there is only one metric on $\mathbf{R}$. The

second reduction, inspired by works by Smith [**27**], Bahouri-Chemin [**8**], Tataru [**29, 30**] and Blair [**12**], consists in smoothing out the coefficients of the operator.

Once this reduction has been achieved, we can construct the parametrix, for which the natural time is the semi-classical one: $s = t|\xi|^{-1/2}$. Here the differences between our two theorems appear. Indeed, in the proof of Theorem 1.1, following the strategy in Burq-Gérard-Tzvetkov [**14**] (see also Staffilani-Tataru [**28**] and Koch-Tataru [**21**]), we construct the parametrix on small times $|s| \leq c$) and the main difficulty is to handle sharp regularity threshold (for smooth enough initial data the proof would be much simpler). In the proof of Theorem 1.2 the difficulties are different: first we have to handle the oscillations generated by the subprincipal part and furthermore we have to prove very large time asymptotics ($|s| \leq c|\xi|^{1/2}$) in the high frequency regime $|\xi| \to +\infty$. Notice that, even for initial data with arbitrarily large smoothness, the analysis would be non trivial. Finally, once the parametrix is constructed, the dispersion estimate is obtained by using non classical stationary phase lemmas involving precise controls on the remainder terms.

Acknowledgements The authors would like to thank the referee for a very carefull reading of the manuscript, which lead to improvements in the presentation.

2. Preliminaries

In this section we recall some notations and results from [**2**] which will be used in the sequel.

2.1. Paradifferential calculus. — In this paragraph we review classical facts about Bony's paradifferential calculus (see [**13**]).

For $\rho \in \mathbf{N}$, according to the usual definition, we denote by $W^{\rho,\infty}(\mathbf{R})$ the Sobolev spaces of L^∞ functions whose derivatives of order $\leq \rho$ are in L^∞. For $\rho \in]0, +\infty[\setminus \mathbf{N}$, we denote by $W^{\rho,\infty}(\mathbf{R})$ the space of bounded functions whose derivatives of order $[\rho]$ are uniformly Hölder continuous with exponent $\rho - [\rho]$.

Definition 2.1. — *Given $\rho \geq 0$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R})$ denotes the space of functions $a(x,\xi)$ on $\mathbf{R} \times (\mathbf{R} \setminus 0)$, which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbf{N}$ and all $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R})$ and there exists a constant C_α such that,*

$$\forall\, |\xi| \geq \frac{1}{2}, \quad \left\|\partial^\alpha_\xi a(\cdot,\xi)\right\|_{W^{\rho,\infty}(\mathbf{R})} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}. \tag{2.1}$$

Definition 2.2. — *$\Sigma^m_\rho(\mathbf{R})$ denotes the space of symbols $a(x,\xi)$ such that*

$$a = \sum_{0 \leq j < \rho} a^{(m-j)} \qquad (j \in \mathbf{N}),$$

where $a^{(m-j)} \in \Gamma^{m-j}_{\rho-j}(\mathbf{R})$ is homogeneous of degree $m-j$ with respect to ξ.

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-1} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\,d\eta, \tag{2.2}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\,dx$ is the Fourier transform of a with respect to the first variable, χ, ψ are two fixed C^∞ functions such that

$$\psi(\eta) = 0 \quad \text{for } |\eta| \le 1, \qquad \psi(\eta) = 1 \quad \text{for } |\eta| \ge 2,$$

$\chi(\theta,\eta)$ is homogeneous of degree 0 and satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad |\theta| \le \varepsilon_1 |\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad |\theta| \ge \varepsilon_2 |\eta|.$$

We shall use quantitative results from Métivier [**22**] about operator norms estimates in symbolic calculus. To do so we introduce the following semi-norms.

Definition 2.3. — *For $m \in \mathbf{R}$, $\rho \ge 0$ and $a \in \Gamma^m_\rho(\mathbf{R})$, we set*

$$M^m_\rho(a) = \sup_{|\alpha| \le \frac{1}{2}+1+\rho} \ \sup_{|\xi| \ge 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R})}. \tag{2.3}$$

The main features of symbolic calculus for paradifferential operators are given by the following theorems.

Definition 2.4. — *Let $m \in \mathbf{R}$. An operator T is said to be of order $\le m$ if, for all $\mu \in \mathbf{R}$, it is bounded from $H^\mu(\mathbf{R})$ to $H^{\mu-m}(\mathbf{R})$.*

Theorem 2.5. — *Let $m \in \mathbf{R}$. If $a \in \Gamma^m_0(\mathbf{R})$, then T_a is of order $\le m$. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \le K M^m_0(a). \tag{2.4}$$

Theorem 2.6 **(Composition)**. — *Let $m \in \mathbf{R}$ and $\rho > 0$. If $a \in \Gamma^m_\rho(\mathbf{R})$ and $b \in \Gamma^{m'}_\rho(\mathbf{R})$ then $T_a T_b - T_{a\#b}$ is of order $\le m + m' - \rho$, where*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial^\alpha_\xi a \partial^\alpha_x b.$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \le K M^m_\rho(a) M^{m'}_\rho(b). \tag{2.5}$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is a called a paraproduct. Paraproducts can also be defined using the Littlewood-Paley decomposition of the frequency space. Indeed, let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a smooth even function with $\phi(t) = 1$ for $|t| \le 1$ and $\phi(t) = 0$ for $|t| \ge 2$. For $k \in \mathbf{N}$, we introduce the symbol

$$\phi_k(\xi) = \phi\Big(\frac{\xi}{2^k}\Big),$$

and then the operators S_k and Δ_k defined by

$$\widehat{S_k f}(\xi) := \phi_k(\xi)\widehat{f}(\xi), \quad \widehat{\Delta_k f}(\xi) := (\phi_k(\xi) - \phi_{k-1}(\xi))\,\widehat{f}(\xi)$$

For all $f \in \mathcal{S}'(\mathbf{R})$, the spectrum of $\Delta_k f$ satisfies $\operatorname{spec} \Delta_k f \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. Hence $\Delta_j \Delta_k = 0$ if $|j-k| \geq 2$. Moreover we have the Littlewood–Paley decomposition:

$$f = S_0 f + \sum_{k \in \mathbf{N}^*} \Delta_k f.$$

With this decompositon, paraproducts can be defined by

$$T_a f = \sum_{k \geq 4} S_{k-3}(a) \Delta_k f.$$

Notice that the difference between paraproducts defined in these two ways is a smoothing operator. Namely, if $a \in W^{\rho,\infty}(\mathbf{R})$ for some $\rho > 0$ then the difference is of order $-\rho$.

Theorem 2.7. — *Let $\alpha, \beta \in \mathbf{R}$ be such that $\alpha + \beta > 0$. If $a \in H^\alpha(\mathbf{R})$ and $b \in H^\beta(\mathbf{R})$ then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{1}{2}}(\mathbf{R})$ and*

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{1}{2}}(\mathbf{R})} \leq K \|a\|_{H^\alpha(\mathbf{R})} \|b\|_{H^\beta(\mathbf{R})}$$

for some positive constant K independent of a, b.

We use the following result which is a consequence of (2.5) with $m = m' = 0, \rho = 1$.

Lemma 2.8. — *Let $s > 2 + \frac{1}{2}$ and $a \in W^{1,\infty}(\mathbf{R})$. Then for all $\sigma \in \mathbf{R}$ there exists a constant $C > 0$ such that for all $j \in \mathbf{N}$,*

$$\|[\Delta_j, T_a]u\|_{H^{\sigma+1}(\mathbf{R})} \leq C \|a\|_{W^{1,\infty}(\mathbf{R})} \|u\|_{H^\sigma(\mathbf{R})}.$$

2.2. The Dirichlet-Neumann operator. —

Lemma 2.9. — *Let $s > 2 + \frac{1}{2}$ and $1 \leq \sigma \leq s$. Then there exists an increasing function $C : \mathbf{R}^+ \to \mathbf{R}^+$ such that for all $(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})$*

$$\|G(\eta)\psi\|_{H^{\sigma-1}(\mathbf{R})} \leq C(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R})}) \|\psi\|_{H^\sigma(\mathbf{R})}.$$

Furthermore, if $(\eta, \psi) \in L^\infty(I; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ is a solution of (1.4), then

$$\partial_t(G(\eta)\psi) = G(\eta)(\partial_t \psi - \mathfrak{B}\partial_t \eta) - \operatorname{div}(V \partial_t \eta) \tag{2.6}$$

where

$$\mathfrak{B}(t,x) := \frac{\partial_x \psi \partial_x \eta + G(\eta)\psi}{1 + |\partial_x \eta|^2}, \qquad V(t,x) := \partial_x \psi - \mathfrak{B}\partial_x \eta \tag{2.7}$$

2.3. Symmetrization. — We consider a solution (η,ψ) of (1.4) on the time interval $I=[0,T]$ with $0<T<+\infty$, satisfying the assumption (1.2) for all $t\in I$ and such that

$$(\eta,\psi)\in C^0\big(I,H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R})\big),$$

for some $s>\frac52$. Then we set

$$U=\psi-T_{\mathfrak{B}}\eta. \tag{2.8}$$

where $\mathfrak{B}$ has been defined in (2.7). It follows from the analysis in [**2**] that we have the following symmetrization of the equations.

Lemma 2.10 ([**2**, Corollary 4.9]). — *Let c, c_1 be defined by*

$$c=\big(1+(\partial_x\eta)^2\big)^{-\frac34},\quad c_1=\big(1+(\partial_x\eta)^2\big)^{-\frac12}.$$

There exists an elliptic symbol $p\in\Sigma^{1/2}_{s-1}$ such that the complex-valued unknown

$$\Phi=T_p\eta+iT_{c_1}U \tag{2.9}$$

satisfies a scalar equation of the form

$$\partial_t\Phi+T_V\partial_x\Phi+i\,|D_x|^{\frac34}\,T_c\,|D_x|^{\frac34}\,\Phi=F, \tag{2.10}$$

where V has been defined in (2.7) *and $F\in L^\infty(I,H^s(\mathbf{R}))$.*

3. Reductions

3.1. Change of variables. — We consider a solution (η,ψ) of (1.4) on the time interval $I=[0,T]$ with $0<T<+\infty$, satisfying the assumption (1.2) for all $t\in I$ and such that

$$(\eta,\psi)\in C^0\big(I,H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R})\big),$$

for some $s>\frac52$.

Our aim in this section is to simplify the equation (2.10) by a change of variable. To compute the effect of a change of variable we shall use Alinhac's paracomposition operators and we refer to [**5**] for the general theory .

Let κ be a C^1 diffeomorphism from $\mathbf{R}$ to $\mathbf{R}$. We define the operator κ^* by,

$$\kappa^*u=u\circ\kappa-T_{(\partial_x u)\circ\kappa}\kappa. \tag{3.1}$$

One of the main properties of κ^* is that there is a symbolic calculus theorem which allows to compute the equation satisfied by κ^*u in terms of the equation satisfied by u (in analogy with the paradifferential calculus).

Theorem 3.1. — *Let $m\in\mathbf{R}$, $r>1$, $\rho>0$ and set $\sigma:=\inf\{\rho,r-1\}$. Consider a diffeomorphism χ such that $\partial_x\chi\in W^{r-1,\infty}(\mathbf{R})$ and set $\kappa=\chi^{-1}$. Let a be a symbol in $\Sigma^m_\rho(\mathbf{R})$. Then there exists $a^*\in\Sigma^m_\sigma(\mathbf{R})$ such that*

$$\kappa^*T_a-T_{a^*}\kappa^*\quad\textit{is order}\ \le m-\sigma.$$

Moreover one can give an explicit formula for a^*. If $a = \sum a_{m-k}$, then

$$a^*(\chi(x),\eta) = \sum_\alpha \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a_{m-k}(x,\chi'(x)\eta)\partial_y^\alpha(e^{i\Psi_x(y)\cdot\eta})|_{y=x}, \tag{3.2}$$

where the sum is taken over all $\alpha \in \mathbf{N}$ such that the summand is well defined, $\chi'(x)$ is the derivative of χ and

$$\Psi_x(y) = \chi(y) - \chi(x) - \chi'(x)(y-x). \tag{3.3}$$

We are now ready to simplify (2.10). Define χ by

$$\chi(t,x) = \int_0^x c(t,y)^{-\frac{2}{3}}\,dy = \int_0^x \sqrt{1+(\partial_y\eta(t,y))^2}\,dy, \tag{3.4}$$

so that

$$\partial_x\chi(t,x) = \sqrt{1+(\partial_x\eta(t,x))^2} = c(t,x)^{-\frac{2}{3}}.$$

Then for each $t \in [0,T]$, $x \mapsto \chi(t,x)$ is a diffeomorphism from $\mathbf{R}$ to $\mathbf{R}$. Introduce its inverse

$$\kappa = \chi^{-1}. \tag{3.5}$$

3.1.1. Notations:— We shall set $I = [0,T]$ and we shall denote

$$A = C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R})}\big) \tag{3.6}$$

where $C : \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function which may change from line to line. Moreover we shall denote by $f\circ\kappa$ the function

$$(f\circ\kappa)(t,x) = f(t,\kappa(t,x)). \tag{3.7}$$

3.1.2. Estimates of χ and κ.— From (3.4), the equation $\partial_t\eta = G(\eta)\psi$, the Lemma 2.9, the Hölder inequality and the fact that $s > 2+\frac{1}{2}$ we deduce,

$$\|\partial_t\chi\|_{L^\infty(I\times\mathbf{R})} \le A. \tag{3.8}$$

Now since

$$\partial_x\chi(t,x) = 1 + f(\partial_x\eta), \quad f\in C^\infty(\mathbf{R}), f(0)=0,$$

we deduce from the assumption $s > 2+\frac{1}{2}$ and the Sobolev embedding that,

$$\|\partial_x\chi(t,x)-1\|_{L^\infty(I,H^{s-\frac{1}{2}}(\mathbf{R}))} + \|\partial_x\chi\|_{L^\infty(I\times\mathbf{R})} \le A. \tag{3.9}$$

Let us consider the function κ.

Since $\partial_t\kappa = -\frac{\partial_t\chi}{\partial_x\chi}\circ\kappa$ we have, using (3.8),

$$\|\partial_t\kappa\|_{L^\infty(I\times\mathbf{R})} \le A. \tag{3.10}$$

On the other hand we have $\partial_x\kappa = 1 + f(\partial_x\eta)$ where $f\in C^\infty(\mathbf{R}), f(0)=0$. It follows that,

$$\|\partial_x\kappa - 1\|_{L^\infty(I,H^{s-\frac{1}{2}}(\mathbf{R}))} \le A. \tag{3.11}$$

Let $p=[s]$ if $s\notin \mathbf{N}$, $\ p=s-1$ if $s\in\mathbf{N}$; then $p\geq s-1$ and it follows from (3.11) that,

$$\|\partial_x\kappa\|_{L^\infty(I,W^{p-1,\infty}(\mathbf{R}))}\leq A. \tag{3.12}$$

To go further we shall need the following elementary lemma.

Lemma 3.2. — *Let $p\in\mathbf{N}^*$ and $\kappa:\mathbf{R}\to\mathbf{R}$ be a diffeomorphism such that $\partial_x\kappa\in W^{p-1,\infty}(\mathbf{R})$. Set $\chi=\kappa^{-1}$. Then for all $F\in H^\mu(\mathbf{R})$ with $0\leq\mu\leq p$ we have $F\circ\kappa\in H^\mu(\mathbf{R})$ and*

$$\|F\circ\kappa\|_{H^\mu(\mathbf{R})}\leq\|\chi'\|_{L^\infty(\mathbf{R})}C\big(\|\partial_x\kappa\|_{W^{p-1,\infty}(\mathbf{R})}\big)\|F\|_{H^\mu(\mathbf{R})}$$

where C is an increasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$.

We deduce from Lemma 3.2 and (3.12) that for $0\leq\mu\leq s-1$ and $F\in L^\infty(I,H^\mu(\mathbf{R}))$ we have,

$$\|F\circ\kappa\|_{L^\infty(I,H^\mu(\mathbf{R}))}\leq A\|F\|_{L^\infty(I,H^\mu(\mathbf{R}))}. \tag{3.13}$$

Coming back to the regularity of χ we deduce from (3.4) that,

$$\partial_x^2\chi=\frac{(\partial_x\eta)(\partial_x^2\eta)}{(1+(\partial_x\eta)^2)^{\frac12}}.$$

It follows from (3.13) that,

$$\|(\partial_x^2\chi)\circ\kappa\|_{L^\infty(I,H^{s-\frac32}(\mathbf{R}))}\leq A. \tag{3.14}$$

On the other hand we have,

$$\partial_x\partial_t\chi=\frac{(\partial_x\eta)\partial_x(G(\eta)\psi)}{(1+(\partial_x\eta)^2)^{\frac12}}.$$

So using Lemma 2.9 and (3.13) we obtain,

$$\|(\partial_x\partial_t\chi)\circ\kappa\|_{L^\infty(I,H^{s-2}(\mathbf{R}))}\leq A. \tag{3.15}$$

Now we would like to estimate $\partial_t^2\chi$. Since $\partial_t\eta=G(\eta)\psi$ we have,

$$\begin{aligned}\partial_t^2\chi(t,x)=&-\int_0^x\frac{[\partial_x\eta\partial_x(G(\eta)\psi)]^2}{(1+(\partial_x\eta)^2)^{\frac32}}dy+\int_0^x\frac{[\partial_x(G(\eta)\psi)]^2}{(1+(\partial_x\eta)^2)^{\frac12}}dy\\&+\int_0^x\frac{\partial_x\eta\partial_x\partial_t(G(\eta)\psi)}{(1+(\partial_x\eta)^2)^{\frac12}}dy.\end{aligned} \tag{3.16}$$

Since $s>2+\frac12$, the Hölder inequality and Lemma 2.9 show that the first two terms are pointwise bounded by A. By the Hölder inequality the last term can be pointwise bounded by

$$\|\partial_x\eta\|_{L^\infty(I,L^2(\mathbf{R}))}\|\partial_x\partial_t\big(G(\eta)\psi\big)\|_{L^\infty(I,L^2(\mathbf{R}))}.$$

Using (2.6) and the equation satisfied by (η,ψ) we find, if $s>3+\frac12$,

$$\|\partial_x\partial_t\big(G(\eta)\psi\big)\|_{L^\infty(I,L^2(\mathbf{R}))}\leq A.$$

Therefore if $s > 3 + \frac{1}{2}$ we obtain,

$$\|\partial_t^2 \chi\|_{L^\infty(I\times\mathbf{R})} \leq A. \tag{3.17}$$

Finally let us estimate the term $\partial_x \partial_t^2 \chi$. Using again (2.6) and the equation satisfied by (η, ψ) we find, if $s > 4$, that,

$$\|\partial_x \partial_t^2 \chi\|_{L^\infty(I, H^{s-\frac{7}{2}}(\mathbf{R}))} \leq A. \tag{3.18}$$

3.1.3. Reduction of the equation. — With V defined in (2.7) and Φ defined in (2.9) let us set (see (3.7)),

$$W = V \circ \kappa(\partial_x \chi \circ \kappa) + \partial_t \chi \circ \kappa, \tag{3.19}$$

$$\Phi^* = \kappa^* \Phi = \Phi \circ \kappa - T_{(\partial_x \Phi)\circ\kappa} \kappa. \tag{3.20}$$

Then we have the following result.

Proposition 3.3. — *Let $s > 2 + \frac{1}{2}$ and $I = [0, T]$. There exists a real valued function g such that $\partial_x g \in \Sigma^0_{s-\frac{3}{2}}$ and the function $u = T_{e^{ig}} \Phi^*$ satisfies the equation*

$$(\partial_t + T_W \partial_x + i\,|D_x|^{\frac{3}{2}})u = F, \tag{3.21}$$

with $F \in L^\infty(I, H^s(\mathbf{R}))$ and W is defined by (3.19).

Proof. — We apply the operator κ^* to the equation (2.10). We first show that

$$\begin{aligned} &\kappa^*(\partial_t + T_V \partial_x)\Phi = (\partial_t + T_W \partial_x)\Phi^* + R(\Phi) \\ &\|R(\Phi)\|_{L^\infty(I, H^s(\mathbf{R}))} \leq C(\|(\eta, \psi)\|_{L^\infty(I, H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))})\|\Phi\|_{L^\infty(I, H^s(\mathbf{R}))}. \end{aligned} \tag{3.22}$$

To estimate the remainders we shall use the estimates (3.8) – (3.15) on κ and χ obtained above. We begin by showing that

$$\kappa^*(\partial_t \Phi) = (\partial_t - T_{(\partial_t \chi)\circ\kappa})\Phi^* + R_1(\Phi) \tag{3.23}$$

where R_1 satisfies the estimate in (3.22).

We have

$$\begin{aligned} \kappa^*(\partial_t \Phi) &= (\partial_t \Phi) \circ \kappa - T_{(\partial_x \partial_t \Phi)\circ\kappa} \kappa \\ &= \partial_t(\Phi \circ \kappa) - (\partial_t \kappa)(\partial_x \Phi \circ \kappa) - T_{(\partial_x \partial_t \Phi)\circ\kappa} \kappa, \end{aligned}$$

therefore,

$$\begin{aligned} \kappa^*(\partial_t \Phi) &= \partial_t(\kappa^* \Phi) + B_1 + B_2, \\ B_1 &= T_{(\partial_x^2 \Phi\circ\kappa)\partial_t \kappa} \kappa \\ B_2 &= T_{(\partial_x \Phi)\circ\kappa} \partial_t \kappa - (\partial_t \kappa)(\partial_x \Phi \circ \kappa) \end{aligned} \tag{3.24}$$

Let us consider the term B_1 in (3.24) and let us set $a = \partial_t \kappa(\partial_x^2 \Phi \circ \kappa)$. We have,

$$T_a \kappa = \sum_{j\geq 4} S_{j-3}(a)\Delta_j(\kappa) = \sum_{j\geq 4} 2^{-j} S_{j-3}(a)\widetilde{\phi}(2^{-j}D)(\partial_x \kappa) = \sum_{j\geq 4} g_j$$

where $\widetilde{\phi} \in C^\infty(\mathbf{R}), \operatorname{supp}\widetilde{\phi} \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. Since $\partial_x\kappa = 1 + f(\partial_x\eta)$ with $f(0) = 0$, we have $\widetilde{\Delta}_j(\partial_x\kappa) = \widetilde{\Delta}_j(f(\partial_x\eta))$ so,

$$\|g_j\|_{L^2(\mathbf{R})} \leq 2^{-j}\|a\|_{L^\infty(\mathbf{R})}2^{-j(s-\frac{1}{2})}c_jC(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R})}), \quad (c_j) \in l^2.$$

On the other hand using (3.8), (3.9) we can write,

$$\begin{aligned}\|a\|_{L^\infty(I\times\mathbf{R})} &\leq \|\Phi\|_{L^\infty(I,H^s(\mathbf{R}))}\|\partial_t\chi(\partial_x\chi)^{-1}\|_{L^\infty(I\times\mathbf{R})}\\ &\leq \|\Phi\|_{L^\infty(I,H^s(\mathbf{R}))}C(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R})}).\end{aligned}$$

It follows that,

$$\|B_1\|_{L^\infty(I,H^s(\mathbf{R}))} \leq C(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R})})\|\Phi\|_{L^\infty(I,H^s(\mathbf{R}))}. \tag{3.25}$$

Let us consider the term B_2.

We have, $\partial_t\kappa = ab$ where $a = \partial_t\chi \in \Gamma^0_1, b = \partial_x\kappa \in \Gamma^0_1$. It follows from Theorem 2.6 that $a\#b = ab$ and $T_{ab} - T_aT_b$ is of order -1. Let us set

$$B_{21} = \|(T_{ab} - T_aT_b)(\partial_x\Phi\circ\kappa)\|_{L^\infty(I,H^s(\mathbf{R}))}. \tag{3.26}$$

Using (2.5) we obtain,

$$B_{21} \leq \|\partial_t\chi\|_{L^\infty(I,W^{1,\infty}(\mathbf{R}))}\|\partial_x\kappa\|_{L^\infty(I,W^{1,\infty}(\mathbf{R}))}\|(\partial_x\Phi\circ\kappa)\|_{L^\infty(I,H^{s-1}(\mathbf{R}))}.$$

Since $s - \frac{3}{2} > 1$, using (3.13) with $\mu = s - 1$ we obtain,

$$B_{21} \leq C\big(\|\eta\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R}))}\big)\|\Phi\|_{L^\infty(I,H^s(\mathbf{R}))}. \tag{3.27}$$

Therefore using (3.24), (3.25), (3.27) and Theorem 2.7 we obtain,

$$\kappa^*(\partial_t\Phi) = \partial_t\kappa^*\Phi - T_{\partial_t\chi}T_{\partial_x\kappa}\partial_x\Phi\circ\kappa + R_2(\Phi), \tag{3.28}$$

where R_2 satisfies (3.22).

Now let us set

$$a = \partial_x\kappa \in L^\infty(I, H^{s-\frac{1}{2}}(\mathbf{R})), b = \partial_x\Phi\circ\kappa \in L^\infty(I, H^{s-1}(\mathbf{R})).$$

It follows from Theorem 2.7 that

$$\|ab - T_ab - T_ba\|_{L^\infty(I,H^{2s-2}(\mathbf{R}))} \leq \|a\|_{L^\infty(I,H^{s-\frac{1}{2}}(\mathbf{R}))}\|b\|_{L^\infty(I,H^{s-1}(\mathbf{R}))}. \tag{3.29}$$

Therefore we obtain

$$\kappa^*(\partial_t\Phi) = \partial_t(\kappa^*\Phi) - T_{\partial_t\chi}\partial_x\kappa\partial_x(\Phi\circ\kappa) + T_{\partial_t\chi}T_{\partial_x\Phi\circ\kappa}\partial_x\kappa + R_3, \tag{3.30}$$

where R_3 satisfies (3.22). Using (3.1) we obtain

$$\kappa^*(\partial_t\Phi) = (\partial_t - T_{\partial_t\chi}\partial_x)(\kappa^*\Phi) - T_{\partial_t\chi}\partial_x(T_{\partial_x\Phi\circ\kappa}\kappa) + T_{\partial_t\chi}T_{\partial_x\Phi\circ\kappa}\partial_x\kappa + R_3,$$

where R_3 satisfies (3.22).

It follows that

$$\kappa^*(\partial_t\Phi) = (\partial_t - T_{\partial_t\chi}\partial_x)(\kappa^*\Phi) - T_{(\partial_x^2\Phi\circ\kappa)\partial_x\kappa}\kappa + R_3.$$

Now the term $T_{(\partial_x^2\Phi\circ\kappa)\partial_x\kappa}\kappa$ can be estimated exactly by the same method as the term B_1, therefore we obtain

$$\kappa^*(\partial_t\Phi) = (\partial_t - T_{\partial_t\chi}\partial_x)(\kappa^*\Phi) + R_4,$$

where R_4 satisfies (3.22). This is precisely (3.23).

Now we claim that

$$\kappa^*(T_V\partial_x\Phi) = T_{(V\partial_x\chi)\circ\kappa}\partial_x\kappa^*\Phi + R_5(\Phi), \tag{3.31}$$

where R_5 satisfies (3.22). But this is precisely a consequence of Theorem 3.1. Indeed we have for (almost all) fixed t, $a(x,\xi) = iV(t,x)\xi \in \Sigma^1_{s-1}$, and the diffeomorphism κ is in $W^{s-\frac{3}{2}}(\mathbf{R})$, so $\sigma = s-\frac{3}{2}$ and the remainder term is of order less than $1-(s-\frac{3}{2}) = \frac{5}{2}-s < 0$. Then (3.22) follows from (3.23) and (3.31).

Let us consider now the principal part. Applying again Theorem 3.1 we find that,

$$\kappa^*(|D_x|^{\frac{3}{4}}T_c|D_x|^{\frac{3}{4}}\Phi) = |D_x|^{\frac{3}{2}}\kappa^*\Phi + T_a\kappa^*\Phi,$$

where a is of order $\frac{1}{2}$.

Finally, it remains to reduce to the case where $a = 0$. Indeed, let g be a real-valued symbol such that $\partial_x g \in \Gamma^0_{s-3/2}(\mathbf{R})$ and

$$\{|\xi|^{3/2}, g\} = -a,$$

then if we set

$$u = T_{e^{ig}}\Phi^*, \tag{3.32}$$

we obtain by symbolic calculus that u satisfies

$$(\partial_t + T_W\partial_x + i\,|D_x|^{\frac{3}{2}} + iT_a + T_b)u = F,$$

with $F \in L^\infty(I, H^s(\mathbf{R}))$ and $b = i\{|\xi|^{3/2}, g\}$. This completes the proof of Proposition 3.3. □

3.1.4. Regularity of W. — The following result gives some informations on the function W defined in (3.19).

Lemma 3.4. — *Let $I = [0,T]$, $E = L^\infty(I\times\mathbf{R})$, $F = L^\infty(I, H^{s-2}(\mathbf{R}))$.*

1. *If $s > 2+\frac{1}{2}$, we have $W \in E$, $\partial_x W \in F$, and*

$$\|W\|_E + \|\partial_x W\|_F \le C(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))}).$$

2. *If $s > 4$, we have $\partial_t W, \partial_x^2 W, \partial_t\partial_x W \in E$ and*

$$\|\partial_t W\|_E + \|\partial_x^2 W\|_E + \|\partial_t\partial_x W\|_E \le C(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))}).$$

Proof. — Let us recall that we have set,

$$A = C\big(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R})}\big) \tag{3.33}$$

where $C : \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function which may change from place to place. Since $s > 2 + \frac{1}{2}$ using (2.7) we obtain,

$$\|V\|_E \leq \|\partial_x \psi\|_{L^\infty(I,H^{s-2}(\mathbf{R}))} + \|\mathfrak{B}\|_{L^\infty(I,H^{s-2}(\mathbf{R}))}\|\partial_x \eta\|_{L^\infty(I,H^{s-2}(\mathbf{R}))} \leq A. \tag{3.34}$$

Then the estimate $\|W\|_E \leq A$ follows fom (3.8) and (3.9) .

Now we have

$$\partial_x W = \partial_x V \circ \kappa + V \circ \kappa(\partial_x^2 \chi \circ \kappa)\partial_x \kappa + (\partial_t \partial_x \chi \circ \kappa)\partial_x \kappa. \tag{3.35}$$

Using (3.13) we see that,

$$\|V \circ \kappa\|_F + \|\partial_x V \circ \kappa\|_F \leq A\|V\|_{L^\infty(I,H^{s-1}(\mathbf{R}))} \leq A. \tag{3.36}$$

Now using (3.11), (3.14) and the fact that $H^{s-2}(\mathbf{R})$ is an algebra we deduce,

$$\|V \circ \kappa(\partial_x^2 \chi \circ \kappa)\partial_x \kappa\|_F \leq A. \tag{3.37}$$

Then the estimate $\|\partial_x W\|_F \leq A$ follows from (3.15) and (3.11).

Let us now prove 2. We have

$$\begin{aligned} \partial_t W &= \partial_t V \circ \kappa(\partial_x \chi \circ \kappa) + \partial_x V \circ \kappa(\partial_x \chi \circ \kappa)\partial_t \kappa + V \circ \kappa(\partial_t \partial_x \chi \circ \kappa) \\ &\quad + V \circ \kappa(\partial_x^2 \chi \circ \kappa)\partial_t \kappa - \partial_t^2 \chi \circ \kappa - \partial_t \partial_x \chi \circ \kappa(\partial_x \kappa) =: \sum_{i=1}^{6} B_i. \end{aligned} \tag{3.38}$$

It follows from (3.13),(3.9),(3.10),(3.14),(3.15), and the Sobolev embedding that

$$|B_2| + |B_3| + |B_4| + |B_6| \leq A. \tag{3.39}$$

Now we have $\partial_t V = \partial_x \partial_t \psi - (\partial_t \mathfrak{B})\partial_x \eta - \mathfrak{B}\partial_x \partial_t \eta$. So using the equations satisfied by (η, ψ), the Sobolev embedding and Lemma 2.9 we obtain

$$\|\partial_t V\|_{L^\infty(I,H^{s-\frac{5}{2}}(\mathbf{R}))} \leq A. \tag{3.40}$$

It follows that

$$|B_1| \leq A. \tag{3.41}$$

The term B_5 is estimated by A using (3.17). Therefore using (3.39) and (3.41) we deduce that $\|\partial_t W\|_E \leq A$.

The claim on $\partial_x^2 W$ follows from the first part of the Lemma and the Sobolev embedding since $s > 3 + \frac{1}{2}$. It remains to consider the quantity $\partial_t \partial_x W$. We go back to (3.38). The term $\partial_t V \circ \kappa(\partial_x \chi \circ \kappa)$ is bounded by A in $L^\infty(I, H^{s-\frac{5}{2}}(\mathbf{R}))$. The third term $V \circ \kappa(\partial_t \partial_x \chi \circ \kappa)$ is bounded by A in $L^\infty(I, H^{s-2}(\mathbf{R}))$. The term $\partial_t \partial_x \chi \circ \kappa(\partial_x \kappa)$ is bounded by A in $L^\infty(I, H^{s-2}(\mathbf{R}))$. Therefore the ∂_x derivative of these three terms are bounded by A in $L^\infty(I, H^{s-\frac{7}{2}}(\mathbf{R}))$. By (3.8) we have,

$$\begin{aligned} \|\partial_x V \circ \kappa(\partial_x \chi \circ \kappa)\partial_t \kappa\|_{L^\infty(I\times\mathbf{R})} &\leq A\|\partial_x V \circ \kappa(\partial_x \chi \circ \kappa)\|_{L^\infty(I\times\mathbf{R})} \\ &\leq A\|\partial_x V \circ \kappa(\partial_x \chi \circ \kappa)\|_{L^\infty(I,H^{s-2})} \leq A. \end{aligned}$$

We can apply the same argument for the term $V\circ\kappa(\partial_x^2\chi\circ\kappa)\partial_t\kappa$. Finally we bound the term $V\circ\kappa(\partial_x\partial_{tt}^2\chi\circ\kappa)$ in the space $L^\infty(I,H^{s-\frac{7}{2}}(\mathbf{R}))$ by using (3.13) and (3.18).This completes the proof of our Lemma.

□

3.2. Symbol Smoothing. — In this section we follow an idea of Smith [**27**] (see also Bahouri-Chemin [**8**]), and we are going to smooth out the coefficients of the function W with respect to x. As already mentioned, here is the main place where the idea of allowing loss in remainder terms enters. We define for $0<\delta\leq 1$,

$$T_W^\delta=\sum_{j\geq 4}S_{[\delta(j-3)]}(W)\Delta_j,$$

The key difference between T_W and T_W^δ is made clear below.

Lemma 3.5. — *The operator $T_W-T_W^\delta$ is of order $-\delta(s-\frac{3}{2})$.*

Proof. — Since for almost all fixed t we have, $\partial_x W(t,\cdot)\in H^{s-2}(\mathbf{R})$ we have,

$$\begin{aligned}\left\|S_j(W)-S_{[\delta j]}(W)\right\|_{L^\infty(\mathbf{R})}&\leq\sum_{n=[\delta j]}^{j}\|\Delta_n(W)\|_{L^\infty(\mathbf{R})}\\&\leq K\sum_{n=[\delta j]}^{j}2^{-n(s-\frac{3}{2})}\leq K2^{-\delta j(s-\frac{3}{2})}.\end{aligned}$$

□

In the sequel we shall set

$$\begin{cases}h=2^{-j},\ j\in\mathbf{N},\\ W_h^\delta=S_{[\delta(j-3)]}(W),\\ a(\xi)=\chi_0(\xi)|\xi|^{\frac{3}{2}},\end{cases}\tag{3.42}$$

where $\chi_0\in C_0^\infty(\mathbf{R}),\operatorname{supp}\chi_0\subset\{\frac{1}{4}\leq|\xi|\leq 4\},\chi_0=1$ in $\{\frac{1}{2}\leq|\xi|\leq 2\}$.

Lemma 3.6. — *Let $s>2+\frac{1}{2}$ and $u\in L^\infty(I,H^s(\mathbf{R}))$ be a solution of* (3.21). *There exist $\delta<\frac{1}{2}$, $\epsilon>0$ and $f_h\in L^\infty(I,H^{s+\epsilon-\frac{1}{2}}(\mathbf{R}))$ such that*

$$\begin{gathered}\|f_h\|_{L^\infty(I,H^{s+\epsilon-\frac{1}{2}}(\mathbf{R}))}\leq C\left(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))}\right),\\ \operatorname{supp}(\widehat{f_h})\subset\{\frac{1}{2}h^{-1}\leq|\xi|\leq 2h^{-1}\}\end{gathered}\tag{3.43}$$

and the functions $u_h=\Delta_j u$ satisfy

$$(\partial_t+\frac{1}{2}(W_h^\delta\partial_x+\partial_x W_h^\delta)+ia(D_x))u_h=f_h.\tag{3.44}$$

Furthermore we have

$$\sum_{j\in\mathbf{N}} \|\Delta_j u\|^2_{L^\infty(I;H^s(\mathbf{R}))} < +\infty. \tag{3.45}$$

Proof. — First of all we remark that we have $|D_x|^{\frac{3}{2}} u_h = a(D_x)u_h$ (and consequently $h^{-\frac{3}{2}}a(hD_x)u_h = a(D_x)u_h$). Now, applying the operator Δ_j to (3.21), we obtain

$$(\partial_t + T_W\partial_x + i\,|D_x|^{\frac{3}{2}})u_h = \Delta_j F - [\Delta_j, T_W]\partial_x u := g^1_h. \tag{3.46}$$

Let us prove (3.45). We deduce from the usual energy estimates that,

$$\begin{aligned}\|u_h\|_{L^\infty(I;H^s(\mathbf{R}))} &\le C\big(\|u_h\,|_{t=0}\,\|_{H^s(\mathbf{R})} + \|g^1_h\|_{L^1(I;H^s(\mathbf{R}))}\big)\\ &\le C'\big(\|u_h\,|_{t=0}\,\|_{H^s(\mathbf{R})} + \|g^1_h\|_{L^2(I;H^s(\mathbf{R}))}\big).\end{aligned}$$

Since by Lemma 3.4 we have $\partial_x W \in L^\infty(I, H^{s-2}(\mathbf{R}))$, using Lemma 2.8 we can write,

$$\begin{aligned}\|g^1_h\|^2_{L^2(I;H^s(\mathbf{R}))} &\le 2\big(\|\Delta_j F\|^2_{L^2(I;H^s(\mathbf{R}))} + \|[\Delta_j, T_W]\partial_x\widetilde{\Delta}_j u\|^2_{L^2(I;H^s(\mathbf{R}))}\big)\\ &\le C\big(\|\Delta_j F\|^2_{L^2(I;H^s(\mathbf{R}))} + \|\widetilde{\Delta}_j u\|^2_{L^2(I;H^s(\mathbf{R}))}\big)\end{aligned}$$

where $\widetilde{\Delta}_j = \sum_{|k-j|\le 3}\Delta_k$. It follows that

$$\sum_{j\in\mathbf{N}} \|g^1_h\|^2_{L^2(I;H^s(\mathbf{R}))} \le C\big(\|F\|^2_{L^2(I;H^s(\mathbf{R}))} + \|u\|^2_{L^2(I;H^s(\mathbf{R}))}\big)$$

from which (3.45) follows easily.

Now we replace T_W by T^δ_W in (3.46) to obtain

$$(\partial_t + \sum_{|k-j|\le 1} S_{[\delta(k-3)]}(W)\Delta_k\partial_x + i\,|D_x|^{\frac{3}{2}})u_h = g^1_h + (T^\delta_W - T_W)\partial_x u_h := g^1_h + g^2_h \tag{3.47}$$

where, according to Lemma 3.5, g^2_h satisfies (3.43) with $\epsilon = \delta(s-\frac{3}{2}) - \frac{1}{2} > 0$ if $\delta < \frac{1}{2}$ is chosen close enough to $\frac{1}{2}$. Now, we have

$$S_{[\delta(j-3)]}(W)\partial_x u_h = \sum_{|k-j|\le 1} S_{[\delta(j-3)]}(W)\Delta_k\partial_x u_h.$$

Consequently, we obtain

$$\begin{aligned}&(\partial_t + S_{[\delta(j-3)]}(W)\partial_x + i\,|D_x|^{\frac{3}{2}})u_h\\ &\quad= g^1_h + g^2_h + \sum_{|k-j|\le 1}(S_{[\delta(j-3)]}(W) - S_{[\delta(k-3)]}(W))\Delta_k\partial_x u_h = g^1_h + g^2_h + g^3_h,\end{aligned} \tag{3.48}$$

and using that for $|k-j|\le 1$,

$$\big\|S_{[\delta(k-3)]}(W) - S_{[\delta(j-3)]}(W)\big\|_{L^\infty} \le C2^{-j\delta(s-\frac{3}{2})},$$

we obtain that g^3_h satisfies (3.43). Finally, we obtain

$$(\partial_t + \frac{1}{2}(W^\delta_h\partial_x + \partial_x W^\delta_h) + i\,|D_x|^{\frac{3}{2}})u_h = g^1_h + g^2_h + g^3_h + g^4_h,$$

where $g_h^4 = \frac{1}{2} S_{[\delta(j-3)]}(\partial_x W) u_h$ satisfies (3.43) (for any $0 < \epsilon \leq \frac{1}{2}$). $\square$

Lemma 3.7. — *Let $s > \frac{11}{2}$ and set*

$$\delta = \frac{1}{s - \frac{3}{2}} \in]0, \frac{1}{4}[.$$

Then there exists $f_h \in L^\infty(I, H^s(\mathbf{R}))$ such that

$$\begin{aligned} \|f_h\|_{L^\infty(I,H^s(\mathbf{R}))} &\leq C \left(\|(\eta, \psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))} \right), \\ \operatorname{supp}(\widehat{f_h}) &\subset \{\frac{1}{2} h^{-1} \leq |\xi| \leq 2h^{-1}\} \end{aligned} \tag{3.49}$$

and the functions $u_h = \Delta_j u$ satisfy

$$(\partial_t + \frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta) + ia(D_x))u_h = f_h \tag{3.50}$$

Proof. — The proof is identical to that of Lemma 3.6, the only difference beeing that now we take δ such that $\delta(s - \frac{3}{2}) = 1$. $\square$

4. Semi-classical parametrix

The purpose of this section is to prove the main step toward Theorem 1.1.

Theorem 4.1. — *Under the assumptions of Theorem 1.1, let u be defined by* (3.32). *Then there exists $C = C(\|(\eta, \psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))})$ such that*

$$\|u\|_{L^4(I,B^{s-\frac{1}{4}}_{\infty,2}(\mathbf{R}))} \leq C.$$

Following [**14**] we shall reduce the analysis to establishing semi-classical estimates. Recall that $2^{-j} = h$ and $W_h^\delta = S_{[\delta(j-3)]}(W) = \phi(h^\delta D_x)W$, $\phi \in C_0^\infty(\mathbf{R})$.

Theorem 4.2. — *Let $\chi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and $t_0 \in \mathbf{R}$. For any initial data $u_{0,h} = \chi(hD_x)u_0$, where $u_0 \in L^1(\mathbf{R})$, let $U_h := S(t, t_0, h)u_{0,h}$ be the solution of*

$$\partial_t U_h + \frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta)U_h + ia(D_x)U_h = 0, \qquad U_h \mid_{t=t_0} = u_{0,h}. \tag{4.1}$$

Then for any $0 < h \leq 1$ and any $|t - t_0| \leq h^{\frac{1}{2}}$,

$$\|S(t, t_0, h)u_{0,h}\|_{L^\infty(\mathbf{R})} \leq \frac{C}{h^{1/4}|t - t_0|^{1/2}} \|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.2}$$

To prove this result, we shall follow a very classical trend and construct a parametrix. Notice that our assumptions being time-translation invariant we can assume $t_0 = 0$. The parametrix will take the following form,

$$\widetilde{U}_h(t, x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}(\Phi(t,x,\xi,h) - z\xi)} \widetilde{B}(t, x, z, \xi, h) u_{0,h}(z) dz d\xi, \tag{4.3}$$

where Φ will satisfy the eikonal equation and

$$\widetilde{B}(t,x,z,\xi,h) = B(t,x,\xi,h)\zeta(x-z-th^{-\frac{1}{2}}a'(\xi)), \tag{4.4}$$

where B will satisfy the transport equations and $\zeta \in C_0^\infty(\mathbf{R})$, $\zeta(s)=1$ if $|s|\le 1$, $\zeta(s)=0$ if $|s|\ge 2$.

In addition to χ (introduced in Theorem 4.2), we shall use two more cut-off functions $\chi_j \in C_0^\infty(\mathbf{R}), j=1,2$, such that

$$\begin{cases} \operatorname{supp}\chi_1 \subset \{\xi : \dfrac{1}{3} \le |\xi| \le 3\}, & \chi_1 = 1 \text{ on } \operatorname{supp}\chi, \\ \operatorname{supp}\chi_0 \subset \{\xi : \dfrac{1}{4} \le |\xi| \le 4\}, & \chi_0 = 1 \text{ on } \operatorname{supp}\chi_1. \end{cases} \tag{4.5}$$

4.1. The eikonal and transport equations. — We introduce some space of symbols in which we shall solve our equations.

Definition 4.3. — *For small h_0 to be fixed, we introduce the sets*

$$\Omega = \left\{(t,x,\xi,h) \in \mathbf{R}^4 \,:\, h \in (0,h_0), |t| < h^{\frac{1}{2}}, 1 < |\xi| < 3\right\},$$
$$\mathcal{O} = \left\{(\sigma,x,\xi,h) \in \mathbf{R}^4 \,:\, h \in (0,h_0), |\sigma| < 1, 1 < |\xi| < 3\right\}.$$

If $m \in \mathbf{R}$ and $\varrho \in \mathbb{R}^+$, we denote by $S^m_\varrho(\Omega)$ (resp.$S^m_{\varrho,}(\mathcal{O})$) the set of all functions f on Ω which are C^∞ with respect to (t,x,ξ) (resp.(σ,x,ξ)) and satisfy the estimate

$$|\partial_x^\alpha f(t,x,\xi,h)| \,(resp.\, |\partial_x^\alpha f(\sigma,x,\xi,h)|) \le C_\alpha h^{m-\varrho\alpha}, \tag{4.6}$$

for all $(t,x,\xi,h) \in \Omega$ (resp.$(\sigma,x,\xi,h) \in \mathcal{O}$).

Remark 4.4. — (i) If $f \in S^m_\varrho$, $g \in S^{m'}_\varrho$ then $fg \in S^{m+m'}_\varrho$; if $f \in S^m_\varrho, (m \ge 0)$ and $F \in C^\infty(\mathbf{C})$ then $F(f) \in S^m_\varrho$; if $f \in S^m_\varrho, (m \le 0)$ and $F \in C_b^\infty(\mathbf{C})$ then $F(f) \in S^0_{\varrho-m}$. Let $f \in S^\mu_\varrho$, then $\partial_x f \in S^{\mu-\varrho}_\varrho$. Moreover $S^m_\varrho \subset S^m_{\varrho'}$ if $\varrho \ge \varrho'$

(ii) Let W be such that $\partial_x W \in H^{s-2}(\mathbf{R})$ with $s > 2+\frac{1}{2}$ and set $W_h^\varrho = \gamma(h^\varrho D_x)W$ where $\gamma \in \mathcal{S}(\mathbf{R})$. Then $\partial_x W_h^\varrho \in S^0_{\varrho,}$.

Let $\delta \in (0,\frac{1}{2})$. We fix

$$\mu_0 = \frac{1}{2}\left(\frac{1}{2}-\delta\right). \tag{4.7}$$

Finally we set,

$$\begin{aligned} \mathcal{L}_0 &= \partial_t + \frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta) + i\chi_0(hD_x)\,|D_x|^{\frac{3}{2}}, \\ a(\xi) &= \chi_0(\xi)\,|\xi|^{\frac{3}{2}}. \end{aligned} \tag{4.8}$$

The main result of this section is the following.

Proposition 4.5. — *There exist a phase Φ of the form*

$$\Phi(t,x,\xi,h) = x\xi - h^{-\frac{1}{2}}ta(\xi) + h^{\frac{1}{2}}\Psi(t,x,\xi,h)$$

with $\partial_x\Psi \in S^0_\delta(\Omega)$ and an amplitude $B \in S^0_\delta(\Omega)$ such that, with $\widetilde{B}$ defined in (4.4),

$$\mathcal{L}_0\left(e^{\frac{i}{h}\Phi}\widetilde{B}\right) = e^{\frac{i}{h}\Phi}R_h. \tag{4.9}$$

and for all $N \in \mathbf{N}$ we have,

$$\Big\| \iint e^{\frac{i}{h}(\Phi(t,x,\xi,h)-z\xi)}R_h(t,x,z,\xi,h)u_{0,h}(z)\,dz\,d\xi\Big\|_{H^1(\mathbf{R}_x)} \le C_N h^N \|u_{0,h}\|_{L^1(\mathbf{R})}, \tag{4.10}$$

for all t in $[0,h^{\frac{1}{2}}]$.

Proof. — We set,

$$\begin{gathered} t = h^{\frac{1}{2}}\sigma, \qquad \varphi(\sigma,x,\xi,h) = \Phi(\sigma h^{\frac{1}{2}},x,\xi,h), \\ \widetilde{b}(\sigma,x,\xi,h) = \widetilde{B}(\sigma h^{\frac{1}{2}},x,\xi,h), \qquad V_h(\sigma,x) = W^\delta_h(\sigma h^{\frac{1}{2}},x), \\ \mathcal{L} = h\partial_\sigma + \frac{1}{2}h^{\frac{1}{2}}\big(V_h(h\partial_x) + h\partial_x V_h\big) + ia(hD_x). \end{gathered} \tag{4.11}$$

Multiplying (4.9) by $h^{\frac{3}{2}}$ we see that it is equivalent to,

$$\mathcal{L}\left(e^{\frac{i}{h}\varphi}\widetilde{b}\right) = e^{\frac{i}{h}\varphi}r(\sigma,x,z,\xi,h), \tag{4.12}$$

and (4.10) becomes,

$$\Big\| \iint e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)}r(\sigma,x,z,\xi,h)u_{0,h}(z)\,dz\,d\xi\Big\|_{H^1(\mathbf{R}_x)} \le C_N h^N \|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.13}$$

In the proof of (4.13), z,ξ,h will be considered as parameters.

We shall take φ of the form

$$\varphi(\sigma,x,\xi,h) = x\xi - \sigma a(\xi) + h^{\frac{1}{2}}\psi(\sigma,x,\xi,h), \tag{4.14}$$

where ψ is the solution of the problem

$$\begin{cases} \partial_\sigma\psi + a'(\xi)\partial_x\psi = -\xi V_h, \\ \psi|_{\sigma=0} = 0. \end{cases} \tag{4.15}$$

Differentiating (4.15) with respect to x and ξ, using an induction on k and the fact that $\partial_x V_h \in S^0_\delta(\mathcal{O})$, we see easily that,

$$|\partial^k_\xi\partial^\alpha_x\psi(\sigma,x,\xi,h)| \le C_{k\alpha}|\sigma|h^{-\delta(\alpha+k-1)^+}, \tag{4.16}$$

for every $(\sigma,x,\xi,h) \in \mathcal{O}$, where $a^+ = \sup(a,0)$. It follows in particular that $\partial_x\psi \in S^0_\delta(\mathcal{O})$, $\partial_\sigma\psi \in S^0_\delta(\mathcal{O})$.

Now, since (see (4.11)) $\widetilde{b} = b\,\zeta$ we have,

$$
\begin{aligned}
e^{-\frac{i}{h}\varphi}\big(h\partial_\sigma + \frac{h^{\frac{3}{2}}}{2}(V_h\partial_x + \partial_x V_h)\big)(e^{\frac{i}{h}\varphi}\widetilde{b}) &= i[h^{\frac{1}{2}}\xi V_h - a(\xi) + h^{\frac{1}{2}}\partial_\sigma\psi + hV_h\partial_x\psi]\widetilde{b} \\
&+ h[\partial_\sigma b + h^{\frac{1}{2}}V_h\partial_x b + \frac{1}{2}h^{\frac{1}{2}}(\partial_x V_h)b]\zeta + h[-a'(\xi) + h^{\frac{1}{2}}V_h]b\,\zeta'.
\end{aligned}
\tag{4.17}
$$

On the other hand recall that we have for all $M \in \mathbf{N}^*$ (see the appendix),

$$
e^{-\frac{i}{h}\varphi}a(hD_x)\left(e^{\frac{i}{h}\varphi}\widetilde{b}\right) = A + r_1 + r_2, \tag{4.18}
$$

where

$$
A = \sum_{k=0}^{M-1}\frac{h^k}{i^k k!}\partial_y^k\left\{(\partial_\xi^k a)\,(\rho(x,y))\,\widetilde{b}(y)\right\}\Big|_{y=x}. \tag{4.19}
$$

with

$$
\rho(x,y) = \int_0^1 \frac{\partial\varphi}{\partial x}(\sigma, \lambda x + (1-\lambda)y, \xi, h)\,d\lambda, \tag{4.20}
$$

and the remainder r_1, r_2 are given by,

$$
r_1 = c\,h^{M-1}\iiint_0^1 e^{\frac{i}{h}(x-y)\eta}\kappa_0(\eta)(1-\lambda)^{M-1}\partial_y^M[a^{(M)}(\lambda\eta + \rho(x,y))\widetilde{b}(y)]d\lambda dy d\eta \tag{4.21}
$$

and

$$
r_2 = \sum_{k=0}^{M-1} c_{k,M}h^{M+k}\iint_0^1 z^M\hat{\kappa}_0(z)(1-\lambda)^{M-1}\partial_y^{M+k}[a^{(k)}(\rho)\widetilde{b}]|_{y=x-\lambda hz}d\lambda dz, \tag{4.22}
$$

where $c_M, c_{k,M} \in \mathbf{C}$, $\kappa_0 \in C_0^\infty(\mathbf{R}), \kappa_0 = 1$ in a neighborhood of the origin. Now since

$$
\widetilde{b}(\sigma, x, z, \xi, h) = b(\sigma, x, \xi, h)\zeta(x - z - \sigma a'(\xi)),
$$

writing for simplicity $b(y) = b(\sigma, y, \xi, h)$ and $\zeta = \zeta(x - z - \sigma a'(\xi))$ we have,

$$
\begin{cases}
A = (\sum_{k=0}^{M-1} A_k)\zeta + r_3, \\
A_k = \dfrac{h^k}{i^k k!}\partial_y^k\left\{(\partial_\xi^k a)\,(\rho(x,y))\,b(y)\right\}|_{y=x}, \\
r_3 = \sum_{k=1}^{M-1}\sum_{j=1}^{k} c_{jk}h^k\partial_y^{k-j}\left\{(\partial_\xi^k a)(\rho(x,y))b(y)\right\}|_{y=x}\zeta^{(j)}.
\end{cases}
\tag{4.23}
$$

The term A_0 in (4.23) is equal to $a(\xi + h^{\frac{1}{2}}\partial_x\psi)b$. Then

$$
A_0 = \left[\sum_{j=0}^{2}\frac{1}{j!}a^{(j)}(\xi)(h^{\frac{1}{2}}\partial_x\psi)^j + \frac{(h^{\frac{1}{2}}\partial_x\psi)^3}{2}\int_0^1(1-\lambda)^2\partial_\xi^3 a(\xi + \lambda h^{\frac{1}{2}}\partial_x\psi)\,d\lambda\right]b. \tag{4.24}
$$

The term A_1 in (4.23) can be written as

$$A_1 = \frac{h}{i}\left[a'(\xi + h^{\frac{1}{2}}\partial_x\psi)\partial_x b + \frac{1}{2}h^{\frac{1}{2}}(\partial_x^2\psi)a''(\xi + h^{\frac{1}{2}}\partial_x\psi)b\right].$$

Therefore

$$\begin{aligned} A_1 = \frac{h}{i}\Bigg[&\left\{a'(\xi) + h^{\frac{1}{2}}\partial_x\psi\int_0^1 a''(\xi + \lambda h^{\frac{1}{2}}\partial_x\psi)\,d\lambda\right\}\partial_x b \\ &+\frac{1}{2}h^{\frac{1}{2}}(\partial_x^2\psi)a''(\xi + h^{\frac{1}{2}}\partial_x\psi)b\Bigg]. \end{aligned} \tag{4.25}$$

Since $\partial_x\psi \in S^0_\delta$, $h^\delta\partial_x^2\psi \in S^0_\delta$, we deduce from (4.24) and (4.25) that

$$\begin{aligned} A_0 + A_1 = \left[a(\xi) + h^{\frac{1}{2}}a'(\xi)\partial_x\psi + \frac{h}{2}a''(\xi)(\partial_x\psi)^2\right]b + \frac{h}{i}a'(\xi)\partial_x b \\ + hh^{\mu_0}(c_1 b + c_2 h^\delta\partial_x b) \end{aligned} \tag{4.26}$$

for some $c_j \in S^0_\delta$, where μ_0 has been defined in (4.7).

Now, consider the term A_k with $k \geq 2$. We have

$$A_k = \frac{h^k}{i^k k!}\sum_{k_1=0}^{k}\binom{k}{k_1}\partial_y^{k_1}\left[(\partial_\xi^k a)(\rho(x,y))\right]\Big|_{y=x}\partial_x^{k-k_1}b.$$

Since $h^{\frac{1}{2}}\partial_x\psi \in S^0_\delta$, we obtain,

$$c_{k,k_1} := h^{k_1\delta}\partial_y^{k_1}\left[(\partial_\xi^k a)(\rho(x,y))\right]\Big|_{y=x} \in S^0_\delta.$$

It follows that the generic term in A_k can be written as

$$hh^{k-1}h^{-k_1\delta}c_{k,k_1}h^{-\delta(k-k_1)}(h^\delta\partial_x)^{k-k_1}b.$$

We have, since $k \geq 2$,

$$\begin{aligned} k - 1 - k_1\delta - \delta(k-k_1) \geq k(1-\delta) - 1 \geq 2(1-\delta) - 1 \\ \geq 2(\frac{1}{2} - \delta) \geq \frac{1}{2} - \delta \geq \mu_0 \end{aligned} \tag{4.27}$$

so that

$$A_k = hh^{\mu_0}h\sum_{\ell=0}^{k}c_\ell(h^\delta\partial_x)^\ell b, \quad c_\ell \in S^0_\delta.$$

We deduce from (4.26) that

$$\sum_{k=0}^{M-1}A_k = \left[\sum_{j=0}^{2}\frac{1}{j!}a^{(j)}(\xi)(h^{\frac{1}{2}}\partial_x\psi)^j\right]b + \frac{h}{i}a'(\xi)\partial_x b + hh^{\mu_0}\sum_{\ell=0}^{M-1}d_\ell(h^\delta\partial_x)^\ell b \tag{4.28}$$

with $d_\ell \in S^0_\delta$.

Then it follows from (4.12),(4.17),(4.18) and (4.28) that

$$\begin{aligned} r = i\left(-a(\xi) + h^{\frac{1}{2}}\xi V_h + h^{\frac{1}{2}}\partial_\sigma\psi + hV_h\partial_x\psi\right) b\,\zeta \\ + h[\partial_\sigma b + h^{\frac{1}{2}}V_h\partial_x b + \frac{1}{2}h^{\frac{1}{2}}(\partial_x V_h)b]\zeta \\ + i\left[a(\xi) + h^{\frac{1}{2}}a'(\xi)\partial_x\psi + \frac{h}{2}a''(\xi)(\partial_x\psi)^2\right] b\,\zeta + ha'(\xi)\partial_x b\,\zeta \\ + hh^{\mu_0}\sum_{\ell=0}^{M-1} d_\ell(h^\delta\partial_x)^\ell b\,\zeta + \sum_{j=1}^{3} r_j. \end{aligned}$$

Gathering the terms in powers of h, noting that the coefficient of h^0 vanishes and using the eikonal equation to see that the coefficient in $h^{\frac{1}{2}}$ vanishes, we are left with

$$r = h\left(\partial_\sigma b + a'(\xi)\partial_x b + ifb + h^{\mu_0}h\sum_{\ell=0}^{M-1} e_\ell(h^\delta\partial_x)^\ell b\right)\zeta + i\sum_{j=1}^{4} r_j. \tag{4.29}$$

where $f = V_h\partial_x\psi + a''(\xi)(\partial_x\psi)^2$ is real valued, $e_\ell \in S^0_\delta$ and

$$r_4 = \frac{1}{i}h[-a'(\xi) + h^{\frac{1}{2}}V_h]b\,\zeta'. \tag{4.30}$$

It follows from (4.16) that

$$|\partial_x^\alpha\partial_\xi^k f(\sigma, x, \xi, h)| \le C_{k\alpha}\sigma h^{-k}h^{-\delta(\alpha+k)}, \tag{4.31}$$

for every $(\sigma, x, \xi, h) \in \mathcal{O}$. In particular $f \in S^0_\delta(\mathcal{O})$.

Now we shall seek b under the form

$$b = \sum_{j=0}^{J-1} h^{j\mu_0} b_j. \tag{4.32}$$

where the $b_j's$ are the solutions of the following problems

$$\begin{cases} \dfrac{\partial b_0}{\partial\sigma} + a'(\xi)\dfrac{\partial b_0}{\partial x} + ifb_0 = 0, \\ b_0|_{\sigma=0} = \chi_1(\xi), \end{cases} \tag{4.33}$$

where $\chi_1 \in C_0^\infty(\mathbf{R})$ has been introduced in (4.5) and

$$\begin{cases} \dfrac{\partial b_j}{\partial\sigma} + a'(\xi)\dfrac{\partial b_j}{\partial x} + ifb_j = -\displaystyle\sum_{\ell=0}^{M-1} e_\ell(h^\delta\partial_x)^\ell b_{j-1}, \\ b_j|_{\sigma=0} = 0. \end{cases} \tag{4.34}$$

It is easy to see that for all j we have,

$$b_j(\sigma, x, \xi, h) = \chi_1(\xi)c_j(\sigma, x, \xi, h). \tag{4.35}$$

For the estimates we shall use the following elementary lemma.

Lemma 4.6. — *If u is a solution of the problem*

$$\partial_\sigma u + a'(\xi)\partial_x u + ifu = g, \qquad u|_{\sigma=0} = z \in \mathbf{C},$$

where f be real-valued, then it satisfies the estimate

$$|u(\sigma, x, \xi, h)| \leq |z| + \int_0^\sigma |g(\sigma', x + (\sigma' - \sigma)a'(\xi), \xi, h)| d\sigma'$$

for every $(\sigma, x, \xi, h) \in \mathcal{O}$.

Proof. — Indeed, the solution is given by

$$u(\sigma, x, \xi, h) = e^{-i\int_0^\sigma f(\sigma', x+(\sigma'-\sigma)a'(\xi), \xi, h)\, d\sigma'} \times$$
$$\times \left\{ z + \int_0^\sigma e^{i\int_0^{\sigma'} f(t, x+(t-\sigma)a'(\xi), \xi, h)\, dt} g(\sigma', x + (\sigma' - \sigma)a'(\xi), \xi, h)\, d\sigma' \right\}.$$

□

Using this lemma we deduce the following.

Lemma 4.7. — *The problems* (4.33), (4.34) *have unique solutions $b_j = \chi_1(\xi)c_j$ where the c_j satisfy the estimates*

$$|\partial_x^\alpha \partial_\xi^k c_j(\sigma, x, \xi, h)| \leq C_{\alpha k j} h^{-(\alpha+k)\delta} \tag{4.36}$$

for all $(\sigma, x, \xi, h) \in \mathcal{O}$, all $\alpha, k \in \mathbf{N}$, and all $j = 0, ..., M$.

In particular $c = \sum_{j=0}^M c_j h^{j\mu_0}$ belongs to $S_\delta^0(\mathcal{O})$.

Proof. — Let us look to the case $j = 0$. Then c_0 satisfies the same equation and $c_0|_{\sigma=0} = 1$. We show first (4.36) for $k = 0$ and all α. By Lemma 4.6 we have $|c_0| \leq C$. So assume that (4.36) is true (for $k = 0$) up to the order $\alpha - 1$ and let us differentiate the equation (4.33) α time with respect to x. It follows that $U = \partial_x^\alpha c_0$ satisfies the equation

$$\frac{\partial U}{\partial \sigma} + a'(\xi)\frac{\partial U}{\partial x} + ifU = -i\sum_{l=1}^{\alpha} C_l^\alpha (\partial_x^l f)\partial_x^{\alpha-l} c_0. \tag{4.37}$$

Using (4.31), Lemma 4.6 and the induction we deduce that

$$|U| \leq C\sum_{l=1}^{\alpha} h^{-l\delta} h^{-(\alpha-l)\delta} \leq Ch^{-\alpha\delta}.$$

This proves (4.36) for $k = 0$ and all α. Then using an induction on k we differentiate the equation (4.37) k times wit respect to ξ we use again (4.31), Lemma 4.6 and the induction to prove (4.36) for all k and α. The proof of (4.36) for $j \geq 1$ is similar. □

It follows from (4.29), (4.33), (4.34) that

$$r = \sum_{j=1}^{5} r_j.$$

where for any $J \in \mathbf{N}$

$$r_5 = h^{J\mu_0} b_{J-1}\zeta. \tag{4.38}$$

4.1.1. End of the proof of Proposition 4.5. — We are left with the proof of (4.13). For $r_j, j = 1, 2$ defined in (4.21), (4.22) we have,

$$\langle x - z - \sigma a'(\xi)\rangle |r_j(\sigma, x, z, \xi, h)| \leq C h^{M(1-\delta)-1} |\chi_1(\xi)|. \tag{4.39}$$

Let us prove (4.39) for r_1. Recall that

$$r_1 = c\, h^{M-1} \iiint_0^1 e^{\frac{i}{h}(x-y)\eta} \kappa_0(\eta)(1-\lambda)^{M-1} \partial_y^M [a^{(M)}(\lambda\eta + \rho(x,y))\widetilde{b}(y)] d\lambda dy d\eta$$

where $\kappa_0 \in C_0^\infty(\mathbf{R}), \kappa_0 = 1$ in a neighborhood of the origin and

$$\widetilde{b}(y) = \chi_1(\xi) c(\sigma, y, \xi, h) \zeta(y - z - \sigma a'(\xi))$$

with $c \in S^0_\delta$ and $\delta < 1$.

We estimate separately $r_1, (x-y)r_1$ and $(y - z - \sigma a'(\xi))r_1$. Since $c \in S^0_\delta$ we can write,

$$|r_1| \leq C h^{M-1-\delta M} \int |\kappa_0(\eta)| d\eta \int |\zeta_1(u)| du$$

where $\zeta_1 \in C_0^\infty(\mathbf{R}), \zeta_1 = 1$ on the support of ζ. The term $(y - z - \sigma a'(\xi))r_1$ is estimated similarly, since $(y - z - \sigma a'(\xi))$ is bounded on the support of $\zeta_1(y - z - \sigma a'(\xi))$. Finally to estimate $(x-y)r_1$we integrate by parts using the fact that $\frac{h}{i}\partial_\eta e^{\frac{i}{h}(x-y)\eta} = (x-y)e^{\frac{i}{h}(x-y)\eta}$.

Recall that

$$r_2 = \sum_{k=0}^{M-1} c_{k,M} h^{M+k} \iint_0^1 z^M \hat{\kappa}_0(z)(1-\lambda)^{M-1} \partial_y^{M+k} [a^{(k)}(\rho)\widetilde{b}]|_{y=x-\lambda hz} d\lambda dz,$$

As above one can estimate separately $r_2, (x - z - \lambda hz - \sigma a'(\xi)) + \lambda hz)r_2$ and $\lambda hz r_2$ to obtain the desired bound. An estimate of the type of (4.39) for r_5 follows immediately from (4.38). Let us prove (4.13) for $r_j, j = 1, 2, 5$. Let us first bound the L^2 norm of the left hand side. Using (4.39) we can see that for any $N \in \mathbf{N}$ one can find M and J so large that

$$\int \|r_j(\sigma, \cdot, z, \xi, h)\|_{L^2(\mathbf{R}_x)} d\xi \leq C_N h^N.$$

It follows that

$$\left\| \iint e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h) - z\xi)} r_j(\sigma, x, z, \xi, h) u_{0,h}(z)\, dz\, d\xi \right\|_{L^2(\mathbf{R}_x)} \leq C_N h^N \int |u_{0,h}(z)| dz. \tag{4.40}$$

The estimate of the L^2 norm of the x derivative is analogous.

The terms corresponding to r_3 and r_4 defined in (4.23) and (4.30) will be treated in the same manner and will use the fact that on the support of a derivative of the function ζ one has $|x - z - \sigma a'(\xi)| \geq 1$. Since by (4.16) we have $h^{\frac{1}{2}}|\partial_\xi \psi| \leq C h^{\frac{1}{2}} |\sigma| \leq C h^{\frac{1}{2}}$ we deduce from (4.14) that $|\partial_\xi(\varphi(\sigma, x, \xi, h) - z\xi)| \geq \frac{1}{2}$ if h is small enough. Therefore we can obtain

an estimate analoguous to (4.40) by integrating N times by parts in the integral appearing in the left hand side of (4.40) using the vector field

$$L = \frac{h}{i(\partial_\xi(\varphi(\sigma, x, \xi, h) - z\xi))}\partial_\xi.$$

The proof of Proposition 4.5 is complete. □

4.2. Refined Van der Corput estimate. — Let us recall that we have set (see (4.3))

$$\widetilde{U}_h(t,x) = \int \widetilde{K}(t,x,z,h)u_{0h}(z)\,dz \tag{4.41}$$

where

$$\widetilde{K}(t,x,z,h) = \frac{1}{2\pi h}\int e^{\frac{i}{h}(\Phi(t,x,\xi,h)-z\xi)}\widetilde{B}(t,x,z,\xi,h)\,d\xi. \tag{4.42}$$

In the variable $\sigma = th^{-\frac{1}{2}}$ we have

$$\widetilde{K}(t,x,z,h) = K(\sigma,x,z,h)$$

where

$$K(\sigma,x,z,h) = \frac{1}{2\pi h}\int e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)}\widetilde{b}(\sigma,x,z,\xi,h)\,d\xi,$$

where φ and b have been determined in (4.14), (4.33) and (4.34).

Proposition 4.8. — *There exists $C > 0$ such that*

$$|K(\sigma,x,z,h)| \le \frac{C}{h}\left(\frac{h}{\sigma}\right)^{\frac{1}{2}}, \tag{4.43}$$

for all (σ,x,z,h) in $]0,1/2]\times\mathbf{R}\times\mathbf{R}\times]0,h_0[$.

Proof. — Since $b \in S^0_\delta$ is bounded with compact support in ξ, the estimate (4.43) is trivial for $|\sigma| \le Ch$. Let us assume that $|\sigma| \ge Ch$. We have by (4.11),

$$\mathcal{L} = \left(h\partial_\sigma + h^{\frac{1}{2}}W^\delta_h(h^{\frac{1}{2}}\sigma,x)(h\partial_x) + \frac{1}{2}h^{\frac{3}{2}}(\partial_x W^\delta_h) + ia(hD_x))\right)$$

By a scaling argument we can assume without loss of generality that $\sigma = \sigma_0 = 1/2$. Indeed, otherwise, setting

$$\tau = \frac{\sigma}{\sigma_0}, \quad \widetilde{x} = \frac{x}{\sigma_0}, \quad \widetilde{h} = \frac{h}{\sigma_0},$$

we see that in the new variables, the operator reads

$$\widetilde{L} = \widetilde{h}\partial_\tau + \widetilde{h}^{1/2}\widetilde{W}^\delta_h\widetilde{h}\partial_x + \widetilde{h}^{\frac{3}{2}}(\partial_{\widetilde{x}}\widetilde{W}^\delta_h) + i|\widetilde{h}D_{\widetilde{x}}|^{3/2}$$

where

$$\widetilde{W}^\delta_h(\tau,\widetilde{x}) = \sigma_0^{1/2}W^\delta_h(\sigma_0\tau,\sigma_0\widetilde{x})$$

and consequently we have

$$\widetilde{W}^\delta_h \in L^\infty(H^{s-1}), \qquad \partial_{\widetilde{x}}\widetilde{W}^\delta_h \in S^0_\delta$$

with bounds uniform with respect to σ_0.

Assume now that the dispersion estimate has been proved for the kernel of the operator $\widetilde{L}$ and $\sigma = 1/2$. Since we have

$$S_h(\sigma)u_0(x) = (\widetilde{S}_{\widetilde{h}}(\frac{\sigma}{\sigma_0})\widetilde{u_0})(\frac{x}{\sigma_0}), \qquad \widetilde{u_0}(\frac{x}{\sigma_0}) = u_0(x),$$

we can write

$$\begin{aligned} (4.44) \quad \|S_h(\sigma_0)u_0\|_{L^\infty(\mathbf{R})} &= \|\widetilde{S}_{\widetilde{h}}(\frac{\sigma}{\sigma_0})\widetilde{u_0}\|_{L^\infty(\mathbf{R})} \\ &\leq \frac{C}{|\widetilde{h}|^{1/2}}\|\widetilde{u_0}\|_{L^1(\mathbf{R})} \leq \frac{C|\sigma_0|^{1/2}}{|h|^{1/2}|\sigma_0|}\|u_0\|_{L^1(\mathbf{R})} \leq \frac{C}{|h\sigma_0|^{1/2}} \end{aligned}$$

which is the dispersion estimate for the kernel of the operator L and $\sigma = \sigma_0$.

Let us set

$$\theta(x, y, \xi, h) = \varphi(\sigma, x, \xi, h) - z\xi = (x - z)\xi + a(\xi)\sigma + h^{\frac{1}{2}}\psi(\sigma, x, \xi, h). \tag{4.45}$$

Then

$$\partial_\xi^2\theta(x, y, \xi, h) = \partial_\xi^2 a(\xi)\sigma + h^{\frac{1}{2}}\partial_\xi^2\psi(\sigma, x, \xi, h).$$

Now by (4.5) and (4.8), on the support of χ_1 we have $a(\xi) = |\xi|^{\frac{3}{2}}$. Therefore $\partial_\xi^2 a(\xi) = \frac{3}{4}|\xi|^{-\frac{1}{2}} \geq c_0 > 0$. On the other hand from (4.16) we have $|\partial_\xi^2\psi| \leq C\sigma h^{-\delta}$ which implies $h^{\frac{1}{2}}|\partial_\xi^2\psi| \leq C\sigma h^{\frac{1}{2}-\delta} \leq C\sigma h^{\mu_0}$. It follows that on the support of χ_1 one can find a constant $c_1 > 0$ such that

$$0 < c_1\sigma \leq \partial_\xi^2\theta(x, y, \xi, h) \leq \frac{1}{c_1}\sigma, \tag{4.46}$$

if h_0 is small enough.

In the sequel we shall omit to note the variables (x, z, h) which are fixed. However, we shall take care of the fact that all the constants are independent of $(x, z, h) \in \mathbf{R}\times\mathbf{R}\times]0, h_0[$.

Let us denote by $[\alpha, \beta] \subset [\frac{1}{3}, 3]$ the support of χ_1.We deduce from (4.46) that the function $\xi \to \partial_\xi\theta(\xi)$ is increasing on $[\alpha, \beta]$. Therefore one can find $\rho \in [\alpha, \beta]$ such that

$$\partial_\xi\theta(\xi) \leq 0 \quad \text{for } \xi \in [\alpha, \rho], \quad \partial_\xi\theta(\xi) \geq 0 \quad \text{for } \xi \in [\rho, \beta].$$

Noting $b(\sigma, x, \xi, h) = b(\xi)$ and assuming that $]\rho, \beta[$ is non empty, we shall estimate

$$K_+(\sigma, x, \xi, h) = \frac{1}{2\pi h}\int_\rho^\beta e^{\frac{i}{h}\theta(\xi)}b(\xi)\zeta(x - z - a'(\xi))d\xi,$$

the estimate corresponding to the intervall $[\alpha, \rho]$ being similar. We write for small h,

$$\begin{cases} K_+ = \dfrac{1}{2\pi h}(I_1 + I_2), \\ I_1 = \displaystyle\int_{\rho}^{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}} e^{\frac{i}{h}\theta(\xi)} b(\xi)\zeta(x - z - a'(\xi))d\xi \\ I_2 = \displaystyle\int_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} e^{\frac{i}{h}\theta(\xi)} b(\xi)\zeta(x - z - a'(\xi))d\xi \end{cases} \tag{4.47}$$

We have obviously,

$$|I_1| \leq C(\frac{h}{\sigma})^{\frac{1}{2}} \tag{4.48}$$

In the integral I_2 using (4.46) and the Taylor formula we see that,

$$\partial_\xi \theta(\xi) \geq c_1 \sigma \big(\frac{h}{\sigma}\big)^{\frac{1}{2}} = C_1 (h\sigma)^{\frac{1}{2}}, \quad \forall \xi \in [\rho + (\frac{h}{\sigma})^{\frac{1}{2}}, \beta]. \tag{4.49}$$

Let us estimate the integral I_2. We can state the following lemma which is a refined version of the well known Van der Corput Lemma.

Lemma 4.9. — *For all $k \in \mathbf{N}$ there exists $C > 0$ such that*

$$\Big| I_2 - (-1)^k \big(\frac{h}{i}\big)^k \int_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} e^{\frac{h}{i}\theta(\xi)} \frac{1}{(\partial_\xi \theta(\xi))^k} \partial_\xi^k q(\xi) d\xi \Big| \leq C\big(\frac{h}{\sigma}\big)^{\frac{1}{2}} \tag{4.50}$$

where $q(\xi) = b(\xi)\zeta(x - z - a'(\xi)$.

Proof. — Let us denote by J_k the integral term in (4.50). The Lemma is true for $k = 0$. Assume it is true up to the order k. Using the fact that $\frac{h}{i\partial_\xi \theta(\xi)} \partial_\xi e^{\frac{i}{h}\theta(\xi)} = e^{\frac{i}{h}\theta(\xi)}$ and integrating by parts in J_k we obtain,

$$\begin{aligned} J_k = (-1)^{k+1} \big(\frac{h}{i}\big)^{k+1} & \int_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} e^{\frac{h}{i}\theta(\xi)} \partial_\xi \Big(\frac{1}{(\partial_\xi \theta(\xi))^{k+1}} \Big) \partial_\xi^k q(\xi) d\xi \\ + (-1)^{k+1} \big(\frac{h}{i}\big)^{k+1} & \int_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} e^{\frac{h}{i}\theta(\xi)} \frac{1}{(\partial_\xi \theta(\xi))^{k+1}} \partial_\xi^{k+1} q(\xi) d\xi \\ + (-1)^{k+1} \big(\frac{h}{i}\big)^{k+1} & [e^{\frac{h}{i}\theta(\xi)} \Big(\frac{1}{(\partial_\xi \theta(\xi))^{k+1}} \Big) \partial_\xi^k q(\xi)]_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} = J_k^1 + J_k^2 + J_k^3. \end{aligned}$$

First of all we have $J_k^2 = J_{k+1}$. Now using (4.36) and (4.49) we can write,

$$|J_k^3| \leq C h^{k+1} \frac{h^{-k(\delta+4)}}{(h\sigma)^{\frac{k+1}{2}}} \leq C\big(\frac{h}{\sigma}\big)^{\frac{1}{2}} h^{k(\frac{1}{2}-\delta-4)} \leq C\big(\frac{h}{\sigma}\big)^{\frac{1}{2}},$$

since $\sigma \geq 1$. Now using again (4.36) we obtain,

$$|J_k^1| \leq C h^{k+1-k(\delta+4)} \int_{\rho+(\frac{h}{\sigma})^{\frac{1}{2}}}^{\beta} \Big| \partial_\xi \Big(\frac{1}{(\partial_\xi \theta(\xi))^{k+1}} \Big) \Big| d\xi.$$

Since by (4.46) the function $\partial_\xi\theta$ is increasing we have

$$\Big|\partial_\xi\Big(\frac{1}{(\partial_\xi\theta(\xi))^{k+1}}\Big)\Big| = -\partial_\xi\Big(\frac{1}{(\partial_\xi\theta(\xi))^{k+1}}\Big).$$

Therefore we can write,

$$|J_k^1| \le Ch^{k+1-k(\delta+4)}[-\Big(\frac{1}{(\partial_\xi\theta(\xi))^{k+1}}\Big)]_{\rho+(\frac{h}{\sigma})^{\frac12}}^{\beta}.$$

We deduce exactly as for J_k^3 that,

$$|J_k^1| \le C\big(\frac{h}{\sigma}\big)^{\frac12}.$$

It follows that $|J_k - J_k^2| \le C\big(\frac{h}{\sigma}\big)^{\frac12}$, which proves our induction. □

Now using Lemma 4.9, (4.36) and (4.49) we can write,

$$|J_k| \le Ch^k\frac{1}{(h\sigma)^{\frac{k}{2}}}h^{-k\delta} \le C\big(\frac{h}{\sigma}\big)^{\frac12}h^{k(\frac12-\delta)-\frac12} \le C\big(\frac{h}{\sigma}\big)^{\frac12}h^{2k\mu_0-\frac12},$$

so taking k such that $2k\mu_0 \ge \frac12$ and using (4.7) we deduce that $|J_k| \le C\big(\frac{h}{\sigma}\big)^{\frac12}$. It follows the from Lemma 4.9 that $|I_2| \le C\big(\frac{h}{\sigma}\big)^{\frac12}$ and from (4.48), (4.47) that $|K_+| \le \frac{C}{h}\big(\frac{h}{\sigma}\big)^{\frac12}$ which completes the proof of Proposition 4.8. □

4.3. End of the proof of Theorem 4.2. — Let us set $J = [0, h^{\frac12}]$. It follows from (4.3) and Proposition 4.5 that

$$\partial_t\widetilde{U}_h + \frac12(W_h^\delta\partial_x + \partial_x W_h^\delta)\widetilde{U}_h + ia(D_x)\widetilde{U}_h = F_h, \quad \widetilde{U}_h|_{t=0} = \widetilde{U}_h(0,x), \tag{4.51}$$

with

$$\sup_{s\in J}\|F_h(s,\cdot)\|_{H^1(\mathbf{R})} \le C_N h^N\|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.52}$$

We claim that,

$$\widetilde{U}_h(0,\cdot) = u_{0,h} + v_{0,h}, \quad \|v_{0,h}\|_{H^1(\mathbf{R})} \le C_N h^N\|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.53}$$

Indeed using (4.3), (4.4),(4.14),(4.33) and (4.34) we see that,

$$v_{0,h}(x) = (2\pi h)^{-1}\iint e^{\frac{i}{h}(x-z)\xi}(\zeta(x-z)-1)\chi_1(\xi)u_{0,h}(z)dzd\xi. \tag{4.54}$$

Since on the support of $1-\zeta(x-z)$ we have $|x-z| \ge 1$ we can integrate by parts as much as we want to obtain that for all $N \ge 1$,

$$v_{0,h}(x) = c_N h^{N-1}\iint e^{\frac{i}{h}(x-z)\xi}\left[\frac{1-\zeta(x-z)}{(x-z)^N}\right](\partial_\xi^N\chi_1)(\xi)u_{0,h}(z)dzd\xi.$$

Using the Hölder inequality we deduce that,

$$|v_{0,h}(x)|^2 \le C_N h^{N-1}\Big(\int\Big|\frac{1-\zeta(x-z)}{(x-z)^N}\Big|^2|u_{0,h}(z)|dz\Big)\|u_{0,h}\|_{L^1(\mathbf{R})}$$

from which we deduce that,

$$\|v_{0,h}\|_{L^2(\mathbf{R})} \leq C_N h^{N-1} \|u_{0,h}\|_{L^1(\mathbf{R})}.$$

Differentiating (4.54) with respect to x and using the same trick we obtain the estimate in (4.53).

Now by (4.51), the Duhamel formula and the definition in Theorem 4.2 we can write,

$$\begin{aligned} &S(t,0,h)u_{0,h} = D_1 + D_2 + D_3 \quad where \quad D_1 = \widetilde{U}_h(t,x), \\ &D_2 = -S(t,0,h)v_{0,h}(x),\ D_3 = -\int_0^t S(t,s,h)[F_h(s,x)]ds. \end{aligned} \tag{4.55}$$

First of all the estimate

$$\|D_1(t)\|_{L^\infty(\mathbf{R})} \leq \frac{C}{h^{1/4}|t|^{1/2}} \|u_{0,h}\|_{L^1(\mathbf{R})} \tag{4.56}$$

follows from Proposition 4.8 and (4.41).

Let us estimate D_2. We have by Sobolev inequality,

$$\|D_2(t)\|_{L^\infty(\mathbf{R})} \leq C_1 \|D_2(t)\|_{H^1(\mathbf{R})} \leq C_2 \|v_{0,h}\|_{H^1(\mathbf{R})},$$

therefore by (4.53),

$$\|D_2(t)\|_{L^\infty(\mathbf{R})} \leq C_N h^N \|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.57}$$

Let us look now to the term D_3. We have,

$$\|D_3(t)\|_{L^\infty(\mathbf{R})} \leq C \int_J \|S(t,s,h)F_h(s,\cdot)\|_{H^1(\mathbf{R})} ds \leq C' \int_J \|F_h(s,\cdot)\|_{H^1(\mathbf{R})} ds,$$

from which we deduce,

$$\|D_3(t)\|_{L^\infty(\mathbf{R})} \leq C_N h^N \|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.58}$$

Then Theorem 4.2 follows from (4.55), (4.56), (4.57), and (4.58).

4.4. The TT^* argument. — Having proved the dispersion estimate the Strichartz estimates for the solution of (4.1) follow very classically.

Proposition 4.10. — *There exist > 0, $C > 0$ such that for any $0 < h < 1$ and any initial data $u_{0,h} = \chi(hD_x)u_0$, we have*

$$\|S(t,0,h)u_{0,h}\|_{L^4((0,h^{\frac{1}{2}}),L^\infty(\mathbf{R}))} \leq C \|u_{0,h}\|_{H^{\frac{1}{8}}(\mathbf{R})}. \tag{4.59}$$

Proof. — Indeed, applying the usual TT^* argument, it suffices to prove that the operator

$$\int_0^{h^{\frac{1}{2}}} S(t,0,h)S(s,0,h)^* f(s)ds$$

maps continuously $L^{\frac{4}{3}}((0,h^{\frac{1}{2}}),L^1(\mathbf{R}))$ to $L^4((0,h^{\frac{1}{2}}),L^\infty(\mathbf{R}))$. But a direct calculation shows that since $\frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta)$ is self adjoint, one has

$$S(s,0,h)^* = S(0,s,h),$$

and consequently, Proposition 4.10 follows from the classical Hardy-Littlewood-Sobolev inequality and the dispersion estimate (4.2). □

Corollary 4.11. — *Let u be a solution of the problem*

$$\partial_t u + \frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta)u + ia(D_x)u = f, \quad u|_{t=0} = 0$$

with $\operatorname{supp} \hat{f} \subset \{\frac{1}{2}h^{-1} \leq |\xi| \leq 2h^{-1}\}$. *Then we have,*

$$\|u\|_{L^4((0,h^{\frac{1}{2}}),L^\infty(\mathbf{R}))} \leq Kh^{-1/8} \|f\|_{L^1((0,h^{\frac{1}{2}}),L^2(\mathbf{R}))}.$$

Proof. — Indeed we have,

$$u(t,\cdot) = \int_0^t S(t,0,h)S^*(s,0,h)f(s,\cdot)\,ds.$$

Let us set $J = [0, h^{\frac{1}{2}}]$. It follows from Proposition 4.10 that,

$$\begin{aligned}\|u\|_{L^4(J,L^\infty(\mathbf{R}))} &\leq C\int_0^{h^{\frac{1}{2}}} \|S(s,0,h)^* f(s,\cdot)\|_{H^{\frac{1}{8}}(\mathbf{R})} ds \\ &\leq C'\int_0^{h^{\frac{1}{2}}} \|f(s,\cdot)\|_{H^{\frac{1}{8}}(\mathbf{R})} ds \leq C'' h^{-1/8} \|f\|_{L^1(J,L^2(\mathbf{R}))},\end{aligned}$$

since $\hat{f}$ is supported in $\{\frac{1}{2}h^{-1} \leq |\xi| \leq 2h^{-1}\}$. □

4.5. Gluying the estimates. — It remains to glue the estimates which up to now have been proved on small time intervals of size $h^{\frac{1}{2}}$. Recall that from Lemma 3.6 we have

$$\partial_t u_h + \frac{1}{2}(W_h^\delta \partial_x + \partial_x W_h^\delta)u_h + i\,|D_x|^{\frac{3}{2}}\, u_h = f_h \in L^\infty((0,T); H^{s+\varepsilon-\frac{1}{2}}).$$

Let $\varphi \in C_0^\infty(0,2)$, equal to 1 on $(\frac{1}{2}, \frac{3}{2})$. For $-1 \leq k \leq Th^{-\frac{1}{2}}$, define

$$u_{h,k} = \varphi\Big(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big)u_h,$$

which satisfies

$$\begin{aligned}(4.60)\quad \partial_t u_{h,k} + \frac{1}{2}(W_h^\delta \partial_x u + \partial_x W_h^\delta)u_{h,k} + i\,|D_x|^{\frac{3}{2}}\, u_{h,k} \\ = \varphi\Big(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big)f_h + h^{-\frac{1}{2}}\varphi'\Big(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big)u_h, \quad u_{h,k}\big|_{t=kh^{\frac{1}{2}}} = 0,\end{aligned}$$

As a consequence, using Corollary 4.11 we obtain,

$$
\begin{aligned}
(4.61)\quad & \|u_{h,k}\|_{L^4((kh^{\frac12},(k+2)h^{\frac12}),\,L^\infty(\mathbf{R}))} \\
&\le h^{-\frac18}\Big\|\varphi\Big(\frac{t-kh^{\frac12}}{h^{\frac12}}\Big)f_h + h^{-\frac12}\varphi'\Big(\frac{t-kh^{\frac12}}{h^{\frac12}}\Big)u_h\Big\|_{L^1((kh^{\frac12},(k+2)h^{\frac12}),\,L^2(\mathbf{R}))} \\
&\le Ch^{\frac12-\frac18}\Big(\|f_h\|_{L^\infty(0,T),\,L^2(\mathbf{R}))} + h^{-\frac12}\|u_h\|_{L^\infty((0,T),\,L^2(\mathbf{R}))}\Big) \\
&\le Ch^{s-\frac18}(h^\varepsilon + \|u_h\|_{L^\infty((0,T),H^s(\mathbf{R}))})
\end{aligned}
$$

where in the last inequality we used that by (3.43) we have $f_h \in L^\infty((0,T),H^{s+\varepsilon-\frac12}(\mathbf{R}))$. Eventually, noticing that,

$$
\|u_h\|^4_{L^4((0,T),\,L^\infty(\mathbf{R}))} \le \sum_{k=-1}^{Th^{-\frac12}} \|u_{h,k}\|^4_{L^4((kh^{\frac12},(k+2)h^{\frac12}),\,L^\infty(\mathbf{R}))},
$$

we obtain

$$
\|u_h\|^4_{L^4((0,T),\,L^\infty(\mathbf{R}))} \le Ch^{-\frac12}h^{4(s-\frac18)}(h^\varepsilon + \|u_h\|_{L^\infty((0,T),H^s(\mathbf{R}))})^4. \tag{4.62}
$$

For $h = 2^{-j}$, let us set

$$
c_j = 2^{-j\varepsilon} + \|u_h\|_{L^\infty((0,T),H^s(\mathbf{R}))}
$$

Notice that according to (3.45), we have

$$
\sum_{j\in\mathbf{N}} c_j^2 < +\infty. \tag{4.63}
$$

We deduce

$$
\begin{aligned}
\|u\|_{L^4((0,T),\,B^{s-\frac14}_{\infty,2}(\mathbf{R}))} &= \|\Big(\sum_{j\in\mathbf{N}} 2^{2j(s-\frac14)}\|\Delta_j u\|^2_{L^\infty(\mathbf{R})}\Big)^{\frac12}\|_{L^4(0,T)} \\
&= \|\sum_{j\in\mathbf{N}} 2^{2j(s-\frac14)}\|\Delta_j u\|^2_{L^\infty(\mathbf{R})}\|^{\frac12}_{L^2(0,T)} \\
&\le \Big(\sum_{j\in\mathbf{N}} \|2^{2j(s-\frac14)}\|\Delta_j u\|^2_{L^\infty(\mathbf{R})}\|_{L^2(0,T)}\Big)^{\frac12} \\
&= \Big(\sum_{j\in\mathbf{N}} 2^{2j(s-\frac14)}\|\Delta_j u\|^2_{L^4((0,T),L^\infty(\mathbf{R}))}\Big)^{\frac12}
\end{aligned} \tag{4.64}
$$

which, by (4.62) and (4.63) completes the proof of Theorem 4.1.

5. Classical time parametrix

In this section we take $s > \frac{11}{2}$ and we prove the usual Strichartz estimates. The main step is, as before, the dispersion estimate. To do so, we seek a parametrix. The main difference with respect to the previous section is that (in the semi-classical framework),

we are looking for a large ($\mathcal{O}(h^{-1/2})$) time parametrix. As a consequence, the lower order term $T_W\partial_x$ induces oscillations. This is reflected in the fact that the new eikonal equation will be quasi-linear.

We begin by an analogue of Theorem 4.1.

Theorem 5.1. — *Under the assumptions of Theorem 1.2, let u be defined by* (3.32). *Then there exists $C = C(\|(\eta,\psi)\|_{L^\infty(I,H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))})$ such that*

$$\|u\|_{L^4(I,B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} \leq C.$$

Theorem 5.2. — *Let $\chi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and $t_0 \in \mathbf{R}$. For any initial data $u_{0,h} = \chi(hD_x)u_0$ where $u_0 \in L^1(\mathbf{R})$ let us denote by $S(t,t_0,h)u_{0,h} := U_h$ the solution of*

$$\partial_t U_h + \frac{1}{2}(W_h^\delta\partial_x + \partial_x W_h^\delta)U_h + ia(D_x)U_h = 0, \qquad U_h\,|_{t=t_0} = u_{0,h}. \tag{5.1}$$

Then there exists $\tau_0 > 0$ such that for any $0 < h \leq 1$ and any $|t-t_0| \leq \tau_0$,

$$\|S(t,t_0,h)u_{0,h}\|_{L^\infty(\mathbf{R})} \leq \frac{C}{h^{1/4}|t-t_0|^{1/2}}\|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{5.2}$$

In the remaining of this section, we shall prove Theorem 5.2. We need first to refine the constructions in Section 4 to handle large times. An important point in the construction of the phase function is that handling large times leads us to *non linear geometric optics*.

Our parametrixe will be of the form (4.3),(4.4) that is,

$$\widetilde{U}_h(t,x) = \frac{1}{2\pi h}\iint e^{\frac{i}{h}(\Phi(t,x,\xi,h)-z\xi)}\widetilde{B}(t,x,z,\xi,h)u_{0,h}(z)dzd\xi, \tag{5.3}$$

where Φ will satisfy the eikonal equation and

$$\widetilde{B}(t,x,z,\xi,h) = B(t,x,\xi,h)\zeta(x-z-th^{-\frac{1}{2}}a'(\xi)), \tag{5.4}$$

where B will satisfy the transport equations and $\zeta \in C_0^\infty(\mathbf{R})$, $\zeta(s) = 1$ if $|s| \leq 1$, $\zeta(s) = 0$ if $|s| \geq 2$.

5.1. Notations. — In this section we fix

$$s > \frac{11}{2} \text{ and } \delta = \frac{1}{s-\frac{3}{2}} < \frac{1}{4}.$$

As before we shall set $2^{-j} = h$, where $j \in \mathbf{N}$ and we shall work with the semiclassical time $\sigma = th^{-\frac{1}{2}}$.

In addition to the function χ introduced in Theorem 5.2, we shall use two more cut-off functions $\chi_j \in C_0^\infty(\mathbf{R}), j = 1,2$, such that,

$$\begin{cases} \operatorname{supp}\chi_1 \subset \{\xi : \dfrac{1}{3} \leq |\xi| \leq 3\}, & \chi_1 = 1 \text{ on the support of } \chi, \\ \operatorname{supp}\chi_0 \subset \{\xi : \dfrac{1}{4} \leq |\xi| \leq 4\}, & \chi_0 = 1 \text{ on the support of } \chi_1. \end{cases} \tag{5.5}$$

Recall that we have

(i) $\quad W \in L^\infty([0,T], W^{2,\infty}(\mathbf{R})), \partial_t W \in L^\infty([0,T], W^{1,\infty}(\mathbf{R}))$ (Lemma 3.4),

(ii) $\quad W_h^\delta = S_{[\delta(j-3)]}(W)$ satisfies $\left\|\partial_x^\alpha W_h^\delta\right\|_{L^\infty_{t,x}} \leq C_\alpha \left\|\partial_x^\alpha W\right\|_{L^\infty_{t,x}}$,

(iii) $\quad a(\xi) = \chi_0(\xi)\,|\xi|^{\frac{3}{2}}$,

Definition 5.3. — *For small h_0, τ_0 to be fixed, we introduce the sets*

$$\Omega = \left\{(t,x,\xi,h) \in \mathbf{R}^4 \,:\, h \in (0,h_0), |t| < \tau_0, 1 < |\xi| < 3\right\}$$
$$\mathcal{O} = \left\{(\sigma,x,\xi,h) \in \mathbf{R}^4 \,:\, h \in (0,h_0), |\sigma| < \tau_0 h^{-\frac{1}{2}}, 1 < |\xi| < 3\right\}.$$

If $m \in \mathbf{R}$ and $\varrho \in \mathbb{R}^+$, we denote by $S^m_\varrho(\Omega)$ (resp. $S^m_\varrho(\mathcal{O})$) the set of all functions f on Ω (resp. $\mathcal{O}$) which are C^∞ with respect to (t,x,ξ) (resp. (σ,x,ξ) and satisfy the estimate

$$|\partial_x^\alpha f(t,x,\xi,h)|\,(resp.\ |\partial_x^\alpha f(\sigma,x,\xi,h)|) \leq C_\alpha h^{m-\varrho\alpha}, \tag{5.6}$$

for all $(t,x,\xi,h) \in \Omega$ (resp. $(\sigma,x,\xi,h) \in \mathcal{O}$).

5.2. The eikonal and transport equations. — In all this section we keep the notations of (4.11),(4.12), and (4.17) to (4.23).

The main result is the following.

Proposition 5.4. — *There exist a phase Φ of the form*

$$\Phi(t,x,\xi,h) = x\xi - h^{-\frac{1}{2}} t a(\xi) + h^{\frac{1}{2}}\Psi(t,x,\xi,h)$$

with $\partial_x \Psi \in S^0_\delta(\Omega)$ and an amplitude $B \in S^0_\delta\Omega)$ such that, with $\widetilde{B}$ defined in (5.4),

$$\mathcal{L}_0\left(e^{\frac{i}{h}\Phi}\widetilde{B}\right) = e^{\frac{i}{h}\Phi} R_h. \tag{5.7}$$

and for all $N \in \mathbf{N}$ we have,

$$\left\| \iint e^{\frac{i}{h}(\Phi(t,x,\xi,h) - z\xi)} R_h(t,x,z,\xi,h) u_{0,h}(z)\, dz\, d\xi \right\|_{H^1(\mathbf{R}_x)} \leq C_N h^N \|u_{0,h}\|_{L^1(\mathbf{R})}, \tag{5.8}$$

for all t in $[0,\tau_0]$.

Proof. — According to (4.23) we have,

$$\begin{cases} A_0 = a(\xi + h^{\frac{1}{2}}\partial_x\psi)b, \\ A_1 = \dfrac{h}{i} a'(\xi + h^{\frac{1}{2}}\partial_x\psi)\partial_x b + \dfrac{1}{2}\dfrac{h^{\frac{3}{2}}}{i}(\partial_x^2\psi) a''(\xi + h^{\frac{1}{2}}\partial_x\psi) b. \end{cases}$$

We deduce (with $\mathcal{L}$ defined in (4.11)) that $\mathcal{L}(e^{\frac{i}{h}\varphi}\widetilde{b}) = e^{\frac{i}{h}\varphi} r$ with

$$
\begin{aligned}
r = i\Big\{ & h^{\frac12}\partial_\sigma\psi - a(\xi) + a(\xi + h^{\frac12}\partial_x\psi) + hV_h\partial_x\psi + h^{\frac12}\xi V_h\Big\}\, b\,\zeta \\
& + h\Big\{\partial_\sigma b + a'(\xi + h^{\frac12}\partial_x\psi)\partial_x b + \frac12 h^{\frac12} V_h\partial_x b + \frac12 h^{\frac12}(\partial_x V_h) b \\
& + h^{\frac12}(\frac12\partial_x^2\psi)a''(\xi + h^{\frac12}\partial_x\psi)b + \frac{i}{h}\sum_{k=2}^{M-1} A_k\Big\}\zeta + i\sum_{j=1}^{4} r_j,
\end{aligned}
\tag{5.9}
$$

where r_1, r_2 are defined in (4.21), (4.22), r_3 in (4.23) and

$$
r_4 = \frac{h}{i}\Big\{-a'(\xi) + h^{\frac12}V_h\Big\}\, b\,\zeta'. \tag{5.10}
$$

5.2.1. The eikonal equation. — As already mentioned, an important point in the construction of the phase function is that handling large times leads us to *non linear geometric optics*.

An important fact in the sequel is that $\partial_\sigma V_h = \mathcal{O}(h^{\frac12})$ which follows from the fact that (see (4.11)) $V_h(\sigma, x) = W_h^\delta(h^{\frac12}\sigma, x)$; that is why we shall keep the notation $W_h^\delta(h^{\frac12}\sigma, x)$ in what follows.

We determine ψ by solving the following nonlinear problem,

$$
\begin{cases}
\partial_\sigma\psi + \dfrac{a(\xi + h^{\frac12}\partial_x\psi) - a(\xi)}{h^{\frac12}} + h^{\frac12}W_h^\delta(h^{\frac12}\sigma, x)\partial_x\psi = -W_h^\delta(h^{\frac12}\sigma, x)\xi, \\
\psi(0, x, \xi, h) = 0.
\end{cases}
\tag{5.11}
$$

In this system, ξ and h are seen as parameters. We begin by establishing that the solutions exist for a time interval of size $h^{-\frac12}$ and satisfy some uniform estimates.

Proposition 5.5. — *There exists $\tau_0 > 0$ such that the problem* (5.11) *has a unique C^∞ solution ψ in the set $\mathcal{O}$ such that $\frac{1}{\sigma}\psi, \partial_x\psi$ and $\partial_x^2\psi$ are uniformly bounded on $\mathcal{O}$ by $C\big(\|(\eta,\psi)\|_{L^\infty(I, H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R}))}\big)$ where C is an increasing function from $\mathbf{R}^+$ to itself.*

Proof. — Let us differentiate the equation (5.11) with respect to x and let us set $\psi_1 = \partial_x\psi$. Then ψ_1 is solution of the quasi-linear equation

$$
\partial_\sigma\psi_1 + A(\sigma, x, h, \xi, \psi_1)\partial_x\psi_1 = B(\sigma, x, h, \xi, \psi_1), \quad \psi_1(0, x, h, \xi) = 0. \tag{5.12}
$$

where

$$
\begin{cases}
A(\sigma, x, h, \xi, z) = a'(\xi + h^{\frac12}z) + h^{\frac12}W_h^\delta(h^{\frac12}\sigma, x), \\
B(\sigma, x, h, \xi, z) = -h^{\frac12}(\partial_x W_h^\delta)(h^{\frac12}\sigma, x)z - \xi(\partial_x W_h^\delta)(h^{\frac12}\sigma, x).
\end{cases}
\tag{5.13}
$$

We shall solve (5.12) by the method of characteristics.
The characteristics are given by the system

$$\begin{cases} \dot\sigma(s) = 1, \quad \sigma(0) = 0, \\ \dot X(s) = A(s, X(s), h, \xi, Z(s)), \quad X(0) = x, \\ \dot Z(s) = B(s, X(s), h, \xi, Z(s)), \quad Z(0) = 0. \end{cases} \tag{5.14}$$

Since A is uniformly bounded and $|B| \le C_1 + C_2|z|$, the above system has a unique global solution (i.e defined for $s \in [0, +\infty[$).

5.2.2. Properties of the flow.— (i) We have,

$$\exists \tau_0 > 0, c_1 > 0, c_2 > 0 : c_1 \le |\dot X(s)| \le c_2, \quad 0 \le s \le \tau_0 h^{-\frac12}. \tag{5.15}$$

We show first that

$$h^{\frac12}|Z(s)| \le C\tau_0 \|\partial_x W\|_{L^\infty} \exp\big(\tau_0 \|\partial_x W\|_{L^\infty}\big), \quad 0 \le s \le \tau_0 h^{-\frac12}. \tag{5.16}$$

To see this we integrate the equation satisfied by Z and use (5.13). We obtain

$$|Z(s)| \le C\|\partial_x W\|_{L^\infty_{t,x}}|s| + h^{\frac12}\|\partial_x W\|_{L^\infty_{t,x}} \int_0^s |Z(\sigma)|d\sigma, \quad 0 \le s \le \tau_0 h^{-\frac12}. \tag{5.17}$$

Then (5.16) follows from the Gronwall inequality.

On the other hand, setting $m(s) = (s, X(s), h, \xi, Z(s))$, we have

$$\begin{aligned} A(m(s)) &= a'(\xi) + \frac12 Z(s) \int_0^1 a''(\xi + \lambda h^{\frac12} Z(s))d\lambda + h^{\frac12} W_h^\delta(h^{\frac12}\sigma, X(s))d\sigma \\ &:= a'(\xi) + R \end{aligned}$$

where

$$|R| \le \tau_0 C(\tau_0, \|\partial_x W\|_{L^\infty_{t,x}})\|a''\|_{L^\infty} + h^{\frac12}\|W\|_{L^\infty_{t,x}})$$

Since for $1/2 \le |\xi| \le 3$ we have $|a'(\xi)| \ge 2c_1 > 0$ we obtain

$$|A(s, X(s), h, \xi, Z(s))| \ge c_1$$

when $0 \le s \le \tau_0 h^{-\frac12}$, ($\tau_0$ and h small enough). This proves (5.15).

(ii) We have,

$$|\ddot X(s)| \le h^{\frac12} C\Big(\|\partial_x W\|_{L^\infty_{t,x}} + \|\partial_t W\|_{L^\infty_{t,x}}\Big), \quad 0 \le s \le \tau_0 h^{-\frac12}. \tag{5.18}$$

Indeed let us set $m(s) = (s, X(s), h, \xi, Z(s))$.Then we have,

$$\ddot X(s) = (\partial_s A)(m(s)) + (\partial_x A)(m(s))\dot X(s) + (\partial_z A)(m(s))\dot Z(s).$$

Moreover we have,

$$\begin{aligned} &(\partial_s A)(m(s)) = h(\partial_s W_h^\delta)(\sigma h^{\frac12}, x), \quad (\partial_x A)(m(s)) = h^{\frac12}(\partial_x W_h^\delta)(\sigma h^{\frac12}, x) \\ &(\partial_z A)(m(s)) = h^{\frac12} a''(\xi + h^{\frac12} Z(s)). \end{aligned}$$

Then (5.18) follows from the expressions of $\dot X(s), \dot Z(s)$ and (5.16).

(iii) We improve now (5.16). We have,

$$|Z(s)| \leq C\Big(\sum_{|\alpha|\leq 1} \|\partial^\alpha W\|_{L^\infty_{t,x}}\Big), \quad \partial = (\partial_t, \partial_x). \tag{5.19}$$

Indeed we can write

$$Z(s) = -\xi \int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))d\sigma - h^{\frac{1}{2}} \int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))Z(\sigma)d\sigma \tag{5.20}$$

Now, using (5.15) we have,

$$(\partial_x W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma)) = \frac{\partial_\sigma[W_h^\delta(\sigma h^{\frac{1}{2}}, X(\sigma))]}{\dot{X}(\sigma)} - \frac{h^{\frac{1}{2}}(\partial_s W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))}{\dot{X}(\sigma)}. \tag{5.21}$$

After an integration by parts we obtain,

$$\begin{aligned} I =: \int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))d\sigma &= \frac{W_h^\delta(sh^{\frac{1}{2}}, X(s))}{\dot{X}(s)} - \frac{W_h^\delta(0, X(0))}{\dot{X}(0)} \\ &+ \int_0^s \frac{\ddot{X}(\sigma)}{(\dot{X}(\sigma))^2} W_h^\delta(\sigma h^{\frac{1}{2}}, X(\sigma))d\sigma - h^{\frac{1}{2}} \int_0^s \frac{1}{\dot{X}(\sigma)} (\partial_s W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))d\sigma. \end{aligned}$$

Using (5.15),(5.18) we deduce that for $0 \leq s \leq \tau_0 h^{-\frac{1}{2}}$ we have,

$$|I| \leq C\Big(\sum_{|\alpha|\leq 1} \|\partial^\alpha W\|_{L^\infty_{t,x}}\Big) \quad \partial = (\partial_s, \partial_x).$$

It follows from (5.20) that,

$$|Z(s)| \leq C\Big(\sum_{|\alpha|\leq 1} \|\partial^\alpha W\|_{L^\infty_{t,x}}\Big) + \|\partial_x W_h^\delta\|_{L^\infty_{t,x}} h^{\frac{1}{2}} \int_0^s |Z(\sigma)d\sigma,$$

which using Gronwall inequality proves (5.19).

We are going now to give some estimates on the x-derivative of the flow.

We claim that,

$$|\frac{\partial X}{\partial x}(s)| + |\frac{\partial Z}{\partial x}(s)| \leq C\Big(\|\partial_t W\|_{W^{1,\infty}} + \|W\|_{W^{2,\infty}}\Big) \quad 0 \leq s \leq \tau_0 h^{-\frac{1}{2}} \tag{5.22}$$

$$|\frac{\partial X}{\partial x}(s) - 1| \leq \frac{1}{2}, \quad 0 \leq s \leq \tau_0 h^{-\frac{1}{2}}, \tag{5.23}$$

if $\tau_0 C\Big(\|\partial_t W\|_{W^{1,\infty}} + \|W\|_{W^{2,\infty}})\Big)$ is small enough.

Indeed using (5.14) we can write,

$$\frac{\partial \dot{X}}{\partial x}(s) = h^{\frac{1}{2}}(\partial_x W_h^\delta)(sh^{\frac{1}{2}}, X(s))\frac{\partial X}{\partial x}(s) + h^{\frac{1}{2}}a''(\xi + h^{\frac{1}{2}}Z(s))\frac{\partial Z}{\partial x}(s),$$
$$\frac{\partial \dot{Z}}{\partial x}(s) = -\xi(\partial_x^2 W_h^\delta)(sh^{\frac{1}{2}}, X(s))\frac{\partial X}{\partial x}(s) - h^{\frac{1}{2}}(\partial_x^2 W_h^\delta)(sh^{\frac{1}{2}}, X(s))\frac{\partial X}{\partial x}(s)Z(s)$$
$$- h^{\frac{1}{2}}(\partial_x W_h^\delta)(sh^{\frac{1}{2}}, X(s))\frac{\partial Z}{\partial x}.$$

From the first equation we deduce

$$|\frac{\partial X}{\partial x}(s)| \leq 1 + h^{\frac{1}{2}}\|(\partial_x W_h^\delta)\|_{L^\infty_{t,x}}\int_0^s |\frac{\partial X}{\partial x}(\sigma)|d\sigma + h^{\frac{1}{2}}\|a''\|_{L^\infty}\int_0^s |\frac{\partial Z}{\partial x}(\sigma)|d\sigma. \tag{5.24}$$

From the second equation we deduce,

$$\begin{cases} \frac{\partial Z}{\partial x}(s) = I_1 = I_2 + I_3 \quad \text{where,} \\ I_1 = -\xi\int_0^s (\partial_x^2 W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))\frac{\partial X}{\partial x}(\sigma)d\sigma, \\ I_2 = -h^{\frac{1}{2}}\int_0^s (\partial_x^2 W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))\frac{\partial X}{\partial x}Z(\sigma)d\sigma, \\ I_3 = -h^{\frac{1}{2}}\int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))\frac{\partial Z}{\partial x}d\sigma. \end{cases} \tag{5.25}$$

We have easily,

$$|I_3| \leq h^{\frac{1}{2}}\|\partial_x W\|_{L^\infty_{t,x}}\int_0^s |\frac{\partial Z}{\partial x}(\sigma)|d\sigma \tag{5.26}$$

Moreover using (5.19) we get,

$$|I_2| \leq h^{\frac{1}{2}}C\Big(\sum_{|\alpha|\leq 1}\|\partial^\alpha W\|_{L^\infty_{t,x}}\Big)\|\partial_x^2 W\|_{L^\infty_{t,x}}\int_0^s |\frac{\partial X}{\partial x}(\sigma)|d\sigma. \tag{5.27}$$

We are left with the estimate of I_1. We use (5.21) applied to $\partial_x W_h^\delta$.We obtain

$$I_1 = -\xi\int_0^s \frac{\partial_\sigma[\partial_x W_h^\delta(\sigma h^{\frac{1}{2}}, X(\sigma))]}{\dot{X}(\sigma)}\frac{\partial X}{\partial x}(\sigma)d\sigma \tag{5.28}$$

$$- h^{\frac{1}{2}}\xi\int_0^s \frac{(\partial_x\partial_s W_h^\delta)(\sigma h^{\frac{1}{2}}, X(\sigma))}{\dot{X}(\sigma)}\frac{\partial X}{\partial x}(\sigma)d\sigma =: A + B. \tag{5.29}$$

We see easily that

$$|B| \leq Ch^{\frac{1}{2}}\|\partial_s\partial_x W\|_{L^\infty_{t,x}}\int_0^s |\frac{\partial X}{\partial x}(\sigma)|d\sigma. \tag{5.30}$$

Let us consider the term A. After an integration by parts one can write,

$$A = -\xi\Big(\frac{(\partial_x W_h^\delta)(sh^{\frac12}, X(s))}{\dot X(s)}\frac{\partial X}{\partial x}(s) - \frac{(\partial_x W_h^\delta)(0,x)}{\dot X(0)}\Big)$$
$$+\xi\int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac12}, X(\sigma))\frac{\partial \dot X}{\partial x}(\sigma)\frac{d\sigma}{\dot X(\sigma)}$$
$$-\xi\int_0^s (\partial_x W_h^\delta)(\sigma h^{\frac12}, X(\sigma))\frac{\ddot X(\sigma)}{(\dot X(\sigma))^2}\frac{\partial X}{\partial x}(\sigma)d\sigma.$$

Using (5.24), (5.15), (5.18) and the equation satisfied by $\frac{\partial X}{\partial x}$ we obtain

$$|A| \le C\Big(\|\partial_x W\|_{L^\infty_{t,x}}\Big)\Big(1 + h^{\frac12}\int_0^s (|\frac{\partial X}{\partial x}(\sigma)| + |\frac{\partial Z}{\partial x}(\sigma)|)d\sigma\Big). \tag{5.31}$$

Using (5.25) to (5.31) we obtain

$$|\frac{\partial Z}{\partial x}(s)| \le C\big(\|\partial_x W\|_{L^\infty_{t,x}}\big)$$
$$+ C\left(\|\partial_t W\|_{L^\infty_t W^{1,\infty}_x} + \|W\|_{L^\infty_t W^{2,\infty}_x}\right)\int_0^s \left(\left|\frac{\partial X}{\partial x}(\sigma)\right| + \left|\frac{\partial Z}{\partial x}(\sigma)\right|\right)d\sigma$$

so using (5.24) and the Gronwall inequality we obtain (5.22).

Then coming back to the equation satisfied by $\frac{\partial X}{\partial x}$ we deduce

$$\left|\frac{\partial X}{\partial x}(s) - 1\right| \le C\left(\|\partial_t W\|_{L^\infty_t W^{1,\infty}_x} + \|W\|_{L^\infty_t W^{2,\infty}_x}\right)h^{\frac12}|s|$$

for $0 \le s \le \tau_0 h^{-\frac12}$. Therefore taking τ_0 small enough we obtain (5.23).

5.2.3. Resolution of the eikonal equation.— We claim now that for all σ in $[0, \tau_0 h^{-\frac12}]$ the map $x \to X(\sigma, x)$ is a global diffeomorphism from $\mathbf{R}$ to $\mathbf{R}$. Indeed we have for such fixed σ,

$$X(\sigma, x) = x + \int_0^\sigma A(\sigma, X(\sigma'), h, \xi, Z(\sigma'))d\sigma.$$

Since A is bounded by $\|a'\|_{L^\infty} + h^{\frac12}\|W\|_{L^\infty_{t,x}}$ we have $\lim_{|x|\to+\infty}|X(\sigma,x)| = +\infty$. Moreover by (5.23) we have, $\frac{\partial X}{\partial x}(\sigma, x) \neq 0$ for all $0 \le \sigma \le \tau_0 h^{-\frac12}$ and all $x \in \mathbf{R}$. Therefore our claim follows from the following classical result (see [**9**] Theorem 5.1.4).

Proposition 5.6. — *Consider a C^1 map $F : \mathbf{R}^d \to \mathbf{R}^d$. Then F is a global C^1 diffeomorphism if and only if F is proper and at any point its Jacobian does not vanish.*

Then

$$X(\sigma, x) = y \Leftrightarrow x = Y(\sigma, x), \quad x, y \in \mathbf{R}, \tag{5.32}$$

and the function $(\sigma, y) \to Y(\sigma, y)$ is C^∞ by the implicit function theorem. Let us consider then the set

$$S = \{(\sigma, X(\sigma, x), Z(\sigma, x),\ 0 \le \sigma \le \tau_0 h^{-\frac12}, x \in \mathbf{R}\}.$$

It follows from (5.32) that S is a submanifold of $\mathbf{R}^3$ of dimension two to which the vector field $L = \frac{\partial}{\partial\sigma} + A(\sigma, x, h, \xi, z)\frac{\partial}{\partial x} + B(\sigma, x, h, \xi, z)\frac{\partial}{\partial z}$ is tangent. It follows then from (5.32) that the function $\psi_1(\sigma, y, h, \xi) = Z(\sigma, Y(\sigma, y))$ is the solution of our eikonal equation (5.12). Then $\psi_1 \in L^\infty$. Moreover we have

$$\frac{\partial\psi_1}{\partial y} = \frac{\partial Z}{\partial x}(\sigma, Y(\sigma, y))\frac{\partial Y}{\partial y}(\sigma, y) = \frac{\partial Z}{\partial x}(\sigma, Y(\sigma, y))(\frac{\partial X}{\partial x}(\sigma, Y(\sigma, y)))^{-1},$$

so, since $\frac{\partial Z}{\partial x}$ is bounded and using (5.23) we deduce,

$$|\frac{\partial\psi_1}{\partial y}| \leq C\Big(\|\partial_t W\|_{W^{1,\infty}} + \|W\|_{W^{2,\infty}} \Big).$$

It follows that the solution ψ of our eikonal equation (5.11) is such that

$$\frac{\partial\psi}{\partial x} \in L^\infty_{\sigma,x}\,, \quad \frac{\partial^2\psi}{\partial x^2} \in L^\infty_{\sigma,x}\,,$$

uniformly in h, ξ. We deduce from the equation (5.11) that $\partial_\sigma\psi$ is uniformly bounded and therefore that $|\psi| \leq C|\sigma|$. □

5.2.4. Properties of the solution.— We investigate in this section futher regularity of the solution ψ.

Proposition 5.7. — *Let ψ be the solution of* (5.11) *given by Proposition* 5.5 *Then we have* $\frac{1}{\sigma}\psi \in L^\infty(\mathcal{O}), \partial_x\psi \in L^\infty(\mathcal{O}), \partial_x^2\psi \in S^0_\delta(\mathcal{O})$.

Proof. — The first two claims have been proved in Proposition 5.5, let us consider the third one. We shall prove that $\partial_x\psi_1 \in S^0_\delta(\mathcal{O})$ where $\psi_1 = \partial_x\psi$. Let us set for $\frac{1}{2} \leq |\xi| \leq 2$,

$$v(\sigma, x, \xi, h) = \partial_x\psi_1(\sigma, x, \xi, h) - \frac{\xi}{a'(\xi)}\partial_x W^\delta_h(\sigma h^{\frac{1}{2}}, x). \tag{5.33}$$

Then according to (5.12) the function v is solution of the equation,

$$\begin{aligned} \partial_\sigma v + (a'(\xi + h^{\frac{1}{2}}\psi_1) + h^{\frac{1}{2}}W^\delta_h)\partial_x v + h^{\frac{1}{2}}a''(\xi + h^{\frac{1}{2}}\psi_1)v^2 \\ +(2h^{\frac{1}{2}}a''(\xi + h^{\frac{1}{2}}\psi_1)\frac{\xi}{a'(\xi)}\partial_x W^\delta_h + 2h^{\frac{1}{2}}\partial_x W^\delta_h)v = f, \end{aligned} \tag{5.34}$$

where,

$$\begin{aligned} f = &- \frac{\xi}{a'(\xi)}h^{\frac{1}{2}}\partial_x\partial_t W^\delta_h + \frac{\xi}{a'(\xi)}\partial_x^2 W^\delta_h\big(a'(\xi + h^{\frac{1}{2}}\psi_1) - a'(\xi)\big) \\ &- h^{\frac{1}{2}}\frac{\xi}{a'(\xi)}W^\delta_h\partial_x^2 W^\delta_h + h^{\frac{1}{2}}(\frac{\xi}{a'(\xi)})^2 a''(\xi + h^{\frac{1}{2}}\psi_1)(\partial_x W^\delta_h)^2 \\ &+ 2h^{\frac{1}{2}}\frac{\xi}{a'(\xi)}(\partial_x W^\delta_h)^2 - h^{\frac{1}{2}}(\partial_x^2 W^\delta_h)\psi_1. \end{aligned} \tag{5.35}$$

Let us set $\Lambda = h^\delta\partial_x$.We shall prove by induction on $k \in \mathbf{N}$ that,

$$\Lambda^j v \in L^\infty(\mathcal{O}), \quad 0 \leq j \leq k. \tag{5.36}$$

This will imply our claim since $\partial_x W^\delta_h \in S^0_\delta(\mathcal{O})$.

Notice that (5.36) is true for $k=0$ by Proposition 5.5. Assume it is true up to the order $k-1$. It follows then, using the Faa-di Bruno formula that,

$$\Lambda^l[b(h^{\frac{1}{2}}\psi_1)] \in h^{\frac{1}{2}}L^\infty(\mathcal{O}), \quad 1 \le l \le k, \tag{5.37}$$

for any C^∞-bounded function b from $\mathbf{R}$ to $\mathbf{R}$.

Applying the operator Λ^k to both sides of (5.34) and using (5.37) and the fact that $\partial^\alpha W_h^\delta \in S_\delta^0(\mathcal{O})$ for $\partial = (\partial_t, \partial_x), |\alpha| \le 2, \alpha \ne (2,0)$ we find that $\Lambda^k v$ is solution of the problem

$$(\partial_\sigma + a'(\xi + h^{\frac{1}{2}}\psi_1)\partial_x + h^{\frac{1}{2}}W_h^\delta\partial_x + h^{\frac{1}{2}}d(\sigma,x,\xi,h))\Lambda^k v \in h^{\frac{1}{2}}L^\infty(\mathcal{O}), \tag{5.38}$$

where $d \in L^\infty(\mathcal{O}_\varepsilon)$.

Let us set $\tilde{v}_k = (\Lambda^k v)(\sigma, x + a'(\xi)\sigma)$. Then $\tilde{v}_k$ is solution of the problem,

$$\Big(\partial_\sigma + h^{\frac{1}{2}}\int_0^1 a''(\xi + \lambda h^{\frac{1}{2}}\tilde{\psi}_1)\,d\lambda\,\tilde{\psi}_1\partial_x + h^{\frac{1}{2}}\tilde{W}_h^\delta\partial_x + h^{\frac{1}{2}}\tilde{d}(\sigma,x,\xi,h)\Big)\tilde{v}_k \in h^{\frac{1}{2}}L^\infty(\mathcal{O}),$$

Then the desired conclusion follows from the following Lemma.

Lemma 5.8. — *Let c_1, c_2 be two real valued functions such that $c_1, \partial_x c_1, c_2$ belong to $L^\infty(\mathcal{O})$ and $P = \partial_\sigma + h^{\frac{1}{2}}c_1(\sigma,x,\xi,h)\partial_x + h^{\frac{1}{2}}c_2(\sigma,x,\xi,h)$. Then for any $F \in L^\infty(\mathcal{O})$, the problem*

$$Pu = F, \quad u|_{\sigma=0} = 0,$$

has a unique solution u which satisfies the estimate

$$|u(\sigma,x,\xi,h)| \le C\int_0^\sigma \|F(s,\cdot,\xi,h)\|_{L^\infty(\mathbf{R})}\,ds,$$

for all (σ,x,ξ,h) in $\mathcal{O}$, uniformly in h.

Proof of Lemma 5.8. — Let us set $t = \sigma h^{\frac{1}{2}}$ and $\widetilde{c}_j(t,x) = c_j(h^{-\frac{1}{2}}t, x)$, $j = 1,2$. Then we are led to the problem

$$\widetilde{P}\widetilde{u} = h^{\frac{1}{2}}\widetilde{F}, \quad \widetilde{u}|_{\sigma=0} = 0 \qquad (t \in [0,\tau_0], x \in \mathbf{R}),$$

where $\widetilde{P} = \partial_t + \widetilde{c}_1(t,x)\partial_x + \widetilde{c}_2(t,x)$. Recall that $\widetilde{c}_1 \in L^\infty$, $\partial_x\widetilde{c}_1 \in L^\infty$, $\widetilde{c}_2 \in L^\infty$. Then the claim of the lemma follows from the classical method of characteristics. Indeed, the characteristics are given by $t(s) = s$ and

$$\dot{X}(s,x) = \widetilde{c}_1(s, X(s,x)), \quad X(0,x) = x.$$

Then $x \mapsto X(t,x)$ is globally invertible for each $t \in [0,\tau_0]$ i.e. $X(t,x) = y \Leftrightarrow x = Y(t,y)$ with $Y \in C^0 \cap L^\infty$. Then

$$\frac{d}{dt}[\widetilde{u}(t,X(t,x))] = \widetilde{c}_2(t,X(t,x))u(t,X(t,x)) + F(t,X(t,x)).$$

Therefore $\widetilde{u}$ given by

$$\widetilde{u}(t,y) = \exp\left(-\int_0^t \widetilde{c}_2(t', X(t', Y(t,y))\,dt'\right)\int_0^t F(t', X(t', Y(t,y))\,dt'$$

is our solution. □

Corollary 5.9. — *Let ψ be defined by Proposition 5.5 and L be the operator*

$$L = \partial_\sigma + a'(\xi + h^{\frac{1}{2}}\partial_x\psi)\partial_x + h^{\frac{1}{2}}d_1\partial_x + h^{\frac{1}{2}}d_2, \tag{5.39}$$

where $d_1, \partial_x d_1, d_2$ are real valued and belong to $S^0_\delta(\mathcal{O})$.
Then for any F such that $\|\Lambda^j F\|_{L^\infty(\mathcal{O})}$ is finite for every $j \in \mathbf{N}$, the problem

$$Lu = F, \quad u|_{\sigma=0} = 0,$$

has a unique solution which satisfies the estimate,

$$|\Lambda^k u(\sigma, x, \xi, h)| \leq C_k \sigma \sum_{j=0}^{k} \|\Lambda^j F\|_{L^\infty(\mathcal{O})}, \tag{5.40}$$

for all $(\sigma, x, \xi, h) \in \mathcal{O}$, where $\Lambda = h^\delta \partial_x$.

Proof. — Since by Proposition 5.7 we have $\partial_x^2\psi \in S^0_\delta(\mathcal{O})$ one can write,

$$L = \partial_\sigma + a'(\xi)\partial_x + h^{\frac{1}{2}}d_3\partial_x + h^{\frac{1}{2}}d_2.$$

where $d_3, \partial_x d_3, d_2$ belong to $S^0_\delta(\mathcal{O})$. Setting

$$U = u(\sigma, x + \sigma a'(\xi), \xi, h), \quad c_1 = d_3(\sigma, x + \sigma a'(\xi), \xi, h),$$
$$c_2 = d_2(\sigma, x + \sigma a'(\xi), \xi, h),$$

we see that $c_1, \partial_x c_1, c_2$ belong to $S^0_\delta(\mathcal{O})$ and U is a solution of the equation

$$L_1 U := (\partial_\sigma + h^{\frac{1}{2}}c_1\partial_x + h^{\frac{1}{2}}c_2)U = F_1. \tag{5.41}$$

We shall prove by induction on k that U satisfies the estimate (5.40). The case $k = 0$ follows immediately from Lemma 5.8. Assume now that (5.40) is true up to the order $k-1, k \geq 1$. Applying Λ^k to (5.41) and using the Leibniz formula we have

$$\begin{aligned} L_1\Lambda^k U + kh^{\frac{1}{2}}(\partial_x c_1)\Lambda^k U = -h^{\frac{1}{2}} &\sum_{i=0}^{(k-2)^+} \binom{k}{i} (\Lambda^{k-i}c_1)\Lambda^i\partial_x U \\ &- h^{\frac{1}{2}} \sum_{i=0}^{k-1} \binom{k}{i} (\Lambda^{k-i}c_2)\Lambda^i U + \Lambda^k F_1 := G_k \end{aligned}$$

where we used that $\binom{k}{k-1}\Lambda c_1 \Lambda^{k-1}\partial_x = k(\partial_x c_1)\Lambda^k$.

The sum in the first line can be written, $-h^{\frac{1}{2}}\sum_{i=1}^{k-1}\binom{k}{i-1}(\Lambda^{k-i}\partial_x c_1)\Lambda^i U$. According to our assumptions on c_1, c_2,we can apply the Lemma 5.8 to the operator $L + kh^{\frac{1}{2}}(\partial_x c_1)$.We obtain, using the induction and the fact that $h^{\frac{1}{2}}\sigma \leq \tau_0$,

$$|\Lambda^k U(\sigma, x, \xi, h)| \leq \sigma\|\Lambda^k F\|_{L^\infty(\mathcal{O})} + C\sigma \sum_{j=0}^{k-1} \|\Lambda^j F\|_{L^\infty(\mathcal{O})},$$

which completes the induction. □

Proposition 5.10. — *Let* $\Lambda = h^\delta \partial_x$. *The solution of* (5.11) *given by Proposition* 5.5 *satisfies the estimates,*

$$\begin{aligned} &|\Lambda^k \partial_\xi \psi(\sigma, x, \xi, h)| + |\Lambda^k \partial_x \partial_\xi \psi(\sigma, x, \xi, h)| \leq C_k \sigma, \\ &|\Lambda^k \partial_\xi^2 \psi(\sigma, x, \xi, h)| \leq C_k \tau_0 h^{-\frac{1}{2}} \sigma, \end{aligned} \tag{5.42}$$

for all $(\sigma, x, \xi, h) \in \mathcal{O}$, *where* C *depends only on the norm in* $L^\infty(I, H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ *of the solution to* (1.4).

Proof. — Differentiating (5.11) with respect to ξ we find that $U = \partial_\xi \psi$ satisfies the equation

$$\partial_\sigma U + (a'(\xi + h^{\frac{1}{2}} \partial_x \psi) + h^{\frac{1}{2}} W_h^\delta) \partial_x U = -(\partial_x \psi) \int_0^1 a''(\xi + h^{\frac{1}{2}} \lambda \partial_x \psi) d\lambda - W_h^\delta. \tag{5.43}$$

Then the estimate of the first term in the first line of (5.42) follows from Corollary 5.9. To estimate the second term we differentiate with respect to ξ the equation (5.12). We find that the fuction $U_1 = \partial_\xi \psi_1 = \partial_x \partial_\xi \psi$ satisfies the equation

$$\partial_\sigma U_1 + (a'(\xi + h^{\frac{1}{2}} \psi_1) + h^{\frac{1}{2}} W_h^\delta) \partial_x U_1 + h^{\frac{1}{2}} a''(\xi + h^{\frac{1}{2}} \psi_1) \partial_x \psi_1 U_1 = -\partial_x W_h^\delta.$$

The second estimate in the first line of (5.42) follows from Corollary 5.9. Finally to estimate $U_2 = \partial_\xi^2 \psi$ we differentiate (5.43) with respect to ξ and we find that U_2 satisfies the equation

$$\partial_\sigma U_2 + (a'(\xi + h^{\frac{1}{2}} \partial_x \psi) + h^{\frac{1}{2}} W_h^\delta) \partial_x U_2 = F$$

where

$$F = -h^{\frac{1}{2}} (\partial_x \partial_\xi \psi)^2 a''(\xi + h^{\frac{1}{2}} \partial_x \psi) + (\partial_x \partial_\xi \psi) a''(\xi + h^{\frac{1}{2}} \partial_x \psi) + \frac{a''(\xi + h^{\frac{1}{2}} \partial_x \psi) - a''(\xi)}{h^{\frac{1}{2}}}.$$

So using the estimate on $\partial_x \psi$ and $\partial_x \partial_\xi \psi$ obtained before, we deduce from Corollary 5.9 the last estimate of (5.42). □

5.2.5. The symbol equations. — According to the formulas (4.18)–(4.23), since the phase ψ now satisfies the eikonal equation (5.11), we are lead to solve the following transport equation

$$\begin{aligned} \partial_\sigma b + a'(\xi + h^{\frac{1}{2}} \partial_x \psi) \partial_x b + h^{\frac{1}{2}} W_h^\delta \partial_x b - h^{\frac{1}{2}} (\partial_x^2 \psi) a''(\xi + h^{\frac{1}{2}} \partial_x \psi) b \\ + \frac{1}{2} h^{\frac{1}{2}} (\partial_x W_h^\delta) b = -\frac{i}{h} \sum_{k=2}^N A_k, \end{aligned} \tag{5.44}$$

with

$$b|_{\sigma=0} = \chi_1(\xi), \tag{5.45}$$

where $\chi_1 \in C_0^\infty(\mathbf{R})$ is equal to one on the support of the function χ given in Theorem 5.2. Let

$$\mu_0 = \frac{1}{2} - 2\delta,$$

where we recall that $\delta < 1/4$.

Let us set

$$\Lambda = h^\delta \partial_x. \tag{5.46}$$

Then according to (4.23) and the Leibniz formula one can write

$$\frac{1}{h} A_k = h^{k(1-\delta)-1} \sum_{k_1=0}^{k} c_{k,k_1} (h^\delta \partial_y)^{k-k_1} \left[(\partial_\xi^k a)(\rho((x,y)) \right] \mid_{y=x} \Lambda^{k_1} b, \tag{5.47}$$

where $c_{k,k_1} \in \mathbf{C}$.

We shall take b of the form

$$b = \chi_1(\xi) e^{\widetilde{\theta}}, \quad \widetilde{\theta} = \sum_{j=0}^{M} \widetilde{\theta}_j, \quad \widetilde{\theta}_j = h^{j\mu_0} \theta_j. \tag{5.48}$$

Then $\Lambda^{k_1} b$ is a finite linear combination of terms of the form

$$\left(\Lambda^{\alpha_0} e^{\widetilde{\theta}_0} \right) \cdots \left(\Lambda^{\alpha_M} e^{\widetilde{\theta}_M} \right), \quad \alpha_0 + \cdots + \alpha_M = k_1.$$

Let

$$\omega = \left\{ \alpha \in \mathbf{N}^{M+1} \, : \, |\alpha| = k_1 \right\},$$

and, for $0 \le p \le M$,

$$\omega_p = \{ \alpha \in \omega, \quad \alpha = (\alpha_0, \cdots, \alpha_p, 0, \cdots, 0) \text{ with } \alpha_p \neq 0 \}.$$

Then ω is the disjoint union of the ω_p. It follows that

$$\Lambda^{k_1} b = \sum_{p=0}^{M} \sum_{\alpha \in \omega_p} d_{p,\alpha} \left(\Lambda^{\alpha_0} e^{\widetilde{\theta}_0} \right) \cdots \left(\Lambda^{\alpha_p} e^{\widetilde{\theta}_p} \right) \exp \left(\sum_{j=p+1}^{M} \widetilde{\theta}_j \right), \quad d_{\alpha,p} \in \mathbf{C}. \tag{5.49}$$

Now by the Faa-di Bruno formula we have for $0 \le \ell \le M$,

$$\Lambda^{\alpha_\ell} e^{\widetilde{\theta}_\ell} = e^{\widetilde{\theta}_\ell} \sum_{s=1}^{\alpha_\ell} E_{s\ell} \tag{5.50}$$

where $E_{s,\ell}$ is a finite linear combintation of terms of the form

$$\prod_{i=1}^{s} \left(\Lambda^{p_i} \widetilde{\theta}_\ell \right)^{q_i} \quad \text{where } 1 \le \sum_{i=1}^{s} q_i \le \alpha_\ell, \quad \sum_{i=1}^{s} p_i q_i = \alpha_\ell.$$

Since $\sum_{i=1}^{s} q_i \ge 1$ and $p(p+1)/2 \ge p$, it follows from (5.49) and (5.48) that

$$\Lambda^{k_1} b = e^{\widetilde{\theta}} \sum_{p=0}^{M} h^{p\mu_0} \sum_{|\beta| \le k_1} G_{p\beta} \left(h, \Lambda^{\beta_0} \theta_0, \ldots, \Lambda^{\beta_p} \theta_p \right), \tag{5.51}$$

where $G_{p,\beta}(h, \zeta_0, \ldots, \zeta_p)$ are bounded in h and polynomial in ζ. Coming back to (5.47) we remark first that since $k \ge 2$ and $\mu_0 = \frac{1}{2} - 2\delta > 0$ we have

$$k(1-\delta) - 1 \ge 2(1-\delta) - 1 = \frac{1}{2} + \mu_0. \tag{5.52}$$

Let us note that this is the only point where we use the fact that $\delta < \frac{1}{4}$.

On the other hand we have,

$$(h^\delta \partial_y)^{k-k_1} \left[(\partial_\xi^k a)(\rho((x,y)) \right] \big|_{y=x} \in S^0_\delta. \tag{5.53}$$

It follows then from (5.47), (5.51), (5.52), (5.53) that for $k \geq 2$

$$\frac{1}{h} A_k = h^{\frac{1}{2}} \sum_{p=0}^{M} \sum_{k_1=0}^{k} \sum_{|\beta| \leq k_1} h^{(p+1)\mu_0} f_{k,k_1} H_{p,\beta} \left(h, \Lambda^{\beta_0}\theta_0, \ldots, \Lambda^{\beta_p}\theta_p \right) e^{\widetilde{\theta}}, \tag{5.54}$$

where $f_{k,k_1} \in S^0_\delta$, $H_{p,\beta}(h, \zeta_0, \ldots, \zeta_p)$ are bounded in h and polynomial in ζ.
Let us denote by L the linear operator appearing in (5.44),

$$L = \partial_\sigma + a'(\xi + h^{\frac{1}{2}} \partial_x \psi)\partial_x + h^{\frac{1}{2}} W_h^\delta \partial_x. \tag{5.55}$$

Since $b = e^{\widetilde{\theta}}$ with $\widetilde{\theta} = \sum_{p=0}^{M} h^{p\mu_0}\theta_p$ we have $Lb = e^{\widetilde{\theta}} L\widetilde{\theta}$. It follows from (5.54) that the transport equation (5.44) can be written, modulo a remainder,

$$\begin{aligned} L\theta_0 - h^{\frac{1}{2}} \big\{ (\partial_x^2 \psi) - \frac{1}{2}(\partial_x W_h^\delta) \big\} \\ + \sum_{p=0}^{M-1} h^{(p+1)\mu_0} \left(L\theta_{p+1} - h^{\frac{1}{2}} G_p(\theta_0, \ldots, \theta_p) \right) = 0, \end{aligned}$$

where $G_j(\theta_0, \ldots, \theta_p)$ are polynomials in $\Lambda^\beta \theta_\ell$ for $|\beta| \leq N$. Therefore we shall take for $\theta_p, 0 \leq p \leq M-1$, the solutions of the problems

$$\begin{cases} L\theta_0 = h^{\frac{1}{2}} \big\{ (\partial_x^2 \psi) a''(\xi + h^{\frac{1}{2}} \partial_x \psi) - \frac{1}{2}(\partial_x W_h^\delta) \big\}, \quad \theta_0|_{\sigma=0} = 0, \\ L\theta_{p+1} = h^{\frac{1}{2}} G_p(\theta_0, \ldots, \theta_p), \quad \theta_{p+1}|_{\sigma=0} = 0 \quad (0 \leq p \leq M-1). \end{cases} \tag{5.56}$$

We have the following result.

Proposition 5.11. — *Let $\Lambda = h^{\frac{1}{2}} \partial_x$. Then* (5.56) *has a unique solution $(\theta_1, ..., \theta_M)$ such that for $0 \leq p \leq M$ and all integers $k \in \mathbf{N}$,*

$$\begin{aligned} &|\Lambda^k \theta_p(\sigma, x, \xi, h)| \leq C_k, \quad |\Lambda^k \partial_\xi \theta_p(\sigma, x, \xi, h)| \leq C_k h^{-\delta} \sigma, \\ &|\Lambda^k \partial_\xi^2 \theta_p(\sigma, x, \xi, h)| \leq C_k h^{-\frac{1}{2}-2\delta} \sigma. \end{aligned} \tag{5.57}$$

Proof. — We proceed by induction on p. If $p = 0$, the estimate of the first term in the first line of (5.57) follows immediately from Proposition 5.7 and Corollary 5.9. Now $\partial_\xi \theta_0$ is solution of the equation

$$\begin{aligned} L\partial_\xi \theta_0 = h^{\frac{1}{2}} \big\{ (\partial_\xi \partial_x^2 \psi) a''(\xi + h^{\frac{1}{2}} \partial_x \psi) + (\partial_x^2 \psi)(1 + h^{\frac{1}{2}} \partial_x \partial_\xi \psi) a'''(\xi + h^{\frac{1}{2}} \partial_x \psi) \big\} \\ - (1 + h^{\frac{1}{2}} \partial_\xi \partial_x \psi) a''(\xi + h^{\frac{1}{2}} \partial_x \psi) \partial_x \theta_0 := F_0. \end{aligned} \tag{5.58}$$

It follows from (5.42) and the first estimate that $\|\Lambda^j F_0\|_{L^\infty(\mathcal{O})} \leq C_j h^{-\delta}$. Using Corollary 5.9 we obtain the estimate of the second term in the first line of (5.57). To estimate $\partial_\xi^2 \theta_0$ we differentiate one more time the equation (5.58) and we find using the same arguments that $L\partial_\xi^2 \theta_0 = F_1$ where F_1 satisfies, $\|\Lambda^j F_1\|_{L^\infty(\mathcal{O})} \leq C_j h^{-\frac{1}{2}-2\delta}$. The estimate of the term

in the second line of (5.57) follows the from Corollary 5.9. Assuming that (5.57) is true up to the order p the estimate θ_{p+1} follows from the second equation in (5.56) and the induction. $\square$

It follows then from (5.9),(5.11), (5.44) and (5.56) that

$$r = \sum_{j=1}^{5} r_j, \tag{5.59}$$

where r_1, r_2, r_3, r_4 are defined in (4.21),(4.22),(4.23),(5.10) and

$$r_5 = h^{\frac{1}{2}} h^{(M+1)\mu_0} G_M(\theta_0, .., \theta_M)\chi_1(\xi)e^{\widetilde{\theta}}\zeta.$$

$\square$

5.2.6. End of the proof of Proposition 5.4. — Since we have

$$\widetilde{b} = \chi_1(\xi)c(\sigma, x, \xi, h)(\zeta(x - z - \sigma a'(\xi))$$

with $c = e^{\widetilde{\theta}} = e^{\sum_{p=0}^{M} h^{p\mu_0}\theta_p} \in S^0_\delta(\Omega)$ the same arguments as those used in section 4.1.1 give the proof of (5.8).

5.3. The stationary phase lemma. — In the sequel we will use the following elementary version of the stationary phase inequality where we allow complex valued phase functions.

Lemma 5.12. — *For any real numbers $\alpha, \beta, \alpha < \beta$, one can find $C_{\alpha\beta} > 0$ such that for all $0 < h \leq 1, \rho > 0$ and for any functions $\phi \in C^2([\alpha, \beta], \mathbf{C})$, $p \in C^1([\alpha, \beta], \mathbf{C})$ such that*

$$\forall \xi \in [\alpha, \beta], \quad |\mathrm{Im}\, \phi(\xi)| \leq h, \quad \left|\mathrm{Im}\, \phi''(\xi)\right| \leq \rho, \quad \frac{\rho}{2} \leq \mathrm{Re}\left(\phi''(\xi)\right) \leq \rho,$$

we have

$$\left|\int_\alpha^\beta e^{\frac{i}{h}\phi(\xi)} p(\xi)\, d\xi\right| \leq C_{\alpha\beta}\left(\|p\|_{L^\infty} + \int_\alpha^\beta \left|p'(\xi)\right|\, d\xi\right)\left(\frac{h}{\rho}\right)^{\frac{1}{2}}.$$

Proof. — Notice that we can assume that $\rho \geq h$ (otherwise, the conclusion is straightforward). Notice now that, using the monotonicity assumption of the real part of the phase ϕ, we can decompose the interval (α, β) into the disjoint union of at most three intervals

$$(\alpha, \beta) = I_1 \cup I_2 \cup I_3,$$

where I_1, I_2 or I_3 are possibly empty and satisfy

$$\begin{aligned}
&\forall \xi \in I_1, \quad \mathrm{Re}(\phi'(\xi)) \leq -(\rho h)^{1/2}, \\
&\forall \xi \in I_2, \quad -(\rho h)^{1/2} \leq \mathrm{Re}(\phi'(\xi)) \leq (\rho h)^{1/2}, \\
&\forall \xi \in I_3, \quad \mathrm{Re}(\phi'(\xi)) \geq (\rho h)^{1/2}.
\end{aligned}$$

Let us first study the contribution of I_3. Either $I_3 = \emptyset$ or I_3 is an interval contained in $[\delta, \beta]$ for some $\delta \in [\alpha, \beta[$. Then, using that

$$\frac{h}{i\phi'(\xi)}\partial_\xi\big(e^{\frac{i}{h}\phi(\xi)}\big) = e^{\frac{i}{h}\phi(\xi)},$$

and integrating by parts, we obtain

$$\begin{aligned}(5.60)\quad \int_\delta^\beta e^{\frac{i}{h}\phi(\xi)}p(\xi)d\xi &= \Big[\frac{h}{i\phi'(\xi)}e^{\frac{i}{h}\phi(\xi)}p(\xi)\Big]_\delta^\beta - \int_\delta^\beta e^{\frac{i\phi(\xi)}{h}}\partial_\xi\Big(\frac{h}{i\phi'(\xi)}p(\xi)\Big)\,d\xi \\ &= \Big[\frac{h}{i\phi'(\xi)}e^{\frac{i}{h}\phi(\xi)}p(\xi)\Big]_\delta^\beta - \int_\delta^\beta e^{\frac{i\phi(\xi)}{h}}\frac{h}{i\phi'(\xi)}\partial_\xi p(\xi)d\xi + \int_\delta^\beta e^{\frac{i\phi(\xi)}{h}}\frac{h\phi''(\xi)}{\phi'(\xi)^2}p(\xi)d\xi\end{aligned}$$

Clearly, the contributions of the two first terms are easily handled by means of the lower bound on $\mathrm{Re}(\phi'(\xi))$ on I_3, and to conclude, it suffices to bound the last term. But according to the assumption on the phase, we now have

$$\mathrm{Re}(\phi'(\delta)) \geq (\rho h)^{1/2} \Rightarrow \mathrm{Re}(\phi'(\xi)) \geq (\rho h)^{1/2} + \frac{\rho}{2}(\xi - \delta),$$

and consequently, the last term is bounded by

$$\|p\|_{L^\infty}\int_\delta^\beta \frac{4h\rho}{(2(h\rho)^{1/2} + \rho(\xi-\delta))^2}\,d\xi \leq \frac{2h\|p\|_{L^\infty}}{(h\rho)^{1/2}},$$

where the last inequality is obtained by a straightforward computation.

Now, of course, the contribution of I_1 is dealt with similarly, and consequently we can focus on the contribution of I_2. Now remark that according to the assumptions on ϕ'', the length of I_2 is smaller that $4(h/\rho)^{1/2}$, which implies

$$\left|\int_{I_2} e^{\frac{i}{h}\phi(\xi)}p(\xi)dt\right| \leq 4\,\|p\|_{L^\infty}\left(\frac{h}{\rho}\right)^{1/2}.$$

This completes the proof. □

5.4. End of the proof of Theorem 5.2. — As in the preceding section we have

$$(5.61)\qquad \begin{cases} S(t,0,h)u_{0,h} = D_1 + D_2 + D_3 \quad where \quad D_1 = \widetilde{U}_h(t,x), \\ D_2 = -S(t,0,h)v_{0,h}(x),\ D_3 = -\displaystyle\int_0^t S(t,s,h)[F_h(s,x)]ds. \end{cases}$$

The terms D_2 and D_3 are estimated exactly as in section 4.3 while D_1 will be estimated differently using Lemma 5.12 instead Van der Corput estimates. Indeed recall that according to (5.4) and (5.48) our amplitude in the parametrix (5.3) is given by

$$\widetilde{b}(\sigma,x,z,\xi,h) = \chi_1(\xi)e^{\widetilde{\theta}(\sigma,x,z,\xi,h)}\zeta(x - z - \sigma a'(\xi)).$$

The new fact here is that we shall glue the term $e^{\widetilde{\theta}}$ with the phase and apply the Lemma 5.12 with the new phase $\varphi - z\xi + \frac{h}{i}\widetilde{\theta}$. Using (5.3) we can write in the variable

$\sigma = th^{-\frac{1}{2}}$,

$$D_1 = \int K(\sigma, x, z, \xi, h) u_{0h}(z)\, dz,$$
$$K = \frac{1}{2\pi h} \int e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h) - z\xi + \frac{h}{i}\widetilde{\theta}(\sigma,x,\xi,h))} \chi_1(\xi)\zeta(x - z - \sigma a'(\xi))\, d\xi.$$

Therefore we shall apply the Lemma 5.12 with,

$$\begin{cases} \phi = \varphi(\sigma, x, \xi, h) - z\xi + \dfrac{h}{i}\widetilde{\theta}(\sigma, x, \xi, h), \quad \varphi = x\xi + \sigma a(\xi) + h^{\frac{1}{2}}\psi(\sigma, x, \xi, h), \\ p = \chi_1(\xi)\zeta(x - z - \sigma a'(\xi)), \\ \rho = C\sigma, \quad C > 0. \end{cases}$$

Let us show that all the hypotheses in this lemma are satisfied. For this we shall use (5.42) and (5.57). First of all, since $\delta \in]0, \frac{1}{4}[$ we have,

$$|\operatorname{Im}\phi| = h|\operatorname{Im}\widetilde{\theta}| \leq Ch, \qquad |\partial_\xi^2 \operatorname{Im}\phi| \leq h|\partial_\xi^2\widetilde{\theta}| \leq Chh^{-\frac{1}{2}-2\delta}\sigma \leq \sigma.$$

Moreover,

$$|\partial_\xi^2 \operatorname{Re}\phi| \leq |a''(\xi)|\sigma + h^{\frac{1}{2}}|\partial_\xi^2\psi| + h|\partial_\xi^2\widetilde{\theta}| \leq C\sigma.$$

Finally,

$$|\operatorname{Re}\phi| \geq |a''(\xi)|\sigma - h^{\frac{1}{2}}|\partial_\xi^2\psi| - h|\partial_\xi^2\widetilde{\theta}| \geq |a''(\xi)|\sigma - C_1\tau_0\sigma - C_2h^{\frac{1}{2}-2\delta}\sigma \geq C_3\sigma,$$

if τ_0 and h are small enough.

It follows the from Lemma (5.12) that,

$$|K| \leq Ch^{-1}\Big(\frac{h}{\sigma}\Big)^{\frac{1}{2}}\Big\{1 + \int \sigma|a''(\xi)||\zeta'(x - z - \sigma a'(\xi)|d\xi\Big\}.$$

Since the last integral is bounded by $C'\int|\zeta'(t)|dt$, we deduce that the term D_1 satisfies the estimate (5.2) which completes the proof of Theorem 5.2.

Proof of Theorem 5.1. — According to Lemma 3.7, Theorem 5.2, the Duhamel formula and the same TT^* argument as in Section 4, we deduce

$$\begin{aligned} \|u_h\|_{L^4((0,\tau),\, L^\infty(\mathbf{R}))} &\leq Ch^{(s-\frac{1}{8})}\Big(\|f_h\|_{L^1((0,\tau);H^s(\mathbf{R}))} + \|u_h\,|_{t=0}\,\|_{H^s(\mathbf{R})})\Big) \\ &\leq Ch^{(s-\frac{1}{8})}\Big(\|f_h\|_{L^2((0,\tau);H^s(\mathbf{R}))} + \|u_h\,|_{t=0}\,\|_{H^s(\mathbf{R})})\Big) \\ &= Ch^{(s-\frac{1}{8})}c_j, \end{aligned}$$

with $(c_j) \in \ell^2$, which (as in the previous section) implies Theorem 5.1. □

6. Back to estimates for (η, ψ)

Notice that up to now, we only proved estimates for the dyadically localized functions $\Delta_j u$. In this section, we shall show how we can recover estimates for (η, ψ), the solutions of the water-wave system (1.4). Recall that the Besov space $B^r_{\infty,2}$ is defined by

$$u \in B^r_{\infty,2}(\mathbf{R}) \Leftrightarrow \sum_{j\in\mathbf{N}} 2^{2jr} \|\Delta_j u\|^2_{L^\infty} < +\infty.$$

We will use the following elementary lemma

Lemma 6.1. — *If the symbol $a \in \Gamma^m_0$, then the operator T_a is bounded from $B^s_{\infty,2}(\mathbf{R})$ to $B^{s-m}_{\infty,2}(\mathbf{R})$.*

We have the slightly stronger result (compared to Theorems 1.1, 1.2)

Proposition 6.2. — *Let $I = [0, T]$. Under the assumptions of Theorem 1.1, there exists $\varepsilon > 0$ such that*

$$(\eta, \psi) \in L^4(I, B^{s-\frac{1}{4}+\frac{1}{2}+\varepsilon}_{\infty,2}(\mathbf{R}) \times B^{s-\frac{1}{4}+\varepsilon}_{\infty,2}(\mathbf{R})). \tag{6.1}$$

Under the assumptions of Theorem 1.2, we have

$$(\eta, \psi) \in L^4(I, B^{s-\frac{1}{8}+\frac{1}{2}}_{\infty,2}(\mathbf{R}) \times B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R})). \tag{6.2}$$

Proof. — For conciseness, we will only prove (6.2), the proof of (6.1) being similar (easier). Recall that the function u is obtained from (η, ψ) through the following steps:

1. $u = T_{e^{ig}}\Phi^*$, where the function g is real and satisfies $\partial_x g \in \Gamma^0_{s-\frac{3}{2}}$ (which implies $e^{ig} \in \Gamma^0_{s-\frac{1}{2}}$) .
2. $\Phi^* = \kappa^*\Phi$ where (see (3.12)) $\kappa \in L^\infty(I, W^{2,\infty}(\mathbf{R}))$.
3. $\Phi = T_p\eta + iT_{c_1}U$, with $p \in \Sigma^{1/2}_{s-1}$ is an elliptic symbol and $c_1 = (1 + (\partial_x\eta)^2)^{-\frac{1}{2}}$.
4. $U = \psi - T_{\mathfrak{B}}\eta$, where $\mathfrak{B} \in L^\infty(I, H^{s-1}(\mathbf{R}))$ is defined in (2.7).

Step 1: Starting from Theorem 5.1, we have

$$\|u\|_{L^4(I,B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty$$

According to the symbolic calculus, since $e^{ig} \in \Gamma^0_0$, we have

$$u = T_{e^{ig}}\Phi^* \Rightarrow \Phi^* = T_{e^{-ig}}u + R_{-1}(\Phi^*)$$

where R_{-1} is of order -1 (i.e. bounded from $H^s(\mathbf{R})$ to $H^{s+1}(\mathbf{R})$) . Since $H^{s+\frac{1}{2}}(\mathbf{R}) \subset B^s_{\infty,2}(\mathbf{R})$ it follows from Lemma 6.1, the boundedness of Φ^* in $L^\infty(I, H^s(\mathbf{R}))$ that

$$\|\Phi^*\|_{L^4(I,B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty.$$

Step 2: We have

$$\Phi^* = \kappa^*\Phi \Leftrightarrow \Phi \circ \kappa = \Phi^* + T_{\partial_x\Phi\circ\kappa}\kappa$$

Notice that $\partial_x \Phi \circ \kappa \in L^\infty(I \times \mathbf{R})$ and $\partial_x \kappa \in L^\infty(I, H^{s-\frac{1}{2}})$. As a consequence, $T_{\partial_x \Phi \circ \kappa} \kappa \in L^\infty(I, H^{s+\frac{1}{2}}) \subset L^\infty(I, B^s_{\infty,2})$. We deduce from Step 1

$$\|\Phi \circ \kappa\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty.$$

We conclude that

$$\|\Phi\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty,$$

by using the following lemma (with $\chi = \kappa^{-1}$, $r = s - \frac{1}{8}$ and $r < \sigma < s$).

Lemma 6.3. — *Let $\sigma > 1$. Consider χ such that $\partial_x \chi \in W^{\sigma-1,\infty}(\mathbf{R})$. Then, for any $0 < r < \sigma$, the map $u \mapsto u \circ \chi$ is continuous on $B^r_{\infty,2}$.*

Indeed, a simple calculation shows that for any $\rho \leq \sigma$, the map $u \mapsto u \circ \chi$ is continuous on $W^{\rho,\infty}$, and we conclude by choosing $r_1 < r < r_2 \leq \sigma$ with $r_i \notin \mathbb{N}$ (notice that this implies $W^{r_i,\infty} = B^{r_i}_{\infty,\infty}$) and using the real interpolation result (see [**10**, Theorem 6.4.5 (1)])

$$[B^{r_1}_{\infty,\infty}, B^{r_2}_{\infty,\infty}]_{\theta,2} = B^r_{\infty,2}, \quad r = (1-\theta)r_1 + \theta r_2.$$

Step 3: Separating real and imaginary parts, we obtain

$$\|T_p \eta\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} + \|T_{c_1} U\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty$$

and the same proof as in Step 1 (using that p is elliptic, $s - \frac{3}{2} \geq 1 \Rightarrow p^{-1} \in \Gamma_1^{-\frac{1}{2}}$ and for fixed t, $c_1^{-1}(t,.) \in W^{1,\infty}(\mathbf{R}) \subset \Gamma_1^0$) gives

$$\|\eta\|_{L^4(I, B^{s-\frac{1}{8}+\frac{1}{2}}_{\infty,2}(\mathbf{R}))} + \|U\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R}))} < +\infty.$$

Step 4: We have $\psi = U + T_{\mathfrak{B}} \eta$. So using the boundedness of η in $L^\infty(I, H^{s+\frac{1}{2}}(\mathbf{R}))$, of $\mathfrak{B}$ in $L^\infty(I, H^{s-1}(\mathbf{R})) \subset L^\infty(I \times \mathbf{R})$ and Sobolev injections, we obtain,

$$\|\psi\|_{L^4(I, B^{s-\frac{1}{8}}_{\infty,2})} < +\infty,$$

which completes the proof of Proposition 6.2 and consequently of Theorems 1.1 and 1.2. □

7. Appendix

In this section we give a proof of (4.18) to (4.22).

Let $a \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp} a \subset \{|\xi| \leq C_0\}, b \in C_0^\infty(\mathbf{R})$ and $\varphi \in C^\infty(\mathbf{R})$ real valued such that $\sup |\frac{\partial \varphi}{\partial x}(x)| \leq C_0$. Let us set

$$I = e^{-\frac{i}{h}\varphi(x)} a(hD)\Big(b\, e^{\frac{i}{h}\varphi}\Big)(x). \tag{7.1}$$

We have

$$I = (2\pi h)^{-1} \iint e^{\frac{i}{h}((x-y)\xi + \varphi(y) - \varphi(x))} a(\xi) b(y) dy d\xi.$$

Moreover we can write

$$\varphi(x)-\varphi(y)=(x-y)\rho(x,y),\quad \rho(x,y)=\int_0^1 \frac{\partial\varphi}{\partial x}(\lambda x+(1-\lambda)y)d\lambda.$$

We have $|\rho|\le C_0$ so, setting $\eta=\xi-\rho(x,y)$ we obtain,

$$I=(2\pi h)^{-1}\iint e^{\frac{i}{h}(x-y)\eta}\kappa_0(\eta)a(\eta+\rho((x,y))b(y)\,dyd\eta$$

where $\kappa_0\in C_0^\infty(\mathbf{R})$ is such that $\kappa_0(\eta)=1$ if $|\eta|\le 2C_0$.

Using the Taylor expansion of the function a at the point $\rho(x,y)$ we obtain $I=I_1+R_1$ where,

$$I_1=(2\pi h)^{-1}\sum_{k=0}^{M-1}\frac{1}{k!}\iint \eta^k e^{\frac{i}{h}(x-y)\eta}\kappa_0(\eta)a^{(k)}(\rho(x,y))b(y)dyd\eta$$

and,

$$R_1=c_M h^{-1}\iint \eta^M e^{\frac{i(x-y)\eta}{h}}\kappa_0(\eta)\int_0^1(1-\lambda)^{M-1}a^{(M)}(\lambda\eta+\rho((x,y))b(y)d\lambda dyd\eta.$$

Now we have $\eta^k e^{\frac{i}{h}(x-y)\eta}=(-\frac{h}{i}\partial_y)^k e^{\frac{i}{h}(x-y)\eta}$, so integrating by parts in the integrals I_1 and R_1 we obtain,

$$I_1=(2\pi h)^{-1}\sum_{k=0}^{M-1}\frac{h^k}{i^k k!}\iint e^{\frac{i}{h}(x-y)\eta}\kappa_0(\eta)\partial_y^k[a^{(k)}(\rho(x,y))b(y)]dyd\eta$$

$$I_1=(2\pi h)^{-1}\sum_{k=0}^{M-1}\frac{h^k}{i^k k!}\int \hat\kappa_0(\frac{x-y}{h})\partial_y^k[a^{(k)}(\rho(x,y))b(y)]dy,$$

$$R_1=c'h^{M-1}\iiint_0^1 e^{\frac{i(x-y)\eta}{h}}\kappa_0(\eta)(1-\lambda)^{M-1}\partial_y^M[a^{(M)}(\lambda\eta+\rho((x,y))b(y)]d\lambda dyd\eta.$$

Let us set

$$f(x,y)=\partial_y^k[a^{(k)}(\rho(x,y))b(y)].$$

Now we set in the integral, $x-y=hz$ and we write,

$$f(x,x-hz)=\sum_{j=0}^{M-1}\frac{(-hz)^j}{j!}(\partial_y^j f)(x,x)$$
$$+\frac{(-hz)^M}{(M-1)!}\int_0^1(1-\lambda)^{M-1}(\partial_y^M f)(x,x-\lambda hz)d\lambda.$$

Then we use the following equality, which reflects the fact that κ_0 is equal to one near the origin. For $j\in\mathbf{N}$ we have,

$$\int z^j\hat\kappa_0(z)dz=2\pi\delta_{j,0},$$

where $\delta_{j,0}$ is the Kronecker symbol. It follows that,

$$I_1 = \sum_{k=0}^{M-1} \frac{h^k}{i^k k!} \partial_y^k [a^{(k)}(\rho(x,y))b(y)] \big|_{y=x} + R_2,$$

where $R_2 = \sum_{k=0}^{M-1} c_k h^{k+M} r_k$ with

$$r_k = \iint_0^1 z^M \hat{\kappa}_0(z)(1-\lambda)^{M-1} \partial_y^{M+k} [a^{(M)}(\rho(x,y))b(y)] \big|_{y=x-\lambda hz} d\lambda dz.$$

Thus we obtain (4.18) to (4.22).

References

[1] T. Alazard and G. Métivier, Paralinearization of the Dirichlet to Neumann operator, and regularity of three dimensional water waves, *Communications in Partial Differential Equations* 34, 1632-1704, 2009.

[2] T. Alazard, N. Burq and C. Zuily, On the Cauchy problem for the water waves with surface tension, A paraître, Duke Math. Journal, 2011.

[3] T. Alazard, N. Burq and C. Zuily, Cauchy problem and Kato smoothing effet for the water waves, *proceeding RIMS,* 2009 to appear.

[4] T. Alazard, N. Burq and C. Zuily, Dispersive estimates for the 3-d water-waves system and applications, In preparation, 2010.

[5] S. Alinhac, Paracomposition et opérateurs paradifférentiels. *Comm. Partial Differential Equations*, 11(1):87–121, 1986.

[6] S. Alinhac, Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations*, 14(2):173–230, 1989.

[7] D. M. Ambrose and N. Masmoudi, The zero surface tension limit of two-dimensional water waves, *Comm. Pure Appl. Math.* **58** (10), 1287–1315, 2005,.

[8] H. Bahouri and J-Y. Chemin, *Equations d'ondes quasi-linaires et estimations de Strichartz. [Quasilinear wave equations and Strichartz estimates]* Amer. J. Math. 121 (6), 1337–1377, 1999.

[9] M.S Berger, Non linearity and functionnal analysis, *Academic Press New York 1977*

[10] J. Bergh, J. Löfström, Interpolation spaces. An introduction. *Grundlehren der Mathematischen Wissenschaften*, No. 223. Springer-Verlag, Berlin-New York, 1976. x+207 pp

[11] K. Beyer and M. Günther, On the Cauchy problem for a capillary drop. I. Irrotational motion, *Math. Methods Appl. Sci.* **21** (1998), no. 12, 1149–1183.

[12] M. Blair, Strichartz estimates for wave equations with coefficients of Sobolev regularity, *Communications in Partial Differential Equations*, 31 (5), 649-688, 2006.

[13] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.

[14] N. Burq, P. Gérard and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrdinger equation on compact manifolds, *Amer. J. Math.* 126 (3), 569–605, 2004.

[15] N. Burq and F. Planchon, On well-posedness for the Benjamin-Ono equation, *Mathematische Annalen*, 340 (3): 497–542, 2008.

[16] J.Y. Chemin Fluides parfaits incompressibles. *Astrisque* No. 230 (1995), 177 pp.

[17] H. Christianson, V. Hur and G. Staffilani, Strichartz estimates for the water-wave problem with surface tension *Preprint* arxiv. http://arxiv.org/abs/0908.3255

[18] D. Coutand and S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, *J. Amer. Math. Soc.***20** (3), 829–930, 2007.

[19] P. Germain, N. Masmoudi, J. Shatah, Global Solutions for the Gravity Water Waves Equation in Dimension 3. Preprint, http://arxiv.org/abs/0906.5343

[20] T. Iguchi, A long wave approximation for capillary-gravity waves and an effect of the bottom, *Comm. Partial Differential Equations*, **32**, 37–85, 2007.

[21] H. Koch and D. Tataru, Dispersive estimates for principally normal pseudodifferential operators, *Comm. Pure Appl. Math.* 58(2):217–284, 2005.

[22] G. Métivier, Para-differential calculus and applications to the Cauchy problem for nonlinear systems, *Ennio de Giorgi Math. Res. Center Publ., Edizione della Normale*, 2008.

[23] M. Ming and Z. Zhang, Well-posedness of the water-wave problem with surface tension, preprint 2008.

[24] F. Rousset and N. Tzvetkov, Transverse instability of the line solitary water-waves, *Discrete Contin. Dyn. Syst. Ser. B*, 13 (4), 859–872, 2010.

[25] B. Schweizer, On the three-dimensional Euler equations with a free boundary subject to surface tension, *Ann. Inst. H. Poincar Anal. Non Linaire* **22** (6), 753–781, 2005.

[26] J. Shatah and C. Zheng, Geometry and a priori estimates for free boundary problems of the Euler equation, *Comm. Pure Appl. Math.* 61 (5), 698–744, 2008.

[27] H.F. Smith, A parametrix construction for wave equations with $C^{1,1}$ coefficients, *Ann. Inst. Fourier (Grenoble)* 48 (3), 797–835 , 1998.

[28] G. Staffilani and D. Tataru, Strichartz estimates for a Schrödinger operator with nonsmooth coefficients, *Comm. Partial Differential Equations*, 27 (7-8), 1337–1372, 2002.

[29] D. Tataru, *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation Amer. J. Math.* 122 (2), 349–376, 2000.

[30] D. Tataru, Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. II *Amer. J. Math.* 123 (3), 385–423, 2001.

[31] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D *J. Amer. Math. Soc.* 12 (2), 445–495, 1999.

[32] S. Wu, Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.* 177 (1), 45–135, 2009.

[33] S. Wu, Global well-posedness of the 3-D full water wave problem preprint http://arxiv.org/abs/0910.2473

March 19, 2014

T. Alazard, CNRS & Univ Paris-Sud 11, Département de Mathématiques, F-91405 Orsay
E-mail : `thomas.alazard@math.u-psud.fr`

N. Burq, Université Paris-Sud 11, Département de Mathématiques; CNRS, F-91405 Orsay
E-mail : `nicolas.burq@math.u-psud.fr`

C. Zuily, Université Paris-Sud 11, Département de Mathématiques; CNRS, F-91405 Orsay
E-mail : `claude.zuily@math.u-psud.fr`