GLOBAL STRICHARTZ ESTIMATES FOR NONTRAPPING GEOMETRIES: ABOUT AN ARTICLE BY H. SMITH AND C. SOGGE

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Abstract. The purpose of this note is to present an alternative proof of a result by H. Smith and C. Sogge showing that in odd dimension of space, local (in time) Strichartz estimates and exponential decay of the local energy for solutions to wave equations imply global Strichartz estimates. Our proof allows to extend the result to the case of even dimensions of space

1. Introduction

Consider $\Theta \subset B(0,R) \subset \mathbb{R}^d$ a smooth obstacle, $\Omega = \Theta^c$, g a smooth metric on $\overline{\Omega}$ equal to $g_{i,j}(x) = \delta_{i,j}$ for $|x| \geq R$, and the wave equation on Ω:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = F(t,x) \\ u\mid_{t=0} = f \in \dot{H}^{\gamma}_D, \partial_t u\mid_{t=0} = g \in \dot{H}^{\gamma-1}_D \\ \qquad u\mid_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where Δ_g is the Laplace Beltrami operator on Ω, and $\dot{H}^\gamma$, $\gamma \in \mathbb{R}$ are the homogeneous Sobolev spaces associated to the square root of the Laplace operator with Dirichlet boundary conditions on Ω, $\sqrt{-\Delta_D}$. In [10] H. Smith and C. Sogge show that in odd space dimensions, if the perturbation (g,Θ) is assumed to be *non trapping*, then local (in time) Strichartz estimates imply global (in time) Strichartz estimates for the solutions of the wave equation. The method of proof relies on the exponential decay for the local energy of solutions of the corresponding wave equation with compactly supported initial data, a localization argument on the wave cone and a result by M. Christ and A. Kiselev [5] allowing to deduce the inhomogeneous estimates from the homogeneous ones. The purpose of this note is to generalize this result to the case of even dimensions of space. In this case, the exponential decay is no longer true and one has to replace it by another form of decay. In fact the decay we will use (global L^2 integrability of the local energy norm for *any* initial data, *i.e.* non necessarily compactly supported) is closely related to the

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exponential decay of the local energy used by H. Smith and C. Sogge, since the proof of our decay relies on a weaker form of the estimates used to prove the exponential decay in scattering theory.

More precisely, the result by H. Smith and C. Sogge reads as follows:

Definition 1.1 ([10]). *We say that* $1 \leq r, s \leq 2 \leq p, q \leq +\infty$ *and* γ *are admissible if the following two mixed norm estimates hold*

- **Local Strichartz estimates.** *For data* f, g, F *supported in* $\{|x| \leq R\}$*, and* u *solution of* (1.1)*, one has*

$$\|u\|_{L^p([0,1]_t;L^q_x(\Omega)} + \|(u, \partial_t u\|_{L^\infty([0,1]_t;\dot{H}^\gamma_D \times \dot{H}^{\gamma-1}_D)} \leq C \Big(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} + \|F\|_{L^r([0,1]_t;L^s_x(\Omega))} \Big) \tag{1.2}$$

- **Global Minkovski Strichartz Estimates.** *For data* f, g, F *and* u *solution of* (1.1) *in the case* $\Omega = \mathbb{R}^d$ *and* $g_{i,j}(x) = \delta_{i,j}$*, one has*

$$\|u\|_{L^p(\mathbb{R}_t;L^q_x(\mathbb{R}^d))} \leq C \Big(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} + \|F\|_{L^r(\mathbb{R}_t;L^s_x(\mathbb{R}^d))} \Big) \tag{1.3}$$

In [10], H. Smith and C. Sogge show:

Theorem 1 ([10, Theorem 1.1]). *Assume that* $d \geq 3$ *is odd,* $1 \leq r, s \leq 2 \leq p, q \leq +\infty$ *and* γ *are admissible,* $p > r$ *and* $\gamma \leq \frac{d-1}{2}$*. Then for data* f, g, F *and* u *solution of* (1.1)*, one has*

$$\|u\|_{L^p(\mathbb{R}_t;L^q_x(\Omega))} \leq C \Big(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} + \|F\|_{L^r(\mathbb{R}_t;L^s_x(\Omega))} \Big) \tag{1.4}$$

In this article we are going to prove:

Theorem 2. *Assume* $d \geq 2$*,* $1 \leq r, s \leq 2 \leq p, q \leq +\infty$ *and* γ *are admissible,* $p > 2$ *and* $\gamma < d/2$*. Assume also if* $d = 2$ *that* $\Theta \neq \emptyset$ *and* $\gamma \leq 1/2$*. Then for data* f, g, F *and* u *solution of* (1.1)*, then*

$$\|u\|_{L^p(\mathbb{R}_t;L^q_x(\Omega))} \leq C \Big(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} + \|F\|_{L^r(\mathbb{R}_t;L^s_x(\Omega))} \Big) \tag{1.5}$$

Remark 1.2. *The assumption* $\Theta \neq \emptyset$ *is used to avoid low frequency problems. It could be replaced by a "0 is not a resonance" assumption (see section 2).*

As explained above, the idea in this article is to replace the exponential decay of the local energy by another (weaker) form of decay. Section 2 is hence devoted to the proof of this decay. Then in section 3 we prove Theorem 2.

Remark 1.3. *At the final stage of this work, I have learned that the results of this article have been obtained independently (by a slightly different method) by Jason Metcalfe* [9]

2. Global L^2-integrability of the local energy

We begin first by recalling (essentially following [10]) the definition of the homogeneous Sobolev spaces (in the boundary case) $\dot{H}^s_D$. Remark that we shall restrict ourselves to the case $s < d/2$, hence the multiplication by a smooth function $\xi \in C_0^\infty(\mathbb{R}^d)$ is continuous from $\dot{H}^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and the two norms are equivalent on functions with fixed compact support. Take R large enough so that $\Theta \subset B(0,R)$ and $g_{i,j}(x) = \delta_{i,j}$ for $|x| \geq R$. Fix $\beta \in C_0^\infty(\mathbb{R}^d)$ equal to 1 on $\{|x| \leq R\}$, and define for $s \in \mathbb{N}$ $\dot{H}^s_D$ to be the set of functions $f \in H^s_{\rm loc}(\Omega)$ such that

$$\|f\|^2_{\dot{H}^s_D} = \|\beta f\|^2_{H^s_D(\Omega)} + \|(1-\beta)f\|^2_{\dot{H}^s(\mathbb{R}^d)} \tag{2.1}$$

is bounded and which satisfy the following compatibility conditions:

$$\Delta_g^j f \mid_{\partial\Omega} = 0, \forall j; s - 2j > \frac{1}{2}. \tag{2.2}$$

For $-s \in \mathbb{N}$, define $\dot{H}^s_D = (\dot{H}^{-s}_D)'$ and finally for $s \in \mathbb{R}$ define $\dot{H}^s_D$ by interpolation. On $\dot{H}^s_D$, the norms $\|(-\Delta_g)^{s/2} f\|_{L^2}$ and $\|f\|_{\dot{H}^s_D}$ are equivalent (it is clear for s an even integer and follows in general by interpolation and duality). Denote by $\dot{\mathcal{H}}^s = \dot{H}^s_D \times \dot{H}^{s-1}_D$ and $\mathcal{H}^s = H^s_D \times H^{s-1}_D$.

In this part we are going to prove:

Theorem 3. *Consider $\chi \in C_0^\infty(\mathbb{R}^d)$. Assume that $d \geq 2$ and if $d = 2$ that $\Theta \neq \emptyset$. Then For data f, g, F such that F* is supported in $B_x(0,R) \cap \Omega$ *and u solution of* (1.1), *one has*

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}_t; \dot{H}^\gamma_D \times \dot{H}^{\gamma-1}_D)} \leq C \left(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} + \|F\|_{L^2(\mathbb{R}_t; \dot{H}^{\gamma-1}_D)} \right) \tag{2.3}$$

Remark 2.1. *Note that in the result above only F (and neither f nor g) is assumed to be compactly supported.*

Remark 2.2. *As will appear below, the proof of this result is closely related to the proof of the exponential decay of the local energy (for compactly supported data). In fact we will use some estimates on the resolvent* weaker *than the estimates required to prove this exponential decay. The result above can be transposed to the framework of Schrödinger equations. It gives then a (global in time) "smoothing effect"* [4].

Remark 2.3. *For $d = 2, \gamma \leq 1/2$ and $g_{i,j}(x) = \delta_{i,j}$, for any u solution of* (1.1) *with $F = 0$, we have (see* [10, Lemma 2.2]*)*

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}_t; \dot{H}^\gamma_D \times \dot{H}^{\gamma-1}_D)} \leq C \left(\|f\|_{\dot{H}^\gamma_D} + \|g\|_{\dot{H}^{\gamma-1}_D} \right) \tag{2.4}$$

To prove Theorem 3, we first show that by a TT^* argument it is enough to prove a non homogeneous estimate. Indeed denote by $A = i \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta & 0 \end{pmatrix}$ the self adjoint operator on $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$ with domain $\mathcal{D} = \{(u,v) \in \mathcal{H} \cap H^2(\Omega) \times H^1_0(\Omega)\}$. The solution of (1.1) with $F = 0$ is given by

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{itA} \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.5}$$

The operator e^{itA} is a group of isometries on $\dot{\mathcal{H}}^s_D$ (equipped with the norm $\|A^s f\|_{L^2}$).

Denote by $T = \chi e^{itA}$. The continuity of T from $\dot{\mathcal{H}}^s$ to $L^2(\mathbb{R}_t; \mathcal{H}^s)$ is equivalent to the continuity of T^* from $L^2(\mathbb{R}_t; \mathcal{H}^s)$ to $\dot{\mathcal{H}}^s$. But

$$T^* G = \int_{s \in \mathbb{R}} A^{-2s} e^{-isA} \chi A^{2s} G(s, \cdot) ds \tag{2.6}$$

and the continuity of T^* is in turn equivalent to the continuity of TT^* from $L^2(\mathbb{R}_t; \dot{\mathcal{H}}^s)$ to $L^2(\mathbb{R}_t; \mathcal{H}^s)$. Finally

$$TT^* G(t, \cdot) = \int_{s \in \mathbb{R}} \chi A^{-2s} e^{i(t-s)A} \chi A^{2s} G(s, \cdot) ds = \int_{s<t} + \int_{t<s} \tag{2.7}$$

And it sufficies to prove the continuity of any of the two operators above, say for example the continuity of χTT^*_1 with

$$TT^*_1 G(t, \cdot) = \int_{s<t} A^{-2s} e^{i(t-s)A} \chi A^{2s} G(s, \cdot) ds \tag{2.8}$$

Suppose that G is supported in $\{s > 0\}$ (which by density and translation invariance is possible). Then $U = A^{2s} TT^*_1 G$ satisfies the equation

$$(i\partial_t + A) U = \chi A^{2s} G(t, \cdot), \ U \mid_{t<0} = 0 \tag{2.9}$$

The Fourier transforms with respect to t of U and G are holomorphic in the half plane $\mathrm{Im} z < 0$ (due to the support property) and satisfy there

$$(A - z)\widehat{U} = \chi A^{2s} \widehat{G} \tag{2.10}$$

We now show how Theorem 3 can be obtained from:

Proposition 2.4. *Under the assumptions of Theorem 2, the resolvent* $\chi(A - (x - i\varepsilon))^{-1}\chi$ *is for* $x \in \mathbb{R}$ *and* $0 < |\varepsilon| < 1$, *uniformly bounded on* $\mathcal{H}^s_D$.

Indeed, taking $z = x - i\varepsilon$ and letting $\varepsilon > 0$ tend to 0 and using that the Fourier transform is an isometry on $L^2(\mathbb{R}_t; \mathcal{H})$ if $\mathcal{H}$ is a Hilbert space we get (using Proposition 2.4 for $-s$)

$$\|\chi U\|_{L^2(\mathbb{R}_t; \mathcal{H}^{-s}_D)} \leq C \|A^{2s} G\|_{L^2(\mathbb{R}_t; \mathcal{H}^{-s}_D)} \leq C \|G\|_{L^2(\mathbb{R}_t; \mathcal{H}^s_D)}$$

and by elliptic regularity (for s integer and therefore for any s by duality and interpolation)

$$\|\chi TT^*_1 G\|_{L^2(\mathbb{R}_t; \mathcal{H}^s_D)} \leq C \|G\|_{L^2(\mathbb{R}_t; \mathcal{H}^s_D)}.$$

Finally, the proof of the nonhomogeneous part of Theorem 3 is slightly simpler since we have to deal with an equation of the form:

$$(i\partial_t + A)V = \begin{pmatrix} 0 \\ F(t,x) \end{pmatrix}.$$

We can take the Fourier transform as above and use Proposition 2.4 directly to conclude.

We are going to deduce Proposition 2.4 from the classical result:

Proposition 2.5. *Suppose that the obstacle Θ is non trapping. Then the resolvent of the operator Δ_D, $(-\Delta_D - \lambda)^{-1}$ (which is analytic in $\mathbb{C} \setminus \mathbb{R}^+$) satisfies:*

$$\begin{aligned} &\forall \chi \in C_0^\infty(\mathbb{R}^2), \exists C > 0; \forall \lambda \in \mathbb{R}, 0 < \varepsilon << 1, \\ &\|\chi(-\Delta_D - (\lambda \pm i\varepsilon))^{-1}\chi\|_{L^2 \to L^2} \leq \frac{C}{1 + \sqrt{|\lambda|}} \end{aligned} \tag{2.11}$$

Remark 2.6. *Proposition 2.5 was proven for $|\lambda| >> 1$ in greater generalities by Lax and Phillips* [6]*, Melrose and Sjöstrand* [7, 8]*), Vainberg*[13]*, Tang and Zworski* [11] *(see also* [3] *for a self contained proof which joined with the results in* [1] *would relax the smoothness assumption required in these papers to a C^2 assumption). The proof for $|\lambda| << 1$ can be found in* [2, Annexe B.2] *(see also* [12, 13]*). Remark that in* [2] *the Poincaré inequality is used to control the local L^2-norm by the local L^2-norm of the gradient of a function (wich is why $\Theta \neq \emptyset$ is required). However, the following inequality*

$$\forall \beta < \frac{d}{2} - 1, \forall u \in C_0^\infty(\mathbb{R}^d), \|\frac{u}{r(1+r^2)^{\beta/2}}\|_{L^2} \leq C\|\frac{\nabla u}{(1+r^2)^{\beta/2}}\|_{L^2} \tag{2.12}$$

allows to handle the argument for $d \geq 4$ (and the result is standard in odd dimensions).

Since

$$(A - z)^{-1} = \begin{pmatrix} -z(\Delta_g + z^2)^{-1} & i(\Delta_g + z^2)^{-1} \\ i\Delta_g(\Delta_g + z^2)^{-1} & -z(\Delta_g + z^2)^{-1} \end{pmatrix} \tag{2.13}$$

to prove Proposition 2.4 we have to estimate uniformly for any $s \in \mathbb{R}$

$$\|\chi(1 + |z|)(\Delta_g + z^2)^{-1}\chi\|_{H^s_D \to H^s_D} \tag{2.14}$$

$$\|\chi(\Delta_g + z^2)^{-1}\chi\|_{H^s_D \to H^{s+1}_D} \tag{2.15}$$

$$\|\chi\Delta_g(\Delta_g + z^2)^{-1}\chi\|_{H^s_D \to H^{s-1}_D} \tag{2.16}$$

Using duality and interpolation it is enough to do this for $s \in \mathbb{N}$.

For $s=0$, (2.14) is Proposition 2.5 and if $u=(\Delta_g+z^2)^{-1}\chi f$, we have

$$\begin{aligned}\int_\Omega \chi f\chi\overline{u} &= \int_\Omega(\Delta+z^2)u\chi\overline{u}\\ &= \int_\Omega -\chi|\nabla u\|^2-\nabla\chi\cdot\nabla u\overline{u}+z^2\chi|u|^2\end{aligned}\tag{2.17}$$

which by Cauchy Schwartz implies (using (2.14) for $s=0$ and a different χ) easily (2.15) for $s=0$. By duality we get also (2.15) for $s=-1$. Applying Δ_g to $\chi(\Delta_g+z^2)^{-1}\chi$ and commuting with χ we obtain by induction (2.15) for $s\in\mathbb{N}$ (hence by duality and interpolation for any s). Applying again Δ_g we deduce (2.16) for any s. Finally, remark that

$$\chi\Delta_g(\Delta_g+z^2)^{-1}\chi=\chi\mathrm{Id}\chi+\chi z^2(\Delta_g+z^2)^{-1}\chi\tag{2.18}$$

from which we estimate uniformly

$$\|\chi z^2(\Delta_g+z^2)^{-1}\chi\|_{H^s_D\to H^{s-1}_D}.\tag{2.19}$$

Interpolating this latter inequality with $(2.15)_s$ we get $(2.14)_s$.

3. From Energy decay to global Strichartz estimates

In this section we are going to show how the local (in time) Strichartz estimates and the global (in time) L^2- energy decay imply global (in time) Strichartz estimates. In fact, taking benefit of the estimates of the previous section, we can essentially follow the strategy in [10]. We consider firstly the homogeneous case ($F=0$) and we also assume that $\gamma\leq 1/2$ if $d=2$.

Consider $\chi\in C_0^\infty(\mathbb{R}^d)$ equal to 1 on $B(0,R)$, u solution of (1.1) with $F=0$ and $v=\chi(x)u$, $w=(1-\chi)(x)u$ solutions of

$$\begin{cases}(\partial_t^2-\Delta_g)v(t,x)=-[\Delta_g,\chi]u\\ v\mid_{t=0}=\chi f\in\dot H^\gamma\\ \partial_t v\mid_{t=0}=\chi g\in\dot H^{\gamma-1}_D\\ v\mid_{\partial\Omega}=0\end{cases}\tag{3.1}$$

$$\begin{cases}(\partial_t^2-\Delta_g)w(t,x)=(\partial_t^2-\Delta_0)w(t,x)=[\Delta_0,\chi]u\\ w\mid_{t=0}=(1-\chi)f\in\dot H^\gamma,\\ \partial_t w\mid_{t=0}=(1-\chi)g\in\dot H^{\gamma-1}\end{cases}\tag{3.2}$$

First we show that w satisfies global Strichartz stimates. Since w is a solution of the *free* wave equation (because it is supported in the set where $g_{i,j}(x)=\delta_{i,i}$), the contributions of $(1-\chi)f$ and $(1-\chi)g$ are handled using (1.3). Consequently it sufficies to study $\widetilde{w}$ the solution of

$$\begin{cases} (\partial_t^2 - \Delta_0)\widetilde{w}(t,x) = [\Delta_0, \chi]u \\ \widetilde{w}\mid_{t=0} = 0 \\ \partial_t \widetilde{w}\mid_{t=0} = 0 \end{cases} \tag{3.3}$$

For this we use the following result by M. Christ and A. Kiselev [5]:

Theorem 4 (M. Christ and A. Kiselev)**.** *Consider a bounded operator $T : L^p(\mathbb{R}; B_1) \to L^q(\mathbb{R}; B_2)$ given by a locally integrable kernel $K(t,s)$ with value operators from B_1 to B_2 where $B_{1;2}$ are Banach spaces. Suppose that $p < q$. Then the operator*

$$\widetilde{T}f(t) = \int_{s<t} K(t,s)f(s)ds \tag{3.4}$$

is bounded from $L^p(\mathbb{R}; B_1)$ to $L^q(\mathbb{R}; B_2)$ by

$$\|\widetilde{T}\|_{L^p(\mathbb{R};B_1)\to L^q(\mathbb{R};B_2)} \leq (1 - 2^{-p^{-1}-q^{-1}})^{-1}\|T\|_{L^p(\mathbb{R};B_1)\to L^q(\mathbb{R};B_2)} \tag{3.5}$$

According to (3.3),

$$(\widetilde{w}, \partial_t \widetilde{w})(t,x) = \int_0^t e^{i(t-s)A_0}\,(0, [\Delta_0, \chi]u(s,\cdot))\,ds \tag{3.6}$$

According to Theorem 3 (and Remark 2.3 since $\gamma \leq 1/2$ if $d = 2$) applied to the free wave operator, for any $\chi \in C_0^\infty(\mathbb{R}^d)$ the operator $T_0 = \chi e^{itA_0}$ maps $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ into $L^2(\mathbb{R}_t^+; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$ and consequently its adjoint

$$T_0^* G = \int_0^\infty e^{-isA_0}\chi G ds \tag{3.7}$$

maps $L^2(\mathbb{R}_t^+; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$ into $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ and since

$$(\widetilde{w}, \partial_t \widetilde{w}) = \int_0^t e^{i(t-s)A_0}\,(0, [\Delta_0, \chi]u)\,ds, \tag{3.8}$$

the global Strichartz estimate follows from Theorem 4 (here we are speaking about adjoints relative to the L^2 in time and $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$ in space norms).

We come back to the analysis of $v = \chi u$. Consider $\varphi \in C_0^\infty(]0,1[)$ equal to 1 on $[1/4, 3/4]$. Denote by $v_n = \varphi(t - n/2)v$ solution of

$$\left\{\begin{aligned} (\partial_t^2 - \Delta_g)v_n(t,x) &= -\varphi(t-n/2)[\Delta_g, \chi]u + [\partial_t^2, \varphi(t-n/2)]\chi u \\ &= u_n(t,x) \\ v_n \mid_{t<n/2} &= 0 \\ \partial_t v_n \mid_{t<n/2} &= 0 \\ v_{n|\partial\Omega} &= 0 \end{aligned}\right. \tag{3.9}$$

With, according to Theorem 3,

$$\sum_{n\in\mathbb{Z}} \|u_n\|^2_{L^2(\mathbb{R}_t; \dot{H}_D^{\gamma-1})} \leq C(\|f\|^2_{\dot{H}_D^\gamma} + \|g\|^2_{\dot{H}_D^{\gamma-1}}) \tag{3.10}$$

According to the (local in time) Strichartz inequality (1.2) (and Minkowski inequality), we obtain

$$\|v_n\|_{L_t^p(\mathbb{R}_t; L_x^q(\Omega))} + \|(v_n, \partial_t v_n\|_{L^\infty([0,1]_t; \dot{H}_D^\gamma \times \dot{H}_D^{\gamma-1})} \leq C\left(\|u_n\|_{L^1([-1+n/2, 2+n/2]_t; H_x^{\gamma-1}(\Omega))}\right) \tag{3.11}$$

Since

$$\|v\|^p_{L^p(\mathbb{R}_t; L_x^q(\Omega))} \sim \sum_{n\in\mathbb{Z}} \|v_n\|^p_{L^p(\mathbb{R}_t; L_x^q(\Omega))} \tag{3.12}$$

and since $p \geq 2$, we obtain, using (3.10) and (3.11)

$$\begin{aligned} \|v\|^p_{L^p(\mathbb{R}_t; L_x^q(\Omega))} &\leq \sum_{n\in\mathbb{Z}} \|v_n\|^p_{L^p(\mathbb{R}_t; L_x^q(\Omega))} \leq C \sum_{n\in\mathbb{Z}} \|u_n\|^p_{L^1(\mathbb{R}_t; H_x^{\gamma-1}(\Omega)} \\ &\leq C\big(\sum_{n\in\mathbb{Z}} \|u_n\|^2_{L^2(\mathbb{R}_t; H_x^{\gamma-1}(\Omega)}\big)^{p/2} \leq C(\|f\|^2_{\dot{H}_D^\gamma} + \|g\|^2_{\dot{H}_D^{\gamma-1}})^{p/2} \end{aligned} \tag{3.13}$$

which proves Theorem 2 for $F = 0$. As recognized by H. Smith and C. Sogge [10], another application of Theorem 4 gives the general case.

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