Low regularity water-waves: canals and swimming pools

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The equations

Incompressible inviscid liquid, having unit density, occupying a moving domain, which at time $t \geq 0$ is of the form

$$\Omega(t) = \{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y < \eta(t, x) \}.$$

η is an unknown. We denote by $\Sigma(t)$ the free surface

$$\Sigma(t) = \{ (x, y) \in \mathbb{R}^d \times \mathbb{R} : y = \eta(t, x) \}.$$

The eulerian velocity field $v \colon \Omega \to \mathbb{R}^3$ solves the incompressible Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \quad \nabla_{x,y} \cdot v = 0 \quad \text{in } \Omega, \tag{1}$$

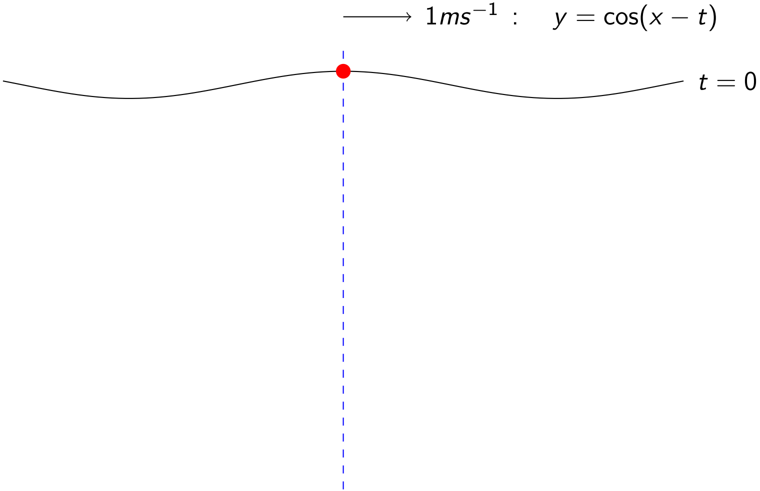
where P is the pressure and $-ge_y$ is the acceleration of gravity ($g > 0$). The domain Ω is deformed by the fluid movement so that the particles of fluid at the interface stay at the interface.

$$\partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, v \cdot n \text{ on } \Sigma(t), \qquad n \text{ normal to } \Sigma(t)$$

There is no surface tension:

$$P = 0 \text{ on } \Sigma(t).$$

Russell : "...the observer near the shore perceives that pieces of wood, or any floating bodies immersed in the water near its surface, and the water in their vicinity, are not carried towards the shore with the rapidity of the wave, but are left nearly in the same place after the wave has passed them..."



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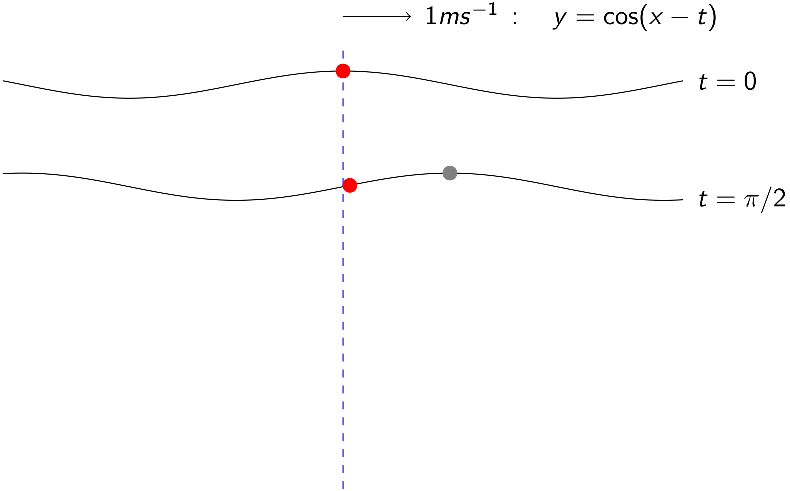

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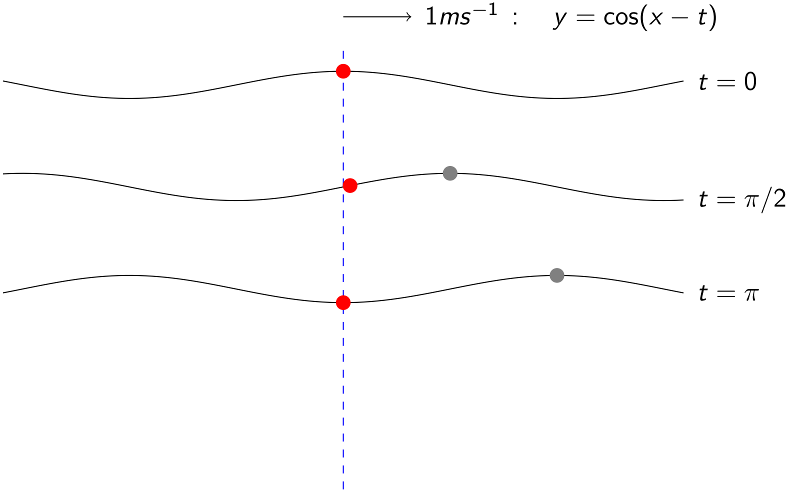

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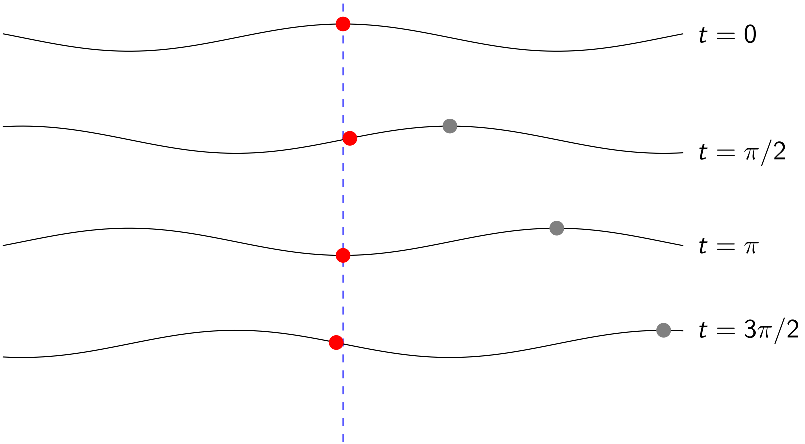
$\longrightarrow 1ms^{-1}$: $\quad y = \cos(x - t)$



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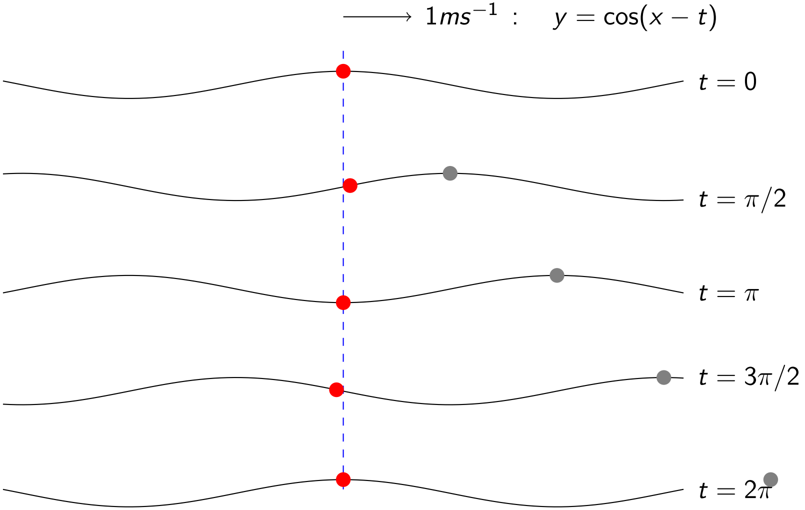

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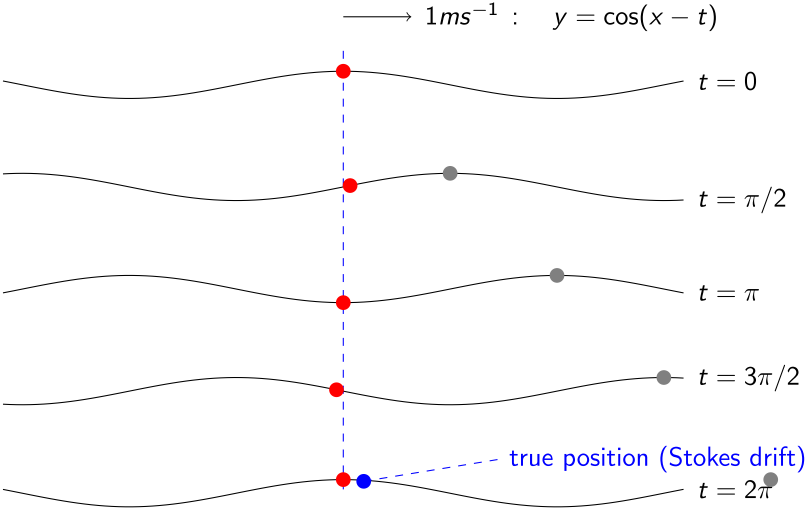

The linearized equation

We have

$$v = \nabla_x \Phi, \qquad \Delta \Phi = 0 \text{ in } \Omega_t, \qquad \Phi\mid_\Sigma = \Psi.$$

The linearized system around $(\Psi = 0, \eta = 0)$ is

$$\begin{cases} \partial_t \eta - |D_x|\, \psi = 0, & \text{where } \psi(t,x) = \phi(t,x,0), \\ \partial_t \psi + g\eta = 0, \end{cases}$$

and hence introducing $u = |D_x|^{\frac{1}{2}}\, \psi + i\sqrt{g}\eta$ we find :

$$\partial_t u + i\sqrt{g}\, |D_x|^{\frac{1}{2}}\, u = 0.$$

Dispersive waves

Basic computation: $u(t,x) = \exp(i(k\cdot x - \omega t))$ solves

$$\partial_t u + i\sqrt{g}\,|D_x|^{\frac{1}{2}}\,u = 0$$

iff $\omega = |k|^{\frac{1}{2}}$.
η_1 propagates at velocity

$$|c| = \frac{|\omega|}{|k|} = \sqrt{\frac{g}{|k|}}.$$

Waves associated to different frequencies propagate at different speeds.
Compare with

Schrödinger:	$\partial_t u + i\,|D_x|^2\,u = 0$	(decay in t^{-1} for $x \in \mathbb{R}^2$)
Wave eq. :	$\partial_t u + i\,|D_x|\,u = 0$	(decay in $t^{-1/2}$ for $x \in \mathbb{R}^2$)
Capillary WW eq :	$\partial_t u + i\,|D_x|^{3/2}\,u = 0$	(decay in t^{-1} for $x \in \mathbb{R}^2$)
Gravity WW eq :	$\partial_t u + i\,|D_x|^{1/2}\,u = 0$	(decay in t^{-1} for $x \in \mathbb{R}^2$)

Lipschitz regularity

- Work in Sobolev spaces.
- Equation for "fluid particles" which are curves $M: [0, T] \to \bar{\Omega}$ satisfying

$$\frac{d}{dt} M(t) = v(t, M(t)), \quad M(0) = (x, y) \in \bar{\Omega},$$

  where v is the eulerian fluid velocity. To solve this ODE one needs $v \in \mathrm{Lip}$.
- Q1 : Prove well posedness under assumptions (in Sobolev spaces) which only ensure that the initial velocity has Lipschitz regularity. Main difficulty: study the elliptic PDE

$$\Delta_{x,y} \phi = 0 \quad \text{in } \Omega = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < \eta(x)\},$$

  in domains with Lipschitz boundary
- The equation $\partial_t + i|D_x|^{1/2}$ enjoys dispersive properties (Strichartz estimates)
- Q2 Prove well posedness under assumptions (in Sobolev spaces) which DO NOT ensure that the initial velocity has Lipschitz regularity This kind of property is typical for quasi-linear wave equations.

Theorem

Let $d \geq 1, s > 1 + \frac{d}{2}$. *Let* $\eta_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ *and let* $\Omega_0 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < \eta_0(x)\}$. *Let* v_0 *be such that* $div\ v_0 = 0$ *and* $curl\ v_0 = 0$ *in* Ω_0 *and* $v_0|_{\partial\Omega_0} \in H^s(\mathbb{R}^d)$.
Then there exist $T > 0$ *and a unique solution* (η, v) *of the gravity water wave system on* Ω *where*

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} : y < \eta(t, x)\}$$

such that $\eta \in C^0([0, T], H^{s+\frac{1}{2}}(\mathbb{R}^d))$ *and* $t \mapsto v|_{\partial\Omega(t)} \in C^0([0, T], H^s(\mathbb{R}^d))$.

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Theorem

Let $d = 2, s > 1 + \frac{d}{2} - \frac{1}{12}$. *Let* $\eta_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ *Let* v_0 *be such that* $\operatorname{div} v_0 = 0$, $\operatorname{curl} v_0 = 0$ *in* Ω_0 *and* $v_0|_{\partial\Omega_0} \in H^s(\mathbb{R}^d)$ *Then there exist* $T > 0$ *and a solution* (η, v) *of the gravity water wave system on*

$$\Omega = \{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} : y < \eta(t,x)\}$$

with $\eta \in C^0([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d))$ *and* $t \mapsto v|_{\partial\Omega_t} \in C^0([0,T], H^s(\mathbb{R}^d))$.
Moreover we have uniqueness in a space X_T *imbedded in* $C^0([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d)) \times C^0([0,T], H^s(\Omega_t))$.

- Pionneering works: Nalimov, Ovsjannikov, Yosihara, Craig, Shinbrot, Beale, T. Hou & J. Lowengrub .
- Geometric analysis: Wu, Christodoulou–Lindblad, Lindblad (vectors fields), Shatah–Zeng (covariant derivative, parallel transport), Coutand–Shkoller (lagrangian identities), Ambrose-Masmoudi.
- Tricky analysis: Beyer-Günther
- Normal forms: Wu, Germain–Masmoudi–Shatah (global existence results for small localized 3D waves), Iooss–Plotnikov.
- Singular integrals analysis: Here we follow the approach initiated by Craig–Schanz–Sulem and further developped by Lannes, Ming-Zhang, Iooss–Plotnikov and Alazard-Métivier. And also works by Alinhac on shocks.

Paraproducts

Let $\psi \in C_0^\infty(\mathbb{R}^d)$, ϕ in $C_0^\infty(\{1/2 < |\xi| < 2)$; $\psi(\xi) + \sum_{j=0}^{+\infty} \phi(2^{-j}\xi) = 1$.

$$\Delta_{-1} a = \mathcal{F}^{-1}(\psi \hat{a}), \quad \Delta_j a = \mathcal{F}^{-1}(\phi(2^{-j}\xi)\hat{a}), \quad j \geq 0, \qquad S_j(a) = \sum_{k=-1}^{j-1} \Delta_k a.$$

so $a = \sum_{j=-1}^{+\infty} \Delta_j a$ and the para multiplication operator is defined by

$$T_a u = \sum_{j \geq 2} S_{j-1}(a) \Delta_j u.$$

$$ab = T_a b + T_b a + R(a,b), \qquad R(a,b) = \sum_{|i-j| \leq 1} \Delta_j a \Delta_i b$$

Theorem (J.M. Bony; paralinearization of a product)

$$\forall \sigma \in \mathbb{R} \qquad a \in L^\infty, \qquad b \in H^\sigma \qquad \Rightarrow \qquad T_a b \in H^\sigma,$$

$$\forall s > 0, \qquad a \in H^s, \qquad b \in H^s \qquad \Rightarrow \qquad R(a,b) \in H^{2s - \frac{d}{2}}.$$

Paradifferential operators

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta)\widehat{a}(\xi - \eta, \eta)\widehat{u}(\eta)\, d\eta, \tag{2}$$

where $\widehat{a}(\theta, \eta) = \int e^{-ix\cdot\theta} a(x, \eta)\, dx$ and where, for some $\varepsilon_1, \varepsilon_2$ small enough,

$$\begin{aligned} \chi(\theta, \eta) &= 0 \quad \text{for} \quad |\xi| \leq 1, \\ \chi(\theta, \eta) &= 1 \quad \text{for} \quad |\theta| \leq \varepsilon_1 |\xi|, \\ \chi(\theta, \eta) &= 0 \quad \text{for} \quad |\theta| \geq \varepsilon_2 |\xi|, \end{aligned}$$

$\chi(\theta, \eta)$ is homogeneous of degree 0.

Definition

The paradifferential operator T_a of Bony is defined by symbol smoothing:

$$T_a = \mathrm{Op}(\sigma) \quad \textit{with } \widehat{\sigma}(\theta, \eta) = \chi(\theta, \eta)\widehat{a}(\theta, \eta).$$

For $\chi = 1$ and $b = a(x)\xi^\alpha$ then $(T_b u)(x) = T_a D_x^\alpha u$.
For $\chi = 1$ and $a = a(x, \xi)$ then $T_a = \mathrm{Op}(a)$:

$$(T_a u)(x) = (2\pi)^{-d} \int e^{ix\cdot\eta} a(x, \eta)\widehat{u}(\eta)\, d\xi.$$

Symbolic calculus

Definition

The class Γ^m_ρ *consists of these symbols* $a = a(x,\xi)$ *of order* m *with limited regularity* C^ρ *in* x, *such that*

$$\forall \alpha \in \mathbb{N}^d, \exists K_\alpha, \forall |\xi| \geq \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{C^\rho} \leq K_\alpha (1+|\xi|)^{m-|\alpha|}.$$

Example: $\eta \in C^1$ implies $\lambda \in \Gamma^1_0$.

Theorem

Let $m \in \mathbb{R}$ *and* $\rho \in [0,1]$.
(i) If $a \in \Gamma^m_0$, *then* T_a *is bounded from* H^s *to* H^{s-m}, $\forall s$
(ii) If $a \in \Gamma^m_\rho, b \in \Gamma^{m'}_\rho$ *then* $T_a T_b - T_{ab}$ *is bounded from* H^s *to* $H^{s-(m+m'-\rho)}$, $\forall s$.

Back to the equations (Zakharov reduction)

$$\begin{cases} \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, & \text{in } \Omega_\eta = \{(x,y); y < \eta(x,t)\}, \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, v \cdot \nu, & \nu \text{ the exterior normal to } \partial\Omega_\eta \end{cases} \quad (3)$$

$$\operatorname{div} v = \operatorname{curl} v = 0 \Rightarrow u = \nabla_x \phi, \qquad \Delta \phi = 0$$

Define $\Psi = \phi \mid_\Sigma = \phi(t, x, \eta(t, x))$. Then the system reads

$$\begin{cases} \partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{\left(\nabla \eta \cdot \nabla \psi + G(\eta)\psi\right)^2}{1 + |\nabla \eta|^2} = 0. \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\, G(\eta)\Psi \end{cases} \quad (4)$$

Where $G(\eta)$ is the Dirichlet-Neumann operator in Ω_η:

$$\Delta \phi = 0 \text{ in } \Omega_\eta, \qquad \phi \mid_{\Sigma_\eta} = \phi(x, \eta(x)) = \Psi(x), \qquad G(\eta)\Psi = \partial_\nu \phi \mid_{\Sigma_\eta}$$

Rk: Zakharov framework is too crude for our purpose (due to the lack of smoothness of η), but it gives a rather good ideas of the difficulties we have to solve.

A nonlinear system

- whose coefficients are given by solving elliptic equations in domain with rough boundaries.
- whose linearized system around the 0 solution can be written as

$$\partial_t u + i\,|D_x|^{\frac{1}{2}}\, u = 0.$$

We shall show a similar diagonalization for the nonlinear equations, by introducing a "good unknown" u which solves a dispersive equation of the form

$$\partial_t u + T_V \cdot \nabla u + i T_\gamma u = f,$$

where V is the horizontal component of the velocity, T_V is a paraproduct and T_γ is a paradifferential operator of order $1/2$.
For 2D waves,

$$T_\gamma = T_{\sqrt{\mathfrak{a}(t,x)}}\, |D_x|^{\frac{1}{2}},$$

where $\mathfrak{a}$ is the so-called Taylor's coefficient.

The Dirichlet-Neumann operator

Using the Fourier transform, it is easily seen that $G(0) = |D_x|$.

More generally, if $\eta \in C^\infty$, then $G(\eta)$ is an elliptic ΨDO of order 1, self-adjoint, whose principal symbol is

$$\lambda(x,\xi) := \sqrt{(1 + |\nabla\eta(x)|^2)\,|\xi|^2 - (\nabla\eta(x)\cdot\xi)^2}.$$

More precisely,

$$G(\eta)f = \mathrm{Op}(\lambda)f + R_0(\eta)f,$$

where the remainder satisfies

$$\exists K \geq 1, \forall s \geq 0, \quad \|R_0(\eta)\psi\|_{H^s} \leq C\left(\|\eta\|_{H^{s+K}}\right)\|\psi\|_{H^s}.$$

Remark (Dichotomy between $d = 1$ and $d = 2$)

If $d = 1$ or $\eta = 0$ then $\lambda(x,\xi) = |\xi|$ and

$$\mathrm{Op}(\lambda) = |D_x|.$$

λ is well-defined for any $\eta \in C^1$. Aim : compare $G(\eta)$ to the para-differential operator T_λ. Namely we want to estimate the operator

$$R(\eta) = G(\eta) - T_\lambda.$$

Proposition (The D-N for "smooth domains")

If $s > 2 + d/2$ *then*

$$\|R(\eta)f\|_{H^s} \leq C\left(\|\eta\|_{H^{s+1}}\right)\|f\|_{H^s},$$

Paralinearization of the DNO

Theorem

Let $d \geq 1$ *and* $\eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \cap C^{\frac{3}{2}}(\mathbb{R}^d)$ *and* $f \in H^s(\mathbb{R}^d) \cap C^r(\mathbb{R}^d)$ *with*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

Then

$$R(\eta)f = G(\eta)f - T_\lambda f \in H^{s-\frac{1}{2}}(\mathbb{R}^d).$$

Moreover

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\right)\left\{1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C^r}\right\}, \qquad (5)$$

for some continuous function $C\colon (\mathbb{R}^+)^2 \to \mathbb{R}^+$ *depending only on* s *and* r *.*

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Moreover

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for some continuous function $C \colon (\mathbb{R}^+)^2 \to \mathbb{R}^+$ *depending only on* s *and* r *.*

Tame estimate : To compare, say, the norms $\left\|\cdot\right\|_{H^{s+\frac{1}{2}}}$ and $\left\|\cdot\right\|_{C^{3/2}}$, for $s < 1 + d/2$, we notice

$$\left\| u\left(\frac{x}{\varepsilon}\right) \right\|_{C^{\frac{3}{2}}} \sim \left(\frac{1}{\varepsilon}\right)^{3/2} >> \left(\frac{1}{\varepsilon}\right)^{s+\frac{1}{2}-\frac{d}{2}} \sim \left\| u\left(\frac{x}{\varepsilon}\right) \right\|_{H^{s+\frac{1}{2}}}.$$

Reduction

Set

$$\zeta = \nabla\eta, \quad V = (v_1, v_2)|_\Sigma, \quad B = v_3|_\Sigma, \quad \mathfrak{a} = -\partial_y P|_\Sigma.$$

γ depends on λ (hence η) and $\mathfrak{a}$.
$\mathfrak{a}$ is given " implicitly" by solving an elliptic boundary value problem.

Proposition (Main reduction)

The scalar complex-valued unknown $u \in C^0([0,T]; H^{s-\frac{1}{2}}(\mathbb{R}^d))$ *defined by*

$$u = T_{\sqrt{a}}\zeta + iT_{\sqrt{\lambda}}(V + T_\zeta B),$$

satisfies

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = F,$$

where

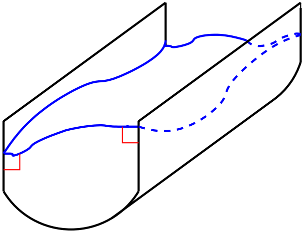
$$\gamma = \sqrt{\mathfrak{a}}\sqrt{\lambda},$$

and

$$\|F\|_{L^\infty(0,T;H^{s-\frac{1}{2}})} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V,B)\|_{H^s}\right)\{1 + \|\eta\|_{C^{3/2}} + \|(V,B)\|_{C^r}\}.$$

Applications: A canal

3D waves in a canal with vertical walls near the free surface: same equations with non-penetration boundary conditions on the boundary: $v \cdot n = 0$, n normal to the boundary of the canal



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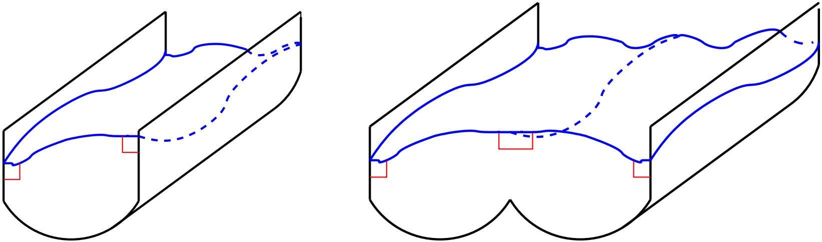

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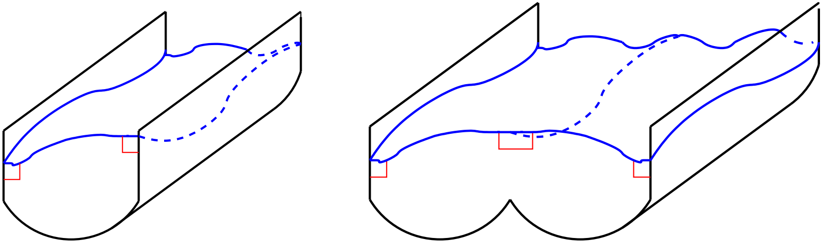
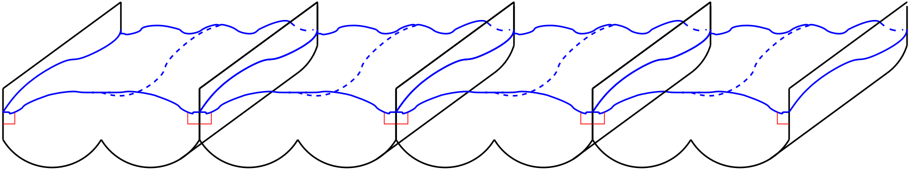


After symmetry and periodization : $(\eta, V) \mapsto (\underline{\eta}, \underline{V})$ the symmetric/periodized:



The canal or a swimming pool

- Ensure that the symmetry/periodization process preserves the Sobolev norms. Right angles at the interface (physical observation, automatically satisfied) imply that $\partial_{x_1}\eta(0, x_2) = 0$, and hence

$$\eta \in H^s((0,1)_{x_1} \times \mathbb{R}_{x_2}) \Rightarrow \underline{\eta} \in H^s((0,2)_{x_1} \times \mathbb{R}_{x_2})$$

$$\eta \in C^\sigma((0,1)_{x_1} \times \mathbb{R}_{x_2}) \Rightarrow \underline{\eta} \in C^\sigma((0,2)_{x_1} \times \mathbb{R}_{x_2})$$

  As long as $s < 7/2$, $\sigma < 3$ Here assumptions are $\eta_0 \in H^{s+\frac{1}{2}}, s > 1 + \frac{d}{2}$, $d = 2$!
- similar calculations for $\underline{V}$.
- Work with rough bottom. Here even though initially the bottom is smooth, after symetry/periodization no more the case
- Work with $\mathbb{T} \times \mathbb{R}$ (canal) or $\mathbb{T}^2$ (swimming pool) instead of $\mathbb{R}^2$.

To prove the right angle at interface,

$$[\partial_t v + (v \cdot \nabla_{x,y})v = -\nabla_{x,y}P] \cdot n(x,y) \Rightarrow -\nabla_{x,y}P \cdot n(x,y) = 0$$

but $P\,|_\Sigma = 0$. Hence $\nabla_{x,y}P$ is (proportional to) the normal to Σ, ν.

Strichartz estimates a brief overview

Lemma

For any $\epsilon > 0$, *there exists* $C > 0$ *such that for any* $u_0 \in H^s(\mathbb{R}^2)$,

$$\text{Strichartz } \|e^{it|D_x|^{1/2}} u_0\|_{L^2((0,1);L^\infty(\mathbb{R}^2))} \leq C\|u_0\|_{H^{\frac{3}{4}+\epsilon}}$$

$$\text{Sobolev } \|e^{it|D_x|^{1/2}} u_0\|_{L^\infty((0,1);L^\infty(\mathbb{R}^2))} \leq CC\|u_0\|_{H^{1+\epsilon}}$$

- Littlewood-Paley decomposition Decompose using a partition of unity in Fourier, with $\chi_n(t) = \chi(2^{-n}t), n \geq 1$

$$u = \sum_n \Delta_n(u)$$

prove estimates uniform w.r.t. $h = 2^{-n}$

$$\|e^{it|D_x|^{1/2}} \chi(h|D_x|) u_0\|_{L^2((0,1);L^\infty(\mathbb{R}^2))} \leq Ch^{-\frac{3}{4}+\epsilon}\|u_0\|_{L^2}$$

Reduction to a dispersive estimate

TT^* argument :

- $T = e^{it|D_x|^{1/2}}\chi(h|D_x|)$ from L^2_x to L^2_t, L^∞_x bounded by $Ch^{-\frac{3}{4}+\epsilon}$
- iff T^* from L^2_t, L^1_x to L^2_x bounded by $Ch^{-\frac{3}{4}+\epsilon}$
- iff TT^* from L^2_t, L^1_x to L^2_x to L^2_t, L^∞_x bounded by $Ch^{-\frac{3}{2}+2\epsilon}$

Dispersion ($L^1_x \to L^\infty_x$) estimate

$$TT^* f = e^{it|D_x|^{1/2}} \int_0^1 e^{-is|D_x|^{1/2}} f(s, \cdot) ds$$

estimate above OK from Hardy-LIttlewood-Sobolev inequality if

$$\|\chi(h|D_x|)e^{i(t-s)|D_x|^{1/2}}\chi(h|D_x|)\|_{L^1_x \to L^\infty_x} \leq \frac{C}{h^{\frac{3}{2}}|t-s|}$$

The parametrix: using Fourier analysis

$$\chi(h|D_x|)e^{i(t-s)|D_x|^{1/2}}\chi(h|D_x|)u_0$$
$$= \frac{1}{(2\pi)^2}\int e^{i(x-y)\cdot\xi+t|\xi|^{1/2}}\chi^2(h|\xi|)u(y)dyd\xi$$
$$= \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}\left((x-y)\cdot\eta+th^{1/2}|\eta|^{1/2}\right)}\chi(\eta)u(y)dyd\eta$$

- Phase $\phi(\eta) = \left((x-y)\cdot\eta + th^{1/2}|\eta|^{1/2}\right)$
- Critical point $(x-y) + th^{1/2}\eta/(2|\eta|^{3/2}) = 0$
- Hessian $Hess\phi \geq cth^{1/2}$

$$\|\chi(h|D_x|)e^{i(t-s)|D_x|^{1/2}}\chi(h|D_x|)\|_{L^1\to L^\infty} = \|K(x,y,t)\|_{L^\infty}$$
$$\leq \frac{C}{h^2}\times\left(\frac{h}{th^{1/2}}\right)^{\frac{d}{2}} = \frac{C}{h^{\frac{3}{2}}|t|}, \qquad d=2$$

Strichartz for the WW system: Difficulties

$$\partial_t + i|D_x|^{1/2} \text{ v.s. } \partial_t + T_V \nabla_x + iT_\gamma$$

- Non constant coefficients (no easy solution in Fourier)
- Non smooth coefficients: $V \in H^s, \gamma \in H^{s-\frac{1}{2}}$ Hence for $s > 1 + d/2$, $V \in C^1, \gamma \in C^{1/2}$
- Dispersion due to the subprincipal term of order $1/2$, T_γ

Sketch of the proof

- step 1: We perform a frequency analysis: $u = \sum_{j=-1}^{\infty} \Delta_j u$

$$(\partial_t + T_V \cdot \nabla + iT_\gamma)\Delta_j u = \Delta_j f + \Big[T_V \cdot \nabla + iT_\gamma, \Delta_j\Big] u =: F_j$$

Main part

$$L = \partial_t + S_j(V) \cdot \nabla + iT_\gamma, \quad S_j(V) = 2^{jd}\hat{\psi}(2^j \cdot) \star V$$

- step 2: We regularize V and γ (inspired by works by Smith (Strichartz for wave equations with C^2 coefficients), Bahouri-Chemin and Tataru (quasilinear wave equations)
  Take $0 < \delta < 1$ (here $\delta = \frac{2}{3}$). Then

$$(\partial_t + S_{j\delta}(V) \cdot \nabla + iT_{\gamma_\delta}\Delta_j u = F_j + g_{j\delta} =: G_j, \qquad \gamma_\delta = \psi(2^{-j\delta}D_x)\gamma$$

where

$$g_{j\delta} = \Big(S_j(V) \cdot \nabla - S_{j\delta}(V) \cdot \nabla + i(T_\gamma - T_{\gamma_\delta})\Big)\Delta_j u$$

- step 3: We straighten the vector field $\partial_t + S_{j\delta}(V)\cdot\nabla$

$$\dot{X}(t) = S_{j\delta}(V)(t, X(t)), \quad X(0) = x.$$

Lemma

$$\|\frac{\partial X}{\partial x}(t,\cdot) - Id\|_{L^\infty(\mathbb{R}^d)} \leq C(\|V\|_{E_0})|t|^{\frac{1}{2}}$$
$$\|(\partial_x^\alpha X)(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C_\alpha(\|V\|_{E_0})h^{-\delta(|\alpha|-1)}|t|^{\frac{1}{2}}, \quad |\alpha| \geq 2, \quad h = 2^{-j}$$

Then we change variables

$$v_h(t,y) = (\Delta_j u)(t, X(t,y)), \quad h = 2^{-j}$$

$$\partial_t v_h + iA_h(t, y, D_y)v_h = g_h$$

for a rather explicit operator A of order $1/2$.

- step 4: semiclassical form We set

$$z = h^{-\frac{1}{2}} y, \quad \tilde{h} = h^{\frac{1}{2}}, \quad w_{\tilde{h}}(t,z) = v_h(t, \tilde{h} y)$$

then multiplying the equation by $\tilde{h}$ we obtain

$$\tilde{L} w_{\tilde{h}} := (\tilde{h}\partial_t + iP(t, \tilde{h}z, \tilde{h}D_z, \tilde{h})) w_{\tilde{h}} = \tilde{h} F_{\tilde{h}}.$$

We look for a parametrix on a time intervall of size $h^\rho = \tilde{h}^{2\rho}$ (where $\rho = \frac{1}{3}$) of the form

$$\mathcal{K} v(t,z) = (2\pi\tilde{h})^{-d} \iint e^{\frac{i}{\tilde{h}}(\phi(t,z,\xi,\tilde{h}) - z'\cdot\xi)} b(t,z,\xi,\tilde{h}) v(z') dz' d\xi$$

where b is a symbol and ϕ a real valued phase such that

$$\phi|_{t=0} = z\cdot\xi, \quad b|_{t=0} = \chi(\xi), \quad \mathrm{supp}\chi \subset \{\xi : \frac{1}{3} \leq |\xi| \leq 3\}$$

- step 5: the parametrix
  - Solve the eikonal equation

$$\frac{\partial \phi}{\partial t} + p(t, z, \frac{\partial \phi}{\partial z}, \tilde{h}) = 0, \quad \phi|_{t=0} = z \cdot \xi$$

  - Solve the transport equations

$$\begin{aligned} &\mathcal{L} b_0 = 0, \quad b_0|_{t=0} = \chi(\xi), \\ &\mathcal{L} b_j = F(b_0, \dots, b_{j-1}), \quad b_j|_{t=0} = 0, \quad j \geq 1 \end{aligned} \tag{6}$$

Strichartz estimates on small time intervals

Using the parametrix, the stationnary phase estimate and coming back to the original variable $z \to y = h^{\frac{1}{2}} z \to x = X(t, y)$ we prove

Proposition

Let $\chi \in C_0^\infty(\mathbb{R}^d), supp\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ *and* $t_0 \in \mathbb{R}$. *Let* $u_0 \in L^1(\mathbb{R}^d)$ *and* $u_{0,h} = \chi(hD_x)u_0$. *Let* $S(t, t_0)u_{0,h}$ *be the solution of*

$$\Big(\partial_t + \frac{1}{2}(T_{V_\delta} \cdot \nabla + \nabla \cdot T_{V_\delta}) + iT_{\gamma_\delta}\Big) U_h = 0, \quad U_h(t_0, x) = u_{0,h}(x).$$

Then there exist constants $C > 0, h_0 > 0$ *such that*

$$\|S(t, t_0)u_{0,h}\|_{L^\infty(\mathbb{R}^d)} \leq Ch^{-\frac{3d}{4}} |t - t_0|^{-\frac{d}{2}} \|u_{0,h}\|_{L^1(\mathbb{R}^d)}$$

for all $h \in]0, h_0]$ *and* $t \in]t_0, t_0 + h^{\frac{2}{3}}]$.