The Gaussian free field, Gibbs measures and NLS on planar domains

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Purpose

Study the dynamics of solutions of non linear Schrödinger equations in a planar domain, $M \subset \mathbb{R}^2$, with Dirichlet or Neumann boundary conditions,

$$(i\partial_t + \Delta)u - |u|^2 u = 0, u \mid_{t=0} = u_0, u \mid_{\partial M} = 0, (\text{resp. } \partial_\nu u \mid_{\partial M} = 0),$$

from a *statistical* point of view, i.e. u_0 is a random variable or (equivalently, endow the space of initial data with a *probability measure*, μ_0.

- Well posedness on the support of the measure (almost sure WP): definition of a flow $\Phi(t)$.
- Statistical properties of the measure propagated by the flow , $\mu(t) = \Phi(t)^*(\mu_0)$: continuity, recurence, growth of Sobolev norms, ...

The Gaussian free field

Consider a sequence of (complex) independent Gaussian random variables $\mathbf{g_k} \sim \mathcal{N}(0,1)$ (of law $\frac{1}{\pi} e^{-|z|^2} |dz|$)and for $e_k, (-\Delta + 1) e_k = \lambda_k^2 e_k, e_k \mid_{\partial M} = 0, \quad n \in \mathbb{N}^*$, the random variable

$$\mathbf{u_0} = \sum_{n \in \mathbb{N}^*} \frac{\mathbf{g_k}}{\lambda_k} e_k(x).$$

equivalently, consider the map

$$\omega \in (\Omega, \mathbf{p}) \mapsto \mathbf{u} = \sum_{n \in \mathbb{N}^*} \frac{\mathbf{g_k}(\omega)}{\lambda_k} e_k(x),$$

and endow the space of distributions $\mathcal{D}'(M)$, with the image of $\mathbf{p}$ by this map.

The GFF continued

equivalently, write

$$u = \sum_{n \in \mathbb{N}^*} u_k e_k(x),$$

identify $\mathcal{D}'(M)$ with the space of sequences $U = (u_k) \in \mathbb{C}^{\mathbb{N}^*}$ (satisfying some temperance growth conditions), and endow $\mathbb{C}^{\mathbb{N}^*}$ with the probability measure

$$\otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} e^{-\lambda_k^2 |u_k|^2} |du_k|.$$

Lemma

The GFF is for any $\epsilon > 0$ supported by $H^{-\epsilon}(M)$:

$$\mu_0(H^{-\epsilon}(M) = 1,$$

but

$$\mu_0(L^2(M)) = 0.$$

Wick re-ordering, Bourgain's result

If u is a solution to NLS, then $v = e^{-it(\|u\|^2_{L^2}-1)}u$ is a solution to

$$(i\partial_t + \Delta - 1)v - (|v|^2 - 2\|v\|^2_{L^2})v = 0, v\,|_{t=0} = u_0. \quad \text{(RNLS)}$$

Theorem (Bourgain 1996)

There exists $\delta > 0$ such that for μ_0-almost every u_0, there exists a unique (global in time) solution of (RNLS) on the torus $\mathbb{T}^2$, $u = \Phi(t)u_0$ in

$$e^{-it\Delta}u_0 + X^{\delta,1/2+}.$$

Furthermore there exists a function $G \in L^1(d\mu_0)$, positive on the support of μ_0 such that the measure $d\nu = G(u)d\mu$ is invariant by the flow $\Phi(t)$. ν is a Gibbs measure.

Recall $\|u\|_{X^{s,b}} = \|\langle D_t\rangle^b e^{-it\Delta}u\|_{L^2_t;H^s_x}$

Our result

Let $M \subset \mathbb{R}^2$ smooth bounded domain, and

$$: \|u\|_{L^4}^4 := \||u|^2 - 2\|u\|_{L^2}^2\|_{L^2}^2 - 4\|u\|_{L^2}^4.$$

Theorem (N.B. L. Thomann, N. Tzvetkov)

$e^{-:\|u\|_{L^4}^4}$ is μ_0 a.s. finite, and in $L^1(d\mu_0)$. Furthermore, there exists for μ_0 a.e. initial data u_0 a global in time solution to (RNLS), with Dirichlet (resp. Neumann) bdry conditions, and the flow $\Phi(t)$ satisfies

$$\Phi(t)^*(e^{-:\|u\|_{L^4}^4} d\mu_0) = e^{-:\|u\|_{L^4}^4} d\mu_0.$$

Rk 1: No uniqueness in our result: weak solutions. But contrarily to usual (deterministic) weak solutions some kind of uniqueness remains: for *any* such flow get same invariant measure

Rk 2 Work in progress: hope to get strong solutions.

Wick re-ordering on the torus

For $\mathbf{u} = e^{it(\Delta)}\mathbf{u}_0 = \sum_{n\in\mathbb{Z}^2} \frac{\mathbf{g_n}}{\langle n\rangle} e^{in\cdot x-|n|^2 t}$, $\langle n\rangle = \sqrt{|n|^2+1}$,

$$|\mathbf{u}|^2\mathbf{u} = \sum_{n_1,n_2,n_3} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} e^{i(n_1-n_2+n_3)\cdot x-(|n_1|^2-|n_2|^2+|n_3|^2)t}$$

$$= \sum_{n_1\neq n_2, n_3\neq n_2} \cdots + 2\sum_{n,m} \frac{|\mathbf{g}_n|^2\mathbf{g}_m}{\langle n\rangle^2\langle m\rangle} e^{im\cdot x-(|m|^2)t} - \sum_n \frac{|\mathbf{g}_n|^2\mathbf{g}_n}{\langle n\rangle^3} e^{in\cdot x-(|n|^2)t}$$

$$(|\mathbf{u}|^2 - 2\|u\|_{L^2}^2)\mathbf{u}$$

$$= \sum_{n_1\neq n_2, n_3\neq n_2} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} e^{i(n_1-n_2+n_3)\cdot x} - \sum_n \frac{|\mathbf{g}_n|^2\mathbf{g}_n}{\langle n\rangle^3} e^{in\cdot x}$$

Wick re-ordering on $\mathbb{T}^2$: analysis

Last term is a.s in H^{2-}:

$$\mathbb{E}(\| \sum_n \frac{|\mathbf{g}_n|^2 \mathbf{g}_n}{\langle n \rangle^3} e^{in\cdot x} \|_{H^\delta}^2) = \sum_{n\in\mathbb{Z}^2} \frac{\mathbb{E}(|\mathbf{g}_n|^6)}{\langle n \rangle^{6-2\delta}} < +\infty \text{ if } 6-2\delta > 2$$

First term is a.s. in $X^{1/2-,-1/2+}$. Indeed,

$$\mathbb{E}(\| \sum_{n_1\neq n_2, n_3\neq n_2} \cdots \|_{X^{s,-b}}^2)$$

$$= \mathbb{E}\Big(\sum_k \Big| \sum_{n_1\neq n_2, n_3\neq n_2, n_1-n_2+n_3=k} \frac{1}{(1+||n_1|^2-|n_2|^2+|n_3|^2-|k|^2|)^b} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} \langle k \rangle^s \Big|^2\Big)$$

$$= \sum_k \mathbb{E}\Big(\sum_{n_1\neq n_2, n_3\neq n_2} \sum_{m_1\neq m_2, m_3\neq m_2} \mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}\overline{\mathbf{g}_{m_1}}\mathbf{g}_{m_2}\overline{\mathbf{g}_{m_3}} \cdots \Big)$$

Since Gaussian are independent, complex and have mean equal to 0, and since $n_1 \neq n_2, n_3 \neq n_2$, expectancy vanishes unless

$$(n_1, n_2, n_3) = (m_1, m_2, m_3) \text{ or } (n_1, n_2, n_3) = (m_3, m_2, m_1)$$

$$\mathbb{E}(\| \sum_{n_1 \neq n_2, n_3 \neq n_2} \cdots \|^2_{X^{s,-b}}) \sim \sum_{\substack{n_1 \neq n_2, n_3 \neq n_2 \\ k = n_1 - n_2 + n_3}} \frac{\langle k \rangle^{2s}}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \times \frac{1}{(1 + ||n_1|^2 - |n_2|^2 + |n_3|^2 - |k|^2|)^{2b}}$$

$\sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^2}$ diverges logarithmically. Factor

$$\frac{1}{(1 + ||n_1|^2 - |n_2|^2 + |n_3|^2 - |k|^2|)^{2b}}$$

gives additional convergence which can be exchanged to compensate the $\langle k \rangle^{2s}$ term, hence end up in $X^{s,b}$, for some $s > 0$ a level at which Cauchy theory well posed! This (plus quite some work) proves Bourgain's Theorem.

The problem on a manifold

- Wick Reordering is in the folklore of quantum field theory. Best (most efficient) approach seems to be via Fock representation
- Fock representation seems to be not so well suited to $X^{s,b}$ analysis (possible development?)
- Take boundary into account (should not be a serious problem in the context of Fock representation though)
- Keep the *elementary* approach and understand at that level the compensations which allow for Wick re-ordering

Back to Wick re-ordering: The case of $\mathbb{S}^2$

Take e_n Hilbert base of spherical harmonics, eigenvalues λ_n^2.

$$
\begin{aligned}
|\mathbf{u}|^2\mathbf{u} &= \sum_{n_1,n_2,n_3} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} e^{-i(\lambda_{n_1}^2-\lambda_{n_2}^2+\lambda_{n_3}^2)t} e_{n_1}(x)e_{n_2}(x)e_{n_3}(x) \\
&= \sum_{n_1\neq n_2, n_3\neq n_2} \cdots + 2\sum_{n,m} \frac{|\mathbf{g}_n|^2\mathbf{g}_m}{\langle n\rangle^2\langle m\rangle} e^{-i\lambda_m^2 t}|e_n|^2(x)e_m(x) \\
&\qquad - \sum_{n} \frac{|\mathbf{g}_n|^2\mathbf{g}_n}{\langle n\rangle^3} e^{-i\lambda_n^2 t}|e_n|^2(x)e_n(x)
\end{aligned}
$$

As in the previous analysis, last term OK.
Main issues:

- $|e_n|^2(x) \neq 1$
- $e_{n_1}\overline{e_{n_2}}e_{n_3}$ is not an eigenfunction

The Weyl formula on $\mathbb{S}^2$

Let $E_k = \text{ Vect } \{e_n; \lambda_n^2 = k(k+1)\}$ an eigenspace, and $e_{n_1}, \cdots e_{n_{2k+1}}$ any orthonormal basis of E_k.

Proposition

Let $e(x,y,k) = \sum_{p=1}^{2k+1} e_{n_p}(x)\overline{e_{n_p}(y)}$.

$$\forall x \in \mathbb{S}^2, e(x,x,k) = \frac{2k+1}{Vol\ (\mathbb{S}^2)}.$$

In a mean value sense, the eigenfunctions are constant on $\mathbb{S}^2$

Proof: $e(x,y,k)$ is the kernel of the orthogonal projector on E_k. Hence for any isometry T, $e(x, Ty) = e(T^{-1}x, y)$.

Theorem (Van der Kam, Zelditch, N.B-G. Lebeau)

There exists orthonormal basis $(e_{n,k})$ *having for any* $p < +\infty$ *uniformly bounded* L^p *norms (actually most ONB are such)*

Back to Wick re-ordering, analysis of 2nd term

$$\sum_{n,m} \frac{|\mathbf{g}_n|^2 \mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} |e_n|^2(x) e_m(x)$$

$$= \sum_{n,m} \frac{(|\mathbf{g}_n|^2 - 1)\mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} |e_n|^2(x) e_m(x)$$

$$+ \sum_{k} \sum_{n, e_n \in E_k, m} \frac{\mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} |e_n|^2(x) e_m(x)$$

$$= I + \sum_{k} \sum_{n, e_n \in E_k, m} \frac{\mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} e_m(x)$$

$$= I + \sum_{n,m} \frac{|\mathbf{g}_n|^2 \mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} e_m(x)$$

$$+ \sum_{n,m} \frac{(1 - |\mathbf{g}_n|^2)\mathbf{g}_m}{\langle n\rangle^2 \langle m\rangle} e^{-i\lambda_m^2 t} e_m(x) = I + II + III$$

$$II = \|e^{it\Delta}u\|_{L^2}^2 e^{it\Delta}u$$

$$I + III = \sum_n \frac{(|\mathbf{g}_n|^2 - 1)}{\langle n\rangle^2}(|e_n|^2(x) - 1)e^{it\Delta}u$$

and $\sum_n \frac{(|\mathbf{g}_n|^2-1)}{\langle n\rangle^2}(|e_n|^2(x) - 1)$ is a.s finite (renormalizable). Indeed, since

$$n \neq m \Rightarrow \mathbb{E}\big((|\mathbf{g}_n|^2-1)(|\mathbf{g}_m|^2-1)\big) = \mathbb{E}(|\mathbf{g}_n|^2-1)\mathbb{E}(|\mathbf{g}_m|^2-1) = 0,$$

$$\mathbb{E}\Big(\Big|\sum_n \frac{(|\mathbf{g}_n|^2 - 1)}{\langle n\rangle^2}(|e_n|^2(x) - 1)\Big|^2\Big)$$

$$= \sum_n \mathbb{E}\Big(\sum_n \frac{(|\mathbf{g}_n|^2 - 1)^2}{\langle n\rangle^4}(|e_n|^2(x) - 1)^2\Big)$$

$$\sim \sum_n \frac{1}{\langle n\rangle^4} < +\infty$$

Back to Wick reordering: analysis of the first term

With $\gamma(n_1, n_2, n_3, p)) = \int_{\mathbb{S}^2} e_{n_1}\overline{e_{n_2}}e_{n_3}\overline{e_p}dx$,

$$\mathbb{E}\Big(\| \sum_{\substack{n_1\neq n_2\\ n_3\neq n_2}} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} e^{-i(\lambda_{n_1}^2-\lambda_{n_2}^2+\lambda_{n_3}^2)t} e_{n_1}(x)e_{n_2}(x)e_{n_3}(x)\|_{X^{s,b}}^2\Big)$$

$$= \mathbb{E}(\sum_{k,p} \frac{\langle p\rangle^{2s}}{\langle k\rangle^{2b}}\Big| \sum_{\substack{n_1\neq n_2, n_3\neq n_2\\ |k-\lambda_{n_1}^2+\lambda_{n_2}^2-\lambda_{n_3}^2|\in[0,1)}} \frac{\mathbf{g}_{n_1}\overline{\mathbf{g}_{n_2}}\mathbf{g}_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle}\gamma(n_1, n_2, n_3, p)\Big|^2)$$

$$\sim \sum_p \sum_{\substack{n_1\neq n_2\\ n_3\neq n_2}} \frac{\langle p\rangle^{2s}}{\langle \lambda_{n_1}^2+\lambda_{n_2}^2-\lambda_{n_3}^2\rangle^{2b}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}|\gamma(n_1, n_2, n_3, p))|^2$$

Difference with $\mathbb{T}^2$: additional sum (in p), but presence of γ^2, which is invariant wrt permutations and in l^1: choice of bases,

$$\sum_p |\gamma(n_1, n_2, n_3, p))^2 = \|e_{n_1}e_{n_2}e_{n_3}\|_{L^2(M)}^2 \leq C < +\infty$$

General manifolds

Use Weyl formula (Volume of manifold normalized to 1)

$$e(x, \lambda, \mu) = \sum_{\mu \leq \lambda_n < \lambda} |e_n|^2(x)$$

Theorem (Hormander Th 17.5.10)

Let $d(x)$ be the distance of the point $x \in M$ to the boundary ∂M. There exists $C > 0$ such that for any $\lambda > 1$ and $x \in M$ satisfying $d(x, \partial M) \geq \lambda^{-1/2}$, any $d \in [0, 1]$, we have

$$|e(x, \lambda + d\lambda^{1/2}, \lambda) - \frac{d}{2\pi}\lambda^{3/2}| \leq C\lambda, \tag{1}$$

Theorem (Sogge)

There exists $C > 0$ such that for any $\lambda > 1$ and $x \in M$

$$|e(x, \lambda + 1, \lambda) \leq C\lambda, \tag{2}$$

The strategy of proof

- Regularize the system by cutting in the non linearity the frequencies higher than N ($\lambda_n > N$) and get global in time solutions. Get a flow $\Phi_N(t)$.
- Define a family $\nu_N = e^{-\frac{1}{2}:\|u_N\|_{L^4}^4:} d\mu_0$ of probability measures invariant by the flows Φ_N.
- Show that these measures converge (in a sense to be precised) to a limit measure $\nu_0 = e^{-\frac{1}{2}:\|u\|_{L^4}^4:} d\mu_0$
- Pass to the limit (in a sense to be precised) in the family of solutions $\Phi_N(t)u_0 \to \Phi(t)u_0$
- Show that the flow $\Phi(t)$ solves (RNLS) and leaves the measure ν_0 invariant
- Strategy quite standard in parabolic settings (see e.g. Da Prato-Debussche),

The approximating systems

Let Π_N be the orthogonal projector on the space

$$\text{Vect}\,(e_n; \lambda_n \leq N).$$

Let $\Phi_N(t)u_0$ be the solution of

$$(i\partial_t + \Delta)u - \Pi_N\big((|\Pi_N(u)|^2 - 2\|\Pi_N(u)\|_{L^2}^2)\Pi_N(u)\big) = 0,$$
$$u\mid_{t=0} = u_0, \qquad u\mid_{\partial M} = 0, \quad (\text{resp.} \quad \partial_\nu u\mid_{\partial M} = 0)$$

Hamiltonian system with Hamiltonian

$$H = \int_M \frac{1}{2}|\nabla_x u|^2 dx + \frac{1}{4}\|\Pi_N(u)\|_{L^4}^4 - \frac{1}{2}\|\Pi_N(u)\|_{L^2}^4$$

Formally, the GFF μ_0 is, in the coordinate system given by the identification $u = \sum_n u_n e_n(x)$ given by

$$d\mu_0 = \otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} e^{-\lambda_k^2 |u_k|^2} |du_k| \text{ " = " } \prod_{k=1}^{+\infty} e^{-\lambda_k^2 |u_k|^2} \otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} |du_k|$$

$$\text{" = " } e^{-\sum_k \lambda_k^2 |u_k|^2} \otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} |du_k| \text{" = " } e^{-\|\nabla_x\|_{L^2}^2} \otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} |du_k|,$$

and

$$\nu_N = e^{-\frac{1}{2} :\|u_N\|_{L^4}^4:} d\mu_0 \text{ " = " } e^{-2H(u)} \otimes_{k=1}^{+\infty} \frac{\lambda_k^2}{\pi} |du_k|$$

is (at least formally) invariant (because the Hamiltonian itself is invariant and a Hamiltonian system can be seen as an ODE with a divergence free vector field hence the (infinite product of) Lebesgue measures is also invariant

Passing to the limit $\nu_N \rightharpoonup \nu$

Definition

S separable complete metric space, $(\rho_N)_{N\geq 1}$ probability measures on Borel $\sigma-$algebra $\mathcal{B}(S)$. (ρ_N) on $(S, \mathcal{B}(S))$ is tight if $\forall \varepsilon > 0, \exists K_\varepsilon \subset S$ compact such that $\rho_N(K_\varepsilon) \geq 1 - \varepsilon$ for all $N \geq 1$.

Theorem (Prokhorov)

$(\rho_N)_{N\geq 1}$ is tight iff it is weakly compact, i.e. there is a subsequence $(N_k)_{k\geq 1}$ and a limit measure ρ_∞ such that for every bounded continuous function $f : S \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_S f(x) d\rho_{N_k}(x) = \int_S f(x) d\rho_\infty(x).$$

Rk. Weak convergence implies convergence in law

A tight sequence of probability measures

Let $T > 0$ and define a probability measure on the space of (space time) functions $f(t, x)$ p_N by the image measure of ν_N by the map

$$u_0 \mapsto \Phi_N(t) u_0.$$

Proposition

The sequence p_N is for any $\epsilon > 0$ tight on $C^0([0, T]; H^{-\epsilon}(M))$.

Proof: Fix $0 < \epsilon' < \epsilon$. the measure ν_0 is supported by $H^{-\epsilon'}$ which embeds compactly in $H^{-\epsilon}$. This gains compactness in space. Then use equation to gain compactness in time.

Passing to the limit $\Phi_N(t)$ " $\rightarrow$ " $\Phi(t)$

Setting: we have a family of r.v. $\mathbf{X}_{\mathbf{N_k}} = \Phi_{N_k}(t)\mathbf{u_0}$ such that the laws of $\mathbf{X}_{\mathbf{N_k}}$, p_{N_k} are weakly convergent to a probability measure p. We'd like to deduce that the r.v. X_{N_k} are a.s. convergent. This is False (take $\mathbf{X}_{\mathbf{N_k}} = (-1)^{N_k}\mathbf{X}$). However

Theorem (Skorohod)

Assume that S is a separable metric space. Let $(\rho_N)_{N\geq 1}$ and ρ_∞ be probability measures on S. Assume that $\rho_N \longrightarrow \rho_\infty$ weakly. Then there exists a probability space on which there are $S-$valued random variables $(\mathbf{Y}_\mathbf{N})_{N\geq 1}$, $\mathbf{Y}_\infty$ such that $\mathcal{L}(\mathbf{Y}_\mathbf{N}) = \rho_N$ for all $N \geq 1$, $\mathcal{L}(\mathbf{Y}_\infty) = \rho_\infty$ and $\mathbf{Y}_\mathbf{N} \longrightarrow \mathbf{Y}_\infty$ a.s.

Passing to the limit in the equation I

Need first to check that the r.v. $\mathbf{Y_N}$ satisfy the same equation as $\mathbf{X_N} = \Phi_N(t)$: Consider the r.v.

$$\mathbf{Z_N} = ((i\partial_t + \Delta)\mathbf{Y_N} - \Pi_N\big((|\Pi_N(\mathbf{Y_N})|^2 - 2\|\Pi_N(\mathbf{Y_N})\|^2_{L^2})\Pi_N(\mathbf{Y_N})\big)$$

All the functions appearing in the r.h.s. are continuous from $C^0((0,T);H^{-\epsilon})$ to $S = H^{-1}((0,T);H^{-2-\epsilon})$. Hence

$$\begin{aligned}
&\mathcal{L}(\mathbf{Z_N})\\
&= \mathcal{L}\Big((i\partial_t + \Delta)\mathbf{Y_N} - \Pi_N\big((|\Pi_N(\mathbf{Y_N})|^2 - 2\|\Pi_N(\mathbf{Y_N})\|^2_{L^2})\Pi_N(\mathbf{Y_N})\big)\Big)\\
&= \mathcal{L}\Big((i\partial_t + \Delta)\mathbf{X_N} - \Pi_N\big((|\Pi_N(\mathbf{X_N})|^2 - 2\|\Pi_N(\mathbf{X_N})\|^2_{L^2})\Pi_N(\mathbf{X_N})\big)\Big)\\
&\qquad = \mathcal{L}(\mathbf{0}) = \delta_0
\end{aligned}$$

Hence $\mathbf{Z_N} = 0$ a.s.

Passing to the limit in the equation II

Need to pass to the limit in

$$\Pi_N\big((|\Pi_N(\mathbf{X_N})|^2 - 2\|\Pi_N(\mathbf{X_N})\|_{L^2}^2)\Pi_N(\mathbf{X_N})\big).$$

Idea is: almost sure convergence in $H^{-\epsilon}$, and estimate in probability of a "stronger term" namely : $\|u\|_{L^4}^4$:. Gives convergence in probability of the non linear term. Hence convergence a.s. for a subsequence.

Toward a generalization of Bourgain's result on any smooth bounded domain?

- Need a "nice" local Cauchy theory at the level of regularity where the measure is supported (rest = more or less automatic)
- Standard ($X^{s,b}$) WP thresholds for cubic NLS:
  - $\mathbb{T}^2$ rational $s > 0$ (Bourgain)
  - $\mathbb{S}^2$ $s > \frac{1}{4}$ (N. B, Gérard, Tzvetkov)
  - $\mathbb{T}^2$ irrational $s > \frac{1}{3}$ (Catoire Wang)
  - Compact surfaces (without bdry) $s > \frac{1}{2}$ (N. B-Gérard-Tzvetkov)
  - Compact surfaces (with bdry) $s > \frac{2}{3}$ (Anton, Blair-Smith-Sogge)
- Control 1st iteration in $X^{s,1/2+0}, s > s_c$
- Perform $X^{s,b}$ analysis in the Fock representation