Control and stabilization from geodesic domains

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Control of waves

Consider the wave equation on a Riemannian manifold $M_g$, $a \in L^\infty(M)$, $a \geq 0$, $T > 0$

$$(\partial_t^2 - \Delta)u = f \times 1_{(0,T)} \times a(x), \quad (u \mid_{t=0}, \partial_t u \mid_{t=0}) = (u_0, u_1)$$

Given $(u_0, u_1) \in \mathcal{H}^1 = H^1(M) \times L^2(M)$ initial data and $(v_0, v_1) \in \mathcal{H}^1$ target data in energy space, can we choose $f$ in suitable space such that

$$(u \mid_{t=T}, \partial_t u \mid_{t=T}) = (v_0, v_1)?$$

Natural space for $f$ is $L^2((0, T) \times M)$. If answer yes: exact controlability
Stabilization for waves

\[ (\partial_t^2 - \Delta + a(x)\partial_t)u = 0, \]
\[ (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in H^1 \times L^2 = \mathcal{H}^1 \]

The natural energy is decaying \((a \geq 0)\)

\[ E(u)(t) = \int_M |\nabla_x u|^2 + |\partial_t u|^2 \, dx, \quad \frac{d}{dt} E(t) = \int_M -a(x)|\partial_t u|^2 \, dx \]

Question: speed of decay of \(E(u)(t)\)?

- The energy of all solutions tend to 0 iff there exists no non trivial stationary equilibrium, i.e.
  \[-\Delta e = \lambda^2 e, \quad a \times e = 0 \Rightarrow e = 0.\]

- Semi-group property: If there exists a uniform rate \(f(t)\),
  \[ \forall (u_0, u_1) \in \mathcal{H}^1, E(u)(t) \leq f(t)E(u)(0), \quad \lim_{t \to +\infty} f(t) = 0, \]
  then can choose \(f(t) = Ce^{-ct}\) (uniform) stabilization.
Observation and HUM duality imply equivalence

- There exists a rate \( f(t) \) such that \( \lim_{t \to +\infty} f(t) = 0 \) and
  \[
  \forall (u_0, u_1) \in H^1(M) \times L^2(M), E(u)(t) \leq f(t)E(u)(0).
  \]
  (and then can choose \( f(t) = Ce^{-ct} \))

- \( \exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M), \) if \( u \) is the solution to the damped wave equation, then
  \[
  E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 \, dx \, dt.
  \]

- \( \exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M), \) if \( u \) is the solution to the undamped wave equation then
  \[
  E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 \, dx \, dt.
  \]

- There exists \( T > 0 \) such that The wave equation is exactly controllable in time \( T \) (and we can take the time given by observation)
The geometric control assumption for waves

\((a \in C^0(M), T)\) controls geometrically \((M, g)\) if every geodesic starting from any point \(x_0 \in M\) in any direction \(\xi_0, \gamma(x_0, \xi_0)(s)\), encounters \(\{a > 0\}\) in time smaller that \(T\).

Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch 88’, N.B- P.G.)

\(a \in C^0(M)\) geometric control is equivalent to observability (and hence control and stabilization) for wave equations. \(a \in L^\infty(M)\)

Strong Geometric Control is sufficient for observability which implies Weak Geometric Control.

\[ \exists T, c > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T), \exists \delta > 0; \]
\[ a \geq c \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta). \]  
\[ (SGCC) \]

\[ \exists T > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T); \gamma_{\rho_0}(s) \in \text{supp}(a) \]
\[ (WGCC) \]

\(\text{supp}(a)\) is the support (in the distributional sense) of \(a\),
The geometric control assumption

Yes

No
Some examples on tori

Figure: Checkerboards: the damping $a$ is equal to 1 in the blue region, 0 elsewhere. The geodesics are (periodized) straight lines. The first example satisfies (SGCC) while all others satisfy (WGCC) but not (SGCC)
Stabilization for wave equations: the result
Theorem (Does Stabilization holds? $a = 1$ in blue region 0 otherwise)

![YES 80’ (Taylor-Rauch)](image1)

![YES (NB-PG 16)](image2)

![NO (NB-PG 16)](image3)

![NO (NB-PG 16)](image4)

![YES (NB-PG 16)](image5)

![NO (NB-PG 16)](image6)
Another geometric condition

When the manifold is a two dimensional torus and the damping $a$ is a linear combination of characteristic functions of rectangles, i.e. there exists $N$ rectangles (or polygons), $R_j, j = 1, \ldots, N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \ldots, N$ such that

$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j},$$

(piecewise smooth domains, no infinite contact with geodesics = much easier)

**Theorem (NB–P. Gérard 15-17)**

*Stabilization holds for the waves on $\mathbb{T}^2$ iff there exists $T > 0$ such that all geodesics (straight lines) of length $T$ either encounters the interior of one of the rectangles or follows for some time one of the sides of a rectangle $R_{j_1}$ on the left and for some time one of the sides of another (possibly the same) rectangle $R_{j_2}$ on the right.*
Stabilization for wave equations: the result

YES

NO

NO

YES

YES

NO
A geometric control condition for control

When the manifold is a two dimensional surface and $a$ is a linear combination of characteristic functions of geodesic polygons, i.e. there exists $N$ polygons, $R_j, j = 1, \ldots, N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \ldots, N$ such that

$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j},$$

(2)

Theorem (NB 17)

Let $T > 0$. Then exact controlability holds for the waves on $M^2$ if and only if a generalized geometric condition (defined in terms of an ODE on a sphere bundle over $S^* M$) is satisfied. Roughly speaking it says that all geodesics of length $T$ either encounters the interior of one of the polygons or follows for some time one of the sides of a polygon $R_j$ and there exists $s > 0$ such all neighbour geodesics spend an amount of time of order at least $s$ in the interior of one of the polygons.
The result on the sphere and the torus

**Theorem (NB 17)**

Let $T > 0$. Then exact controllability holds for the waves on $M^2$ if and only if there exists $\alpha > 0$ such that for almost every $(x_0, \xi_0) \in S^* M$,

$$\int_0^T a(x(s, x_0, \xi_0)) ds \geq \alpha > 0.$$  

Here $(x(s, x_0, \xi_0), \xi(s, x_0, \xi_0))$ is the bicharacteristic starting from $(x_0, \xi_0)$ at $s = 0$.
Control for wave equations: the result on the sphere

On spheres, geodesics are great circles and the generalized geometric condition reduces to checking

– The geodesic enters the *interior of the control region* \( \omega = \{ a(x) > 0 \} \) or

– The geodesic follows (some) sides of (some) polygons and we check the following algorithm. Write the whole oriented geodesic circle

\[
\Gamma = \Gamma^u \cup \Gamma^d \cup \Gamma^0,
\]

(parts of \( \Gamma \) which encounter the side of a polygon on the upper (lower) hemisphere—or not). Then any choice of oriented diameter \( D \) separates any piece of geodesic \( \gamma(0, T) \) into three (possibly empty) pieces

\[
\gamma^l \cup \gamma^r \cup \gamma^c,
\]
corresponding to the part on the left (right) of the diameter or on the diameter. Then we assume that we never have

\[
(\gamma^l \subset \Gamma^u \text{ and } \gamma^r \subset \Gamma^d).
\]
Contradiction argument

Want to prove observation estimates for half wave solutions with spectrally localized initial data (h small enough)

\[(\partial_t^2 - \Delta)u = 0, \, u_0 = 1_{a < -h^2 \Delta < b}u_0, \, u_1 = 1_{a < -h^2 \Delta < b}u_1 \quad a < 1 < b\]

\[\|u_0\|_{L^2}^2 \leq C \int_0^T \int_\omega |u|^2(x, t) dx dt \quad \omega = \{a > 0\}\]

Assume false then there exists sequences

\[a_n, b_n \to 1, \, u_n \in L^2, \, 1_{a_n < -h^2 \Delta < b_n}u_n = u_n,\]

such that

\[\|u_n\|_{L^2} = 1, \int_0^T \int_\omega |u_n|^2(x, t) dx dt = o(1)\]
First microlocalization

Scales: \( t, X \sim 1, \tau, \Xi \sim h^{-1} \). Consider operators

\[ a(t, X, hD_t, hD_X), a \in C_0^\infty(T^*M). \]

If \( a \geq 0 \) then (Gårding) \( a(t, X, hD_t, hD_X) \geq -Ch. \)

Proposition

There exists a subsequence (now we drop all sub-indexes) and a positive measure \( \mu \) (on continuous functions on \( T^*M \)) such that

\[
\lim_{n \to +\infty} \left( a(t, X, h_nD_t, h_nD_X)u_n, u_n \right)_{L^2_{t,X}} = \langle \mu, a \rangle.
\]

\[
\text{supp } (\mu) \subset \{(t, \tau, X, \Xi); 1 = \tau^2 = \|\Xi\|^2_{g(x)} = p(X, \Xi) \}
\]

\[
\mu(T^*(0, Y) \times M) = T, \quad \partial_t \mu = H_p \mu, \quad \mu \big|_{(0, T) \times \omega} = 0.
\]

As a consequence, \( \mu \) is supported on bicharacteristics which do not encounter \( \omega \) but hence graze \( \partial \omega \) on left or right
Second microlocalization

Understand at finer scales how the mass can concentrate on the geodesic from left or right. Work in a geodesic coordinate system \((x, y)\) where

\[-\Delta = -\partial_x^2 - \partial_y^2 (1 + x^2 \kappa(y) + O(x^3)),\]

where the geodesic is given by \(\{x = 0\}\) (and the bicharacteristic by \(\{(x = 0, \xi = 0)\}\)) and \(\kappa(y)\) is the gauss curvature of the surface at point \(y\).

Scales: 2 different regimes

- Transversal HF
  \[t, y \sim 1, \tau = \eta = 1 + o(1), h^{1/2} \ll \|x, \xi\| = o(1)\]

- Transversal LF
  \[t, y \sim 1, \tau = \eta = 1 + o(1), \|x, \xi\| \leq Ch^{1/2}\]

Describe concentration at these scales and conclude contradiction.
2-pseudodifferential operators

– Symbols: functions $a(t, y, z, \tau, \eta, \zeta) \in S$ the class of smooth compactly supported in the $(y, \eta)$ variables and polyhomogeneous of degree 0 near infinity in the $(z, \zeta)$ variables

$$|\partial_{t,y,\tau,\eta}^\alpha \partial_z^\gamma \partial_\zeta^\delta a| \leq C(1 + |z| + |\zeta|)^{-(\gamma + \delta)}.$$ 

– Operators: $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 near 0,

$$\text{Op}_h(a) = a(t, y, hD_t, hD_y, h^{-1/2}x, h^{1/2}D_x)$$

$$\text{Op}_{h,\epsilon}(a) = \text{Op}(a \times \chi(\epsilon z, \epsilon \zeta)),$$

$$\text{Op}^\epsilon_h(a) = \text{Op}(a \times (1 - \chi)(\epsilon z, \epsilon \zeta)),$$

– Bad pseudodifferential calculus for $\text{Op}(a)$ and $\text{Op}_\epsilon(a)$ $L^2$ boundedness but no symbolic calculus, no Gårding
– Good pseudodifferential calculus $\text{Op}^\epsilon(a)$ (gain $\epsilon^2$) symbolic calculus, and Gårding

Approach inspired from works by Fermanian, Nier and Anantharaman-Macia for Schrödinger on tori. Here $S_{1,1/2}$, $S_{0,0}$ calculus, Nier $S_{1,1}$, A–M, $S_{0,0}$ calculus
2-microlocal measure: transversal HF (for tori $\epsilon \to h^\epsilon$)

$t, y \sim 1, \tau = \eta = 1 + o(1), h^{1/2} \ll \|x, \xi\| = o(1)$,

If $a \geq 0$ then (Gårding) $\text{Op}^\epsilon(a) \geq -C\epsilon$.

**Proposition**

There exists a subsequence and a positive measure $\nu^+$ (on continuous functions on $T^*N \times \mathbb{R}^2$ homogeneous of degree 0 at infinity in $(z, \zeta)$ ) such that

$$\lim_{\epsilon \to 0} \lim_{n \to +\infty} \left( \text{Op}_{h_n}^\epsilon(a) u_n, u_n \right)_{L^2_{t,y,x}} = \langle \nu^+, a \rangle.$$ 

$$\text{supp} (\nu^+) \subset \{ (t, \tau, y, z, \eta, \zeta); 1 = \tau = \eta \}$$

The projection of $\nu$ on the $(t, y, \tau, \eta)$ variables is bounded by the previous (1)-microlocal measure and additional propagation holds

$$\left( \partial_t - \partial_y - \zeta \partial_z + z\kappa(y)\partial_\zeta \right) \nu^+ = 0.$$
Proof of propagation

Key remark

\[- h^2 \Delta = -h^2 \partial_y^2 (1 - x^2 \kappa(y) - h^2 \partial_x^2 + O(x^3)) \]
\[= \text{Op}(\eta^2 (1 + h z^2 \kappa(y) + h \zeta^2 + O(h z^2 x))) \quad (3)\]

Compute

\[\frac{i}{2h} \left[ (h^2 \partial_t^2 - h^2 \Delta, \text{Op}_{h_n}^\epsilon (a)) \right] = \text{Op}_{h_n}^\epsilon (-\tau \partial_t (a) + \eta \partial_y (a) + \zeta \partial_z (a) - \eta^2 \kappa(y) z \partial_\zeta (a) + O(\epsilon^2) + O(x) \]

implies

\[0 = \lim_{\epsilon \to 0} \lim_{n \to +\infty} \frac{i}{2h} \left( \left[ (h^2 \partial_t^2 - h^2 \Delta, \text{Op}_{h_n}^\epsilon (a)) u_n, u_n \right]_{L^2_{t,y,x}} \right) = \langle (\partial_t - \partial_y - \zeta \partial_z + \kappa(y) z \partial_\zeta) \nu, a \rangle\]
Conclusion in the transversal HF regime

• The measure $\nu^+$ is invariant by the flow defined by previous equation
• It is supported on geodesics grazing $\partial \omega$
• By contradiction assumption

$$\int_{(0,T)} \int_{\omega} |u|^2(t,x) dx dt = o(1),$$

We deduce that if near points $(t, y, x = 0)$ such that $(y, x = 0) \in \partial \omega^r$ then $\nu_+$ is supported in $\{z \leq 0\}$
• The geometric hypothesis implies that $\nu_+ \equiv 0$
2-microlocal measure: transversal LF

We are looking at \((\chi_\epsilon = \chi(\epsilon \cdot))\)

\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \left( (a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, h_n^{-1/2} x, h_n^{1/2} D_x) u_n, u_n \right)_{L^2}
\]

change variables in \(x, z = h^{-1/2} x, \nu_n(z) = h^{1/4} u_n(h^{1/2} z)\) We are now looking at

\[
\lim_{\epsilon \to 0} \lim_{n \to +\infty} \left( (a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, z, D_z) \nu_n, \nu_n \right)_{L^2}
\]

Due to the presence of the cut off \(\chi_\epsilon\) for any fixed \(\epsilon\), the operators

\((a \times \chi_\epsilon)(t, y, h_n D_t, h_n D_y, z, D_z)\)

are semi-classical operators in the \((t, y)\) variable with values compact operators on \(L^2_x = H\)

The sequence \(\nu_n\) is bounded in \(L^2_{loc}(\mathbb{R}^2_{t,y}; H)\).
2-microlocal measure: transversal LF

Proposition (P. Gérard 90’)

There exists a subsequence and a positive measure $\nu^-$ on continuous functions on $T^*\mathbb{R}^2$ with values trace class operators

$$\lim_{n \to +\infty} \left( (a \times \chi_\varepsilon)(t, y, h_n D_t, h_n D_y, z, D_z) \nu_n, \nu_n \right)_{L^2} = \text{Tr}(\nu^-, (a\chi_\varepsilon)(t, y, \tau, \eta, z, D_z)).$$

Radon-Nikodym: $\nu^- = A(t, y, \tau, \eta) d\rho$, where $A$ is trace class

Proposition (Saut Scheurer?)

Consider the following classical one dimensional harmonic oscillator

$$(i\partial_s - \frac{\partial_z^2}{2} + \frac{\kappa(-s)z^2}{2})u = 0.$$ 

Assume that $u$ vanishes on $(\alpha, \beta)_s \times \mathbb{R}_z^+$. Then $u \equiv 0$.

Theorem

As soon as $\Gamma^+$ or $\Gamma^-$ is non empty, the measure $\nu^-$ is identically 0.
The harmonic oscillator

\[ t, y \sim 1, \tau = \eta = 1 + o(1), |z| + |D_z| \leq \varepsilon^{-1}, \]

\[ (i\hbar_n \partial_t + \sqrt{-\hbar_n^2 \Delta})u_n = 0, \]

\[ -\hbar^2 \Delta = -\hbar^2 \partial_y^2 (1 + h\kappa(y)z^2) - h\partial_z^2 + O_\varepsilon(h^{3/2}) \]

\[ \sqrt{-\hbar^2 \Delta} = -i\hbar \partial_y \sqrt{(1 + h\kappa(y)z^2) - h\frac{\partial_z^2}{-\hbar^2 \partial_y^2} + O_\varepsilon(h^{3/2})} \]

\[ = -i\hbar \partial_y + h\left(-\frac{\partial_z^2}{2} + \frac{\kappa(y)z^2}{2}\right) + o_\varepsilon(h) \quad (4) \]

\[ (i\hbar_n(\partial_t - \partial_y) + h_n(-\frac{\partial_z^2}{2} + \frac{\kappa(y)z^2}{2})v_n = o_\varepsilon(h). \]

\[ s = \frac{t - y}{2}, \ r = \frac{t + y}{2} \Rightarrow (i\partial_s + H_{r-s})w_n = o_\varepsilon(h) \]

\[ \Leftrightarrow \ w_n = S_r(s, 0)(w_n \mid_{s=0}) + o(1). \quad (5) \]
An elementary argument in infinite dimension

Conjugate (micro-locally) with the inverse of the evolution to replace the equation \((i\partial_s + H_{r\to s})w_n = 0\) by \(i\partial_s \tilde{w}_n = 0\).

\[
\tilde{w}_n = S_r(0, s)w_n = S_r(s, 0)^* w_n,
\]

Let \(\tilde{\nu}^-\) be the measure of the new sequence \(\tilde{w}_n\). Then

\[
\tilde{\nu}^- = S_r(0, s)\nu S_r(0, s)^*
\]

is independent of the variable \(s\). It writes

\[
\tilde{\nu}^- = A(r)d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},
\]

Where \(A(r)\) is a family of hermician trace class operators and \(d\lambda_r\) a non negative measure. Let \(e_n(r) \in H\) a Hilbert Basis diagonalizing \(A(r)\) (eigenvalues \(\lambda_n\)). We get

\[
\tilde{\nu}^- = \sum_n \lambda_n \langle \cdot, e_{n,r} \rangle_H e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},
\]

\[
\nu^- = \sum_n \lambda_n \langle \cdot, S(s, 0)e_{n,r} \rangle_H S(s, 0)e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2},
\]
Conclusion in the transversal LF regime

\[ \nu^{-} = \sum \lambda_n \langle \cdot, S(s, 0)e_{n,r} \rangle_H S(s, 0)e_{n,r} d\lambda_r \otimes ds \otimes \delta_{\sigma=0} \otimes \delta_{\rho=2}, \]

If \( \zeta(s, r) \) is supported in the region where the bicharacteristic grazes a part of \( \Gamma_r^\prime \), then for any \( \epsilon > 0 \),

\[ \lim_{n \to +\infty} \int_{s, r} \chi(s, r) \int_{z \in (0, \delta h^{-1/2})} |w_n|^2 d\lambda r \otimes ds = 0, \]

We deduce that for any \( \epsilon > 0 \),

\[ \langle \nu^{-}, \zeta(s, r) 1_{x>0} \chi(\epsilon(z, Dz)) \rangle = 0 \]

\[ \Rightarrow 0 = \int_{r, s} \zeta(r, s) \operatorname{Tr} \left( \sum_n \lambda_n \langle \cdot, S(s, 0)e_{n,r} \rangle_H 1_{z>0} S(s, 0)e_{n,r} d\lambda_r ds \right), \]

\[ = \int_{r, s} \zeta(r, s) \sum_n \lambda_n \int_{z>0} |S(s, 0)e_{n,r}|^2 ds d\lambda_r \quad (6) \]

(we used that \( S(s, o)e_n \) is a Hilbert basis of \( H \)). Hence implies \( \forall n, \lambda_n = 0 \) and \( \nu^{-} = 0 \).