Groups and Geometry

Daniel Monclair [1]

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[1] Institut de Mathématique d'Orsay, Université Paris-Saclay. daniel.monclair@universite-paris-saclay.fr

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Part I

Lie groups and Lie algebras

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Chapter 1

Basics of Lie groups and Lie algebras

1.1 Lie groups

A Lie group is just on group on which one can do calculus. They were first considered as models for the symmetry groups of several physical systems, but they quickly gained a purely mathematical interest as the symmetry groups of many different geometries.

Definition 1.1.1. A **Lie group** is a smooth manifold G endowed with a group structure such that the group operation

$$\left\{ \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \right.$$

is smooth.

Remarks.

- A manifold is not necessarily connected.
- This is the definition of a real Lie group. One can also define a complex Lie group using complex manifolds and requiring holomorphicity of the group operation. This class only presents real Lie groups, but it is a good exercise to check which properties and which proofs also hold in the complex setting.
- Here smooth means $\mathcal{C}^\infty$. Requiring everything to be real analytic or only $\mathcal{C}^1$ leads to the same theory.
- Quite often, smoothness of the inverse map $x \mapsto x^{-1}$ is also required. However, we will see that it is a consequence of this definition.

Examples 1.1.2.

1. Countable groups are exactly 0-dimensional Lie groups.
2. $(\mathbb{R},+)$, $(\mathbb{R}^*,\times)$ and $(\mathbb{U},\times)$ where $\mathbb{U}=\{z\in\mathbb{C}:|z|=1\}$ are 1-dimensional Lie groups. Similarly, $(\mathbb{C},+)$ and $(\mathbb{C}^*,\times)$ are 1-dimensional complex Lie groups (and 2-dimensional real Lie groups).
3. The additive group of a real (resp. complex) vector space is a real (resp. complex) Lie group.
4. A complex Lie group can be seen as a real Lie group.
5. The product of finitely many Lie groups is a Lie group.
6. Given a finite dimensional real vector space V, the general linear group $\mathrm{GL}(V)$ is a Lie group. Indeed, it is a open subset of $\mathrm{End}(V)$, hence a manifold. The group operation is smooth because it is the restriction of a bilinear map between finite dimensional vector spaces.
7. For the same reasons, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are Lie groups ($\mathrm{GL}(n,\mathbb{C})$ being a complex Lie group).
8. Classical Lie groups: let n, p and q be positive integers. Let $I_n\in\mathcal{M}_n(\mathbb{R})$ be the identity matrix, and

$$I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}\in\mathcal{M}_{p+q}(\mathbb{R})\,;\quad J_n=\begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}\in\mathcal{M}_{2n}(\mathbb{R})\,.$$

$\mathrm{SL}(n,\mathbb{R})$	$=$	$\{g\in\mathrm{GL}(n,\mathbb{R})\,\vert\,\det g=1\}$
$\mathrm{SL}(n,\mathbb{C})$	$=$	$\{g\in\mathrm{GL}(n,\mathbb{C})\,\vert\,\det g=1\}$
$\mathrm{O}(n,\mathbb{R})$	$=$	$\left\{g\in\mathrm{GL}(n,\mathbb{R})\,\middle\vert\,{}^t g\,g=I_n\right\}$
$\mathrm{SO}(n,\mathbb{R})$	$=$	$\mathrm{O}(n,\mathbb{R})\cap\mathrm{SL}(n,\mathbb{R})$
$\mathrm{O}(p,q)$	$=$	$\left\{g\in\mathrm{GL}(p+q,\mathbb{R})\,\middle\vert\,{}^t g\,I_{p,q}\,g=I_{p,q}\right\}$
$\mathrm{SO}(p,q)$	$=$	$\mathrm{O}(p,q)\cap\mathrm{SL}(p+q,\mathbb{R})$
$\mathrm{O}(n,\mathbb{C})$	$=$	$\left\{g\in\mathrm{GL}(n,\mathbb{C})\,\middle\vert\,{}^t g\,g=I_n\right\}$
$\mathrm{SO}(n,\mathbb{C})$	$=$	$\mathrm{O}(n,\mathbb{C})\cap\mathrm{SL}(n,\mathbb{C})$
$\mathrm{U}(n)$	$=$	$\left\{g\in\mathrm{GL}(n,\mathbb{C})\,\middle\vert\,{}^t\overline{g}\,g=I_n\right\}$
$\mathrm{SU}(n)$	$=$	$\mathrm{U}(n)\cap\mathrm{SL}(n,\mathbb{C})$
$\mathrm{Sp}(2n,\mathbb{R})$	$=$	$\left\{g\in\mathrm{GL}(2n,\mathbb{R})\,\middle\vert\,{}^t g\,J_n\,g=J_n\right\}$
$\mathrm{Sp}(2n,\mathbb{C})$	$=$	$\left\{g\in\mathrm{GL}(2n,\mathbb{C})\,\middle\vert\,{}^t g\,J_n\,g=J_n\right\}$

Table 1.1: Lie groups you should remember

Here $\mathbb{H}$ is the quaternion algebra: the associative 4-dimensional $\mathbb{R}$-algebra possessing a basis $(1,i,j,k)$ such that 1 is a unit, $i^2=j^2=k^2=-1$ and $ij=-ji=k$.

$\mathrm{SO}(n,\mathbb{H})$	$=$	$\left\{g \in \mathrm{SL}(n,\mathbb{H}) \,\middle|\, {}^t\overline{g}\, g = I_n\right\}$
$\mathrm{SO}^*(2n)$	$=$	$\left\{g \in \mathrm{SO}(2n,\mathbb{C}) \,\middle|\, {}^t\overline{g}\, J_n\, g = J_n\right\}$
$\mathrm{U}(p,q)$	$=$	$\left\{g \in \mathrm{GL}(p+q,\mathbb{C}) \,\middle|\, {}^t\overline{g}\, I_{p,q}\, g = I_{p,q}\right\}$
$\mathrm{SU}(p,q)$	$=$	$\mathrm{U}(p,q) \cap \mathrm{SL}(p+q,\mathbb{C})$
$\mathrm{SU}^*(2n)$	$=$	$\{g \in \mathrm{SL}(2n,\mathbb{C}) \mid J_n\, g = \overline{g} J_n\}$
$\mathrm{Sp}(p,q)$	$=$	$\left\{g \in \mathcal{M}_{p+q}(\mathbb{H}) \,\middle|\, {}^t\overline{g}\, I_{p,q}\, g = I_{p,q}\right\}$
$\mathrm{Sp}(n)$	$=$	$\left\{g \in \mathcal{M}_n(\mathbb{H}) \,\middle|\, {}^t\overline{g}\, g = I_n\right\}$
$\mathrm{Sp}_*(n)$	$=$	$\mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)$

Table 1.2: More classic Lie groups

1.1.1 Multiplication and inverse maps

Given a Lie group G and an element $g \in G$, we let L_g (resp. R_g) denote the left (resp. right) multiplication by g, i.e. the maps:

$$L_g : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & gx \end{array} \right. \text{ and } R_g : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & xg \end{array} \right. .$$

Note that L_g and R_g are diffeomorphisms, their inverses being $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. For any $g, h \in G$, we have that $L_g \circ R_h = R_h \circ L_g$ (this is a rewriting of the associativity of G).

Consider the maps:

$$m : \left\{ \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \right. \text{ and } \mathrm{inv} : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array} \right.$$

Smoothness of m is part of the definition of a Lie group. The map inv is also smooth.

Proposition 1.1.3. *Let G be a Lie group. The inverse map* inv $: G \to G$ *is smooth.*

Proof. Let $x_0 \in G$. The partial differential of m with respect to its second variable at the point (x_0, x_0^{-1}) is $d_{x_0^{-1}} L_{x_0}$, therefore invertible. According to the Implicit Function Theorem, there are an open neighbourhood $U \subset G$ of x_0 and a smooth map $\varphi : U \to G$ such that $m(x, \varphi(x)) = e$ for any $x \in U$. Since $x\varphi(x) = e$, we find that $\varphi(x) = x^{-1}$, and inv is smooth on U. $\square$

This implies that a Lie group is a topological group.

Definition 1.1.4. A **topological group** is a topological space G endowed with a group structure such that the maps

$$\left\{ \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \right. \text{ and } \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array} \right.$$

are continuous.

Note that in the topological setting, continuity of the inverse map is not a consequence of the continuity of the multiplication map.

Corollary 1.1.5. *Every Lie group is a topological group.*

Remark. The fact that the smoothness of inv is usually included in the definition of a Lie group is a habit coming from topological groups.

It should also be a part of the definitions of some infinite dimensional generalizations of Lie groups (especially if we want a group such as $\mathrm{Diff}(M)$ to be a Lie group, where M is a manifold), but we will not discuss this further.

We will now see that up to order one, there is no distinction between $\mathbb{R}^n$ and any other Lie group.

Proposition 1.1.6. *Let G be a Lie group. For $X, Y \in T_eG$, we have that:*

$$d_{(e,e)}m(X,Y) = X + Y \text{ and } d_e\mathrm{inv}(X) = -X$$

Proof. Differentiating the identity $m(x,e) = x$ (resp. $m(e,y) = y$), we find that $d_{(e,e)}m(X,0) = X$ (resp. $d_{(e,e)}m(0,Y) = Y$), hence:

$$d_{(e,e)}m(X,Y) = d_{(e,e)}m(X,0) + d_{(e,e)}m(0,Y) = X + Y$$

Differentiating $m(x,\mathrm{inv}(x)) = e$ at e now yields $X + d_e\mathrm{inv}(X) = 0$. □

This tells us that in order to tell Lie groups apart using infinitesimal quantities, we will need to work with second order differentials, which is why the study of Lie groups has a geometric nature: as we will see in the other chapters of this course, differential geometry treats second order differentials as curvature.

Before going further, we should also have in mind that only few manifolds can carry a Lie group structure, as there are some strong topological obstructions.

Proposition 1.1.7. *If G is a Lie group, then the tangent bundle TG is trivialisable (i.e. G is parallelisable).*

Proof. The map

$$\begin{cases} G \times T_eG & \to & TG \\ (g,v) & \mapsto & (g, d_eL_g(v)) \end{cases}$$

is a trivialisation of TG. It is smooth because $d_eL_g(v) = d_{(g,e)}m(0,v)$, and its inverse is $(g,w) \mapsto (g, d_{(g^{-1},g)}m(0,w))$ which is also smooth because m and inv are. □

As a consequence, there is no Lie group structure on $\mathbb{S}^2$.

1.1.2 Lie group morphisms

Since we will talk a lot about morphisms between Lie groups, we need to make the definitions clear.

Definition 1.1.8. Let G and H be Lie groups. A **Lie group morphism** from G to H is a smooth map $f : G \to H$ which is a group homomorphism.

A **linear representation** of G is a Lie group morphism $f : G \to \mathrm{GL}(V)$ where V is a finite dimensional real vector space.

A **matrix representation** of G is a Lie group morphism $f : G \to \mathrm{GL}(n, \mathbb{R})$ or $f : G \to \mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$.

A **representation** of G is either a linear representation or a matrix representation.

A **Lie group isomorphism** between G and H is a smooth diffeomorphism $f : G \to H$ which is a group isomorphism. When $G = H$, we call f a **Lie group automorphism**.

Remark. The fact that f is a group homomorphism translates as $f \circ L_g = L_{f(g)} \circ f$ for all $g \in G$. Note that we use the same notation $L_g : G \to G$ and $L_{f(g)} : H \to H$ as there is very little risk for confusion.

Recall that a smooth map $f : M \to N$ has **constant rank** if the rank of $d_x f : T_x M \to T_{f(x)} N$ does not depend on $x \in M$. Recall that a constant rank map is **linearisable** (i.e. for any $x_0 \in M$, there are diffeomorphisms φ, ψ such that $f = \psi \circ d_{x_0} f \circ \varphi$ near x_0).
A constant rank map is a **submersion** if its differential is always onto, an **immersion** if its differential is always injective.

Exercise.

1. If $f : M \to N$ has constant rank and is injective, then it is an immersion.

2. If $f : M \to N$ has constant rank and is surjective, then it is a submersion.

Hint: show that the image of a constant rank map which is not a submersion has empty interior using Baire's theorem.

Proposition 1.1.9.

1. *A Lie group morphism has constant rank.*

2. *A bijective Lie group morphism is a Lie group isomorphism.*

Proof.

1. Given $g \in G$, differentiating the expression $f \circ L_g = L_{f(g)} \circ f$ at e yields:

$$d_g f \circ d_e L_g = d_{f(g)} L_{f(g)} \circ d_e f$$

   Since L_g and $L_{f(g)}$ are diffeomorphisms, it follows that $d_g f$ and $d_e f$ have the same rank.

2. As a consequence of the previous statement and of the result of the exercise, a bijective Lie group morphism is a local diffeomorphism, therefore a diffeomorphism.

□

1.1.3 The adjoint representation of a Lie group

Let G be a Lie group. Given $g \in G$, we denote by $i_g = L_g \circ R_{g^{-1}}$ the conjugacy by g (i.e. $i_g(x) = gxg^{-1}$). It is a Lie group automorphism of G.

Definition 1.1.10. Let G be a Lie group. Given $g \in G$, we set $\mathrm{Ad}(g) = d_e i_g : T_e G \to T_e G$. The map

$$\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$$

is called the **adjoint representation** G.

Note that Ad is indeed a representation.

Definition 1.1.11. The **centre** of G is:

$$Z(G) = \{g \in G \,|\, \forall x \in G \; gx = xg\}$$

Note that we always have $Z(G) \subset \ker \mathrm{Ad}$. However it is not always an equality.

1.2 Lie algebras

Definition 1.2.1. Let $\mathbb{K}$ be a field. A **Lie algebra** $\mathfrak{g}$ over $\mathbb{K}$ is a pair $(V, [\cdot,\cdot])$ where V is a vector space over $\mathbb{K}$ and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map satisfying:

1. $\forall X, Y \in V \; [Y, X] = -[X, Y]$

2. $\forall X, Y, Z \in V \; [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

We will only consider the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. A bilinear map $[\cdot,\cdot]$ satisfying *1.* and *2.* is called a **Lie bracket**. Condition *2.* is called the **Jacobi identity**.

Remarks.

- The vector space V needs not be finite dimensional. We will actually consider one infinite dimensional example.

- We will use the same notation for $\mathfrak{g}$ and V.

Definition 1.2.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over a field $\mathbb{K}$. A **Lie algebra morphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a $\mathbb{K}$-linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\forall X, Y \in \mathfrak{g} [\varphi(X), \varphi(Y)] = [X, Y]$.
A **Lie algebra isomorphism** is a bijective Lie algebra morphism (its inverse is also a Lie algebra morphism). When $\mathfrak{g} = \mathfrak{h}$, a Lie algebra isomorphism is called a **Lie algebra automorphism**.
A **Lie subalgebra** of $\mathfrak{g}$ is a vector subspace $V \subset \mathfrak{g}$ such that $[X, Y] \in V$ for all $X, Y \in V$.

Examples 1.2.3.

1. Any vector space V can be endowed with the Lie bracket defined by $[X, Y] = 0$ for all $X, Y \in V$. Such a Lie algebra $(V, [\cdot, \cdot])$ is called **abelian**.

2. Let $\mathcal{A}$ be an associative algebra over $\mathbb{K}$. For $X, Y \in \mathcal{A}$, we set $[X, Y] = XY - YX$. Tedious yet easy computations show that it is a Lie bracket. This applies to the associative algebras $\mathrm{End}_{\mathbb{K}}(V)$ (where V is a vector space over $\mathbb{K}$) and $\mathcal{M}_n(\mathbb{K})$. These Lie algebras will be denoted respectively by $\mathfrak{gl}(V)$ ans $\mathfrak{gl}(n, \mathbb{K})$.
   A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra morphism $\varphi : \mathfrak{g} \to \mathrm{End}(V)$ where V is a vector space (or $\varphi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{K})$).

   Remark. Note that we do not ask representations of Lie algebras to be finite dimensional, while we do for Lie groups (so that a Lie group representation is a Lie group morphism).

3. The classical Lie algebras (see table 3).

4. If $\mathfrak{g}$ is a complex Lie algebra, we denote by $\mathfrak{g}_{\mathbb{R}}$ the underlying Lie algebra (obtained by considering $\mathfrak{g}$ as a real vector space). This operation is known as *restriction of scalars*. If $\mathfrak{g}$ is a real Lie algebra, then $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra when endowed with the only Lie bracket satisfying $[X \otimes z, X' \otimes z'] = [X, X'] \otimes zz'$ for all $X, X' \in \mathfrak{g}$ and $z, z' \in \mathbb{C}$.

5. The product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ of Lie algebras is a Lie algebra with bracket:

$$[(X_1, \dots, X_k), (Y_1, \dots, Y_k)] = ([X_1, Y_1], \dots, [X_k, Y_k])$$

6. The image and the kernel of a Lie algebra morphism are Lie subalgebras.

$\mathfrak{sl}(n,\mathbb{R})$	$=$	$\{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} X = 0\}$
$\mathfrak{sl}(n,\mathbb{C})$	$=$	$\{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} X = 0\}$
$\mathfrak{so}(n,\mathbb{R})$	$=$	$\left\{X \in \mathfrak{gl}(n,\mathbb{R}) \,\middle|\, {}^tX + X = 0\right\}$
$\mathfrak{so}(p,q)$	$=$	$\left\{X \in \mathfrak{gl}(p+q,\mathbb{R}) \,\middle|\, {}^tX\, I_{p,q} + I_{p,q}\, X = 0\right\}$
$\mathfrak{so}(n,\mathbb{C})$	$=$	$\left\{X \in \mathfrak{gl}(n,\mathbb{C}) \,\middle|\, {}^tX + X = 0\right\}$
$\mathfrak{u}(n)$	$=$	$\left\{X \in \mathfrak{gl}(n,\mathbb{C}) \,\middle|\, {}^t\overline{X} + X = 0\right\}$
$\mathfrak{su}(n)$	$=$	$\mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{sp}(2n,\mathbb{R})$	$=$	$\left\{X \in \mathfrak{gl}(2n,\mathbb{R}) \,\middle|\, {}^tX\, J_n + J_n\, X = 0\right\}$
$\mathfrak{sp}(2n,\mathbb{C})$	$=$	$\left\{X \in \mathfrak{gl}(2n,\mathbb{C}) \,\middle|\, {}^tX\, J_n + J_n\, X = 0\right\}$

Table 1.3: Lie algebras you should remember

1.2.1 The Lie algebra of a Lie group

Let G be a Lie group. We have already defined the adjoint representation $\operatorname{Ad} : G \to \operatorname{GL}(T_eG)$ of G. We set $\operatorname{ad} = d_e \operatorname{Ad} : T_eG \to \operatorname{End}(T_eG) = T_{\operatorname{Id}}\operatorname{GL}(T_eG)$.

Proposition 1.2.4. *Let G be a Lie group. The map*

$$[\cdot,\cdot] : \left\{ \begin{array}{ccc} T_eG \times T_eG & \to & T_eG \\ (X,Y) & \mapsto & \operatorname{ad}(X)Y \end{array} \right.$$

is a Lie bracket.

Examples 1.2.5.

- If G is Abelian, then $i_g = \operatorname{Id}$ for any $g \in G$, which yields $\operatorname{Ad}(g) = \operatorname{Id}$, hence $\operatorname{ad} = 0$. Therefore $[X,Y] = \operatorname{ad}(X)Y$ is a Lie bracket, and $(T_eG, [\cdot,\cdot])$ is an Abelian Lie algebra.
- If $G = \operatorname{GL}(V)$, then $T_{\operatorname{Id}}G = \operatorname{End}(V)$ (since $\operatorname{GL}(V)$ is an open subset of $\operatorname{End}(V)$). Given $g \in G$, the map $i_g : G \to G$ is the restriction to G of a linear map defined on $\operatorname{End}(V)$, its differential is $\operatorname{Ad}(g)Y = gYg^{-1}$. Differentiating with respect to g at Id, yields $\operatorname{ad}(X)Y = XY - YX$, the usual Lie bracket on $\operatorname{End}(V)$.

Proof. The bracket is bilinear because differentials are linear maps. To prove skew-symmetry, consider elements $X, Y \in T_eG$ and paths $x, y : \mathbb{R} \to G$ such that $x(0) = y(0) = e$, $\dot{x}(0) = X$ and $\dot{y}(0) = Y$.

Consider the function:

$$c : \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & G \\ (s,t) & \mapsto & x(s)y(t)x(s)^{-1}y(t)^{-1} \end{array} \right. .$$

The element $c(s,t)$ is the commutator of $x(s)$ and $y(t)$ (usually denoted by $[x(s), y(t)]$).

Notice that for all $s, t \in \mathbb{R}$ we have $c(s,0) = c(0,t) = e$. It follows that the first derivatives $\frac{\partial c}{\partial s}(0,t)$ and $\frac{\partial c}{\partial t}(s,0)$ are elements of T_eG, and so are the crossed second order derivatives $\frac{\partial^2 c}{\partial s \partial t}(0,0)$ and $\frac{\partial^2 c}{\partial t \partial s}(0,0)$. The Schwarz Lemma ensures that they are equal.

Let us compute $\frac{\partial c}{\partial t}(s,0)$. By rewriting

$$c(s,t) = i_{x(s)}(y(t))y(t)^{-1} = m(i_{x(s)}(y(t)), \mathrm{inv}(y(t)))$$

we can apply Proposition 1.1.6 to get:

$$\frac{\partial c}{\partial t}(s,0) = \mathrm{Ad}(x(s))Y - Y$$

It follows that $\frac{\partial^2 c}{\partial s \partial t}(0,0) = \mathrm{ad}(X)Y$.

Let us now compute the same second order derivative in the opposite order. We now use that $c(s,t) = x(s)i_{y(t)}(x(s)^{-1}) = m(x(s), i_{y(t)}(\mathrm{inv}(x(s))))$.

$$\frac{\partial c}{\partial s}(0,t) = X - \mathrm{Ad}(y(t))X.$$

It follows that $\frac{\partial^2 c}{\partial s \partial t}(0,0) = -\mathrm{ad}(Y)X$, and the equality between crossed derivatives yields $\mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$, i.e. $[X,Y] = -[Y,X]$.

In order to prove the Jacobi identity, we will start by considering the behaviour of the bracket under Lie group morphisms.

Lemma 1.2.6. *If G, H are Lie groups and $f : G \to H$ is a Lie group morphism, then:*

- $\forall g \in G\ \mathrm{Ad}(f(g)) \circ d_e f = d_e f \circ \mathrm{Ad}(g)$.
- $\forall X \in T_eG\ \mathrm{ad}(d_e f(X)) \circ d_e f = d_e f \circ \mathrm{ad}(X)$.
  i.e. $\forall Y \in T_eG\ [d_e f(X), d_e f(Y)] = d_e f([X,Y])$.

Proof of Lemma 1.2.6. Since f is a Lie group morphism, we have that $f \circ i_g = i_{f(g)} \circ f$. Differentiating these two maps at e yields the first point. Differentiating with respect to g at e yields the second. □

The second point of Lemma 1.2.6 applied to the adjoint representation $f = \mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ yields (thanks to Examples 1.2.5)

$$\underbrace{\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)}_{=[\mathrm{ad}(X), \mathrm{ad}(Y)] \in \mathrm{End}(T_eG)} = \mathrm{ad}([X,Y])$$

Applied to $Z \in T_eG$, we get

$$[X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z]$$

Reordering the terms and using skew-symmetry leads to the Jacobi identity. □

1.2.2 The adjoint representation of a Lie algebra

Definition 1.2.7. Let $\mathfrak{g}$ be a Lie algebra. For $X \in \mathfrak{g}$, we let $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ be the map defined by $\operatorname{ad}(X)Y = [X,Y]$. We call $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the **adjoint representation** of $\mathfrak{g}$.

A simple rewriting of the Jacobi identity shows that ad is a Lie algebra morphism.

In the case of the Lie algebra of a Lie group, both definitions coincide.

Definition 1.2.8. The **centre** of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g}\ [X,Y] = 0\}$$

By definition of ad, we have that $\mathfrak{z}(\mathfrak{g}) = \ker \operatorname{ad}$.

1.2.3 Structural constants

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $e = (e_1, \dots, e_d)$ be a basis of $\mathfrak{g}$. We can decompose brackets between elements of e in the basis e:

$$[e_i, e_j] = \sum_{k=1}^{d} C_{i,j}^k e_k$$

The numbers $(C_{i,j}^k)_{1 \le i,j,k \le d}$ are called the **structural constants** of $\mathfrak{g}$ (in the basis $(e_1, \dots, e_d)$).

Knowing the structural constants of a Lie algebra makes for easy computations (especially if there is a basis in which they have very simple expressions, e.g. most of them vanish).

One can use a basis to show that a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie bracket. Skew-symmetry has a simple expression:

$$\forall i,j,k \in \{1, \dots, d\} \quad C_{i,j}^k + C_{j,i}^k = 0,$$

The Jacobi identity is more difficult to handle in this form:

$$\forall i_1, i_2, i_3, j \in \{1, \dots, d\} \quad \sum_{k=1}^{d} \left(C_{i_1,i_2}^k C_{i_3,k}^j + C_{i_2,i_3}^k C_{i_1,k}^j + C_{i_3,i_1}^k C_{i_2,k}^j \right) = 0.$$

However, small dimensions allow for simpler proofs. In dimension 2, skew-symmetry implies the Jacobi identity. In dimension 3, once skew-symmetry is proven, it is sufficient to prove the Jacobi identity for one linearly independent family (X,Y,Z). In terms of structural constants, we only have to check three equations:

$$\forall j \in \{1,2,3\} \quad \sum_{k=1}^{3} \left(C_{1,2}^k C_{3,k}^j + C_{2,3}^k C_{1,k}^j + C_{3,1}^k C_{2,k}^j \right) = 0.$$

Examples 1.2.9.

1. In $\mathfrak{sl}(2,\mathbb{R})$ (or $\mathfrak{sl}(2,\mathbb{C})$), consider the matrices:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

They form a basis in which the structural constants are given by the following relations:

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H.$$

2. In $\mathfrak{su}(2)$, consider the following matrices:

$$u_1 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad u_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad u_3 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

They form a basis in which the structural constants are given by the following relations:

$$[u_1,u_2] = u_3,\ [u_2,u_3] = u_1,\ [u_3,u_1] = u_2.$$

3. In $\mathfrak{so}(3,\mathbb{R})$, consider the following matrices:

$$r_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} ; \quad r_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

They form a basis in which the structural constants are given by the following relations:

$$[r_1,r_2] = r_3,\ [r_2,r_3] = r_1,\ [r_3,r_1] = r_2.$$

This shows that the linear map $\varphi : \mathfrak{su}(2) \longrightarrow \mathfrak{so}(3,\mathbb{R})$ satisfying $\varphi(u_j) = r_j$ for $j = 1,2,3$ is a Lie algebra isomorphism, so the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3,\mathbb{R})$ are isomorphic.

4. Consider the cross product on $\mathbb{R}^3$, defined by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - y_1x_2 \end{pmatrix}.$$

The relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$ satisfied by the canonical basis not only show that $(\mathbb{R}^3,\times)$ is a Lie algebra, but also that it is isomorphic to $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

5. Consider the Lie subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ defined by

$$\mathfrak{heis}(3) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| (x,y,z) \in \mathbb{R}^3 \right\}.$$

Consider the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They form a basis in which the structural constants are given by the following relations:

$$[X,Y] = Z, \ [Z,X] = [Z,Y] = 0.$$

6. Consider the Lie subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ defined by

$$\mathfrak{sol}(3) = \left\{ \begin{pmatrix} t & 0 & a \\ 0 & -t & b \\ 0 & 0 & 0 \end{pmatrix} \middle| (a,b,t) \in \mathbb{R}^3 \right\}.$$

Consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They form a basis in which the structural constants are given by the following relations:

$$[T,A] = A, \ [T,B] = -B, \ [A,B] = 0.$$

1.2.4 Automorphisms and derivations of a Lie algebra

In order to understand the nature of the map $\mathrm{ad}(X)$ for a given $X \in \mathfrak{g}$, we have to look at derivations of Lie algebras.

Definition 1.2.10. Let $\mathfrak{g}$ be a Lie algebra. A **derivation** of $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\forall X, Y \in \mathfrak{g} \ \delta[X,Y] = [\delta X, Y] + [X, \delta Y]$$

We denote by $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ the set of derivations of $\mathfrak{g}$.

Proposition 1.2.11.

- $\mathrm{Der}(\mathfrak{g})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{g})$.

- *For any* $X \in \mathfrak{g}$, $\mathrm{ad}(X) \in \mathrm{Der}(\mathfrak{g})$.

Proof. If $\delta, \delta' \in \mathrm{Der}(\mathfrak{g})$ ans $X, Y \in \mathfrak{g}$, we find:

$$\delta \circ \delta'[X,Y] = [\delta \circ \delta' X, Y] + [\delta X, \delta' Y] + [\delta' X, \delta Y] + [X, \delta \circ \delta' Y]$$

This leads to $[\delta, \delta'] \in \mathrm{Der}(\mathfrak{g})$.
The fact that $\mathrm{ad}(X)$ is a derivation is a rewriting of the Jacobi identity. □

The Lie algebra $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of a Lie group.

Proposition 1.2.12. *The automorphism group of* $\mathfrak{g}$

$$\mathrm{Aut}(\mathfrak{g}) = \{\varphi \in \mathrm{GL}(\mathfrak{g}) | \forall X, Y \in \mathfrak{g}\ \varphi([X,Y]) = [\varphi(X), \varphi(Y)]\}$$

is a Lie group, with Lie algebra $\mathrm{Der}(\mathfrak{g})$.

Proof. For $X, Y \in \mathfrak{g}$, consider the map

$$F_{X,Y} : \left\{ \begin{array}{ccc} \mathrm{GL}(\mathfrak{g}) & \to & \mathfrak{g} \\ \varphi & \mapsto & \varphi^{-1}([\varphi(X), \varphi(Y)]) \end{array} \right. .$$

Let $(X_1, \ldots, X_d)$ be a vector basis of $\mathfrak{g}$, and consider the map

$$F : \left\{ \begin{array}{ccc} \mathrm{GL}(\mathfrak{g}) & \to & \mathfrak{g}^{\frac{d(d-1)}{2}} \\ \varphi & \mapsto & (F_{X_i,X_j}(\varphi))_{1 \le i < j \le d} \end{array} \right. .$$

In order to show that $\mathrm{Aut}(\mathfrak{g}) = F^{-1}\left(([X_i, X_j])_{1 \le i < j \le d}\right)$ is a submanifold of $\mathrm{GL}(\mathfrak{g})$, let us show that the smooth map F has constant rank. For this we first notice that

$$\forall \varphi \in \mathrm{GL}(\mathfrak{g}) \quad \ker d_\varphi F = \bigcap_{1 \le i < j \le d} \ker d_\varphi F_{X_i, X_j} = \bigcap_{X,Y \in \mathfrak{g}} \ker d_\varphi F_{X,Y}.$$

These differentials can be computed:

$$\begin{aligned} \forall \varphi \in \mathrm{GL}(\mathfrak{g})\ \forall \dot\varphi \in \mathrm{End}(\mathfrak{g})\ d_\varphi F_{X,Y}(\dot\varphi) = & \varphi^{-1}([\dot\varphi(X), \varphi(Y)]) \\ & + \varphi^{-1}([\varphi(X), \dot\varphi(Y)]) \\ & - \varphi^{-1} \circ \dot\varphi \circ \varphi^{-1}([\varphi(X), \varphi(Y)]). \end{aligned}$$

Substituting X and Y with $\varphi^{-1}(X)$ and $\varphi^{-1}(Y)$, we find:

$$\dot\varphi \in \ker d_\varphi F \iff \dot\varphi \circ \varphi^{-1} \in \mathrm{Der}(\mathfrak{g}).$$

It follows that $\dim \ker d_\varphi F = \dim \mathrm{Der}(\mathfrak{g})$, so F has constant rank and $\mathrm{Aut}(\mathfrak{g})$ is a submanifold of $\mathrm{GL}(\mathfrak{g})$. It is also a subgroup, and the composition map is smooth, so it is a Lie group. It tangent space at Id is $\ker d_{\mathrm{Id}} F = \mathrm{Der}(\mathfrak{g})$ and the Lie bracket is the restriction of the Lie bracket of $\mathrm{End}(\mathfrak{g})$ (the same proof as for $\mathrm{GL}(V)$ works). □

This means that a derivation of a Lie algebra should be thought of as an infinitesimal automorphism.

In the case of the Lie algebra of a Lie group, an important family of automorphisms is given by the adjoint representation.

Proposition 1.2.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $g \in G$, the map $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ is a Lie algebra automorphism of $\mathfrak{g}$, i.e.* $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$.

Proof. Since $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation, it is enough to check that $\mathrm{Ad}(g)$ is a Lie algebra morphism. The second point of Lemma 1.2.6 states that the differential $d_e f$ of a Lie group morphism $f : G \to H$ is a Lie algebra morphism. Applied to $f = i_g$ for some $g \in G$, we get the desired result. □

1.3 The Lie algebra of a Lie group

Definition 1.3.1. Let G be a Lie group. The Lie algebra $(T_e G, [\cdot,\cdot])$ defined by $[X,Y] = \mathrm{ad}(X)Y$ is called the **Lie algebra of** G, it will be denoted by $\mathrm{Lie}(G)$ or $\mathfrak{g}$.

We now wish to understand how a Lie group and its Lie algebra work together. The main goal is see how complicated calculations in a Lie group can boil down to some simple linear algebra in its Lie algebra.

1.3.1 Locally isomorphic Lie groups

A Lie group morphism induces a Lie algebra morphism.

Proposition 1.3.2. *Let G, H be Lie groups with respective Lie algebras $\mathfrak{g}, \mathfrak{h}$, and $f : G \to H$ a Lie group morphism. The map $d_e f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism.*
Moreover, if f is local diffeomorphism (e.g. if f is a Lie groups isomorphism), then $d_e f$ is a Lie algebra isomorphism.

Proof. The fact that $d_e f$ is a Lie algebra morphism is exactly the second point of lemma 1.2.6. If f is a local diffeomorphism, then $d_e f$ is an invertible Lie algebra morphism, hence a Lie algebra isomorphism. □

This tells us that isomorphic Lie groups have isomorphic Lie algebras. The converse is false: $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not isomorphic, but their Lie algebras are.

Definition 1.3.3. Two Lie groups are called **locally isomorphic** if their Lie algebras are isomorphic.

Example 1.3.4. We have seen that the Lie algebras $\mathfrak{so}(3,\mathbb{R})$ and $\mathfrak{su}(2)$ are isomorphic (Examples 1.2.9), however the Lie groups $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$ are not isomorphic. This can be proved by using topology (they are not diffeomorphic to each other, $\mathrm{SO}(3,\mathbb{R})$ is diffeomorphic to $\mathbb{RP}^3$ but $\mathrm{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$) or algebra (the centre of $\mathrm{SO}(3,\mathbb{R})$ is trivial, while $-I_2 \in \mathrm{SU}(2)$ is a central element).

1.3.2 The identity component of a Lie group

Given a Lie group G and its neutral element e, we denote by $G_\circ$ the connected component of e in G, called the **neutral component** or **identity component**.

Proposition 1.3.5. *Let G be a Lie group. The neutral component $G_\circ$ is a normal subgroup of G. It is both open and closed in G.*

Remarks.

- Since it is open, it is also a manifold of the same dimension as G, and a Lie group.
- The quotient group $G/G_\circ$ can be identified with the set of connected components of G. It is usually denoted by $\pi_0(G)$. The quotient topology is the discrete topology (because $G_\circ$ is open).

Proof. The image of $G_\circ \times G_\circ$ under the continuous map $(x,y) \mapsto xy^{-1}$ is connected, hence a subset of $G_\circ$, which shows that $G_\circ$ is a subgroup.
Given $g \in G$, the set $gG_\circ g^{-1} = L_g \circ R_{g^{-1}}(G_\circ)$ is connected, hence a subset of $G_\circ$, which shows that $G_\circ$ is a normal subgroup.
Closedness and openness of $G_\circ$ are consequences of the more general facts that connected components of a topological space are always closed, and connected components of a manifold are always open. □

By studying the Lie algebra, we cannot see the difference between a Lie group and its identity component.

Proposition 1.3.6. *Let G be a Lie group. The identity component $G_\circ$ is locally isomorphic to G.*

Proof. One can apply Proposition 1.3.2 to the inclusion $G_\circ \hookrightarrow G$. □

Studying the Lie algebra of a Lie group will, at first glance, only allow us to understand the group near the identity element. However, for connected groups, this will be enough to recover the whole algebraic structure.

Proposition 1.3.7. *If G is a connected Lie group, then any neighbourhood of e generates G as a group.*

The proof will use a well known fact about open subgroups of topological groups.

Lemma 1.3.8. *If G is a topological group and $H \subset G$ is an open subgroup, then G is closed.*

Proof. Simply notice that $G \setminus H = \bigcup_{g \notin H} gH$ is also open. □

Proof of Proposition 1.3.7. Let $V \subset G$ be a neighbourhood of e, and H be the subgroup of G generated by V. It is an open subgroup of G, hence closed because of lemma 1.3.8. Since G is connected, we find that $H = G$. □

1.3.3 Reminders on vector fields

Let M be a manifold and $X \in \mathcal{X}(M) = \Gamma(TM)$ a vector field M. The flow φ_X^t of X is the maximal solution to the ordinary differential equation:

$$\begin{cases} \frac{d}{ds}\big|_{s=t} \varphi_X^s(x) & = & X(\varphi_X^t(x)) \\ \varphi_X^0(x) & = & x \end{cases}$$

A vector field is **complete** it its flow is defined for all times. Among the important properties of the flow of a vector field, you should remember the **semi-group property** $\varphi_X^{t+s} = \varphi_X^t \circ \varphi_X^s$ (whenever defined) and the **scaling invariance** $\varphi_{sX}^t = \varphi_X^{ts}$.

If $\psi : M \to N$ is a local diffeomorphism and $X \in \mathcal{X}(N)$, the **pull-back** $\psi^* X \in \mathcal{X}(M)$ is defined by $\psi^* X(x) = (d_x\psi)^{-1}(X(\psi(x)))$.

The flow of $\psi^* X$ is related to the flow of X by the formula:

$$\varphi_{\psi^* X}^t = \psi^{-1} \circ \varphi_X^t \circ \psi$$

The **Lie bracket of vector fields** $X, Y \in \mathcal{X}(M)$ is:

$$[X, Y] = \frac{d}{dt}\bigg|_{t=0} \left(\varphi_X^t\right)^* Y$$

The right hand term is also the Lie derivative $\mathcal{L}_X Y$ (which can be defined for any type of tensor replacing Y).

In local coordinates, the Lie bracket of vector fields can be computed through the formula:

$$[X, Y](x) = d_x Y(X(x)) - d_x X(Y(x))$$

A simple computation shows that for any $f \in \mathcal{C}^\infty(M)$, we have that:

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$$

This formula shows that $\mathcal{X}(M)$ endowed with the Lie bracket of vector fields is a Lie algebra (more precisely, a Lie subalgebra of $\mathrm{End}_{\mathbb{R}}(\mathcal{C}^\infty(M))$). It also allows for an easy proof of the following identity:

$$\psi^*[X, Y] = [\psi^* X, \psi^* Y]$$

1.3.4 Left invariant vector fields

If G is a Lie group, then G acts on the space of vector fields $\mathcal{X}(G)$ by $g.X = L_g^* X$ for $g \in G$ and $X \in \mathcal{X}(G)$. Vector fields which are invariant under this action are called **left-invariant**. We will denote by ${}^G\mathcal{X}(G)$ the space of left-invariant vector fields on G.

Proposition 1.3.9. *The G be a lie group. The space of left-invariant vector fields ${}^G\mathcal{X}(G)$ is a Lie subalgebra of $\mathcal{X}(G)$.*

Proof. Simply notice that $L_g^*[X,Y] = [L_g^* X, L_g^* Y]$ (more generally, $f^*[X,Y] = [f^*X, f^*Y]$ for any diffeomorphism f). □

Left-invariant vector fields provide a second way of associating a Lie algebra to a Lie group. We will see that they are isomorphic, but first let us check that they have the same dimension:

Proposition 1.3.10. *Let G be a Lie group. The map*

$$\varphi : \begin{cases} {}^G\mathcal{X}(G) & \to & T_eG \\ X & \mapsto & X(e) \end{cases}$$

is a linear isomorphism of vector spaces.

Proof. It is linear by definition of the vector space structure on the space of vector fields. Given $g \in G$, the identity $L_g^* X = X$ evaluated at e yields $X(g) = d_e L_g(X(e))$. The injectivity of φ follows.
Given $X \in T_eG$, we denote by $\overline{X} \in \mathcal{X}(G)$ the vector field $\overline{X}(g) = d_e L_g(X)$. For $g, x \in G$, we compute:

$$\begin{aligned}
(L_g^* \overline{X})(x) &= (d_x L_g)^{-1}(\overline{X}(gx)) \\
&= d_{gx} L_{g^{-1}}(d_e L_{gx}(X)) \\
&= d_e(L_{g^{-1}} \circ L_{gx})(X) \\
&= d_e L_x(X) \\
&= \overline{X}(x)
\end{aligned}$$

Therefore $\overline{X} \in {}^G\mathcal{X}(G)$ and $\varphi(\overline{X}) = X$, so φ is also onto. □

We will use the notation $\overline{X} = \varphi^{-1}(X)$ for $X \in \mathfrak{g}$ repeatedly in this section.

We will see later on that φ is also a Lie algebra isomorphism. This will be done through the study of the most important tool relating a Lie group and its Lie algebra: the exponential map.

The exponential map encodes the flow of left-invariant Lie groups, and we will first need to check that it is defined for all times.

Lemma 1.3.11. *A left-invariant vector field on a Lie group is complete.*

Proof. Let $X \in {}^G\mathcal{X}(G)$ and $x \in G$. If $y = \varphi_X^t(x)$ is defined for some $t \in \mathbb{R}$, we set $g = yx^{-1}$, and notice that $L_g^* X = X$ yields:

$$\varphi_X^s(x) = g^{-1}\varphi_X^s(y)$$

This formula shows that $\varphi^s(x)$ is defined if and only if $\varphi_X^s(y) = \varphi_X^{t+s}(x)$ is defined. $\square$

1.3.5 The exponential map of a Lie group

Definition 1.3.12. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **exponential map of** G is defined as follows:

$$\exp_G : \left\{ \begin{array}{ccc} \mathfrak{g} & \to & G \\ X & \mapsto & \varphi_{\overline{X}}^1(e) \end{array} \right.$$

where $\overline{X} \in {}^G\mathcal{X}(G)$ satisfies $\overline{X}(e) = X(e)$.

Remark. It is well defined because of Lemma 1.3.11.

Examples 1.3.13.

1. For $G = \mathbb{U} = \{z \in \mathbb{C} | \, |z| = 1\}$, the Lie algebra is $\mathfrak{g} = i\mathbb{R}$. For $i\theta \in \mathfrak{g}$, we find that $\exp_{\mathbb{U}}(i\theta) = e^{i\theta}$.
2. For $G = \mathrm{GL}(n, \mathbb{K})$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the Lie exponential is equal to the matrix exponential $\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}$.

The exponential map is sufficient to retrieve to whole flow of a left-invariant vector field.

Proposition 1.3.14. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $X \in \mathfrak{g}$. The flow of $\overline{X} \in {}^G\mathcal{X}(G)$ is given by $\varphi_{\overline{X}}^t = R_{\exp_G(tX)}$.*

Proof.

$$\begin{aligned} \varphi_{\overline{X}}^t(g) &= \varphi_{L_{g^{-1}}^* \overline{X}}^t(g) \\ &= L_g \circ \varphi_{\overline{X}}^t \circ L_{g^{-1}}(g) \\ &= g\varphi_{\overline{X}}^t(e) \\ &= g\varphi_{t\overline{X}}^1(e) \end{aligned}$$

$\square$

Note the use of the scaling invariance of the flow of a vector field in order to get $\exp_G(tX) = \varphi_{\overline{X}}^t(e)$. This will be used repeatedly.

Most of the properties of the matrix exponential also hold for the exponential map of a Lie group.

Proposition 1.3.15. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp_G : \mathfrak{g} \to G$ is smooth, and $d_0 \exp_G = \mathrm{Id}_{\mathfrak{g}}$.*

Proof. For the smoothness, consider the vector field $L \in \mathcal{X}(G \times T_eG)$ defined by $L(g, X) = (\overline{X}(x), 0)$. It is smooth thanks to the formula $\overline{X}(x) = d_eL_x(X) = d_{(x,e)}m(0, X)$. It follows that the flow φ_L is smooth, and the formula $\varphi_L^t(x, X) = (\varphi_{\overline{X}}^t(x), X)$ implies the smoothness of $\exp_G$.
Proposition 1.3.14 yields $\exp_G(tX) = \varphi_{\overline{X}}^t(e)$. Differentiating at $t = 0$ gives $d_0 \exp_G(X) = \overline{X}(e) = X$. □

The Local Inverse Function Theorem guarantees that $\exp_G$ is a local diffeomorphism between a neighbourhood of 0 in $\mathfrak{g}$ and a neighbourhood of e in G. Because of Proposition 1.3.7, we get the following:

Proposition 1.3.16. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The subgroup generated by $\exp_G(\mathfrak{g})$ is $G_\circ$.*

However, the exponential map of a Lie group is not always onto, even for a connected Lie group (such as $\mathrm{SL}(2, \mathbb{R})$).

Proposition 1.3.17. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the map:*

$$\left\{ \begin{array}{ccc} \mathbb{R} & \to & G \\ t & \mapsto & \exp_G(tX) \end{array} \right. .$$

is a Lie group morphism. Every Lie group morphism from $\mathbb{R}$ to G has this form for a unique $X \in \mathfrak{g}$.

Such a Lie group morphism is called a **one-parameter subgroup.**

Proof. Since $\exp_G(tX) = \varphi_{\overline{X}}^t(e)$, the additive property of the flow of a vector field shows that $t \mapsto \exp_G(tX)$ is a group morphism. It is smooth because $\exp_G$ is smooth.
Let $\varphi : \mathbb{R} \to G$ be a Lie group morphism. Set $X = \varphi'(0) \in \mathfrak{g}$. Differentiating the expression $\varphi(t+s) = \varphi(t)\varphi(s)$ with respect to s at $s = 0$, we find:

$$\varphi'(t) = d_eL_{\varphi(t)}(X) = \overline{X}(\varphi(t))$$

Since $\varphi(0) = e$, uniqueness of the solution of ordinary differential equations (Cauchy-Lipschitz Theorem) yields $\varphi(t) = \exp_G(tX)$ for any $t \in \mathbb{R}$. As $Y = \frac{d}{dt}\Big|_{t=0} \exp_G(tY)$ for all $Y \in \mathfrak{g}$, the uniqueness of X follows. □

Note that a simple consequence of Proposition 1.3.17 is that $\exp_G(nX) = \left(\exp_G(X)\right)^n$ for $n \in \mathbb{Z}$.

Proposition 1.3.18. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is a Lie group morphism, then:*

$$\forall X \in \mathfrak{g} \; f(\exp_G(X)) = \exp_H(d_ef(X))$$

Proof. The map $t \mapsto f(\exp_G(tX))$ is a one parameter subgroup of H. □

Corollary 1.3.19. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$ and $X \in \mathfrak{g}$, we have that*

$$\exp_G(\mathrm{Ad}(g)(X)) = i_g(\exp_G(X))$$

and

$$\mathrm{Ad}(\exp_G(X)) = \exp(\mathrm{ad}(X))$$

(the right-hand term being the matrix exponential in $\mathfrak{gl}(\mathfrak{g})$).

Proof. Apply Proposition 1.3.18 first to $f = i_g$, then to $f = \mathrm{Ad}$. □

The exponential map of a Lie group is in general not a group morphism. There is no explicit formula for the exponential of the sum of two elements of the Lie algebra. There is however an asymptotic formula.

Proposition 1.3.20. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, we have that:*

$$\exp_G(X+Y) = \lim_{n\to+\infty} \left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n$$

Proof. Following Proposition 1.3.15 and the Local Inverse Function Theorem, we can consider a neighbourhood U of 0 in $\mathfrak{g}$ and a neighbourhood V of e in G such that $\exp_G$ restricts to a diffeomorphism from U to V. Let $\mathrm{Log}_G : V \to U$ be its inverse.

Consider a neighbourhood $U' \subset U$ of 0 in $\mathfrak{g}$ such that $\exp_G(X)\exp_G(Y) \in V$ for all $X, Y \in U'$. Consider the map:

$$f : \left\{ \begin{array}{ccc} U' \times U' & \to & U \\ (X,Y) & \mapsto & \mathrm{Log}_G(\exp_G(X)\exp_G(Y)) \end{array} \right.$$

Let $X, Y \in \mathfrak{g}$. For large enough n, we have that $\frac{X}{n} \in U'$ and $\frac{Y}{n} \in U'$, so we can consider $f\left(\frac{X}{n}, \frac{Y}{n}\right)$.

Since $f(0,0) = 0$ and $d_{(0,0)}f(X,Y) = X + Y$, we find:

$$f\left(\frac{X}{n}, \frac{Y}{n}\right) = \frac{X}{n} + \frac{Y}{n} + \underset{n\to+\infty}{o}\left(\frac{1}{n}\right)$$

Therefore:

$$\lim_{n\to+\infty} nf\left(\frac{X}{n}, \frac{Y}{n}\right) = X + Y$$

The continuity of $\exp_G$ leads to:

$$\lim_{n\to+\infty} \exp_G\left(nf\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \exp_G(X+Y)$$

Using Proposition 1.3.17, we get:

$$\exp_G\left(nf\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \left(\exp_G\left(f\left(\frac{X}{n}, \frac{Y}{n}\right)\right)\right)^n$$

The definition of f simplifies the right-hand term:

$$\exp_G\left(f\left(\frac{X}{n},\frac{Y}{n}\right)\right) = \exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)$$

Finally:

$$\exp_G\left(nf\left(\frac{X}{n},\frac{Y}{n}\right)\right) = \left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n.$$

□

Remark. There is an explicit formula (as a formal series) for the function f that we used, called the Baker-Campbell-Hausdorff formula.

Proposition 1.3.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$ and $[X, Y] = 0$, then* $\exp_G(X + Y) = \exp_G(X)\exp_G(Y)$.

Proof. Since $\mathrm{ad}(X)Y = 0$, we have that $\exp(\mathrm{ad}(X))Y = Y$, and Corollary 1.3.19 yields $\mathrm{Ad}(\exp_G(X))Y = Y$.
We also have that $\exp_G(\mathrm{Ad}(\exp_G(X))Y) = i_{\exp_G(X)}(\exp_G(Y))$ according to the same corollary, therefore $i_{\exp_G(X)}(\exp_G(Y)) = \exp_G(Y)$, i.e. $\exp_G(X)$ and $\exp_G(Y)$ commute.
The same applies to $\frac{X}{n}$ and $\frac{Y}{n}$ for $n > 0$, and we find:

$$\begin{aligned}\left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n &= \left(\exp_G\left(\frac{X}{n}\right)\right)^n\left(\exp_G\left(\frac{Y}{n}\right)\right)^n \\ &= \exp_G(X)\exp_G(Y)\end{aligned}$$

Proposition 1.3.20 shows that $\exp_G(X + Y) = \exp_G(X)\exp_G(Y)$. □

While we are still on the subject of the exponential map, let us mention that there is an explicit formula for the differential of the exponential map of a Lie group at any point of the Lie algebra. We will not use nor prove this formula.

Proposition 1.3.22. *Let $\Theta : \mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$ be defined by $\Theta(f) = \sum_{n=1}^{+\infty} \frac{f^{n-1}}{n!}$ (i.e. $\Theta(z) = \frac{e^z-1}{z}$). For $X \in \mathfrak{g}$, the differential $d_X \exp_G$ is given by:*

$$d_X \exp_G = d_e L_{\exp_G(X)} \circ \Theta(-\mathrm{ad}(X))$$

We now have everything needed to show the equivalence between the two Lie algebras associated to a Lie group.

Theorem 1.3.23. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The map $X \mapsto \overline{X}$ is a Lie algebra isomorphism between $\mathfrak{g}$ and ${}^G\mathcal{X}(G)$.*

Proof. It only remains to show that it is a Lie algebra morphism. For $X, Y \in \mathfrak{g}$, we can compute:

$$\begin{aligned}
[\overline{X}, \overline{Y}](e) &= \left.\frac{d}{dt}\right|_{t=0} \left(\varphi^t_{\overline{X}}\right)^* \overline{Y}(e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e\varphi^t_{\overline{X}}\right)^{-1} \left(\overline{Y}(\varphi^t_{\overline{X}}(e))\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e R_{\exp_G(tX)}\right)^{-1} \left(\overline{Y}(\exp_G(tX))\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e R_{\exp_G(tX)}\right)^{-1} \left(d_e L_{\exp_G(tX)}(Y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} d_e i_{\exp_G(tX)}(Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp_G(tX))Y \\
&= \mathrm{ad}(X)Y
\end{aligned}$$

□

Chapter 2

Correspondence between Lie groups and Lie algebras

Our goal is now to understand up to which extent the classification of Lie groups reduces to the classification of Lie algebras. We will work on two distinct aspects: the relationship between subgroups of a given Lie group and subalgebras of its Lie algebra, and the relationship between Lie groups whose Lie algebras are isomorphic.

2.1 Correspondence between Lie subgroups and subalgebras

The natural definition for a Lie subgroup would be a subgroup which is a submanifold.

Definition 2.1.1. Let G be a Lie group. An **embedded Lie subgroup** of G is a submanifold $H \subset G$ which is also a subgroup.

An embedded Lie subgroup $H \subset G$ is a Lie group itself, since restrictions of smooth functions to submanifolds are smooth. Since the inclusion map is a Lie group morphism, we see that the Lie algebra $\mathfrak{h} = T_eH$ is a subalgebra of $\mathfrak{g} = T_eG$. However we will see that if we want all Lie subalgebras of $\mathfrak{g}$ to correspond to a Lie subgroup, it is necessary to work with immersed submanifolds.

2.1.1 Immersed submanifolds and foliations

Definition 2.1.2. Let M be a manifold. An **immersed submanifold** is the data of a subset $N \subset M$ and a manifold structure on N for which the inclusion $i : N \to M$ is a smooth immersion.

Remark. The manifold structure may not be unique, i.e. there can be two manifold structures on $N \subset M$ such that the inclusion $i : N \to M$ is an immersion for both structures on N, yet the identity map Id_N seen from N with one structure to N with the other is not smooth.
In order to avoid this confusion, it is better to work with an abstract manifold S and an injective immersion $f : S \to M$, so the immersed manifold is the set $f(S)$ with the manifold structure that turns f into a diffeomorphism.
One can actually define an immersed submanifold in this way, as an equivalence class of pairs (S, f) where $f : S \to M$ is an injective immersion, identifying (S, f) with (S', f') if there is a diffeomorphism $\varphi : S \to S'$ such that $f = f' \circ \varphi$.

An immersed submanifold $N \subset M$ has two topologies: the topology induced from M (just as any subset of M), and the manifold topology (sometimes called the intrinsic topology). Unless otherwise specified, any topological condition on N (e.g. connectedness) is considered for the manifold topology.

For $x \in N$, we will identify the tangent space T_xN (for the manifold structure on N) with its image $d_xi(T_xN) \subset T_xM$.

In order to avoid confusion, we will refer to a submanifold (i.e. with the usual definition) as an embedded submanifold. Note that an immersed submanifold is embedded if and only if the subset topology is equal to the manifold topology.

Examples 2.1.3.

- Consider $M = \mathbb{R}^2/\mathbb{Z}^2$ (the 2-dimensional torus), and $\pi : \mathbb{R}^2 \to M$ the canonical projection. Given $\alpha \in \mathbb{R}^*$, we consider

$$D_\alpha = \{\pi(x, \alpha x) | x \in \mathbb{R}\}$$

  If $\alpha \notin \mathbb{Q}$, then D_α is an immersed submanifold which is not embedded. Indeed, the map $x \mapsto \pi(x, \alpha x)$ is an injective immersion, and D_α is dense in M, therefore not embedded.

- Another example is Bernoulli's lemniscate: consider the maps (see Figure 2.1)

$$f : \begin{cases} \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & \left(\frac{t+t^3}{1+t^4}, \frac{t-t^3}{1+t^4}\right) \end{cases}, \quad g : \begin{cases} \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & \left(\frac{t+t^3}{1+t^4}, \frac{t^3-t}{1+t^4}\right) \end{cases}$$

  They are injective immersions with the same image, but the composition $\varphi = f \circ g^{-1}$ is given by $\varphi(t) = \frac{1}{t}$ if $t \neq 0$, and $\varphi(0) = 0$, so it is not continuous. This means that the set $f(\mathbb{R}) = g(\mathbb{R})$ has two distinct differentiable structures for which it is an immersed submanifold.

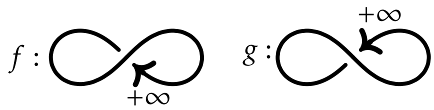


Figure 2.1: Two different structures on an immersed submanifold.

Definition 2.1.4. Let M be a manifold of dimension d. A **smooth foliation** of dimension k of M is a partition $\mathcal{F}$ of M into connected subsets such that for all $x_0 \in M$, there is an open set $U \subset M$ containing x_0 and a smooth submersion $\varphi : U \to \mathbb{R}^{d-k}$ satisfying the following:
For all $x \in U$, the set $\varphi^{-1}(\{\varphi(x)\})$ is the connected component of $\mathcal{F}(x) \cap U$ containing x.
Such a pair (U, φ) is called a **local equation** of $\mathcal{F}$.

Proposition 2.1.5. *Let M be a manifold and $\mathcal{F}$ a foliation of M. For all $x \in M$, the* ***leaf*** *$\mathcal{F}(x)$ carries a unique immersed submanifold structure such that for every local equation (U, φ) of $\mathcal{F}$ with $x \in U$, we have $T_x\mathcal{F}(x) = \ker d_x\varphi$.*

Definition 2.1.6. Let M be a smooth d-dimensional manifold, and let $p \in \{1, \dots, d\}$. A **distribution** Δ of rank p on M is a collection of p-dimensional vector subspaces $\Delta_x \subset T_xM$ for each $x \in M$ with the following regularity property: M can be covered by open sets U on which there are vector fields $X_1, \dots, X_p$ such that:

$$\forall x \in U \ \Delta_x = \mathrm{Vect}(X_1(x), \dots, X_p(X))$$

We say that Δ is **integrable** if there is a foliation $\mathcal{F}$ of M such that $\Delta_x = T_x\mathcal{F}(x)$ for all $x \in M$.

Remarks.

- A distribution Δ is integrable if and only if it is locally spanned by the fundamental vector fields $X_i = \partial_i$ of a local coordinate system $(x^1, \dots, x^d)$.
- Any foliation $\mathcal{F}$ defines a distribution by setting $\Delta_x := T_x\mathcal{F}(x)$.

Lemma 2.1.7. *Let $\mathcal{F}$ be a foliation of a manifold M. If $\gamma : [0,1] \to M$ is a smooth path such that $\dot\gamma(t) \in T_{\gamma(t)}\mathcal{F}(\gamma(t))$ for all $t \in [0,1]$, then $\gamma(1) \in \mathcal{F}(\gamma(0))$.*

Remark. This means that the whole path γ is included in some leaf.

Proof. If the whole range of γ is included in the domain U of a local equation (U, φ) of $\mathcal{F}$, then the derivative of $\varphi \circ \gamma$ vanishes, so $\varphi \circ \gamma$ is constant and the range of γ lies in some leaf. For the general case, we cover the range of γ by finitely many domains of local equations for $\mathcal{F}$ and use the transitivity of the equivalence relation defined by the partition $\mathcal{F}$. $\square$

Corollary 2.1.8. *Let $\mathcal{F}$ and $\mathcal{F}'$ be foliations on a manifold M. If $T_x\mathcal{F}(x) = T_x\mathcal{F}'(x)$ for all $x \in M$, then $\mathcal{F} = \mathcal{F}'$.*

More generally, one can prove that any connected immersed submanifold $N \subset M$ such that $T_xN = \Delta_x$ for all $x \in N$ is an open subset of a leaf.

The leaves of a foliation have an additional property compared to general immersed submanifolds, which allows us to check the smoothness of some functions without working out the differentiable structure of the leaves.

Definition 2.1.9. A smooth map $\varphi : N \to M$ is called a **weak embedding** if it is an injective immersion, and if for any manifold P and map $f : P \to N$, the smoothness of $\varphi \circ f$ implies the smoothness of f.

Definition 2.1.10. Let M be a manifold and $N \subset M$ an immersed submanifold. We say that N is **weakly embedded** if the inclusion $i : N \to M$ is a weak embedding, i.e. any smooth map $f : P \to M$ whose range lies in N is also smooth seen as a map from P to N.

Remarks.

- Note that when $f : P \to N$ is smooth, the differentials $df : TP \to TN \subset TM$ and $TP \to TN$ coincide.
- The figure eight in $\mathbb{R}^2$ is a typical example of a non weakly embedded immersed submanifold.

Proposition 2.1.11. *The leaves of a foliation are weakly embedded.*

The idea behind the proof is very simple: locally, we can construct a smooth projection p onto each leaf, so a function $f : P \to M$ whose values are in a single leaf can be written as $p \circ f$, so is smooth for the differentiable structure of the leaf.

Note that the restriction of a smooth map $f : M \to P$ to an immersed submanifold $N \subset M$ is always smooth for the differentiable structure on N (it is the composition of f with the inclusion map).

Remark. In this course we only consider smooth foliations, i.e. the coordinate system appearing in the definition is smooth. In other contexts (especially in the study of hyperbolic dynamical systems), we treat separately the regularity of the leaves (they usually are smooth) and the transverse regularity (that of the coordinate system, which is usually less regular).

Definition 2.1.12. Let M be a manifold. A distribution Δ of M is called **involutive** if it is stable under the Lie bracket of vector fields, i.e. the space $\Gamma(\Delta) = \{X \in \mathcal{X}(M) \mid \forall x \in M \; X(x) \in \Delta_x\}$ is a Lie subalgebra of $\mathcal{X}(M)$.

Theorem 2.1.13 (Frobenius Theorem). *Let M be a manifold, and Δ a distribution on M. Then Δ is integrable if and only if it is involutive.*

Here again let us discuss the idea of the proof rather than the proof itself. The most natural construction of a foliation $\mathcal{F}$ would be by setting $\mathcal{F}(x) = \left\{\varphi_X^t(x) \,\middle|\, X \in \Gamma(\Delta), t \in \mathbb{R}\right\}$. This always defines (at least locally) a submanifold with $T_x\mathcal{F}(x) = \Delta_x$. However, at some point $y = \varphi_X^t(x) \in \mathcal{F}(x)$, it is not clear that we still have $T_y\mathcal{F}(y) = \Delta_y$. This is the case if we can prove that the flow φ_X^t of $X \in \Gamma(\Delta)$ preserves Δ. The fact that Δ is involutive is exactly the infinitesimal version of this statement, and must then be (very carefully) integrated to get global invariance.

2.1.2 Lie subgroups

Definition 2.1.14. Let G be a Lie group. A **Lie subgroup** of G is the data of a subset $H \subset G$ and a Lie group structure on H for which the inclusion in G is a Lie group morphism.
An **embedded Lie subgroup** is a subset $H \subset G$ which is a subgroup and an embedded submanifold.

Remarks.

- An embedded Lie subgroup is a Lie subgroup.
- A Lie subgroup is an immersed submanifold (because an injective Lie group morphism is an immersion).
- Whenever we consider an immersed Lie subgroup $H \subset G$, we have implicitly fixed a Lie group structure on H for which the inclusion is a Lie group morphism and an immersion.

Examples 2.1.15.

- The example $D_\alpha \subset \mathbb{R}^2/\mathbb{Z}^2$ previously described (for $\alpha \notin \mathbb{Q}$) is an immersed Lie subgroup which is not embedded.
- The kernel of a Lie group morphism is an embedded Lie subgroup (this is a consequence of Proposition 1.1.9).

Let us see how the exponential map of a Lie group restricts to a subgroup. If $H \subset G$ is an immersed Lie subgroup, then $\mathrm{Lie}(H) = T_eH \subset T_eG = \mathrm{Lie}(G)$. Since the inclusion of H into G is a Lie group morphism, we find:

Proposition 2.1.16. *Let G be a Lie group, and $H \subset G$ an immersed Lie subgroup. For $X \in T_eH \subset T_eG$, we have $\exp_G(X) = \exp_H(X)$ (hence $\exp_G(X) \in H$).*

We will see more implications of this later, once we know how to recover a Lie subgroup from a Lie subalgebra.

2.1.3 From Lie subalgebras to Lie subgroups

We will now see that there is a one on one correspondence between connected Lie subgroups (i.e. connected for the intrinsic topology) and Lie subalgebras.

Theorem 2.1.17. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The map $H \mapsto \mathrm{Lie}(H)$ is a bijection from the set of connected immersed Lie subgroups $H \subset G$ to the set of Lie subalgebras of $\mathfrak{g}$.*

The main ingredient of the proof is that any Lie subalgebra is associated to an integrable distribution.

Lemma 2.1.18. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then $\Delta_g = d_eL_g(\mathfrak{h})$ for $g \in G$ defines an involutive distribution Δ on g.*

Proof. Given a vector basis $(X_1, \dots, X_p)$ of $\mathfrak{h}$, we consider the associated left-invariant vector fields $\overline{X}_1, \dots, \overline{X}_p$. Notice that $(\overline{X}_1(x), \dots, \overline{X}_p(x))$ is a vector basis of Δ_x for all $x \in G$, therefore Δ is a distribution of rank $p = \dim \mathfrak{h}$.

If $X, Y \in \mathcal{X}(G)$ take values in Δ, we can consider functions $f_1, \dots, f_p$ and $g_1, \dots, g_p$ in $\mathcal{C}^\infty(G)$ such that $X = \sum_{i=1}^p f_i \overline{X}_i$ and $Y = \sum_{i=1}^p g_i \overline{X}_i$. By developing the expression of $[X, Y]$, we see that it is a linear combination of $[\overline{X}_i, \overline{X}_j]$, $\overline{X}_i$ and $\overline{X}_j$, which all take values in Δ because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{h}$. □

Proof of Theorem 2.1.17. Let us start with the surjectivity. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Define the distribution Δ on G by:

$$\forall x \in G \ \Delta_x = d_eL_x(\mathfrak{h})$$

It is involutive by Lemma 2.1.18, therefore integrable because of Frobenius' Theorem. We let $\mathcal{F}$ be the associated foliation (it is unique because of Corollary 2.1.8), and let $H = \mathcal{F}(e)$. It is a connected immersed submanifold of G.

By construction, the distribution Δ is left-invariant, i.e. $\Delta_{gx} = d_xL_g(\Delta_x)$ for all $g, x \in G$. It follows that for any $g \in G$, the foliation $\mathcal{F}'$ defined by $\mathcal{F}'(x) = L_g(\mathcal{F}'(g^{-1}x))$ is also tangent to Δ, so by Corollary 2.1.8 it is equal to $\mathcal{F}$, i.e. $\mathcal{F}(gx) = L_g(\mathcal{F}(x))$ for all $g, x \in G$.

For $x, y \in H = \mathcal{F}(e)$, we find:

$$\mathcal{F}(xy) = L_x(\mathcal{F}(y)) = L_x(\mathcal{F}(e)) = \mathcal{F}(x) = \mathcal{F}(e)$$

This shows that $xy \in \mathcal{F}(e) = H$.

$$\mathcal{F}(e) = \mathcal{F}(x^{-1}x) = L_{x^{-1}}(\mathcal{F}(x)) = L_{x^{-1}}(\mathcal{F}(e))$$

This implies that $x^{-1} = L_{x^{-1}}(e) \in \mathcal{F}(e) = H$, so H is a subgroup of G.

The fact that the group operation on H is smooth is a consequence of Proposition 2.1.11, so H is a Lie subgroup.

We now tackle the injectivity. Consider a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a connected immersed Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$. We wish to show that H is the leaf $\mathcal{F}(e)$ of the foliation $\mathcal{F}$ above.

It suffices to notice that setting $\mathcal{F}'(x) = L_x(H)$ defines a foliation $\mathcal{F}'$ tangent to Δ, so $\mathcal{F}' = \mathcal{F}$ thanks to Corollary 2.1.8. □

One of the consequences of this proof is that a connected Lie subgroup is a leaf of a foliation. Associated to Proposition 2.1.11, we get that they are weakly embedded.

Corollary 2.1.19. *Connected Lie subgroups are weakly embedded.*

This allows for a description of the Lie algebra of an immersed Lie subgroup that does not involve any differentiation.

Proposition 2.1.20. *Let G be a Lie group, and $H \subset$ an immersed Lie subgroup. Then*

$$\mathrm{Lie}(H) = \left\{ X \in \mathrm{Lie}(G) \,\middle|\, \forall t \in \mathbb{R}\ \exp_G(tX) \in H \right\}.$$

Proof. We have seen in Proposition 2.1.16 that $\exp_G(X) = \exp_H(X)$ for all $X \in \mathrm{Lie}(H)$, hence $\exp_G(tX) \in H$ for all $(t, X) \in \mathbb{R} \times \mathrm{Lie}(H)$.
Let $X \in \mathrm{Lie}(G)$ be such that: $\forall t \in \mathbb{R}\ \exp_G(tX) \in H$. Naively, we want to say that $X = \frac{d}{dt}\big|_{t=0} \exp_G(tX) \in T_eH = \mathrm{Lie}(H)$. This is valid because $t \mapsto \exp_G(tX)$ is smooth for the manifold structure on H and has the same derivative as in G, a consequence of Proposition 2.1.11 and of the proof of Theorem 2.1.17. □

2.1.4 Closed subgroups of Lie groups

The difference between immersed and embedded Lie subgroups is purely topological. Just as Lie groups have restrictions on their topology, so do Lie subgroups.

Proposition 2.1.21. *An embedded Lie subgroup is closed.*

Proof. Let G be a Lie group, and $H \subset G$ a Lie subgroup. Since H is embedded, it is locally closed (i.e. open in its closure $\overline{H}$). But $\overline{H}$ is a subgroup of G, and an open subgroup of $\overline{H}$ is also closed in $\overline{H}$ (see Lemma 1.3.8) hence closed in G. □

One of the most remarkable facts in Lie theory is that the converse holds: any closed subgroup is a Lie subgroup. This may seem extremely useful, as closed subgroups appear all the time, e.g. as stabilizers for some group action, but most of the time they are clearly level set of constant rank maps.

Theorem 2.1.22 (Cartan-Von Neumann Theorem)**.** *Let G be a Lie group. If $H \subset G$ is a closed subgroup, then H is an embedded Lie subgroup.*

Proof. With Proposition 2.1.20 in mind, we set:

$$V = \{X \in \mathfrak{g} | \forall t \in \mathbb{R} \ \exp_G(tX) \in H\}.$$

First step: Show that V is a vector subspace of $\mathfrak{g}$.

We have $0 \in V$. It is also straightforward that if $X \in V$ and $\lambda \in \mathbb{R}$ then $\lambda X \in V$.
Consider $X, Y \in V$, and $t \in \mathbb{R}$. According to Proposition 1.3.20, we have:

$$\exp_G(t(X+Y)) = \lim_{n \to +\infty} \left(\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right)\right)^n$$

Since $X \in V$ and $Y \in V$, we find:

$$\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right) \in H$$

Since H is a subgroup of G, we get:

$$\left(\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right)\right)^n \in H$$

Finally, because H is closed, we find:

$$\lim_{n \to +\infty} \left(\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right)\right)^n \in H$$

It follows that $X + Y \in V$, and V is a vector subspace of $\mathfrak{g}$.

Second step: Let $W \subset \mathfrak{g}$ be a supplementary subspace of V (i.e. $V \oplus W = \mathfrak{g}$). Prove the existence of a neighbourhood U of 0 in W such that $\exp_G(X) \notin H$ for all $X \in U \setminus \{0\}$.

Let us prove this by contradiction: if it were not the case, we could find $X_n \in W \setminus \{0\}$ such that $X_n \to 0$ and $\exp_G(X_n) \in H$.
Consider a norm $\|\cdot\|$ on W, and set $\alpha_n = \frac{1}{\|X_n\|}$. Using the compactness of spheres in finite dimension normed spaces, up to considering subsequences we can assume that $\alpha_n X_n \to X \in W \setminus \{0\}$. Let us show that $X \in V$

(which is a contradiction with $X \in W \setminus \{0\}$).
Let $t \in \mathbb{R}$. Set $k_n = \lfloor t\alpha_n \rfloor \in \mathbb{Z}$ and $r_n = \{t\alpha_n\} \in [0,1[$, so that $t\alpha_n = k_n + r_n$. Since $tX = \lim_{n\to+\infty} t\alpha_n X_n$, we find:

$$\begin{aligned}\exp_G(tX) &= \lim_{n\to+\infty} \exp_G(t\alpha_n X)\\ &= \lim_{n\to+\infty} \left(\exp_G(X_n)\right)^{k_n} \exp_G(r_n X_n)\end{aligned}$$

Since (r_n) is bounded and $X_n \to 0$, we have that $r_n X_n \to 0$, hence $\exp_G(r_n X_n) \to e$. We get:

$$\exp_G(tX) = \lim_{n\to+\infty} \left(\exp_G(X_n)\right)^{k_n}$$

As $\exp_G(X_n) \in H$, and H is a closed subgroup of G, we know that $\exp_G(tX) \in H$, hence $X \in V$, which is the aforementioned contradiction.

Third step: Build a trivialising chart for H on a neighbourhood of e.

Consider the map:

$$\varphi : \begin{cases} \mathfrak{g} = V \oplus W & \to & G \\ X + Y & \mapsto & \exp_G(X)\exp_G(Y) \end{cases}$$

Since $d_0\varphi = \mathrm{Id}$, the Local Inverse Function Theorem provides us with an open subset $U_V \subset V$ (resp. $U_W \subset W$, $U_G \subset G$) containing 0 (resp. 0, e) such that φ restricts to a diffeomorphism from $U_V + U_W$ onto U_G. According to the previous step, we can assume that $\exp_G(X) \notin H$ for all $X \in U_W \setminus \{0\}$. It follows that $\varphi(U_V) = H \cap U_G$, therefore φ is a trivialising chart for H on a neighbourhood of e.

Fourth step: For all $g \in H$, the map $L_g \circ \varphi$ is a trivialising chart for H around g, which shows that H is an embedded submanifold of G. □

2.2 Lie groups with a given Lie algebra

We will now study Lie groups that have the same Lie algebra (up to isomorphism). Note that according to Proposition 1.3.6, a Lie group and its neutral component share the same Lie algebra, so the problem reduces to connected Lie groups.

The aim of this section is to partially prove the following result.

Theorem 2.2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. There is a simply connected Lie group $\widetilde{G}$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*
Moreover, if G is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$, then G is isomorphic to a quotient $\widetilde{G}/\Gamma$ where Γ is a discrete subgroup of $Z(\widetilde{G})$.

2.2.1 Reminders on covering maps

Covering maps have a definition close to that of a local diffeomorphism, except that the locality should be considered on the target manifold rather than on the source.

Definition 2.2.2. Let M,N be smooth manifolds. A smooth map $p : M \to N$ is called a **covering map** if every $y \in N$ has a open neighbourhood V such that, for every connected component U of $p^{-1}(V)$, the restriction $p|_U : U \to V$ is a diffeomorphism.

Remarks.

- For $y \in N$, the set $p^{-1}(\{y\})$ is called the **fibre over** y**.**
- If N is connected, then all fibres have the same cardinality. This number (eventually infinite) is called the **order** of p.

An important family of covering maps is quotients by actions of discrete groups.

Definition 2.2.3. An action of a group Γ on a manifold M is called **properly discontinuous** if for every compact subset $K \subset M$, the set

$$\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$$

is finite.
The action is called **free** if

$$\forall x \in M \forall \gamma \in \Gamma \ \gamma.x = x \Rightarrow \gamma = e.$$

Proposition 2.2.4. *Let M be a manifold and Γ a group that acts on M by diffeomorphisms. If the action is free and properly discontinuous, then there is a unique manifold structure on the quotient $\Gamma\backslash M$ for which the quotient map $\pi : M \to \Gamma\backslash M$ is a covering map.*

Definition 2.2.5. Let $p : M \to N$ and $\varphi' : M' \to N$ be covering maps. A **covering isomorphism** is a diffeomorphism $\varphi : M \to M'$ such that $p' \circ \varphi = p$. If $p = p'$, we call it a **deck transformation**.

Proposition 2.2.6. *Let $p : M \to N$ be a covering map between connected manifolds, and let $\Gamma \subset \mathrm{Diff}(M)$ be set of deck transformations. Then Γ is a group and its action on M is free and properly discontinuous.*

Both properties are a consequence of the following lemma.

Lemma 2.2.7. *Let $p : M \to N$ be a covering map. Let $y \in N$, and let $V \subset N$ be an open neighbourhood of z such that the restriction of p to every connected component of $p^{-1}(V)$ is a diffeomorphism onto V.*

For $x \in p^{-1}(\{y\})$, let U_x be the connected component of $p^{-1}(V)$ containing x, and $\psi_x : V \to U_x$ the inverse of p.

If a deck transformation φ satisfies $\varphi(U_x) \cap U_{x'} \neq \emptyset$ for some $x, x' \in p^{-1}(\{y\})$, then $\varphi(x) = x'$ and $\varphi = \psi_{x'} \circ p$ on U_x.

Proof. Consider $x, x' \in p^{-1}(\{y\})$ and $z \in U_x$ such that $\varphi(z) \in U_{x'}$. Note that $\varphi(z) \in U_{x'}$ and $p(\varphi(z)) = p(z)$, so $\varphi(z) = \psi_{x'} \circ p(z)$.

Reciprocally, if $\varphi(z) = \psi_{x'} \circ p(z)$ for some $z \in U_x$, then $\varphi(z) \in \psi_{x'}(V) = U_{x'}$. It follows that $U_x \cap \varphi^{-1}(U_{x'}) = \{z \in U_x \,|\, \varphi(z) = \psi_{x'} \circ p(z)\}$, and this set is both open and closed in U_x. If it is not empty, it must be equal to U_x, hence the result. □

Proof of Proposition 2.2.6. The fact that Γ is a group is quite straightforward. Lemma 2.2.7 applied to $x' = x$ shows that the fixed point set $\mathrm{Fix}(\varphi)$ of $\varphi \in \Gamma$ is open and closed. Since M is connected, we find that the action is free.

Let $K \subset M$ be a compact set, and consider a sequence $(\varphi_i)_{i \in \mathbb{N}}$ of elements of Γ such that $\varphi_i(K) \cap K \neq \emptyset$. Let $x_i \in K$ be such that $\varphi_i(x_i) \in K$. Up to considering a subsequence, we may assume that x_i converges to some $x \in K$ and $\varphi_i(x_i)$ converges to some $x' \in K$. Let $y = p(x) = p(x')$, and let us use the same notations as in Lemma 2.2.7. If i is large enough, then $x_i \in U_x$ and $\varphi_i(x_i) \in U_{x'}$, thus $\varphi_i(x) = x'$ thanks to Lemma 2.2.7. It follows from the freeness of the action that there all these elements φ_i for i large enough are equal to each other. We have shows that any sequence in $\{\varphi \in \Gamma \,|\, \varphi(K) \cap K \neq \emptyset\}$ has an eventually constant subsequence, so this set must be finite.

□

The quotient manifold of M by the group of deck transformations is not always diffeomorphic to N.

Definition 2.2.8. A **Galois covering** is a covering map $p : M \to N$ such that the group of deck transformations acts transitively on each fibre.

The covering maps that are obtained by quotients by free and properly discontinuous actions are exactly the Galois coverings.

Let us now recall the notion of universal covering.

Definition 2.2.9. A manifold M is **simply connected** M if it is connected and any continuous map $f : \mathbb{S}^1 \to M$ extends to a continuous map $\overline{\mathbb{D}} \to M$.

Definition 2.2.10. A universal cover of a manifold N is a covering map $f : M \to N$ where M is simply connected.

Theorem 2.2.11. *Let N be a connected manifold. Then N admits a universal cover $\pi = \widetilde{N} \to N$. If $p : M \to N$ is another universal cover, then for all $(x, x', y) \in \widetilde{N} \times M \times N$ satisfying $\pi(x) = p(x') = y$, there is a unique covering isomorphism $\varphi : \widetilde{N} \to M$ such that $\varphi(x) = x'$.*

In particular, the universal cover is Galois, thus the identification $M \approx \pi_1(M)\backslash\widetilde{M}$.

Definition 2.2.12. Let N be a connected manifold. Its **fundamental group** $\pi_1(N)$ is the group of deck transformations of its universal cover.

Remarks.

- The fundamental group is only well defined up to isomorphism.
- If M is simply connected and Γ acts freely and properly discontinuously on M, then $\pi_1(\Gamma\backslash M) \approx \Gamma$.

This is a geometric definition of the fundamental group: we define it through one of its actions (compare with the topological definition through homotopy classes).

Covering maps are nice to work with because of the lifting property.

Theorem 2.2.13. *Let M, N, P be smooth manifolds, $p : M \to N$ a covering map and $f : P \to N$ a smooth map. Assume that P is simply connected. Then for all $(x, y) \in P \times M$ satisfying $p(y) = f(x)$, there is a unique smooth map $\widetilde{f} : P \to M$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(x) = y$ (such a map $\widetilde{f}$ is a called a **lift** of f).*

2.2.2 Coverings of Lie groups

We will now study coverings of Lie groups.

Definition 2.2.14. Let G and $\widehat{G}$ be Lie groups. A map $f : \widehat{G} \to G$ is a **Lie group covering** if it is both a Lie group morphism and a covering map.

We will see that a Lie group covering $\widehat{G} \to G$ is always a quotient by a subgroup $\Gamma \subset \widehat{G}$. Such a subgroup Γ must be normal (so that the quotient $\Gamma\backslash\widehat{G}$ inherits a group structure) and discrete (for the quotient to have the same dimension as that of G, the subgroup Γ must have dimension 0). We first notice that such a group must be central, i.e. included in the centre $Z(\widehat{G})$.

Proposition 2.2.15. *Let G be a connected Lie group, and $\Gamma \subset G$ a discrete subgroup. Then Γ is normal if and only if $\Gamma \subset Z(G)$.*

Proof. Assume that Γ is normal. Let $\gamma \in \Gamma$. The image of G by the continuous map $g \mapsto g\gamma g^{-1}$ is connected and included in the discrete set Γ, hence reduced to γ. This shows that $\gamma \in Z(G)$.
If $\Gamma \subset Z(G)$, we find $g\Gamma g^{-1} = \Gamma$ for all $g \in G$, so Γ is normal. □

The quotient by such a subgroup is always a Lie group covering.

Proposition 2.2.16. *Let G be a connected Lie group, and let $\Gamma \subset G$ be a discrete normal subgroup. There is a unique Lie group structure on $\Gamma\backslash G$ for which the projection $\pi : G \to \Gamma\backslash G$ is a Lie group covering. The Lie algebras of G and $\Gamma\backslash G$ are isomorphic to each other.*

Proof. Let us first show that the left action of Γ on G is free and properly discontinuous.
It is free because $gx = x \Rightarrow g = xx^{-1} = e$.
For any subset $K \subset G$, and $g \in G$, we have that:

$$gK \cap K \neq \emptyset \iff g \in KK^{-1}$$

where $KK^{-1} = \{xy^{-1} | x, y \in K\}$. If K is compact, then so is KK^{-1}. Since Γ is discrete, the intersection $\Gamma \cap KK^{-1}$ is finite, and it follows that the action on G is properly discontinuous.

Following Proposition 2.2.4, there is a unique manifold structure on $\Gamma\backslash G$ for which π is a smooth covering map.

Since the group operation in $\Gamma\backslash G$ can be expressed through local inverses of π, it is smooth, i.e. $\Gamma\backslash G$ is a Lie group, and π a Lie group covering. □

It happens that every Lie group covering can be obtained in this way.

Proposition 2.2.17. *Let $\widehat{G}$ and G be connected Lie groups, and $p : \widehat{G} \to G$ a Lie group covering. There exists a discrete subgroup $\Gamma \subset Z(\widehat{G})$ and a Lie group isomorphism $\varphi : \Gamma\backslash\widehat{G} \to G$ such that $\varphi \circ \pi = p$ where $\pi : \widehat{G} \to \Gamma\backslash\widehat{G}$ is the projection.*

Proof. Set $\Gamma = p^{-1}(\{e\})$. It is a discrete normal subgroup of $\widehat{G}$, and $p(x) = p(y) \iff xy^{-1} \in \Gamma$, which allows us to construct φ using the universal property of the quotient. □

We will see that any smooth covering of a Lie group is a Lie group covering. Let us start with the universal cover.

Lemma 2.2.18. *Let G be a Lie group, and let $p : \widetilde{G} \to G$ be its universal cover. There is a Lie group structure on $\widetilde{G}$ for which p is a Lie group morphism. Moreover, the Lie algebra of $\widetilde{G}$ is isomorphic to G.*

Proof. We first need to choose a point $\widetilde{e}$ in the fibre of e, which we will set to be the neutral element of $\widetilde{G}$.

Consider the map $f : \widetilde{G} \times \widetilde{G} \to G$ defined by $f(x, x') = p(x)p(x')$. Since $\widetilde{G} \times \widetilde{G}$ is simply connected, according the the Lifting Theorem (Theorem 2.2.13), there is a unique smooth map $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $p \circ \widetilde{m} = f$ and $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$. We now want to check that $\widetilde{f}$ is a group operation that meets the requirements.

Since $f(e, \cdot) = p$, and $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$, it follows from the uniqueness of the lift to the universal cover that $\widetilde{m}(\widetilde{e},\cdot) = \mathrm{Id}$, i.e. $\widetilde{e}$ is neutral for multiplication on the left: $\forall x \in \widetilde{G}\ \widetilde{m}(\widetilde{e},x) = x$. A similar argument shows that $\widetilde{e}$ is neutral for multiplication on the right.

We now have to prove associativity, i.e.

$$\forall x,y,z \in \widetilde{G}\ \widetilde{m}(\widetilde{m}(x,y),z) = \widetilde{m}(x,\widetilde{m}(y,z))$$

For this, consider both expressions as functions in z, with x and y fixed. They both coincide at $z = \widetilde{e}$, and they are both lifts of $z \mapsto p(x)p(y)p(z)$ (using the associativity of G). It follows from the uniqueness of the lift that they are equal, i.e. the law $\widetilde{m}$ is associative.

We now have to prove the existence of an inverse. For this consider the map $\theta : \widetilde{G} \to \widetilde{G}$ that lifts $x \mapsto p(x)^{-1}$, such that $\theta(\widetilde{e}) = \widetilde{e}$. The map $x \mapsto \widetilde{m}(x,\theta(x))$ is a lift of a constant map, so it must be constant. It follows that $\widetilde{m}(x,\theta(x)) = \widetilde{e}$ for all $x \in \widetilde{G}$. Similarly we get $\widetilde{m}(\theta(x),x) = \widetilde{e}$, i.e. $\theta(x)$ is an inverse of x.

The fact that p is a group morphism is exactly the lifting property $p\circ\widetilde{m} = f$.

Finally, since p is a Lie group morphism, its differential $d_e p$ is a Lie algebra morphism from $\mathrm{Lie}(\widetilde{G})$ to $\mathrm{Lie}(G)$. Since p is a local diffeomorphism, it is a Lie algebra isomorphism. □

Examples 2.2.19.

- For $n \geq 3$, the universal cover of $\mathrm{SO}(n,\mathbb{R})$ is called the **spin group** $\mathrm{Spin}(n)$. Note that $\mathrm{Spin}(3) \approx \mathrm{SU}(2)$.

- The universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is an interesting Lie group that doesn't have a simpler definition.

Corollary 2.2.20. *Let G be a connected Lie group. Its fundamental group $\pi_1(G)$ is abelian.*

Proof. According to Lemma 2.2.18 and Proposition 2.2.17, the fundamental group $\pi_1(G)$ is isomorphic to a discrete normal subgroup of $\widetilde{G}$, hence abelian because of Proposition 2.2.15. □

Proposition 2.2.21. *Let G be a connected Lie group, and let $p : M \to G$ be a smooth covering. There is a Lie group structure on M for which p is a Lie group morphism. Moreover, the Lie algebra of M is isomorphic to the Lie algebra of G.*

Proof. Every covering of G is isomorphic to $\widetilde{G}/\Gamma'$ where Γ' is a subgroup of $\pi_1(G)$. The result is a consequence of Proposition 2.2.16. □

Proposition 2.2.22. *Let $f : G \to H$ be a Lie group morphism between connected Lie groups. If $d_e f$ is invertible, then f is a covering map.*

Proof. According to Proposition 1.1.9, f is a local diffeomorphism, so $\ker f$ is an embedded Lie subgroup, with Lie algebra $\ker d_e f = \{0\}$. It follows that $\ker f$ is a discrete subgroup of G, so by Proposition 2.2.16 there is a (unique) Lie group structure on $G/\ker f$ for which the projection π is a Lie group covering. Now f factorises to a Lie group morphism $\overline{f} : G/\ker f \to H$. It is injective by the universal property of quotients, and a local diffeomorphism because $d_e\overline{f} = d_e f \circ (d_e\pi)^{-1}$. It follows that $f(G) = \overline{f}(G/\ker f)$ is an open subgroup of the connected Lie group H, so $\overline{f}$ is also onto, and it is a diffeomorphism. Finally $f = \overline{f} \circ \pi$ is the composition of a covering map and a diffeomorphism, so it is a covering map. □

2.2.3 Lie's third theorem

In order to finish the proof of Theorem 2.2.1, we need to prove that any finite dimensional Lie algebra is the Lie algebra of a Lie group. We will use the following result without proof.

Theorem 2.2.23 (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. Then $\mathfrak{g}$ possesses a finite dimensional faithful linear representation.*

Remark. The equivalent statement for Lie groups is false. The Lie group $\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ (which can be shown using Iwasawa decomposition), so its fundamental group is isomorphic to $\mathbb{Z}$. One can show that any finite dimensional linear representation of the universal cover of $\mathrm{SL}(2,\mathbb{R})$ factorizes through $\mathrm{SL}(2,\mathbb{R})$, so it cannot be faithful.

Since a Lie subalgebra of $\mathfrak{gl}(V)$ is the Lie algebra of a Lie subgroup of $\mathrm{GL}(V)$ according to Theorem 2.1.17, Ado's Theorem has the following consequence:

Theorem 2.2.24. *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. There is a Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$.*

Combining Theorem 2.2.24, Proposition 2.2.21 and Proposition 2.2.17, we get a proof of Theorem 2.2.1.

Chapter 3

The structure of Lie algebras

3.1 Ideals of a Lie algebra

We can define an ideal of a Lie algebra just as we would for a ring.

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie algebra. An **ideal** of $\mathfrak{g}$ is a vector subspace $\mathfrak{I} \subset \mathfrak{g}$ such that:

$$\forall X \in \mathfrak{g}\ \forall Y \in \mathfrak{I}\ [X,Y] \in \mathfrak{I}$$

Note that an ideal is a Lie subalgebra (but the converse is false). Ideals are related to normal subgroups.

Proposition 3.1.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and H a Lie subgroup with Lie algebra $\mathfrak{h}$.*

If H is a normal subgroup, then its Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, and G and H are connected, then H is normal in G.

Proof. First assume that H is normal in G. Consider $g \in G$ and $Y \in \mathfrak{h}$. According to Corollary 1.3.19, we have $\exp_G(t\,\mathrm{Ad}(g)Y) = i_g(\exp_G(tY))$. By differentiation we find

$$\mathrm{Ad}(g)Y = \left.\frac{d}{dt}\right|_{t=0} i_g(\exp_G(tY)).$$

Since H is normal in G, the right hand term is in T_eH, hence $\mathrm{Ad}(g)Y \in \mathfrak{h}$. Applying this formula to $g = \exp_G(sX)$ for $X \in \mathfrak{g}$ and differentiating at $s = 0$ gives $[X,Y] \in \mathfrak{h}$.

Now assume that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and that G and H are connected. For any $g \in G$, $\mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}$. Indeed, this is true for $g = \exp_G(X)$ with $X \in \mathfrak{g}$ since $\mathrm{Ad}(\exp_G(X)) = \exp(\mathrm{ad}(X))$ (Corollary 1.3.19), hence for all elements because G is connected (Proposition 1.3.16). Consequently, the Lie algebra of the connected Lie subgroup gHg^{-1} is $\mathfrak{h}$. By uniqueness in Theorem 2.1.17, we find that $gHg^{-1} = H$, i.e. H is normal in G. □

Quotients of Lie algebras by ideals are Lie algebras.

Proposition 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{I} \subset \mathfrak{g}$ an ideal. There is a unique Lie bracket on the vector space $\mathfrak{g}/\mathfrak{I}$ for which the projection is a Lie algebra morphism.*
If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then $\ker f$ is an ideal and $\mathrm{im} f$ is a Lie algebra isomorphic to $\mathfrak{g}/\ker f$.
The intersection and the sum of ideals are ideals.

This last property allows us to define the ideal generated by a subset $A \subset \mathfrak{g}$ as the smallest ideal of $\mathfrak{g}$ containing A.

Proposition 3.1.4. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{I}, \mathfrak{J} \subset \mathfrak{g}$ be ideals. The vector space $[\mathfrak{I}, \mathfrak{J}]$ spanned by all $[X, Y]$ for $X \in \mathfrak{I}$ and $Y \in \mathfrak{J}$ is an ideal of $\mathfrak{g}$.*

Proof. Let $(X, Y, Z) \in \mathfrak{g} \times \mathfrak{I} \times \mathfrak{J}$. The Jacobi identity yields:

$$[X,[Y,Z]] = [Y,\underbrace{[X,Z]}_{\in \mathfrak{J}}] + [\underbrace{[X,Y]}_{\in \mathfrak{I}},Z] \in [\mathfrak{I},\mathfrak{J}]$$

□

Definition 3.1.5. The **derived algebra** of a Lie algebra $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

3.2 The Killing form of a Lie algebra

A very remarkable property of finite dimensional Lie algebras is that they come with a symmetric bilinear form, which will be an important tool in the structure theory of Lie algebras.

Definition 3.2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The Killing form of $\mathfrak{g}$ is the bilinear form B on $\mathfrak{g}$ defined by

$$\forall X, Y \in \mathfrak{g} \; B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Proposition 3.2.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and B its Killing form. Then B is symmetric and* ad*-invariant, i.e.*

$$\forall X, Y, Z \in \mathfrak{g} \; B(\mathrm{ad}(X)Y, Z) + B(Y, \mathrm{ad}(X)Z) = 0.$$

Proof. The symmetry of B is a consequence of the symmetry of the bilinear map $A, B \mapsto \mathrm{Tr}(AB)$ on $\mathfrak{gl}(\mathfrak{g})$. The ad-invariance of B is a consequence of the fact that ad is a Lie algebra morphism:

$$\begin{aligned}
B(\mathrm{ad}(X)Y,Z) &= \mathrm{Tr}(\mathrm{ad}([X,Y]) \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}((\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)) \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z)) - \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(Z) \circ \mathrm{ad}(X)) - \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}(\mathrm{ad}(Y) \circ (\mathrm{ad}(Z) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Z))) \\
&= \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}([Z,X])) \\
&= -B(Y, \mathrm{ad}(X)Z)
\end{aligned}$$

□

Remark. Here ad-invariance means that $\mathrm{ad}(\mathfrak{g})$ is included in the Lie algebra $\mathfrak{o}(B)$ of the Lie group $\mathrm{O}(B)$ of linear isomorphisms of $\mathfrak{g}$ preserving B.

Examples 3.2.3.

- The Killing form of $\mathfrak{gl}(n,\mathbb{R})$ is $B(X,Y) = 2n\,\mathrm{Tr}(XY) - 2\,\mathrm{Tr}(X)\,\mathrm{Tr}(Y)$.
- The Killing form of $\mathfrak{sl}(n,\mathbb{R})$ is $B(X,Y) = 2n\,\mathrm{Tr}(XY)$.
- The Killing form of an Abelian Lie algebra vanishes.
- The Killing form of $\mathfrak{heis}(3)$ vanishes.
- The Killing form of $\mathfrak{sol}(3)$, the 3-dimensional Lie algebra spanned by T, A, B satisfying

  $$[T,A] = A,\ [T,B] = -B,\ [A,B] = 0,$$

  has the following matrix in the basis (T,A,B):

  $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the Lie algebra of a Lie group is finite dimensional, so it is possible to define its Killing form (Definition 3.2.1).

Proposition 3.2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $g \in G$, the map* $\mathrm{Ad}(g)$ *preserves the Killing form of $\mathfrak{g}$, i.e.*

$$\forall X, Y \in \mathfrak{g}\ B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X,Y).$$

Proof. The result of proposition 1.2.13 translates as

$$\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}$$

for $g \in G$ and $X \in \mathfrak{g}$. Invariance of the Killing form by $\mathrm{Ad}(g)$ is a consequence of the invariance of the trace by conjugation in $\mathrm{GL}(\mathfrak{g})$. □

Remark. This means that the adjoint representation of a Lie group G lands in the subgroup $\mathrm{O}(B)$ of $\mathrm{GL}(\mathfrak{g})$.

In general, the Killing form of a subalgebra is not the restriction of the Killing form of the ambient Lie algebra (e.g. compare the Killing form of $\mathfrak{sol}(3)$ with the restriction of the Killing form of $\mathfrak{gl}(3,\mathbb{R})$). However, they coincide for ideals.

Proposition 3.2.5. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $\mathfrak{I} \subset \mathfrak{g}$ an ideal. The Killing form of $\mathfrak{I}$ is the restriction to $\mathfrak{I} \times \mathfrak{I}$ of the Killing form of $\mathfrak{g}$.*

Proof. Consider a vector basis of $\mathfrak{g}$ adapted to a decomposition $\mathfrak{g} = \mathfrak{I} \oplus V$. The matrices of operators $\mathrm{ad}(X)$ for $X \in \mathfrak{I}$ in this basis are bloc diagonal with a vanishing bloc, so the trace of the whole matrix is equal to the trace of the (potentially) non-vanishing bloc. □

Proposition 3.2.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The kernel of the Killing form*

$$\ker B = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g}\ B(X,Y) = 0\}$$

is an ideal of $\mathfrak{g}$.

Proof. Let $X \in \ker B$, and $Y \in \mathfrak{g}$. For $Z \in \mathfrak{g}$, we find:

$$\begin{aligned} B([X,Y],Z) &= -B(\mathrm{ad}(Y)X,Z) \\ &= B(X,\mathrm{ad}(Y)Z) \\ &= 0 \end{aligned}$$

This yields $[X,Y] \in \ker B$. Since $\ker B$ is also a vector subspace of $\mathfrak{g}$, it is an ideal. □

3.3 Solvable Lie algebras

Definition 3.3.1. Let $\mathfrak{g}$ be a Lie algebra. The lower central series $C_i(\mathfrak{g})$ is defined by $C_0(\mathfrak{g}) = \mathfrak{g}$ and $C_{i+1}(\mathfrak{g}) = [\mathfrak{g}, C_i(g)]$.
The derived series $D_i(\mathfrak{g})$ is defined by $D_0(\mathfrak{g}) = \mathfrak{g}$ and $D_{i+1}(\mathfrak{g}) = [D_i(\mathfrak{g}), D_i(\mathfrak{g})]$.
A Lie algebra $\mathfrak{g}$ is called **nilpotent** if there is some n such that $C_n(\mathfrak{g}) = \{0\}$.
A Lie algebra $\mathfrak{g}$ is called **solvable** if there is some n such that $D_n(\mathfrak{g}) = \{0\}$.

A quick induction argument shows that $D_n(\mathfrak{g}) \subset C_n(\mathfrak{g})$ for all n, so any nilpotent Lie algebra is solvable.

Examples 3.3.2.

- Consider the 3-dimensional Lie algebra $\mathfrak{heis}(3)$ spanned by X, Y, Z satisfying $[X,Y] = Z$ and $[T,X] = [T,Y] = 0$. Then $C_1(\mathfrak{heis}(3)) = \mathbb{R}.Z$ and $C_2(\mathfrak{heis}(3)) = \{0\}$, so $\mathfrak{heis}(3)$ is nilpotent (and therefore solvable).

- Consider the 3-dimensional Lie algebra $\mathfrak{sol}(3)$ spanned by T, A, B satisfying $[T,A] = A$, $[T,B] = -B$ and $[A,B] = 0$. For the derived series we find $D_1(\mathfrak{sol}(3)) = \mathbb{R}.A \oplus \mathbb{R}.B$ and $D_2(\mathfrak{sol}(3)) = \{0\}$, so $\mathfrak{sol}(3)$ is solvable. However we find $C_n(\mathfrak{sol}(3)) = \mathbb{R}.A \oplus \mathbb{R}.B$ for any $n \geq 1$, so $\mathfrak{sol}(3)$ is not nilpotent.

- For $n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, consider the following Lie algebras:

$$\mathfrak{b}(n,\mathbb{K}) = \left\{ \begin{pmatrix} * & \dots & \dots & * \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \right\} \subset \mathfrak{gl}(n,\mathbb{K}) \ ;$$
$$\mathfrak{n}(n,\mathbb{K}) = \left\{ \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}(n,\mathbb{K}).$$

  Then $\mathfrak{n}(n,\mathbb{K})$ is nilpotent, and $\mathfrak{b}(n,\mathbb{K})$ is solvable but not nilpotent. Note that $\mathfrak{n}(n,\mathbb{K}) = [\mathfrak{b}(n,\mathbb{K}), \mathfrak{b}(n,\mathbb{K})]$.

Let us start with a few elementary properties of solvable Lie algebras.

Proposition 3.3.3. *Any subalgebra or quotient of a solvable Lie algebra is solvable. Given a short exact sequence* $0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$ *of Lie algebras, if* $\mathfrak{a}$ *and* $\mathfrak{c}$ *are solvable then so is* $\mathfrak{b}$*.*

Remark. Solvable Lie algebras form the smallest family of Lie algebras that contains Abelian Lie algebras and that is stable under short exact sequences.

Proof. If $\mathfrak{a} \subset \mathfrak{b}$ is a Lie subalgebra, then $D_n(\mathfrak{a}) \subset D_n(\mathfrak{b})$ for any n, so a subalgebra of a solvable Lie algebra is solvable. If $\mathfrak{a}$ is an ideal, then $d_n(\mathfrak{b}/\mathfrak{a})$ is the projection of $D_n(\mathfrak{b})$, this proves the statement about quotients.

Now let $0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$ be a short exact sequence of Lie algebras, and assume that $\mathfrak{a}$ and $\mathfrak{c}$ are solvable. Let n be such that $D_n(\mathfrak{c}) = \{0\}$. Then $D_n(\mathfrak{b}) \subset \mathfrak{a}$, and $D_{n+k}(\mathfrak{b}) \subset D_k(\mathfrak{a})$ for all $k \in \mathbb{N}$, so $\mathfrak{b}$ is also solvable. □

Consequently, any subalgebra of $\mathfrak{b}(n,\mathbb{K})$ is solvable. It happens that any complex solvable Lie algebra is isomorphic to a subalgebra of $\mathfrak{b}(n,\mathbb{C})$.

Theorem 3.3.4 (Lie's Theorem)**.** *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over* $\mathbb{C}$*. If* $\mathfrak{g}$ *is solvable, there is an injective Lie algebra morphism* $\pi : \mathfrak{g} \to \mathfrak{b}(n,\mathbb{C})$ *for some* $n \in \mathbb{N}$*.*

A non trivial consequence is that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. Note that this also holds for real solvable Lie algebras (as $\mathfrak{g} \otimes \mathbb{C}$ is also solvable).

Solvability can be expressed in terms of the Killing form.

Theorem 3.3.5 (Cartan's criterion)**.** *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. Then* $\mathfrak{g}$ *is solvable if and only if* $B(X,Y) = 0$ *for all* $X \in \mathfrak{g}$ *and* $Y \in [\mathfrak{g},\mathfrak{g}]$*.*

In particular, any Lie algebra with a vanishing Killing form is solvable. This is not the case for all solvable Lie algebras, as is shown by $\mathfrak{sol}(3)$.

Theorem 3.3.6 (Engel's Theorem). *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if* $\mathrm{ad}(X)$ *is nilpotent for all $X \in \mathfrak{g}$.*

In order to prove that a Lie algebra is nilpotent, i.e. that $C_n(\mathfrak{g}) = \{0\}$ for some $n \in \mathbb{N}$, one must show that all Lie brackets $[X_1,[X_2,[\dots X_n]\dots]] = 0$ for any $X_1,\dots,X_n \in \mathfrak{g}$. Engel's Theorem states that it is sufficient to check this for the specific case $X_1 = \cdots = X_{n-1}$.

Proposition 3.3.7. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{I},\mathfrak{J} \subset \mathfrak{g}$ are solvable ideals, then $\mathfrak{I}+\mathfrak{J}$ is solvable.*

Proof. Applying Proposition 3.3.3 twice, we see that the quotient $(\mathfrak{I}+\mathfrak{J})/\mathfrak{I} \approx \mathfrak{J}/(\mathfrak{I}\cap\mathfrak{J})$ is solvable then that $\mathfrak{I}+\mathfrak{J}$ is solvable because of the short exact sequence

$$0 \to \mathfrak{I} \to \mathfrak{I}+\mathfrak{J} \to (\mathfrak{I}+\mathfrak{J})/\mathfrak{I} \to 0.$$

□

Proposition 3.3.8. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There is a unique solvable ideal* $\mathrm{Rad}(\mathfrak{g}) \subset \mathfrak{g}$, *called the* ***solvable radical*** *of $\mathfrak{g}$, such that any solvable ideal of $\mathfrak{g}$ is included in* $\mathrm{Rad}(\mathfrak{g})$.

Proof. Uniqueness is a consequence of the existence as two such ideals must contain each other.

Let $k = \max\{\dim\mathfrak{I} \,|\, \mathfrak{I}\subset\mathfrak{g} \text{ solvable ideal }\}$, and let $\mathfrak{I}\subset\mathfrak{g}$ be a solvable ideal of dimension k. If $\mathfrak{J}$ is a solvable ideal of $\mathfrak{g}$, then so is $\mathfrak{I}+\mathfrak{J}$ by Proposition 3.3.7, and the maximality of $\dim\mathfrak{I}$ implies that $\mathfrak{I}=\mathfrak{I}+\mathfrak{J}$, i.e. $\mathfrak{J}\subset\mathfrak{I}$. □

Note that the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ cannot contain any non trivial solvable ideal, as its preimage would be a solvable ideal containing $\mathrm{Rad}(\mathfrak{g})$. This means that there is always a short exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0$$

where $\mathrm{Rad}(\mathfrak{g})$ is solvable and $\mathrm{Rad}\left(\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})\right) = 0$. This means that the classification of finite dimensional Lie algebras can be broken down into solvable Lie algebras and Lie algebras with a trivial solvable radical.

Proposition 3.3.9. *If $\mathfrak{g}$ is a finite dimensional Lie algebra and B is its Killing form, then* $\ker B \subset \mathrm{Rad}(\mathfrak{g})$.

Proof. We have seen in Proposition 3.2.6 that $\ker B$ is an ideal, so its Killing form is the restriction of B by Proposition 3.2.5, i.e. its Killing form vanishes. According to Cartan's criterion (Theorem 3.3.5), $\ker B$ is solvable. □

3.4 Semi-simple Lie algebras

Definition 3.4.1. A Lie algebra $\mathfrak{g}$ is called **simple** if $\dim \mathfrak{g} \geq 2$ and the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. It is called **semi-simple** if the only abelian ideal is $\{0\}$.

3.4.1 Cartan's semi-simplicity criterion

Theorem 3.4.2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following are equivalent:*

1. $\mathfrak{g}$ *is semi-simple.*
2. *The solvable radical of $\mathfrak{g}$ is trivial (i.e.* $\mathrm{Rad}(\mathfrak{g}) = \{0\}$*).*
3. *The Killing form of $\mathfrak{g}$ is non-degenerate (i.e.* $\ker B = \{0\}$*).*
4. *There are simple ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_k \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ for $i \neq j$.*

Remark. In particular, for any finite dimensional Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semi-simple.

The third property, known as **Cartan's semi-simplicity criterion**, is by far the one that we will use most often, as it turns a semi-simple Lie algebra into a geometric object.

Examples 3.4.3.

- The Killing form of $\mathfrak{sl}(n, \mathbb{K})$ is given by $B(X,Y) = 2n\,\mathrm{Tr}(XY)$, it is non degenerate for $n \geq 2$.
- The Killing form of $\mathfrak{so}(n, \mathbb{K})$ is given by $B(X,Y) = (n-2)\,\mathrm{Tr}(XY)$. It is non degenerate for $n \geq 3$.
- The Killing form of $\mathfrak{sp}(2n, \mathbb{K})$ is given by $B(X,Y) = 2(n+1)\,\mathrm{Tr}(XY)$, it is non degenerate for $n \geq 1$.

There is however no easy way of knowing if a Lie algebra is simple.

Examples 3.4.4.

- The Lie algebra $\mathfrak{sl}(n, \mathbb{K})$ is simple for $n \geq 2$.
- The Lie algebra $\mathfrak{so}(n, \mathbb{K})$ is simple for $n = 3$ or $n \geq 5$.
- The Lie algebra $\mathfrak{sp}(2n, \mathbb{K})$ is simple for $n \geq 1$.

For $\mathbb{K} = \mathbb{C}$, this list of simple Lie algebras is almost complete.

Theorem 3.4.5. *Up to five exceptions, any finite dimensional Lie algebra over* $\mathbb{C}$ *is isomorphic to some* $\mathfrak{sl}(n,\mathbb{C})$ *(for* $n \geq 2$*),* $\mathfrak{so}(n,\mathbb{C})$ *(for* $n = 3$ *or* $n \geq 5$*) or* $\mathfrak{sp}(2n,\mathbb{C})$ *(for* $n \geq 1$*).*

There are some redundancies in this list, given by the **accidental isomorphisms:**

$$\mathfrak{sl}(2,\mathbb{C}) \approx \mathfrak{so}(3,\mathbb{C}) \approx \mathfrak{sp}(2,\mathbb{C}) \; ; \quad \mathfrak{so}(5,\mathbb{C}) \approx \mathfrak{sp}(4,\mathbb{C}) \; ; \quad \mathfrak{sl}(4,\mathbb{C}) \approx \mathfrak{so}(6,\mathbb{C}) \, .$$

The missing $\mathfrak{so}(4,\mathbb{C})$ is also involved in such an isomorphism:

$$\mathfrak{so}(4,\mathbb{K}) \approx \mathfrak{so}(3,\mathbb{K}) \oplus \mathfrak{so}(3,\mathbb{K}) \, .$$

There is also a classification of real simple Lie algebras, but the list is longer (and there are more exceptions). If $\mathfrak{g}$ is a real semi-simple Lie algebra, then $\mathfrak{g}\otimes\mathbb{C}$ is a complex semi-simple Lie algebra (this is more tricky with simple Lie algebras, as $\mathfrak{so}(3,1)$ is simple yet $\mathfrak{so}(3,1) \otimes \mathbb{C} \approx \mathfrak{so}(4,\mathbb{C})$ is not). But there can be several non isomorphic real Lie algebras with the same complexification.

$$\begin{aligned} &\mathfrak{sl}(n,\mathbb{R}) \otimes C \approx \mathfrak{su}(n) \otimes \mathbb{C} \approx \mathfrak{sl}(n,\mathbb{C}); \\ &\mathfrak{so}(p,q) \otimes \mathbb{C} \approx \mathfrak{so}(p+q,\mathbb{C}). \end{aligned}$$

There are many more accidental isomorphism for real algebras, here are a few:

$$\begin{array}{lll} \mathfrak{so}(2,1) \approx \mathfrak{sl}(2,\mathbb{R}) \; ; & \mathfrak{so}(3,\mathbb{R}) \approx \mathfrak{su}(2) \; ; & \mathfrak{so}(3,1) \approx \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \; ; \\ \mathfrak{so}(2,2) \approx \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \; ; & \mathfrak{so}(3,2) \approx \mathfrak{sp}(4,\mathbb{R}) \; ; & \mathfrak{so}(3,3) \approx \mathfrak{sl}(4,\mathbb{R}) \, . \end{array}$$

3.4.2 The structure of semi-simple Lie algebras

One nice way of describing a Lie algebra is to find a vector basis with simple structural constants, meaning that most of them are zero. This happens to be possible with semi-simple Lie algebras.

Definition 3.4.6. Let $\mathfrak{g}$ be a semi-simple Lie algebra. A **Cartan subalgebra** of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that:

1. $\mathfrak{h}$ is maximal abelian.

2. For every $X \in \mathfrak{h}$, $\mathrm{ad}(X)$ is semi-simple.

Recall that a linear map is semi-simple if and only if it is diagonalisable in the algebraic closure of the base field (in characteristic zero).

Example 3.4.7. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$, we set:

$$\mathfrak{h} = \left\{ \left(\begin{array}{ccc} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{array} \right) \middle| h_1 + \cdots + h_n = 0 \right\} \subset \mathfrak{sl}(n, \mathbb{K})$$

Let us show that it is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{K})$. It is an abelian subalgebra. It is maximal because if $\mathfrak{h}' \supset \mathfrak{h}$ is abelian, then any element of $\mathfrak{h}'$ must commute with the matrix:

$$\left(\begin{array}{ccccc} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n-1 & \\ & & & & -\frac{n(n-1)}{2} \end{array} \right),$$

and therefore be diagonal, i.e. be an element of $\mathfrak{h}$.

Consider the canonical basis $(E_{i,j})_{1 \leq i,j \leq n}$ of $\mathcal{M}_n(\mathbb{K})$. For

$$H = \left(\begin{array}{ccc} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{array} \right)$$

we get $\mathrm{ad}(H)E_{i,j} = (h_i - h_j)E_{i,j}$. Therefore $\mathrm{ad}(H)$ is diagonalisable for every $H \in \mathfrak{h}$, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{K})$.

Proposition 3.4.8. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$. If $\mathfrak{g}$ is semi-simple, then it has a Cartan subalgebra.*

Moreover, if $\mathbb{K}$ is algebraically closed, then $\mathrm{Aut}(\mathfrak{g})$ *acts transitively on the set of Cartan subalgebras.*

Definition 3.4.9. The **rank** of a finite dimensional complex semi-simple Lie algebra is the dimension of its Cartan subalgebras.

Consider a complex finite dimensional semi-simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. For every $H \in \mathfrak{h}$, the linear map $\mathrm{ad}(H) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalisable, and they all commute with each other. This implies the existence of a vector basis $(X_1, \ldots, X_n)$ of $\mathfrak{g}$ in which the matrices of all the $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$ are diagonal, i.e. $\mathrm{ad}(H)X_i = \alpha_i(H)X_i$.

Note that the eigenvalues $\alpha_i(H)$ are linear forms on $\mathfrak{h}$, which leads us to the concept of roots.

Definition 3.4.10. Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.
For $\alpha \in \mathfrak{h}^*$, we set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall H \in \mathfrak{h}\ \mathrm{ad}(H)X = \alpha(H)X\}$$

A **root** is a linear form $\alpha \in \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$. We call $\mathfrak{g}_\alpha$ the **root space** associated to α.

Proposition 3.4.11. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $\Phi \subset \mathfrak{h}^*$ the set of roots. We have the following decomposition of $\mathfrak{g}$:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Proof. It is a consequence of the previous discussion, and of the fact that the eigenspace $\mathfrak{g}_0$ (i.e. the **centralizer** of $\mathfrak{h}$) is equal to $\mathfrak{h}$ (because $\mathfrak{h}$ is maximal abelian). □

In order to find the structural constants in a basis adapted to this decomposition, we must also understand the brackets between elements of the root spaces. They happen to have a very nice property.

Proposition 3.4.12. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. For any $\alpha, \beta \in \mathfrak{h}^*$,*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

Proof. It is a consequence of the Jacobi identity. Consider $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$. We calculate:

$$\begin{aligned}
[H,[X_\alpha,X_\beta]] &= -[X_\alpha,[X_\beta,H]] - [X_\beta,[H,X_\alpha]] \\
&= -[X_\alpha, -\beta(H)X_\beta] - [X_\beta, \alpha(H)X_\alpha] \\
&= (\alpha(H)+\beta(H))[X_\alpha,X_\beta]
\end{aligned}$$

□

3.5 Abstract root systems

3.5.1 Manipulations of root systems

Let $(V, \langle\cdot|\cdot\rangle)$ be a (finite dimensional) Euclidean vector space. For $x \in V \setminus \{0\}$, we let $x^\vee = \frac{2x}{\langle x|x\rangle}$ and denote by $s_x : V \to V$ the orthogonal reflection with respect to $x^\perp$, which writes as:

$$s_x(y) = y - \langle y|x^\vee\rangle x.$$

Note that $(x^\vee)^\vee = x$, $s_{x^\vee} = s_x$ and $s_x(y^\vee) = (s_x(y))^\vee$ for all $x, y \in V \setminus \{0\}$.

Definition 3.5.1. Let $(V, \langle\cdot|\cdot\rangle)$ be a Euclidean vector space. A **root system** of V is a subset $\Phi \subset V$ such that:

1. Φ is finite, $0 \notin \Phi$ and $\mathrm{Vect}(\Phi) = V$.

2. $\forall \alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$.

3. $\forall \alpha, \beta \in \Phi$, $\langle \alpha | \beta^\vee \rangle \in \mathbb{Z}$.

We call Φ **reduced** if it also satisfies:

4. $\forall \alpha \in \Phi \forall t \in \mathbb{R}$, $t\alpha \in \Phi \Rightarrow t = \pm 1$.

The elements of Φ are called **roots**. The **rank** of Φ is $\dim V$.
The group $W \subset \mathrm{O}(V)$ generated by the s_α for $\alpha \in \Phi$ is called the **Weyl group** of Φ.
For $\alpha, \beta \in \Phi$ we note $n(\alpha, \beta) = \langle \alpha | \beta^\vee \rangle$.

The notion of isomorphism between root systems may seem unnecessarily complicated at first, but it happens to be the right one.

Definition 3.5.2. Let V, V' be Euclidean vector spaces, and let $\Phi \subset V$, $\Phi' \subset V'$ be root systems. An **isomorphism** from Φ to Φ' is a linear isomorphism $f : V \to V'$ such that $f(\Phi) = \Phi'$ et $f \circ s_\alpha \circ f = s_{f(\alpha)}$ pour tout $\alpha \in \Phi$.

Proposition 3.5.3. *Let V be a Euclidean vector space, and $\Phi \subset V$ a root system.*

1. *The Weyl group W is finite.*

2. *If $\alpha \in \Phi$ then $-\alpha \in \Phi$.*

3. *If $\alpha \in \Phi$ and $t \in \mathbb{R}$ satisfies $t\alpha \in \Phi$ then $t \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. We cannot have $\frac{1}{2}\alpha \in \Phi$ and $2\alpha \in \Phi$ at the same time.*

Proof. 1. The group W acts on the finite set Φ, and this action is faithful because $\mathrm{Vect}\Phi = V$.

2. We have $s_\alpha(\alpha) = -\alpha$.

3. We have $n(\alpha, t\alpha) = \frac{2}{t} \in \mathbb{Z}$ and $n(t\alpha, \alpha) = 2t \in \mathbb{Z}$, which yields $t \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. Since $n(\frac{1}{2}\alpha, 2\alpha) = \frac{1}{2}$, these two elements cannot be simultaneously in Φ.

□

Exercise. Classify rank one root systems.

Let $\Phi \subset V$ be a root system. Given two roots $\alpha, \beta \in \Phi$, let φ be the angle between α and β. We have:

$$n(\alpha, \beta) = 2\frac{\|\alpha\|}{\|\beta\|}\cos\varphi.$$

It follows that $n(\alpha, \beta)n(\beta, \alpha) = 4\cos^2(\varphi) \in \mathbb{Z}$, so the only possible values for the integer $n(\alpha, \beta)$ are $0, \pm 1, \pm 2, \pm 3, \pm 4$.

Notice that the case $n(\alpha,\beta) = \pm 4$ corresponds exactly to proportional roots ($\varphi = \pm\pi$). Let us now assume that it is not the case.

Replacing α by $-\alpha \in \Phi$ if necessary, we can assume that $n(\alpha,\beta) \leq 0$ (geometrically, this means that $\varphi \geq \frac{\pi}{2}$, i.e. the angle between α and β is obtuse). Switching α and β if necessary, we can also assume that $\|\alpha\| \leq \|\beta\|$. We now have that $n(\alpha,\beta) \in \{0,-1\}$, and only the four following cases are possible:

$$\begin{array}{llll}
n(\alpha,\beta) = 0 & n(\beta,\alpha) = 0 & \varphi = \frac{\pi}{2} & \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -1 & \varphi = \frac{2\pi}{3} & \|\beta\| = \|\alpha\| \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -2 & \varphi = \frac{3\pi}{4} & \|\beta\| = \sqrt{2}\|\alpha\| \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -3 & \varphi = \frac{5\pi}{6} & \|\beta\| = \sqrt{3}\|\alpha\|
\end{array}$$

Corollary 3.5.4. *Let V be a Euclidean vector space and $\Phi \subset V$ a root system. If two roots $\alpha,\beta \in \Phi$ form and obtuse (resp. acute) angle, then $\alpha+\beta$ (resp. $\alpha-\beta$) is a root.*

Proof. If the angle is obtuse, then $n(\alpha,\beta) \leq 0$. According to the previous comments, up to switching α and β, we have that $n(\alpha,\beta) = -1$, hence $\beta+\alpha = \beta - n(\alpha,\beta)\alpha = s_\alpha(\beta) \in \Phi$.
If the angle is acute, then substituting $-\beta$ for β leads to the previous case. □

In particular, if the roots α,β are not proportional nor orthogonal, then either $\alpha+\beta$ or $\alpha-\beta$ is a root.

Exercise. Classify rank two root systems.

3.5.2 Classification of root systems

Definition 3.5.5. Let V be a Euclidean vector space, and $\Phi \subset V$ a root system. A **basis** of Φ is a subset $\Pi \subset \Phi$ such that:

- Π is a vector basis of V.
- Any root $\alpha \in \Phi$ decomposes as $\alpha = \sum_{\pi\in\Pi} \alpha_\pi \pi$ where all the coefficients α_π are integer and have the same sign.

Elements of Π are called **simple roots**. A root $\alpha = \sum_{\pi\in\Pi} \alpha_\pi \pi$ is called **positive** (resp. **negative**) if all α_π are non negative (resp. non positive).
The **Cartan matrix** is the matrix $(n(\alpha,\beta))_{\alpha,\beta\in\Pi}$.

We will admit the following fact.

Proposition 3.5.6. *Any root system has a basis.*

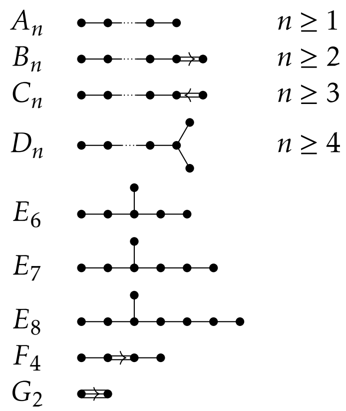
Definition 3.5.7. Let V be a Euclidean vector space, $\Phi \subset V$ a root system, Π a basis of Φ, and $(n(\alpha,\beta))_{\alpha,\beta\in\Pi}$ the Cartan matrix. The **Dynkin diagram** of Φ (relatively to Π) is the oriented multi-edged graph Π, and two vertices $s,t \in \Pi$ are linked by:

- A single edge if $n(s,t) = n(t,s) = -1$.
- A double edge if $n(s,t) = -1$ and $n(t,s) = -2$.
- A triple edge if $n(s,t) = -1$ et $n(t,s) = -3$.

The edges are oriented from s towards t if $\|s\| > \|t\|$ (or).

Theorem 3.5.8 (Classification of root systems)**.**

1. *Any non empty connected Dynkin diagram is isomorphic to exactly one of the following diagrams:*



2. *For every diagram in this list, there is, up to isomorphism, a unique irreducible reduced root system of which it is the Dynkin diagram.*

3. *For every $n \geq 1$, there is, up to isomorphism, a unique irreducible non reduced root system of rank n, denote by BC_n.*

3.5.3 Root systems of complex semi-simple Lie algebras

Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Recall that for $\alpha \in \mathfrak{h}^*$, we set:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall H \in \mathfrak{h}\ [H,X] = \alpha(H)X\}.$$

We write $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq \{0\}\}$ the set of roots of $\mathfrak{g}$ relatively to $\mathfrak{h}$, and $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ the $\mathbb{R}$-vector subspace spanned by Φ. We now wish to endow $\mathfrak{h}^*_{\mathbb{R}}$ with an inner product, so that Φ is a root system. It will be constructed by using the Killing form of $\mathfrak{g}$. There is actually a strong relationship between the decomposition of $\mathfrak{g}$ into root spaces and the Killing form (it is almost an orthogonal decomposition).

Proposition 3.5.9. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ the set of roots and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the root space decomposition.*

1. $\mathrm{Vect}_{\mathbb{C}}\Phi = \mathfrak{h}^*$.
2. $\forall \alpha, \beta \in \mathfrak{h}^*$ $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
3. $\forall \alpha \in \Phi$, $\mathfrak{g}_\alpha \perp \mathfrak{h}$ *for the Killing form B of $\mathfrak{h}$.*
4. $B|_{\mathfrak{h}\times\mathfrak{h}}$ *is non degenerate.*

Proof.

1. We have that $\cap_{\alpha\in\Phi} \ker \alpha \subset \mathfrak{z}(\mathfrak{g}) = \{0\}$. Considering duals, we find that $\mathrm{Vect}_{\mathbb{C}}\Phi = \mathfrak{h}^*$.

2. It is a consequence of the Jacobi identity. Consider $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$. We calculate:

$$\begin{aligned}[H,[X_\alpha,X_\beta]] &= -[X_\alpha,[X_\beta,H]] - [X_\beta,[H,X_\alpha]] \\ &= -[X_\alpha, -\beta(H)X_\beta] - [X_\beta, \alpha(H)X_\alpha] \\ &= (\alpha(H)+\beta(H))[X_\alpha,X_\beta]\end{aligned}$$

3. If $H \in \mathfrak{h}$ and $X_\alpha \in \mathfrak{g}_\alpha$, the matrix of $\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)$ in a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta\in\Phi} \mathfrak{g}_\beta$ has vanishing diagonal coefficients, since $\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\alpha+\beta}$. Hence $\mathrm{Tr}(\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)) = 0$, i.e. $B(H, X_\alpha) = 0$.

4. Let $H \in \ker(B|_{\mathfrak{h}\times\mathfrak{h}})$. Since H is orthogonal to all the root spaces, we find that $H \in \ker B$, hence $H = 0$ because $\mathfrak{g}$ is semi-simple.

□

The Killing form induces a non degenerate bilinear form $\langle\cdot|\cdot\rangle$ on $\mathfrak{h}^*$ by duality.

Theorem 3.5.10. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ the set of roots, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the root space decomposition and $\langle\cdot|\cdot\rangle$ the bilinear form induced on $\mathfrak{h}^*$ by the Killing form B of $\mathfrak{g}$.*

1. *The restriction of $\langle\cdot|\cdot\rangle$ to $\mathfrak{h}^*_{\mathbb{R}}$ is a scalar product.*
2. *The set Φ is a reduced root system of $(\mathfrak{h}^*_{\mathbb{R}}, \langle\cdot|\cdot\rangle)$.*
3. *Up to isomorphism, this root system only depends on $\mathfrak{g}$.*

So we can talk about the root system of a complex finite dimensional semi-simple Lie algebra, and its Dynkin diagram.

3.5.4 The classification theorem

Theorem 3.5.11 (Classification of complex semi-simple Lie algebras)**.**

1. *Two complex finite dimensional semi-simple Lie algebras are isomorphic if and only if their root systems are isomorphic.*
2. *A complex finite dimensional semi-simple Lie algebra is simple if and only if its Dynkin diagram is connected.*
3. *Every Dynkin diagram can be obtained by a complex finite dimensional semi-simple Lie algebra.*

Example 3.5.12. Let us pick up the case of $\mathfrak{sl}(n,\mathbb{C})$ where we left it. We saw that the subalgebra $\mathfrak{h}$ of diagonal traceless matrices is a Cartan subalgebra. We also saw that the roots are the forms $\alpha_{i,j}$ defined by $\alpha_{i,j}(H) = h_i - h_j$ for $H = \mathrm{diag}(h_1,\dots,h_n)$ (with $i \neq j$), and that the root spaces are $\mathfrak{sl}(n,\mathbb{C})_{\alpha_{i,j}} = \mathbb{C}E_{i,j}$.

Let us prove that $\alpha_{1,2},\dots,\alpha_{n-1,n}$ is a basis of Φ. It is a family of $n-1$ linearly independent elements of $\mathfrak{h}^*$, hence a vector basis. For $i < j$, we find:

$$\alpha_{i,j} = \alpha_{i,i+1} + \cdots + \alpha_{j-1,j}$$

This shows that $\alpha_{i,j}$ and $\alpha_{j,i} = -\alpha_{i,j}$ have integer coefficients all sharing the same sign in the basis Π.

In order to compute scalar products and establish the Dynkin diagram, we need to find the vectors $H_1,\dots,H_{n-1} \in \mathfrak{h}$ such that $B(H_i,\bullet) = \alpha_{i,i+1}$. Recall that $B(X,Y) = 2n\,\mathrm{Tr}(XY)$. It follows that

$$H_i = \frac{1}{2n}\mathrm{diag}(0,\dots,1,-1,0,\dots)$$

where the 1 is in the i^{th} position.

We have that $\left\langle \alpha_{i,i+1} \middle| \alpha_{j,j+1} \right\rangle = B(H_i,H_j)$, which yields:

$$\langle \alpha_{i,i+1} | \alpha_{j,j+1} \rangle = \begin{cases} \frac{1}{n} & \text{if } i = j \\ \frac{-1}{2n} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

We now can find the coefficients $n(\alpha,\beta) = \frac{2\langle \alpha|\beta\rangle}{\langle \beta|\beta\rangle}$, which are:

$$n(\alpha_{i,i+1},\alpha_{j,j+1}) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

This shows that the Dynkin diagram of $\mathfrak{sl}(n,\mathbb{C})$ is of type A_{n-1}. Similar calculations allows us to find the Dynkin diagrams of all classical semi-simple Lie algebras.

Proposition 3.5.13 (Dynkin diagrams of classical Lie algebras)**.**
The Dynkin diagram of the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ *is of type* A_n*.*
The Dynkin diagram of the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ *is of type* B_n*.*
The Dynkin diagram of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ *is of type* C_n*.*
The Dynkin diagram of the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ *is of type* D_n*.*

Types E_6, E_7, E_8, F_4 and G_2, called **exceptional** correspond to Lie algebras that are much more complicated to describe.

Chapter 4

Actions of Lie groups on manifolds

4.1 Smooth actions of Lie groups

We will now start to relate Lie groups and the geometry of manifolds, through actions of Lie groups. We will consider both left and right actions (a right action $X \curvearrowleft G$ is a map $X \times G \to X$ satisfying $(x.g).h = x.(gh)$ and $x.e = x$).

The two standard examples are obtained from a group G and a subgroup $H \subset G$. The group H acts on G by right multiplication ($g.h = gh$, this is a right action). One can consider the quotient G/H of G by this action, it consists of cosets gH for $g \in G$. The group G acts on G/H by $g.g'H = gg'H$ (this is a left action).

4.1.1 Some vocabulary

All the vocabulary will be defined for left actions, but is also used for right actions.

Definition 4.1.1. Let G be a Lie group, and X a smooth manifold. An action $G \curvearrowright X$ is called **smooth** if the map

$$\begin{cases} G \times X & \to & X \\ (g,x) & \mapsto & gx \end{cases}$$

is smooth.

Definition 4.1.2. Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$.

For $g \in G$, we denote by $m_g : X \to X$ the diffeomorphism defined by $m_g(x) = gx$.

For $x \in X$, the **orbit map** is $\varphi_x : G \to X$ defined by $\varphi_x(g) = gx$.

The **orbit** $G.x$ of x is the range of φ_x.

The **stabiliser** of x (also called its **isotropy subgroup**) is the subgroup $G_x = \{g \in G | gx = x\}$.

We denote by $\Theta_x : G/G_x \to X$ the map induced by the orbit map φ_x.

Definition 4.1.3. Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$.

We say that $G \curvearrowright X$ is **transitive** if there is $x \in X$ such that $G.x = X$.

We say that $G \curvearrowright X$ is **free** if for all $x \in X$, we have $G_x = \{e\}$.

We say that $G \curvearrowright X$ is **proper** if for all compact subset $K \subset X$, the set $\{g \in G | gK \cap k \neq \emptyset\}$ is compact.

Lemma 4.1.4. *Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$. Let $x \in X$.*

1. *The orbit map $\varphi_x : G \to X$ has constant rank.*
2. *The stabiliser G_x is an embedded Lie subgroup of G, and $T_e G_x = \ker d_e \varphi_x$.*

Remark. The rank of φ_x may depend on $x \in X$.

Proof.

1. Differentiating $\varphi_x \circ L_g = m_g \circ \varphi_x$ (i.e. the equivariance of $\varphi_x : G \to X$) at e yields $d_g \varphi_x \circ d_e L_g = d_x m_g \circ d_e \varphi_x$, which implies that $d_g \varphi_x$ and $d_e \varphi_x$ have the same rank since L_g and m_g are diffeomorphisms.
2. It is a level set of a constant rank map, hence an embedded submanifold with tangent space the kernel of $d_e \varphi_x$.

□

4.1.2 Topology of the quotient by a smooth action

We will now discuss manifolds that are obtained as quotients by a smooth (right) action of a Lie group. Before discussing manifold structures on quotients, we need to discuss the topology, which will always be the quotient topology. Given a right action $X \curvearrowleft G$, if $\pi : X \to X/G$ is the canonical projection, then we know that π is surjective and continuous (recall that $V \subset X/G$ is open if and only if $\pi^{-1}(V)$ is open), which is the case for all quotient topologies. In the case of the quotient by a smooth group action, we have an additional property (which only uses the continuity of the action).

Proposition 4.1.5. *Let G be a Lie group, X a manifold, and consider a smooth action $X \curvearrowleft G$. The quotient map $\pi : X \to X/G$ is open, i.e. the image of an open set is open.*

Proof. Let $U \subset X$ be open. Then $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} R_g(U)$ is open because R_g is a homeomorphism, therefore $\pi(U)$ is open. □

Lemma 4.1.6. *Let G be a Lie group, X a manifold, and consider a smooth proper action $X \curvearrowleft G$. For all compact subset $K \subset X$, the orbit of K, i.e. the set*

$$K.G = \{x.g \,|\, (g,x) \in G \times K\} \subset X,$$

is closed. In particular, orbits of points are closed.

Proof. Recall that the topology of a manifold is metrisable, so we can use sequences to show that a subset is closed. Si $(g_n) \in G^{\mathbb{N}}$, $(x_n) \in K^{\mathbb{N}}$ and $x_n.g_n \to y \in X$, we wish to show that $y \in K.G$. Since K is compact, we can assume that $x_n \to x \in K$. Since X is locally compact, there is a compact set $L \subset X$ such that $y \in \mathring{L}$ and $x \in \mathring{L}$. For large enough n, we find that $x_n.g_n \in L.g_n \cap L$, therefore (g_n) stays in a compact set of G (because the action is proper), and we can assume that $g_n \to g \in G$. We get $y = gx \in K.G$. □

Lemma 4.1.7. *Let G be a Lie group, and $H \subset G$ a Lie subgroup. The right action $G \curvearrowleft H$ is proper if and only if G is closed.*

Remark. The same holds for the left action.

Proof. If $G \curvearrowleft H$ is proper, then the orbit $e.H$ is closed according to Lemma 4.1.6, i.e. H is closed.

Reciprocally, if H is closed and $K \subset G$ is compact, then the set

$$\{h \in H \,|\, K.h \cap K \neq \emptyset\} = H \cap (K \cap K^{-1})$$

is the intersection of a closed and a compact subset, hence compact, i.e. the action is proper. □

Lemma 4.1.8. *Let G be a Lie group, X a manifold, and consider a smooth proper action $X \curvearrowleft G$. The quotient topology on X/G is Hausdorff and second countable.*

Proof. First notice that the image of a second countable space under a continuous and open map is also second countable, so it is the case for X/G.

Let $x, y \in X$ be such that $\pi(x) \neq \pi(y)$. Since $x \notin y.G$ and $y.G$ is closed according to Lemma 4.1.6, we can consider an open set $U \subset X$ containing x, such that $U \cap y.G = \emptyset$ (because the topology of X is metrisable). Up to shrinking U, we can assume that $\overline{U}$ is compact, and that $\overline{U} \cap y.G = \emptyset$. This directly implies that $\overline{U}.G \cap y.G = \emptyset$.

According to Lemma 4.1.6, the set $\overline{U}.G$ is closed. Since $\overline{U}.G \cap y.G = \emptyset$, we can consider open sets $V, W \subset X$ such that $\overline{U}.G \subset V$, $y.G \subset W$ and $V \cap W = \emptyset$ (once again, because the topology of X is metrisable).

Since π is an open map, the sets $\pi(U)$ and $\pi(W)$ are open in X/G. We have that $\pi(x) \in \pi(U)$ and $\pi(y) \in \pi(W)$. The inclusion $\pi^{-1}(\pi(U)) \subset V$ shows that $\pi(U) \cap \pi(W) = \emptyset$, therefore X/G is Hausdorff. □

Before we try to turn the quotient into a manifold, let us prove a simple result on submersions.

Lemma 4.1.9. *Let M, N, P be manifolds, $p : M \to N$ a surjective submersion, and $f : N \to P$ a map. Then f is smooth if and only if $f \circ p$ is smooth.*

Proof. The Constant Rank Theorem associated to the surjectivity of p implies that every point in N has a neighbourhood U on which we can find a smooth function $\varphi : U \to M$ satisfying $p \circ \varphi = \mathrm{Id}_U$. It follows that $f = f \circ p \circ \varphi$ on U. □

Theorem 4.1.10. *Consider a smooth free proper action of a Lie group $X \curvearrowleft G$. There is a unique manifold structure on the quotient X/G such that the quotient map $\pi : X \to X/G$ is a submersion. Furthermore,* $\dim X/G = \dim X - \dim G$.

Remarks.

- In the zero-dimensional case, this is Proposition 2.2.4.
- Since π is a submersion, its level sets are closed embedded submanifolds, i.e. orbits are closed embedded submanifolds.

Proof. Lemma 4.1.8 assures that the quotient topology is Hausdorff and second countable. Let $d = \dim X$ and $p = \dim G$.

The uniqueness of the differentiable structure comes from Lemma 4.1.9 applied to the identity map of X/G, so we focus on the existence.

First step: Given $x \in X$, we look for a $d-p$-dimensional submanifold $\mathcal{W}_x \subset X$ such that:

1. $x \in \mathcal{W}_x$,
2. $\pi|_{\mathcal{W}_x}$ is injective,
3. $\forall z \in \mathcal{W}_x \;\; \mathfrak{Im}(d_e\varphi_z) \oplus T_z\mathcal{W}_x = T_zX$.

First note that the orbit maps are immersions (they are injective because the action is free, and they have constant rank by Lemma 4.1.4). It follows that $\dim \mathfrak{Im}(d_e\varphi_x) = p$. Let $W \subset T_xX$ be a vector subspace such that $T_xX = \mathfrak{Im}(d_e\varphi_x) \oplus W$, and let $\mathcal{W}_x \subset X$ a $d-p$-dimensional submanifold such that $x \in \mathcal{W}_x$ and $T_x\mathcal{W}_x = W$.

Transversality (condition 3.) is open. Indeed, given local coordinates $(x^1, \ldots, x^k)$ on $\mathcal{W}_x$, a local volume form ω on X and a basis $X_1, \ldots, X_p$ of $\mathfrak{g}$, it is equivalent to

$$\omega_z\left(d_e\varphi_z(X_1), \ldots, d_e\varphi_z(X_p), \frac{\partial}{\partial x^1}(z), \ldots, \frac{\partial}{\partial x^k}(z)\right) \neq 0.$$

So up to shrinking $\mathcal{W}_x$, we may assume that the third condition is satisfied.

Consider the action map $\alpha : X \times G \to X$. The differential $d_{(x,e)}\alpha : T_xX \times \mathfrak{g} \to T_xX$ induces an isomorphism from $T_x\mathcal{W}_x \times \mathfrak{g}$ to T_xX, so up to shrinking $\mathcal{W}_x$, the Local Inverse Function Theorem provides us with open sets $U_x \subset G$ and $V_x \subset X$ respectively containing e and x such that α restricts to a diffeomorphism $\psi_x : \mathcal{W}_x \times U_x \to V_x$.

Let us now show that we can shrink $\mathcal{W}_x$ to make $\pi|_{\mathcal{W}_x}$ injective. Were it not the case, we could find sequences $(g_k) \in G^{\mathbb{N}}$ and $(w_k) \in \mathcal{W}_x^{\mathbb{N}}$ such that $w_k \to x$, $w_kg_k \in \mathcal{W}_x$, $w_kg_k \to x$ and $w_kg_k \neq w_k$. The properness of the action implies that g_k lies in a compact subset of G, so up to considering a subsequence we can assume that $g_k \to g \in G$. But $w_kg_k \to xg$ and $w_kg_k \to x$, so $xg = x$ and $g = e$ because the action of G is free. So we find that $g_k \to e$, hence $g_k \in U_x$ for large enough k. But $\psi_x(w_k, g_k) = \psi_x(w_kg_k, e)$ is a contradiction with the fact that ψ_x is a diffeomorphism.

Second step: Build a chart around $\pi(x)$.

Consider the diffeomorphism $\psi_x : \mathcal{W}_x \times U_x \to V_x$ from the first step. Notice that $\pi(V_x) = \pi(\mathcal{W}_x)$, therefore $\pi(\mathcal{W}_x)$ is an open subset of X/H (since π is an open map) containing $\pi(x)$. Let $\Phi_x : \pi(\mathcal{W}_x) \to \mathcal{W}_x$ be defined by $\Phi_x(\pi(w)) = w$ for $w \in \mathcal{W}_x$ (it is well defined because $\pi|_{\mathcal{W}_x}$ is injective).

Let us show that Φ_x is a homeomorphism. Its construction makes it a bijection, and its inverse $\pi|_{\mathcal{W}_x}$ is continuous. Continuity of Φ_x is a consequence of the fact that π is an open map: if $\mathcal{O} \subset \mathcal{W}_x$ is open, then $\Phi_x^{-1}(\mathcal{O}) = \pi(\psi_x(\mathcal{O} \times U_x))$ is open.

Third step: Build an atlas on X/G.

For every $x \in X$, consider a chart $(\pi(\mathcal{W}_x), \Phi_x)$ defined as above (axiom of choice haters will find a way to avoid using it). Let $x, x' \in X$, and consider the transition function

$$\Phi_{x'} \circ \Phi_x^{-1} : \Phi_x(\pi(\mathcal{W}_x) \cap \pi(\mathcal{W}_{x'})) \to \Phi_{x'}(\pi(\mathcal{W}_x) \cap \pi(\mathcal{W}_{x'})).$$

The domain can be simplified:

$$\begin{aligned}
\Phi_x(\pi(\mathcal{W}_x) \cap \pi(\mathcal{W}_{x'})) &= \{z \in \mathcal{W}_x \,|\, \pi(z) \in \pi(\mathcal{W}_x) \cap \pi(\mathcal{W}_{x'})\} \\
&= \{z \in \mathcal{W}_x \,|\, \pi(z) \in \pi(\mathcal{W}_{x'})\} \\
&= \{z \in \mathcal{W}_x \,|\, z \in \mathcal{W}_{x'}G\} \\
&= \mathcal{W}_x \cap \mathcal{W}_{x'}G
\end{aligned}$$

Let $z \in \mathcal{W}_x \cap \mathcal{W}_{x'}G$, and let $g(z) \in G$ be such that $z.g(z) \in \mathcal{W}_{x'}$. It is unique (because $\pi|_{\mathcal{W}_{x'}}$ is injective), and moreover

$$\Phi_{x'} \circ \Phi_x^{-1}(z) = z.g(z).$$

It remains to show that the map $z \mapsto g(z)$ is smooth. This is a classic arguments from dynamical systems (it is a Poincaré first return map). Since we are working locally, we may assume that $\mathcal{W}_{x'} = F^{-1}(\{0\})$ for some submersion $F : \mathcal{O} \to \mathbb{R}^p$ where $\mathcal{O} \subset X$ is open. Then $z \mapsto g(z)$ is the solution of the implicit equation $F(z.g(z)) = 0$. Let $z_0 \in \mathcal{W}_x \cap \mathcal{W}_{x'}G$, and write $g_0 = g(z_0)$. Consider the function G defined on a neighbourhood of (z_0, g_0) in $\mathcal{W}_x \times G$ by $G(z, g) = F(z.g)$. When differentiating with respect to g, we find:

$$d_{(z_0,g_0)}G(0,\bullet) = d_{z_0\cdot g_0}F \circ d_{g_0}\varphi_{z_0}.$$

This map is an isomorphism:

$$\begin{aligned}
\ker d_{(z_0,g_0)}G(0,\bullet) &= \ker d_{z_0\cdot g_0}F \cap \operatorname{Im} d_{g_0}\varphi_{z_0} \\
&= T_{z_0 g_0}\mathcal{W}_{x'} \cap \operatorname{Im} d_{g_0}\left(\varphi_{z_0\cdot g_0} \circ L_{g_0^{-1}}\right) \\
&= T_{z_0 g_0}\mathcal{W}_{x'} \cap \operatorname{Im} d_e\varphi_{z_0\cdot g_0} \\
&= \{0\}.
\end{aligned}$$

We can use the Implicit Function Theorem to show that $z \mapsto g(z)$ is smooth, and so is the transition function $\Phi_{x'} \circ \Phi_x^{-1}$. □

4.2 Quotients of Lie groups

We will now study quotients of Lie groups by Lie subgroups. Note that the quotient by an immersed subgroup need not be a manifold (think of an irrational line in the torus).

Theorem 4.2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ an embedded Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and $\pi : G \to G/H$ the projection. There is a unique manifold structure on G/H for which π is a submersion. Moreover, the action of G on G/H is smooth.*

If H is a normal subgroup of G, then G/H is a Lie group, π is a Lie group morphism, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the Lie algebra of G/H is isomorphic to $\mathfrak{g}/\mathfrak{h}$.

Remarks.

- $\dim G/H = \dim G - \dim H$.
- $T_{\pi(e)}G/H \approx T_eG/T_eH$, more generally $T_{\pi(g)}G/H \approx T_gG/T_g(gH)$.
- If $f : G \to G'$ is a Lie group morphism, then $f(G)$ is an immersed Lie subgroup of G'. Indeed, the quotient $G/\ker f$ is a Lie group, and $f(G)$ is the image of $\overline{f} : G/\ker f \to G'$ which is an injective Lie group morphism, hence an immersion.

Proof. Since the action of H is proper (Lemma 4.1.7) and free, we can apply Theorem 4.1.10 to get the manifold structure on G/H.

The smoothness of the action is a consequence of Lemma 4.1.9 applied to the action map $G \times G/H \to G/H$ and the submersion $G \times G \to G \times G/H$.

If H is a normal subgroup, then G/H carries a group structure, and by applying Lemma 4.1.9 to the projection $G \times G \to G/H \times G/H$ and the multiplication map of G/H, we find that G/H is a Lie group and that π is a Lie group morphism. Finally the differential $d_e\pi$ of the projection gives a short exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$.

□

4.3 The manifold structure of orbits

Proposition 4.3.1. *Let G be a Lie group, X a manifold, and consider a smooth action $G \curvearrowright X$. For all $x \in X$, the canonical map $\Theta_x : G/G_x \to X$ is an injective immersion.*

Remark. In particular, the orbit $G.x$ is an immersed submanifold, and $T_xG.x = d_e\varphi_x(\mathfrak{g})$.

Proof. Let $\pi : G \to G/G_x$ be the projection. The map Θ_x is smooth because $\Theta_x \circ \pi = \varphi_x$ is smooth.

It is also injective, and its range is that of φ_x, i.e. $G.x$.

We are left to show that $d_{\pi(g)}\Theta_x$ is injective for all $g \in G$. Using the equivariance, we only need to prove it for $g = e$.

We know that $T_eG_x = \ker d_e\varphi_x$, and $\varphi_x = \Theta_x \circ \pi$ where π is a submersion satisfying $\ker d_e\pi = T_eG_x$, so the formula $d_e\varphi_x = d_{\pi(e)}\Theta_x \circ d_e\pi$ yields $\ker d_{\pi(e)}\Theta_x = \{0\}$. □

Proposition 4.3.2. *Let G be a Lie group, and X a manifold. If a smooth action $G \curvearrowright X$ is proper, then orbits are embedded submanifolds.*

Remarks.

- According to Proposition 4.3.1 and Lemma 4.1.6, we know that the orbits are closed immersed submanifolds. But that is not enough to guarantee that they are embedded submanifolds.
- If G is compact, then all smooth actions are proper, so orbits are always embedded submanifolds.

Proof. Since $\Theta_x : G/G_x \to X$ is an injective immersion with range $G.x$, we only need to show that it is a homeomorphism onto $G.x$.

Let $\Phi : Gx \to G/G_x$ be its inverse. Consider a sequence $(y_k) \in (Gx)^{\mathbb{N}}$ and $y \in Gx$ such that $y_k \to y$. We can write $y_k = g_kx$ and $y = gx$ where $g_k, g \in G$.

It follows that $\Phi(y_k) = \pi(g_k)$ and $\Phi(y) = \pi(g)$ where $\pi : G \to G_x$ is the projection.

Since $g_k x \to y$, properness of the action implies that (g_k) lies in a compact subset of G. Up to considering a subsequence, we can assume that $g_k \to h \in G$.

We find that $hx = y$, hence

$$\Phi(y) = \pi(h) = \lim_{k \to +\infty} \pi(g_k) = \lim_{k \to +\infty} \Phi(y_k)$$

and Φ is continuous. □

Theorem 4.3.3. *Let $G \curvearrowright X$ be a smooth action of a Lie group G on a manifold X, and let $x \in X$.*

1. *The orbit Gx is an embedded submanifold if and only if it is locally closed.*
2. *If Gx is an embedded submanifold, then Θ_x is a diffeomorphism from G/G_x to Gx.*

Remark. In particular, every closed orbit is an embedded submanifold.

Proof.

1. Any embedded submanifold is locally closed. Reciprocally, assume that $G.x$ is locally closed. Then $G.x$ is locally compact, and satisfy Baire's property. Since Θ_x is an injective immersion, we only need to show that it is a homeomorphism onto its image. It is enough for this to show that φ_x is an open map. Using the equivariance, we are left to show that for every open neighbourhood U of e, the image $\varphi_x(U) = U.x$ is an open subset of $G.x$. Let V be a compact neighbourhood of e such that $V^{-1}V \subset U$. Consider a dense sequence $(g_i)_{i \in \mathbb{N}}$ in G. We have that $G = \bigcup_{i \in \mathbb{N}} g_i V$, and $G.x = \bigcup_{i \in \mathbb{N}} g_i(V.x)$. Since V is compact, so is $V.x = \varphi_x(V)$, and it must be closed in $G.x$. According to Baire's property, there is some $i \in \mathbb{N}$ such that $g_i(V.x)$ has non empty interior. If $g \in V$ is such that gx lies in the interior of $g_i(V.x)$, then $g^{-1}Vx$ is a neighbourhood of x in $G.x$ that lies in $\varphi_x(U)$.
2. A proper injective immersion is a diffeomorphism onto its image.

□

4.4 Transitive actions of Lie groups

Theorem 4.4.1. *Let G be a Lie group and X a manifold. If a smooth action $G \curvearrowright X$ is transitive, then for every $x \in X$, the canonical map $\Theta_x : G/G_x \to X$ is a diffeomorphism.*

Proof. According to Proposition 4.3.1, Θ_x is an injective immersion. Since the action is transitive, it is also surjective. A bijective immersion is a diffeomorphism. □

Definition 4.4.2. A **homogeneous space** is a smooth manifold M equipped with a smooth transitive action of a Lie group $G \curvearrowright M$.

Examples 4.4.3.

- Spheres.
$$\mathbb{S}^n \approx \mathrm{O}(n+1)/\mathrm{O}(n) \approx \mathrm{SO}(n+1)/\mathrm{SO}(n)$$
- Projective spaces.
$$\mathbb{RP}^n \approx \mathrm{PO}(n+1)/\mathrm{O}(n) \approx \mathrm{PSL}(n,\mathbb{R})/P$$
- Grassmannians.
$$\mathcal{G}_k(\mathbb{R}^n) \approx \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k) \approx \mathrm{GL}(n,\mathbb{R})/P_{n,k}$$
- The upper half plane.
SL$(2,\mathbb{R})$ acts on $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$.
$$\mathcal{H} \approx \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \approx \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2,\mathbb{R})$$

Part II

Vector bundles and connections

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When first learning about functions, your attention was drawn to the importance of figuring out its domain, formally this means the set E on which a map $f : E \to F$ is defined. For the purpose of calculus, you first learned that E should be an interval in $\mathbb{R}$, then an open subset of $\mathbb{R}^d$, later on that E should be a manifold.

But the target space F received little attention so far, with the exception of the study of local inverses, where you learned that you can always replace F with the range of f in order to make it artificially surjective.

A strange situation tends to happen when working on the geometry of manifolds (or in physics): we may define maps f, for which the source is a manifold M, but the point $f(x)$ belongs to a set F_x which depends on x. We will mostly focus on the case where each F_x is a vector space.

From a set theoretic point of view, this is not a huge problem, as you can always consider such maps to have image in the disjoint union $\sqcup_{x\in M} F_x$. From a differential point of view, it is not so straightforward, as we naturally want to define a differential as $d_x f(v) = \lim_{t\to 0} \frac{1}{t}(f(\gamma(t)) - f(x))$ where $\gamma : \mathbb{R} \to M$ is a smooth curve satisfying $\gamma(0) = x \in M$ and $\dot\gamma(0) = v \in T_x M$. The problem is that the difference between $f(\gamma(t))$ and $f(\gamma(0))$ is not defined as they live in different vector spaces. To make some sense of this formula, we need to find a way of connecting $F_{\gamma(0)}$ and $F_{\gamma(t)}$, which is possible, but not canonical.

This situation happens naturally when we want to consider second order derivatives of maps defined on manifolds. Start with a smooth function $f : M \to \mathbb{R}$. Then for $x \in M$, we have a linear map $d_x f : T_x M \to \mathbb{R}$, i.e. $d_x f \in (T_x M)^* = T_x^* M$. So df can be seen as a map of M, but the image $d_x f$ lies in a set $T_x^* M$ that depends on x.

Chapter 5

Fibre bundles

5.1 General fibre bundles

5.1.1 Submersions and trivialisations

Consider two manifolds M and F. A fibre bundle over M with fibre F is the assignment to each $x \in M$ of a manifold ξ_x which is diffeomorphic to F in a way that "depends smoothly on x". The whole point of the following definitions is to make some sense of this smooth dependence.

Let us first focus on the task of assigning a manifold ξ_x to each $x \in M$. Since constructing a manifold structure on a set is a tedious task, we want to define ξ_x as a submanifold of a given manifold E. The simplest setting is to consider a submersion $p : E \to M$, and to set $\xi_x = p^{-1}(x)$.

For this to actually produce a manifold for every $x \in M$, we require p to be surjective. In this case, all manifolds ξ_x have the same dimension, but they need not be diffeomorphic to each other.

Even if they are all diffeomorphic to the same manifold F, just knowing it abstractly is not enough, we really need the diffeomorphism to "depend smoothly on x". If we consider a diffeomorphism $\theta_x : F \to \xi_x$ for each $x \in M$, then one can simply express the smooth dependence of θ_x in x by requiring the smoothness of the map $(x, y) \mapsto \theta_x(y)$. This is what we will call a trivialisation.

Definition 5.1.1. Consider a surjective submersion $p : E \to M$ and a manifold F. A **trivialisation of** p **with respect to** F is a collection of diffeomorphisms $(\theta_x : F \to p^{-1}(x))_{x \in M}$ such that the map

$$\Theta : \begin{cases} M \times F & \to & E \\ (x, z) & \mapsto & \theta_x(z) \end{cases}$$

is smooth. We say that p is **trivialisable with respect to** F if it possesses a trivialisation with respect to F.

The simplest example of a submersion onto M which is trivialisable with respect to F is the projection $\pi_1 : M \times F \to M$ onto the first factor. Here we assign to each $x \in M$ the manifold $\xi_x = \pi_1^{-1}(\{x\}) = \{x\} \times F$ which is diffeomorphic to F, and by setting $\theta_x(y) = (x, y)$ we find a diffeomorphism $\theta_x : F \to \xi_x$ which is a smooth function on $M \times F$.

Up to a diffeomorphism, it is the only example.

Proposition 5.1.2. *Consider manifolds E, M, F, a surjective submersion $p : E \to M$, and $\theta = (\theta_x)_{x \in M}$ a trivialisation of p with respect to F. Then the map*

$$\Theta : \begin{cases} M \times F & \to & E \\ (x, z) & \mapsto & \theta_x(z) \end{cases}$$

is a diffeomorphism.

Remark. The important property of Θ is that $p \circ \Theta = \pi_1$ where $\pi_1(x, z) = x$.

Proof. The map Θ is smooth by definition of a trivialisation of p with respect to F. Note that Θ is bijective, and that $\Theta^{-1}(y) = (p(y), \theta^{-1}_{p(y)}(y))$ for all $y \in E$.

The fact that $p \circ \Theta(x, z) = x$ for all $(x, z) \in M \times F$ differentiates to:

$$\forall \dot{x} \in T_x M \ \forall z \in T_z F \quad d_{(x,z)}(p \circ \Theta)(\dot{x}, \dot{z}) = \dot{x}.$$

If $d_{(x,z)}\Theta(\dot{x}, \dot{z}) = 0$ for some $x \in M$, $\dot{x} \in T_x M$, $z \in F$, $\dot{z} \in T_z F$, it follows from the differentiation of $p \circ \Theta$ that $\dot{x} = 0$. Now $d_{(x,z)}\Theta(0, \dot{z}) = d_z \theta_x(\dot{z}) = 0$, therefore $\dot{z} = 0$ since θ_x is a diffeomorphism.

We have seen that Θ is a bijective immersion, hence a diffeomorphism. □

For now it may seem that we are stuck in a dead end, since the assignment to each $x \in M$ of a manifold ξ_x which is diffeomorphic to F in a way that depends smoothly on x is the same as considering the product $M \times F$.

For this to lead to a rich theory, we only need to ask for the trivialisations to be defined locally.

To make sense of this, we need to consider restrictions of surjective submersions. It is important to notice that the notion of surjective submersions behaves well under restrictions to open sets of the target space: if $p : E \to M$ is a surjective submersion, and $U \subset M$ is open, then the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is also a surjective submersion.

Definition 5.1.3. A **fibre bundle** $\xi = (E, p, M, F)$ is the data of manifolds E, M, F and a map $p : E \to M$ such that:

- The map $p : E \to M$ is a surjective submersion.
- For every $x \in M$, there is an open neighbourhood $U \subset M$ such that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is trivialisable with respect to F.

The **total space** of ξ is the manifold E, the **projection** of ξ is the map $p : E \to M$, the **base** of ξ is the manifold M and the **fibre** of ξ is the manifold F. For $x \in M$, the **fibre over** x is the manifold $\xi_x = p^{-1}(x)$.

We say that ξ is **trivialisable** if p is trivialisable with respect to F.

Remarks.

- As mentionned above, given any open set $U \subset M$, the map $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a surjective submersion, which is important for this definition to make sense.
- For every $x \in M$, the fibre $\xi_x = p^{-1}(x)$ is a submanifold of E diffeomorphic to F.
- A map $p : E \to M$ is called a **locally trivial fibration** if there is a manifold F such that (E, p, M, F) is a fibre bundle.

The standard example is the **trivial bundle** $\underline{F}_M$ defined by:

$$\underline{F}_M = (M \times F, \pi_1, M, F).$$

We will have many examples of fibre bundles in this course that will not be equivalent to the trivial bundle in any relevant way.

Just as mentioned for surjective submersions, the notion of restriction of fibre bundles behave well when restricting to open subsets of the base.

Definition 5.1.4. Let $\xi = (E, p, M, F)$ be a fibre bundle, and $U \subset M$ an open set. The **restriction** of ξ to U is $\xi|_U = (p^{-1}(U), p|_{p^{-1}(U)}, U, F)$.

Remark. It is also a fibre bundle.

Definition 5.1.5. Let $\xi = (E, p, M, F)$ be a fibre bundle.

An open set $U \subset M$ is a **trivialising domain** if $\xi|_U$ is trivialisable.

A **trivialising chart** is a pair (U, θ) where $U \subset M$ is a trivialisation domain and θ is a trivialisation of $\xi|_U$.

A **trivialising atlas** $\mathcal{A}$ is a set of trivialising charts whose domains cover M.

Remark. For all the set theory nerds out there, a trivialising chart can be seen as a subset of $M \times F \times E$, so an atlas is a subset of $\mathcal{P}(M \times F \times E)$, and all set theoretic manipulations are valid.

Fibre bundles are not all trivialisable, however by definition they possess a trivialising atlas.

5.1.2 Constructing fibre bundles

Constructing fibre bundles, even the most basic examples such as the tangent bundle of a manifold, can be quite tricky and rather annoying because of one aspect: the manifold structure on the total space. It turns out that is usually the most technical and time consuming part of the proof that something is a fibre bundle. As a result, this manifold structure is usually quite poorly understood. Moreover, this differentiable structure ends up getting more attention than it deserves, as the study of fibre bundles basically boils down to techniques that allow us to avoid manipulating this differentiable structure in an abstract way. For this reason, we will now see how this manifold structure can systematically be derived from local trivialisations.

Since we usually consider that the base M and the fibre F are familiar manifolds, it is much more convenient to only work with the issue of differentiabily for maps defined on and with values in open subsets of the familiar M, F, or $M \times F$ rather than the very abstract total space E.

Theorem 5.1.6. *Let M,F be manifolds. Consider a collection of manifolds $(\xi_x)_{x\in M}$, an open cover $\mathcal{U}$ of M, and for each $U \in \mathcal{U}$ a collection of diffeomorphisms $(\theta_x^U : F \to \xi_x)_{x\in U}$.*

Assume that for every $U,V \in \mathcal{U}$, the map

$$\begin{cases} (U\cap V)\times F & \to & F \\ (x,z) & \mapsto & \left(\theta_x^U\right)^{-1} \circ \theta_x^V(z) \end{cases}$$

is smooth. Then there is a unique manifold structure on the disjoint union $E = \sqcup_{x\in M}\xi_x$ satisfying the following properties:

- *The quadruple (E,p,M,F) is a fibre bundle, where $p : E \to M$ is defined by $p(z) = x$ when $z \in \xi_x$.*
- *For each $U \in \mathcal{U}$, the map $\Theta^U : \begin{cases} U\times F & \to & E \\ (x,y) & \mapsto & \theta_x^U(y) \end{cases}$ is smooth.*

Proof. We start with the existence. We wish to show that the maps θ^U for $U \in \mathcal{U}$ provide an atlas for E. Note that they are injective, and that $\Theta^U(U\times F) = p^{-1}(U) = \sqcup_{x\in U}\xi_x$.

First step: define a topology on E.

Declare that a set $O \subset E$ is open if for every $y \in O$ there are:

- An element $U \in \mathcal{U}$, and an open subset $V \subset U$ containing $p(y)$.
- An open subset $W \subset F$ containing $\left(\theta_{p(y)}^U\right)^{-1}(y)$.

Such that $\Theta^U(V \times W) \subset O$. One easily checks that it defines a topology on E: $\emptyset$ is open because the condition is then empty, E is open because $\bigcup_{U \in \mathcal{U}} U = M$, stability by union is a tautology, and stability by finite intersections is a consequence of the same properties for the topologies of M and F.

Second step: show that E has the right topological properties.

First notice that p is continuous: given $z \in E$, we pick $U \in \mathcal{U}$ such that $p(z) \in \mathcal{U}$, so that for any open subset $V \subset M$ containing $p(z)$, we have that $p^{-1}(U \cap V)$ is open, so $p^{-1}(V)$ is a neighbourhood of z.

We now wish to show that E is Hausdorff. Let $z, z' \in E$ be distinct. Set $x = p(z)$ and $x' = p(z')$. If $x \neq x'$, then the continuity of p and the fact that M is Hausdorff provide disjoint open sets containing z and z'. If $x = x'$, we consider $U \in \mathcal{U}$ that contains x, and set $y = \left(\theta_x^U\right)^{-1}(z), y' = \left(\theta_x^U\right)^{-1}(z')$. Since $z \neq z'$, we also have $y \neq y'$, and we can consider disjoint open sets $W, W' \subset F$ such that $y \in W$ and $y' \in W'$. We now have that $\Theta^U(U \times W)$ and $\Theta^U(U \times W')$ are disjoint open subsets of E containing y and y' respectively.

In order to show that E is secound countable, we first use the fact that M is Lindelöf (i.e. every open cover has a countable subcover) to consider a sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}$ such that $\bigcup_{i \in \mathbb{N}} U_i = M$. For every $i \in \mathbb{N}$, we consider a countable base $(U_{i,j})_{j \in \mathbb{N}}$ of the topology of U_i. We also consider a countable base $(W_k)_{k \in \mathbb{N}}$ of the topology of F. For $i, j, k \in \mathbb{N}$, set:

$$O_{i,j,k} = \Theta^{U_i}(U_{i,j} \times W_k)$$

One easily checks that $(O_{i,j,k})_{(i,j,k) \in \mathbb{N}^3}$ is a countable base of the topology of E.

Third step: Find an atlas.

Given $U \in \mathcal{U}$, the map Θ^U is injective and its image is $p^{-1}(U)$. Let $\Phi_U : p^{-1}(U) \to M \times F$ be its inverse. Note that its domain $p^{-1}(U)$ is an open subset of E, and that its image $U \times F$ is an open subset of $M \times F$.

Let $V \subset U$ and $W \subset F$ be open subsets. It follows from the definition of the topology on E that $\Phi_U^{-1}(U \times W)$ is open. This shows that Φ_U is continuous. Also, if $O \subset E$ is open and $z \in O$, then by definition we have that $\Phi_U(O)$ contains some $V \times W$ which contains $\Phi_U(z)$, therefore $\Phi_U(O)$ is open, and Φ_U is a homeomorphism.

The transition maps are the maps:

$$\begin{cases} (U \cap V) \times F & \to & (U \cap V) \times F \\ (x, y) & \mapsto & \left(x, \left(\theta_x^U\right)^{-1} \circ \theta_x^V(y)\right) \end{cases}$$

Their smoothness is one of the assumptions.

Fourth step: Show that (E,p,M,F) is a fibre bundle.

Since $p \circ \Theta^U(x,y) = x$, and Θ^U is a local diffeomorphism, we see that p is a submersion. It is surjective by definition of the disjoint union. The family $(\theta^U_x)_{x\in U}$ provide a trivialization of $p|_{p^{-1}(U)}$ with respect to F, which shows that (E,p,M,F) is a fibre bundle.

Fifth step: Prove uniqueness.

Because the maps Θ^U are smooth, the charts φ_U define an atlas, and the uniqueness follows from the uniqueness of a maximal atlas containing a given atlas.

□

5.1.3 Topological aspects of fibre bundles

Let us finish by discussing the gap between surjective submersions and locally trivial fibrations. The projection $p : \mathbb{R}\times\mathbb{R}\setminus\{(0;0)\} \to \mathbb{R}$ onto the first factor is a surjective submersion but not a locally trivial fibration, since the fibres are not all diffeomorphic to each other. There are also examples of surjective submersions whose fibres are all diffeomorphic to each other, but that are not locally trivial fibrations, however their constructions are much more involved.

In the specific case where the fibres are compact, things are more simple. If $\xi = (E,p,M,F)$ is a fibre bundle and F is compact, then p is proper. Reciprocally, proper surjective submersions are locally trivial fibrations.

Theorem 5.1.7 (Ehresmann)**.** *If $f : M \to N$ is a proper surjective submersion, then (M,f,N,F) is a fibre bundle (where $F = f^{-1}(x)$ for some $x \in N$).*

Note that if the total space of a fibre bundle is compact, then so are the base and the fibre. The other direction is also true.

Proposition 5.1.8. *Let $\xi = (E,p,M,F)$ be a fibre bundle. If M and F are compact, then so is E.*

Proof. Since the topology of a manifold is metrizable, we can carry a sequential proof. Let $(z_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$, and set $x_n = p(z_n) \in M$. Since M is compact, there is a subsequence (z_{n_k}) such that x_{n_k} converges to some $x \in M$. Let (U,θ) be a trivialising chart around x, and set $y_n = \theta^{-1}_{x_n}(z_n) \in F$ when it is defined. Since F is compact, up to replacing n_k with another subsequence we can assume that y_{n_k} converges to some $y \in F$. Now set $z = \theta_x(y) \in E$, and notice that $z_{n_k} = \Theta(x_{n_k}, y_{n_k}) \to z$. □

5.1.4 Morphisms and subbundles

Definition 5.1.9. If $\xi = (E, p, M, F)$ and $\xi' = (E', p', M, F')$ are fibre bundles, a **fibre bundle morphism** $\varphi = (\varphi_x)_{x\in M}$ is a family of smooth maps $\varphi_x : \xi_x \to \xi'_x$ such that the map

$$\Phi : \left\{ \begin{array}{ccc} E & \to & E' \\ z & \mapsto & \varphi_{p(z)}(z) \end{array} \right.$$

is smooth. It is a **fibre bundle isomorphism** if each φ_x is a diffeomorphism.

Remark. Here we only define morphisms for bundles with the same base. There is also a definition for bundles over different bases, involving a map between the bases. A fibre bundle morphism as defined above is called a fibre bundle morphism over the identity in this general setting. This general case can still be treated in our terminology as a morphism from a bundle over some manifold M to the pull-back of a bundle over a manifold N by some map $M \to N$.

Proposition 5.1.10. *Let $\xi = (E, p, M, F)$ and $\xi' = (E', p', M, F')$ be fibre bundles over the same base M, and $\varphi = (\varphi_x)_{x\in M}$ a fibre bundle morphism. Then φ is a fibre bundle isomorphism if and only if the map*

$$\Phi : \left\{ \begin{array}{ccc} E & \to & E' \\ z & \mapsto & \varphi_{p(z)}(z) \end{array} \right.$$

is a diffeomorphism.

Remark. This is a generalisation of Proposition 5.1.2, which is the case where ξ is the trivial bundle. The proof is exactly the same.

Proof. If Φ is a diffeomorphism, then $\varphi_x = \Phi|_{\xi_x}$ is a diffeomorphism with inverse $\Phi^{-1}|_{\xi'_x}$.

Now assume that φ is a fibre bundle isomorphism. Note that Φ is a bijection, with $\Phi^{-1}(z') = \left(\varphi_{p'(z')}\right)^{-1}(z')$ for all $z' \in E'$. Let us show that Φ is an immersion. Since this is a local notion, we may assume that ξ is the trivial bundle $\underline{F}_M$. If $(x, y) \in E = M \times F$ and $(\dot{x}, \dot{y}) \in \ker d_{(x,y)}\Phi$, then $d_{(x,y)}\pi_1 \circ \Phi(\dot{x}, \dot{y}) = 0$, which shows that $\dot{x} = 0$. Now we have that $d_{(x,y)}\Phi(0, \dot{y}) = d_y\varphi_x(\dot{y}) = 0$, so $\dot{y} = 0$.

We have shown that Φ is a bijective immersion, so it is a diffeomorphism. $\square$

Proposition 5.1.11. *Let $\xi = (E, p, M, F)$ be a fibre bundle, and $f : N \to M$ a smooth map. Define the set*

$$f^*E = \{(x, z) \in N \times E \,|\, p(z) = f(x)\}$$

and the map

$$f^*p : \begin{cases} f^*E & \to & N \\ (x,z) & \mapsto & x \end{cases}$$

Then f^*E is a submanifold of $N \times E$, and $f^*\xi = (f^*E, f^*p, N, F)$ is a fibre bundle, called the ***pulled back bundle*** *of ξ by f.*

Remark. The pulled back bundle should be thought of as the fibre bundle over N, for which the fibre over $x \in N$ is the fibre of ξ over $f(x) \in M$, i.e. $(f^*\xi)_x = \xi_{f(x)}$. This is how we will prove that it is a fibre bundle: given a local trivialisation (U,θ) of ξ, we show that

$$\left(f^{-1}(U), (\theta_{f(x)})_{x \in f^{-1}(U)}\right)$$

is a local trivialisation of $f^*\xi$.

Proof. The fact that f^*E is a submanifold of $N \times E$ is a basic example of transversality: consider the map $f \times p : N \times E \to M \times M$ defined by $f \times p(x,z) = (f(x), p(z))$. Then $f^*E = f \times p^{-1}(\Delta)$, where $\Delta \subset M \times M$ is the diagonal, and $f \times p$ is transverse to Δ because p is a submersion. It follows that f^*E is a submanifold of $N \times E$ and that $T_{(x,z)} f^*E = \{(v,w) \in T_xN \times T_zE \,|\, d_xf(v) = d_zp(w)\}$. It also follows from this characterization of the tangent space that f^*p is a submersion. It is surjective because p is surjective.

Finally, given a local trivialisation (U,θ) of ξ, set $\theta'_x(y) = (x, \theta_{f(x)}(y)) \in (f^*p)^{-1}(\{x\})$, so that $(f^{-1}(U), \theta')$ is a local trivialisation of $f^*\xi$. □

Definition 5.1.12. Let $\xi = (E,p,M,F)$ be a fibre bundle. A **subbundle** of ξ is a fibre bundle $\xi' = (E',p',M,F')$ such that:

- $E' \subset E$ is a submanifold, and $p' = p|_{E'}$.
- $F' \subset F$ is a submanifold.
- For every $x \in M$, there is a local trivialisation (U,θ) with $x \in U$ such that:
$$\forall y \in U \quad \theta_y(F') = p^{-1}(y) \cap E$$

Note that the other requirements imply that E' must be a submanifold of E.

Proposition 5.1.13. *Let $\xi = (E,p,M,F)$ be a fibre bundle. Consider a submanifold $F' \subset F$, and a collection of submanifolds $\xi'_x \subset \xi_x$ for all $x \in M$. Assume that for every $x \in M$, there is a local trivialisation (U,θ) with $x \in U$ such that:*

$$\forall y \in U \quad \theta_y(F') = p^{-1}(y) \cap E$$

Then $E' = \sqcup_{x \in M} \xi'_x$ is a submanifold of E, and $\xi' = (E',p',M,F')$ is a subbundle of ξ.

The proof is left as an exercise.

5.2 Reductions of the structural group

5.2.1 Transition functions

Definition 5.2.1. Let $\xi = (E, p, M, F)$ be a fibre bundle, and (U, θ), (U', θ') two trivialising charts. The **transition function** is the map:

$$\begin{cases} U \cap U' & \to & \mathrm{Diff}(F) \\ x & \mapsto & \theta_x^{-1} \circ \theta'_x \end{cases}$$

Given a subgroup $G \subset \mathrm{Diff}(F)$, two trivialising charts are called G-compatible if the transition function has values in G. A trivialising atlas is called G-compatible if it consists of trivialising charts that are pairwise G-compatible.

A **reduction of the structural group of** $\xi = (E, p, M, F)$ **to** G is a maximal G-compatible atlas (i.e. maximal amongst G-compatible atlases).

The data of a reduction of the structural group of $\xi = (E, p, M, F)$ to $G \subset \mathrm{Diff}(F)$ means that fibres ξ_x for $x \in M$ inherit any algebraic or geometric structure of F which is invariant by G.

The data of the transition functions $\tau_{U,V} : U \cap V \to \mathrm{Diff}(F)$ on an open cover $\mathcal{U}$ of M trivialising domains characterizes a fibre bundle up to isomorphism. Moreover, one can construct fibre bundles from the data of such transitions.

Theorem 5.2.2. *Let M and F be manifolds, and $\mathcal{U}$ an open cover of M. For every pair $U, V \in \mathcal{U}$, consider a map $\tau_{U,V} : U \cap V \to \mathrm{Diff}(F)$. Assume the following:*

1. *The maps* $\begin{cases} U \cap V \times F & \to & F \\ (x, z) & \mapsto & \tau_{U,V}(x)(z) \end{cases}$ *are smooth.*
2. *For every $U, V, W \in \mathcal{U}$ and $x \in U \cap V \cap W$, we have:*

$$\tau_{U,V}(x) \circ \tau_{V,W}(x) \circ \tau_{W,U}(x) = \mathrm{Id}_F$$

Then there exists a fibre bundle $\xi = (E, p, M, F)$, and a local trivialisation (θ^U) of $\xi|_U$ for each $U \in \mathcal{U}$ such that $\tau_{U,V}(x) = \left(\theta_x^U\right)^{-1} \circ \theta_x^V$ for all $x \in U \cap V$.

We will discuss two examples of reductions of the structure group: the case where F is a vector space, and G is the general linear group $\mathrm{GL}(F)$, known as vector bundles, and the case where F is a Lie group, and G is the group of right translations in F, known as principal bundles.

Definition 5.2.3. Let H be a Lie group. A H**-principal bundle** (or principal bundle with structural group H) is the data of a fibre bundle $\xi = (E, p, M, H)$ and a reduction of the structural group to $\left\{ L_g \,\middle|\, g \in H \right\}$.

Definition 5.2.4. Let M be a manifold, and $r \in \mathbb{N}$. A real (resp. complex) **vector bundle** of rank r over M is the data of a fibre bundle $\xi = (E, p, M, \mathbb{R}^r)$ (resp. $\xi = (E, p, M, \mathbb{C}^r)$) and a reduction of the structural group to $\mathrm{GL}(\mathbb{R}^n)$ (resp. $\mathrm{GL}(\mathbb{C}^n)$).

5.2.2 A transition-free approach

Any notion defined on a manifold should have a description that does not involve coordinate changes! For reduction of structural groups of fibre bundles, it is possible, but not so obvious. Given a reduction of the structural group of a fibre bundle $\xi = (E, p, M, F)$ to a group $G \subset \mathrm{Diff}(F)$, and a point $x \in M$, there is a well defined group $G_x \subset \mathrm{Diff}(\xi_x)$, and one could hope to use this family of groups of diffeomorphisms of the fibres to get a trivialisation free definition. But there is no natural identification of G_x with G, which makes this tricky. Instead, we can define a reduction of the structural group to G as the assignment to each point $x \in M$ of an orbit $\mathcal{O}_x$ for the right action of G on the set of diffeomorphisms from F to ξ_x, such that around every point there is a local trivialisation $(\theta_x)_{x \in U}$ with $\theta_x \in \mathcal{O}_x$ for all $x \in U$.

5.2.3 Principal bundles

The archetypal principal bundle is a quotient G/H of a Lie group by an embedded Lie sugroup.

Proposition 5.2.5. *Let G be a Lie group, $H \subset G$ an embedded Lie subgroup and $\pi : G \to G/H$ the projection. Then $(G, \pi, G/H, H)$ is a H-principal bundle.*

Proposition 5.2.5 is a particular case of the following characterisation of principal bundles in terms of actions of the structural group. Recall that the quotient by a smooth free proper action is a manifold (Theorem 4.1.10).

Theorem 5.2.6 (Principal bundles)**.** *Let $X \curvearrowleft H$ be a smooth, free and proper action of a Lie group H on a manifold X, and let $\pi : X \to X/H$ be the projection.*

Then $(X, \pi, X/H, H)$ is a H-principal bundle.

Reciprocally, if $\xi = (E, p, M, H)$ is a H-principal fibre bundle, then there is a smooth, free and proper action $X \curvearrowleft H$ whose orbits are the fibres of ξ.

Proof. Start with a smooth, free and proper action $X \curvearrowleft H$. Consider an open set $U \subset X/H$ and a smooth map $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{Id}$ (such open sets exist around each point of X/H as a consequence of the Submersion Theorem).

For $x \in U$, we consider the orbit map $\varphi_{\sigma(x)} : H \to \pi^{-1}(\{x\})$, and we wish to show that the family $(\varphi_{\sigma(x)})_{x \in U}$ is a trivialisation of $\pi|_{\pi^{-1}(U)}$ with respect to H.

The map $(x,h) \mapsto \varphi_{\sigma(x)}(h) = \sigma(x).h$ is smooth because σ and the action are smooth. Given $x \in U$, the map $\varphi_{\sigma(U)}$ is injective because the action is free, and surjective by definition of an orbit. It is an immersion as shows Proposition 4.3.1. It follows that it is a diffeomorphism, so $(\varphi_{\sigma(x)})_{x\in U}$ is a trivialisation of $\pi|_{\pi^{-1}(U)}$ with respect to H and $(X,\pi,X/H,H)$ is a fibre bundle.

Let $U' \subset X/H$ be another open set and $\sigma' : U' \to X$ another smooth map satisfying $\pi \circ \sigma' = \mathrm{Id}$. Because the action is free, for any $x \in U \cap U'$ there is a unique $h(x) \in H$ such that $\sigma'(x) = \sigma(x).h(x)$. The transition function is $x \mapsto L_{h(x)}$, since $\varphi_{\sigma'(x)} = \varphi_{\sigma(x).h(x)} = \varphi_{\sigma(x)} \circ L_{h(x)}$. This proves that $(X,\pi,X/H,H)$ is a H-principal bundle.

Now given a H-principal bundle $\xi = (E,p,M,H)$, consider a trivialising atlas $\mathcal{A}$ of G-compatible trivialising charts. Define the (right) action of H on M in the following way: for $z \in E$ and $h \in H$, let $(U,\theta) \in \mathcal{A}$ be such that $x \in U$, and set $z.h = \Theta\left(p(z), \theta^{-1}_{p(z)}(z)h\right)$. This action is well defined, i.e. does not depend on the choice of $(U,\theta) \in \mathcal{A}$. The smoothness of p, Θ, Θ^{-1} and the group operation on H imply the smoothness of the action. It is free because the action by right multiplication of H on itself, and the orbits are the fibres of ξ. Finally, one can check that it is proper by using the properness of the right action of H on itself and the continuity of Θ^{-1}. $\square$

Examples 5.2.7.

- Principal bundles with 0-dimensional fibre are Galois coverings.
- If M is a d-dimensional manifold, the **frame bundle** $\mathcal{R}(M)$ of M is defined by

  $$\mathcal{R}(M)_x = \left\{\varphi \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^d, T_xM) \,\middle|\, \varphi \text{ is an isomorphism}\right\}$$

  for $x \in M$. The group $\mathrm{GL}(\mathbb{R}^d)$ acts on $\mathcal{R}(M)_x$ by $\varphi.f = \varphi \circ f$. It is a principal bundle with structural group $\mathrm{GL}(\mathbb{R}^d)$.
- The **Hopf fibration** of $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 |\ |z|^2 + |w|^2 = 1\}$. Consider the $\mathbb{S}^1$ action on $\mathbb{S}^3$ given by $e^{i\theta}.(z,w) = (e^{i\theta}z, e^{i\theta}w)$. It is free and proper. The quotient is diffeomorphic to $\mathbb{S}^2$ (because the projection is the restriction of the canonical projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 \approx \mathbb{S}^2$).

5.3 Sections of fibre bundles

Definition 5.3.1. Let $\xi = (E,p,M,F)$ be a fibre bundle. We set:

$$\Gamma(\xi) = \{\sigma \in \mathcal{C}^\infty(M,E) \mid \forall x \in M\ \sigma(x) \in \xi_x\}$$

A **section** of ξ is an element $\sigma \in \Gamma(\xi)$.

Remark. For people who are into commutative diagrams, $\sigma \in \Gamma(\xi)$ translates as $p \circ \sigma = \mathrm{Id}_M$.

General fibre bundles need not have sections. For principal bundles, the existence of a section is equivalent to triviality.

Proposition 5.3.2. *A principal bundle $\xi = (X, \pi, X/H, H)$ is trivialisable if and only if it admits a section $\sigma : X/H \to X$.*

Proof. Let $\sigma : X/H \to X$ be a section. Then $(\varphi_{\sigma(x)})_{x \in X/H}$ is a trivialisation of π with respect to H. □

Remarks.

- Given a principal bundle, fibres can be identified with H up to choosing a point in the fibre that we associate with e (this is exactly what a section does).
- The H-principal bundle $(G, \pi, G/H, H)$, where $H \subset G$ is a closed Lie subgroup, is not necessarily trivial.

Vector bundles, on the other end of the sprectrum, have many sections. For a start, you can always define the **zero section**. However trivialisability can still be expressed in terms of sections.

Proposition 5.3.3. *Let $\xi = (E, p, M, \mathbb{K}^r)$ be a vector bundle of rank r (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If there are r sections $\sigma_1, \ldots, \sigma_r$ of ξ such that $(\sigma_1(x), \ldots, \sigma_r(x))$ is a vector basis of ξ_x for all $x \in M$, then ξ is trivialisable.*

Proof. For $x \in M$, consider the linear isomorphism

$$\varphi_x : \left\{ \begin{array}{ccc} \mathbb{K}^r & \to & \xi_x \\ (v^1, \ldots, v^r) & \mapsto & \sum_{i=1}^r v^i \sigma_i(x) \end{array} \right.$$

Then $(\varphi_x)_{x \in M}$ is a trivialisation of ξ. □

We will see later that this is an equivalence.

Chapter 6

Vector bundles

Remarks.

- We only consider smooth vector bundles over real manifolds (the fibres can be complex vector spaces), not holomorphic vector bundles over complex manifolds, which are a science apart.
- A vector bundle will just be denoted by $\xi = (E, p, M)$.
- Given a vector bundle $\xi = (E, p, M)$ of rank r and $x \in M$, the fibre ξ_x is a vector space of dimension r (every trivialisation chart gives an isomorphism with $\mathbb{R}^r$ or $\mathbb{C}^r$).

6.1 Morphisms of vector bundles and vector subbundles

All the operations described on fibre bundles make sense within vector bundles.

Definition 6.1.1. Let M be a manifold, and $\xi = (E, p, M), \xi' = (E', p', M)$ be vector bundles over M. A **vector bundle morphism** from ξ to ξ' is a smooth map $\varphi : E \to E'$ such that $p' \circ \varphi = p$, and for all $x \in M$ the restriction $\varphi_x : \xi_x \to \xi'_x$ is linear.
It is a **vector bundle isomorphism** if moreover the restrictions to fibres are isomorphisms.

One can easily check that a vector bundle isomorphism is a diffeomorphism between the total spaces (e.g. by using Proposition 5.1.2). We want to call a a vector bundle **trivialisable** if it is isomorphic as a vector bundle to the trivial vector bundle $\underline{\mathbb{K}^r}_M$. This happens to be equivalent to being trivialisable as a fibre bundle.

Proposition 6.1.2. *Let $\xi = (E,p,M)$ be a vector bundle of rank r over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). There is a vector bundle isomorphism between ξ and the trivial vector bundle $(M \times \mathbb{K}^r, \pi_1, M)$ if and only if the fibre bundle $(E,p,M,\mathbb{K}^r)$ is trivialisable.*

Proof. It is a straightforward consequence of the definitions that the trivialisability as a vector bundle implies the trivialisability as a fibre bundle. If the fibre bundle $(E,p,M,\mathbb{K}^r)$ is trivialisable, then consider a trivialisation $(\varphi_x)_{x\in M}$ of p with respect to $\mathbb{K}^r$. Then $(d_0\varphi_x)_{x\in M}$ is also a trivialisation of p with respect to $\mathbb{K}^r$, made of linear maps, so the map $\psi : M \times \mathbb{K}^r \to E$ defined by $\psi(x,v) = d_0\varphi_x(v)$ is a vector bundle isomorphism between ξ and the trivial vector bundle $(M \times \mathbb{K}^r, \pi_1, M)$. □

This means that there is no possible confusion on what we mean by a trivialisable vector bundle. Note that the proof works in a more general setting: two vector bundles that are isomorphic as fibre bundles are also isomorphic as vector bundles.

Pulling back a vector bundle also yields a vector bundle.

Proposition 6.1.3. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $f : N \to M$ a smooth map. There is a unique vector bundle structure on the pulled-back bundle $f^*\xi$ such that the vector space operations on $(f^*\xi)_x$ for $x \in N$ coincide with those on $\xi_{f(x)}$.*

Definition 6.1.4. Let $\xi = (E,p,M)$ be a vector bundle. A **vector subbundle** of ξ is a fibre subbundle ξ' of ξ such that ξ'_x is a vector subspace of ξ_x for all $x \in M$.

6.2 Sections of vector bundles

6.2.1 Frame fields

Instead of working with vector bundle isomorphisms or some more or less sophisticated types of charts, it is more practical to deal with vector bundles through frame fields, i.e. a vector basis of each fibre that depends smoothly on the base point.

Definition 6.2.1. Let $\xi = (E,p,M)$ be a vector bundle of rank r. A **frame field** of ξ is a r-tuple $(\varepsilon_1,\dots,\varepsilon_r) \in \Gamma(\xi)^r$ such that, for any $x \in M$, the family $(\varepsilon_1(x),\dots,\varepsilon_r(x))$ is a basis of ξ_x.

Frame fields need not exist globally, but only locally, since they are directly related to the trivialisability of the vector bundle.

Proposition 6.2.2. *Let $\xi = (E,p,M)$ be a vector bundle of rank r. Then ξ is trivialisable if and only if it possesses a frame field.*

Proof. If φ is a vector bundle isomorphism from the trivial bundle to ξ and $(e_1,\dots,e_r)$ is a vector basis of $\mathbb{K}^r$, then $\varepsilon_i(x) = \varphi(x,e_i)$ defines a frame field.

Reciprocally, if $(\varepsilon_1,\dots,\varepsilon_r)$ is a frame field, we set $\theta_x(v) = v^1\varepsilon_1(x) + \cdots + v^r\varepsilon_r(x)$ for $x \in M$ and $v = (v^1,\dots,v^r) \in \mathbb{K}^r$. For each $x \in M$, the map $\theta_x : \mathbb{K}^r \to \xi_x$ is a linear isomorphism, in particular a diffeomorphism. Since $\varepsilon_1,\dots,\varepsilon_r$ are smooth maps, the map $(x,v) \mapsto \theta_x(v)$ is smooth, and it is a trivialisation of ξ. □

Most of the local computations on vector bundles will be done through the choice of a local frame field, i.e. a frame field for a restriction $\xi|_U$ to an open set $U \subset M$. This is practical because we then treat sections of ξ as a r-tuple of smooth functions.

Lemma 6.2.3. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and let $(\varepsilon_1,\dots,\varepsilon_r)$ be a frame field of ξ. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i\varepsilon_i$ where $\sigma_1,\dots,\sigma_r \in \mathcal{C}^\infty(M)$.*

Proof. The uniqueness is a consequence of the uniqueness of the decomposition of a vector in a vector basis. The existence of functions $\sigma_1,\dots,\sigma_r$ follows from the same fibrewise consideration. To prove their smoothness, consider the trivialisation $(\theta_x)_{x\in M}$ used in the proof of Proposition 6.2.2, i.e. $\theta_x(v) = \sum_{i=1}^r v^i\varepsilon_i(x)$ for $x \in \mathbb{K}^r$. The map $\Theta : M \times \mathbb{K}^r \to E$ defined by $\Theta(x,v) = \theta_x(v)$ is a diffeomorphism according to Proposition 5.1.2. Now notice that $(x,(\sigma_1(x),\dots,\sigma_r(x))) = \Theta^{-1}(x,\sigma(x))$, the smoothness of the functions $\sigma_1,\dots,\sigma_r$ follows. □

6.2.2 The space of sections of a vector bundle

Given a vector bundle ξ, the space of sections $\Gamma(\xi)$ is a vector space, as one can add and multiply by scalars on each fibre. First, notice that sections of vector bundles are plentiful (which is the main difference with the holomorphic setting).

Lemma 6.2.4. *Si $\xi = (E,p,M)$ be a vector bundle of rank r. For all $x \in M$ and $v \in \xi_x$, there is a section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$.*

Proof. Let $U \subset M$ be a trivializing domain containing x and $(\varepsilon_1,\dots,\varepsilon_r)$ a frame field of $\xi|_U$. Every $v \in \xi_x$, decomposes as $v = \sum_{i=1}^r v^i\varepsilon_i(x)$. Using a bump function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi(x) = 1$ and $\varphi = 0$ outside U, we can define $\sigma_i = \varphi\varepsilon_i \in \Gamma(\xi)$, and $\sigma = \sum_{i=1}^r v^i\sigma_i \in \Gamma(\xi)$ satisfies $\sigma(x) = v$. □

Of course, such a section is far from being unique (it depends strongly on all the choices made). One can easily check that the vector space $\Gamma(\xi)$ of all sections is infinite dimensional. We will make use of the following technical result relating sections that agree at some given point.

Lemma 6.2.5. *Let $\xi = (E, p, M)$ be a vector bundle of rank r. If $\sigma, \sigma' \in \Gamma(\xi)$ satisfy $\sigma(x) = \sigma'(x)$ for some $x \in M$, then there are $s_1, \dots, s_r \in \Gamma(\xi)$ and $f_1, \dots, f_r \in \mathcal{C}^\infty(M)$ such that $f_i(x) = 0$ and $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ on a neighbourhood of x.*

Proof. Let $U \subset M$ be a trivializing domain containing x and $(\varepsilon_1, \dots, \varepsilon_r)$ a frame field of $\xi|_U$.

If $\varphi \in \mathcal{C}^\infty(M)$ is a plateau function, equal to 1 on a neighbourhood of x and 0 outside of U, then the sections $s_i = \varphi\varepsilon_i \in \Gamma(\xi)$ are well defined.

There are smooth functions $g_1, \dots, g_r \in \mathcal{C}^\infty(U)$ such that $\sigma - \sigma' = \sum_{i=1}^r g_i \varepsilon_i$ on U.

The functions $f_i = \varphi g_i \in \mathcal{C}^\infty(M)$ are well defined, and we have that $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ near x. □

The space $\Gamma(\xi)$ has an additional algebraic structure: given a smooth function $f \in \mathcal{C}^\infty(M)$ and a section $\sigma \in \Gamma(\xi)$, we can define the fibrewise product $f\sigma \in \Gamma(\xi)$. This means that $\Gamma(\xi)$ is a $\mathcal{C}^\infty(M)$-module. Proposition 6.2.2 can be restated as saying that ξ is trivialisable if and only if $\Gamma(\xi)$ is a free $\mathcal{C}^\infty(M)$-module (necessarily of rank r).

6.2.3 Constructing vector bundles from frame fields

It is actually possible to construct vector bundles via frame fields, and this is what we will use for all algebraic constructions.

Theorem 6.2.6. *Let M be a manifold, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $r \in \mathbb{N}$. Consider a collection of $\mathbb{K}$-vector spaces $(\xi_x)_{x \in M}$ of dimension r, an open cover $\mathcal{U}$ of M, and for each $U \in \mathcal{U}$ and $x \in U$ a vector basis $(\sigma_{U,1}(x), \dots, \sigma_{U,r}(x))$ of ξ_x.*

Assume that for every $U, V \in \mathcal{U}$, the maps $\tau_{U,i}^{V,j} : U \cap V \to \mathbb{K}$ defined by

$$\sigma_{U,i}(x) = \sum_{j=1}^r \tau_{U,i}^{V,j}(x) \sigma_{V,j}(x)$$

are smooth. Then there is a unique manifold structure on the disjoint union $E = \sqcup_{x \in M} \xi_x$ satisfying these two conditions:

- *(E, p, M) is a vector bundle of rank r, where $p : E \to M$ is defined by $p(z) = x$ when $z \in \xi_x$.*
- *The functions $\sigma_{U,i}$ are smooth for all $U \in \mathcal{U}$ and $1 \le i \le r$.*

Proof. The existence and uniqueness of a manifold structure on E for which $(E, p, M, \mathbb{K}^n)$ is a fibre bundle is given by Theorem 5.1.6, when considering the diffeomorphisms $\theta_x^U : \mathbb{K}^r \to \xi_x$ defined by $\theta_x^U(v) = \sum_{i=1}^r v^i \sigma_{U,i}(x)$. Since the transition functions are linear, with matrices $\left(\tau_{U,i}^{V,j}(x)\right)_{1 \le i,j \le r}$ that depend smoothly on x, it is a vector bundle. □

Vector subbundles can be defined in terms of frame fields.

Proposition 6.2.7. *Let $\xi = (E, p, M)$ be a vector bundle of rank r. Consider a vector subbundle $\xi' = (E', p', M, \mathbb{K}^{r'})$ of ξ. Then around every $x \in M$ there is a local frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of ξ such that $(\varepsilon_1, \dots, \varepsilon_{r'})$ is a local frame field of ξ'.*

Proof. The idea is very similar to Proposition 6.1.2. By definition of a vector subbundle, for every $x \in M$ the submanifold $\xi'_x \subset \xi_x$ is a vector subspace.

Consider a local trivialisation $(\theta_x)_{x \in U}$ of ξ (as a fibre bundle, i.e. each $\theta_x : \mathbb{K}^r \to \xi_x$ is a diffeomorphism, but not necessarily linear) such that $\xi'_x = \theta_x(\mathbb{K}^{r'} \times \{0\})$ (the existence of such trivialisations around every point of M comes from the definition of a fibre subbundle).

Now let $(e_1, \dots, e_r)$ be the canonical basis of $\mathbb{K}^r$, and simply check that $(\varepsilon_1, \dots, \varepsilon_r)$ is a local frame field of ξ satisfying the requirements where $\varepsilon_i(x) = d_0\theta_x(e_i)$. □

Remark. If $\xi = (E, p, M)$ is a a vector bundle, and $E' \subset E$ is a subset such that for all $x \in M$, the intersection $E' \cap \xi_x$ is a vector subspace of dimension r', and if around every $x \in M$ there are sections $(\varepsilon_1, \dots, \varepsilon_{r'})$ of ξ such that $(\varepsilon_1(y), \dots, \varepsilon_{r'}(y))$ is a vector basis of $E' \cap \xi_y$ for all y near x, then $\xi' = (E', p|_{E'}, M)$ is a vector bundle, and a vector sub-bundle of ξ.

6.3 Examples of vector bundles

6.3.1 The tangent bundle

Recall that given a manifold M and $x \in M$, the tangent space T_xM is defined as:

$$T_xM = \{D \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathbb{R}) \,|\, \forall f, g \in \mathcal{C}^\infty(\mathbb{R})\ D(fg) = D(f)g(x) + f(x)D(g)\}$$

Theorem 6.2.6 provides a vector bundle structure on $TM = \sqcup_{x \in M} T_xM$ by requiring that for every chart (U, φ) the local sections $\partial_1, \dots, \partial_d$ given by $\partial_i(x).f = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(x))$ are smooth.

A section of TM is called a **vector field**. We write $\mathcal{X}(M) = \Gamma(TM)$. A manifold M is called **parallelisable** if its tangent bundle TM is trivialisable.

6.3.2 Tautological bundles

Definition 6.3.1. Let V be a finite dimensional vector space over $\mathbb{K}$ (= $\mathbb{R}$ of $\mathbb{C}$), and let $k \in \{0, \dots, \dim V\}$. The **tautological bundle** of the Grassmannian $\mathcal{G}_k(V) = \{W \subset V \,|\, \dim W = k\}$ is the vector subbundle $\tau_k(V)$ of the trivial bundle $\mathcal{G}_k(V) \times V$ with total space $\{(x, v) \in \mathcal{G}_k(V) \,|\, v \in x\}$.

Proposition 6.3.2. *Up to vector bundle isomorphism, the only vector bundles of rank 1 over the circle are the trivial bundle and $\tau_1(\mathbb{R}^2)$.*

6.3.3 Flat bundles

Consider a manifold M, and a linear representation $\rho : \pi_1(M) \to \mathrm{GL}(V)$. The fundamental group $\pi_1(M)$ acts $\widetilde{M} \times V$ via the diagonal action $\gamma.(x,v) = (\gamma.x, \rho(\gamma)v)$.

Lemma 6.3.3. *Let M be a manifold, V a finite dimensional vector space and $\rho : \pi_1(M) \to \mathrm{GL}(V)$ a representation. The diagonal action $\pi_1(M) \curvearrowright \widetilde{M} \times V$ is free and properly discontinuous.*

Proof. It follows from the fact that the action on $\widetilde{M}$ is free and properly discontinuous. If $\gamma.(x,v) = (x,v)$, then $\gamma.x = x$ so $\gamma = e$. If $K \subset \widetilde{M} \times V$ is compact, then $K' = \left\{ x \in \widetilde{M} \,\middle|\, \exists v \in V\ (x,v) \in K \right\}$ is also compact, any any $\gamma \in \pi_1(M)$ satisfying $\gamma.K \cap K \neq \emptyset$ must also satisfy $\gamma.K' \cap K' \neq \emptyset$, so the set of such elements is compact. □

We can consider the quotient manifold $E = \pi_1(M) \backslash (\widetilde{M} \times V)$, and denote by $\pi_E : \widetilde{M} \times V \to E$ the projection. The composition of the projection on the first factor $\pi_1 : \widetilde{M} \times V \to \widetilde{M}$ with the universal covering map $\pi_M : \widetilde{M} \to M$ induces a map $p : E \to M$ satisfying $p \circ \pi_E = \pi_M \circ \pi_1$.

Proposition 6.3.4. *Let M be a manifold, V a finite dimensional vector space and $\rho : \pi_1(M) \to \mathrm{GL}(V)$ a representation. Set $E = \pi_1(M) \backslash (\widetilde{M} \times V)$, and $p : E \to M$ the map induced by the projection on the first factor. Then $\xi = (E, p, M)$ is a vector bundle.*

Proof. The fact that p is a surjective submersion comes from the identity $p \circ \pi_E = \pi_M \circ \pi_1$. Let $U \subset E$ be an open set and $\sigma : U \to \widetilde{M} \times V$ a local section of π_E (i.e. $\pi_E \circ \sigma = \mathrm{Id}$).For $y \in U$, let φ_y^σ be the map:

$$\varphi_y^\sigma : \left\{ \begin{array}{ccc} V & \to & p^{-1}(\{y\}) \\ v & \mapsto & \pi_E(\pi_1 \circ \sigma(y), v) \end{array} \right. .$$

Then $(\varphi_y)_{y \in U}$ is a local trivialisation of p with respect to V. Given a second local section $\sigma' : U' \to \widetilde{M} \to V$ of π_E, the transition function at $y \in U \cap U'$ is $\rho(\gamma)$ where $\gamma \in \pi_1(M)$ satisfies $\pi_1 \circ \sigma'(y) = \gamma.\pi_1 \circ \sigma(y)$. Since ρ is a linear representation, we find that ξ is a vector bundle. □

This construction works in a more general setting. First of all, instead of working with the universal cover $\widetilde{M}$, any Galois covering and representation of the Galois group Γ would do. Moreover, given a manifold F and a group homomorphism $\Gamma \to \mathrm{Diff}(F)$ (i.e. a smooth action of Γ on F), we can define a fibre bundle with fibre F over M.

One can also replace Γ with a Lie group G, and construct bundles over quotients by free proper actions of G (called **associated bundles**).

6.4 Vector subbundles, quotients and direct sums

6.4.1 Supplementary vector subbundles

Just as any manifold can be seen as a submanifold of the Euclidean space, any vector bundle can be seen as a subbundle of a trivial bundle.

Proposition 6.4.1. *Any smooth vector bundle is isomorphic to a vector subbundle of a trivial vector bundle.*

Proof. Let $\xi = (E,p,M)$ be a vector bundle. Let $T_0\xi$ be the vector bundle over M defined by $(T_0\xi)_x = T_0\xi_x$ (it is the pull-back of the tangent bundle TE by the zero section $0_\xi : M \to E$). Since each fibre ξ_x is a vector space, $T_0\xi$ is isomorphic to ξ.

Using Whitney's Embedding Theorem, we may assume that E is a submanifold of some $\mathbb{R}^N$, which makes $T_0\xi$ a subbundle of the trivial bundle $\underline{\mathbb{R}^N}_M$. □

Definition 6.4.2. Let $\xi = (E,p,M)$ be a vector bundle, and consider vector subbundles $\xi_1 = (E_1,p_1,M), \xi_2 = (E_2,p_2,M)$ of ξ. We say that ξ_1 and ξ_2 are **supplementary** in ξ if $\xi_x = (\xi_1)_x \oplus (\xi_2)_x$ for all $x \in M$.

The fibrewise projection is a vector bundle morphism.

Proposition 6.4.3. *Let $\xi = (E,p,M)$ be a vector bundle, and consider two supplementary vector subbundles $\xi_1 = (E_1,p_1,M), \xi_2 = (E_2,p_2,M)$ of ξ. The map $\pi_1 : E \to E_1$ such that $(\pi_1)|_{\xi_x}$ is the projection of ξ_x onto $(\xi_1)_x$ parallel to $(\xi_2)_x$ is a vector bundle morphism.*

Proposition 6.4.4. *Every vector subbundle of a smooth vector bundle has a supplementary subbundle.*

Let us postpone the proof of Proposition 6.4.4 to Chapter 8, as it will be a straightforward consequence of Proposition 8.1.2 and Proposition 8.1.5. The reader can check that it does not create any logical inconsistency.

6.4.2 Quotients of vector bundles

Given a vector space V and a subspace $W \subset V$, we let $\overline{x} \in V/W$ be the image of $x \in V$ as long as there is no possible confusion.

Proposition 6.4.5. *Let $\xi = (E,p,M)$ be a vector bundle, and $\eta = (F,q,M)$ be a vector subbundle of ξ. The* ***quotient bundle*** *ξ/η is the vector bundle defined by $(\xi/\eta)_x = \xi_x/\eta_x$ for all $x \in M$, and such that for any local frame $(\varepsilon_1,\ldots,\varepsilon_r)$ of ξ for which $(\varepsilon_1,\ldots,\varepsilon_k)$ is a local frame of η, the maps $(\overline{\varepsilon}_{k+1},\ldots,\overline{\varepsilon}_r)$ form a frame field.*

The fibre wise projection defines a vector bundle morphism $\xi \to \xi/\eta$. A section $\sigma \in \Gamma(\xi)$ projects to a section $\overline{\sigma} \in \Gamma(\xi/\eta)$. Note that any section of the quotient can be lifted.

Lemma 6.4.6. *Let $\xi = (E, p, M)$ be a vector bundle, and $\eta = (F, q, M)$ be a vector subbundle of ξ. Any section of ξ/η is the projection of a section of ξ.*

Remark. This means that there is an isomorphism $\Gamma(\xi/\eta) \approx \Gamma(\xi)/\Gamma(\eta)$, both as vector spaces and as $\mathcal{C}^\infty(M)$-modules.

Proof. Start locally by using a trivialisation to produce a supplementary subbundle, then extend with a partition of unity. □

6.4.3 The direct sum of vector bundles

Definition 6.4.7. Let $\xi = (E, p, M)$ and $\eta = (F, q, M)$ be vector bundles over a same manifold. The **direct sum** $\xi \oplus \eta$ is the vector bundle defined by $(\xi \oplus \eta)_x = \xi_x \oplus \eta_x$ for all $x \in M$, and such that for local frame fields $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ and $(\zeta_1, \ldots, \zeta_s)$ of η over a same open set $U \subset M$, the maps $(\varepsilon_1 \oplus 0, \ldots, \varepsilon_r \oplus 0, 0 \oplus \zeta_1, \ldots, 0 \oplus \zeta_s)$ form a frame field.

The existence and uniqueness of such a vector bundle is given by Theorem 6.2.6.

One can also recover its total space $E \oplus F$ as follows:

$$E \oplus F = \{(v, w) \in E \times F \mid p(v) = q(w)\}$$

Given sections $\sigma \in \Gamma(\xi)$ and $\tau \in \Gamma(\eta)$, we can define $\sigma \oplus \tau \in \Gamma(\xi \oplus \eta)$. Any section of $\xi \oplus \eta$ can be obtained in this way, so we have an isomorphism $\Gamma(\xi \oplus \eta) \approx \Gamma(\xi) \oplus \Gamma(\eta)$ (this works both as real vector spaces and as $\mathcal{C}^\infty(M)$-modules).

6.5 Algebraic operations on vector bundles

6.5.1 The dual bundle

Recall the definition of a dual basis.

Proposition 6.5.1. *Let V be a vector space and $e = (e_1, \ldots, e_r)$ be a basis of V. There is a unique basis $e^* = (e^1, \ldots, e^r)$ of V^* such that $e^i(e_j) = \delta^i_j$ for all $1 \leq i, j \leq r$.*

The basis e^* is called the **dual basis** of e. Given some vector $v \in V$, the scalar $e^i(v)$ is the coordinate along e_i of its decomposition in the basis e.

Definition 6.5.2. Let $\xi = (E, p, M)$ be a vector bundle of rank r. The **dual bundle** ξ^* of ξ is the vector bundle defined by $(\xi^*)_x = (\xi_x)^*$ for all $x \in M$, and such that for a local frame $(\varepsilon_1, \ldots, \varepsilon_r)$, the family $(\varepsilon^1, \ldots, \varepsilon^r)$ is a local frame of ξ^*, where $(\varepsilon^1(x), \ldots, \varepsilon^r(x))$ is the dual basis of $(\varepsilon_1(x), \ldots, \varepsilon_r(x))$ when defined.

If $\lambda \in \Gamma(\xi^*)$ and $\sigma \in \Gamma(\xi)$, we can define $\lambda(\sigma) \in \mathcal{C}^\infty(M)$. Note that while a section $\sigma \in \Gamma(\xi)$ is usually written with a functional notation, i.e. $\sigma(x) \in \xi_x$ for $x \in M$, for a section $\lambda \in \Gamma(\xi^*)$ it is more convenient to use a subscript notation $\lambda_x \in \xi_x^*$ for $x \in M$, so that given $v \in \xi_x$ we can write $\lambda_x(v) \in \mathbb{K}$ instead of $\lambda(x)(v)$.

The dual bundle T^*M of the tangent bundle TM of a manifold M is called the **cotangent bundle**.

Since we are working with finite dimensional vector spaces, there is a natural identification between a vector space V and its **bidual** $V^{**} = (V^*)^*$, sending $x \in V$ to the evaluation map $\lambda \mapsto \lambda(x)$. This property, which may seem rather trivial at a first glance, is a key feature of finite dimensional vector spaces and it makes the description of tensor products dramatically simpler.

Let us now see how to construct sections of a dual bundle. If $\xi = (E, p, M)$ is a vector bundle of rank r, then a section $\lambda \in \Gamma(\xi^*)$ defines a linear map $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ by setting $\Lambda(\sigma)(x) = \lambda_x(\sigma(x))$. However, not all linear maps $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$, as the value of $\Lambda(\sigma)(x)$ could also depend on the values of σ at other points of M, or on derivatives of σ. The following result, called the **Tensoriality Lemma**, states that the defining condition of the maps $\Gamma(\xi) \to \mathcal{C}^\infty(M)$ thus obtained is $\mathcal{C}^\infty(M)$-linearity.

Lemma 6.5.3. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, and let $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ be a linear map. The following are equivalent:*

1. $\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(M)\ \ \Lambda(f\sigma) = f\Lambda(\sigma)$
2. $\exists \lambda \in \Gamma(\xi^*)\ \ \forall \sigma \in \Gamma(\xi)\ \forall x \in M\ \ \Lambda(\sigma)(x) = \lambda_x(\sigma(x))$

Remark. This means that there is an isomorphism of $\mathcal{C}^\infty(M)$-modules between $\Gamma(\xi^*)$ and $\mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(\xi), \mathcal{C}^\infty(M))$, i.e. between $\Gamma(\xi^*)$ and the dual of $\Gamma(\xi)$ seen as a $\mathcal{C}^\infty(M)$-module.

Proof. The implication *(2)⇒(1)* is straightforward, let us prove *(1)⇒(2)*. Consider $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ which is $\mathcal{C}^\infty(M)$-linear. For $x \in M$ and $v \in \xi_x$, consider a section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$ (the existence being given by Lemma 6.2.4). Set $\lambda_x(v) = \Lambda(\sigma)(x)$.

Let us first check that $\lambda_x(v)$ does not depend on the choice of σ.

Let $\sigma, \sigma' \in \Gamma(\xi)$ be such that $\sigma(x) = \sigma'(x) = v$. According to Lemma 6.2.5, there are sections $s_1, \ldots, s_r \in \Gamma(\xi)$ and functions $f_1, \ldots, f_r \in \mathcal{C}^\infty(M)$ such that $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ on a neighbourhood V of x.

If $\varphi \in \mathcal{C}^\infty(M)$ is a plateau function such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside V, then we find:

$$\varphi\sigma = \varphi\sigma' + \sum_{i=1}^{r} \varphi f_i s_i \tag{6.1}$$

This equality stands on all of M. Evaluating Λ on the left hand side of 6.1, we find:

$$\begin{aligned}\Lambda(\varphi\sigma)(x) &= \underbrace{\varphi(x)}_{=1} \Lambda(\sigma)(x)\\ &= \Lambda(\sigma)(x)\end{aligned}$$

Evaluating Λ on the right hand side of 6.1 yields:

$$\begin{aligned}\Lambda\left(\varphi\sigma' + \sum_{i=1}^{r} \varphi f_i s_i\right)(x) &= \underbrace{\varphi(x)}_{=1} \Lambda(\sigma')(x) + \sum_{i=1}^{r} \varphi(x) \underbrace{f_i(x)}_{=0} \Lambda(s_i)(x)\\ &= \Lambda(\sigma')(x)\end{aligned}$$

In the end, we do find that $\Lambda(\sigma)(x) = \Lambda(\sigma')(x)$.

In order to prove the linearity and regularity of λ, consider a trivializing domain $U \subset M$ containing x, and $(\varepsilon_1, \ldots, \varepsilon_r)$ a frame field of $\xi|_U$.

We once again consider a plateau function $\psi \in \mathcal{C}^\infty(M)$ such that $\psi = 1$ on a neighbourhood W of x and $\psi = 0$ outside of U.

On W, we find:

$$\lambda = \sum_{i=1}^{r} \Lambda(\psi\varepsilon_i)\varepsilon^i$$

This shows both the linearity and the smoothness of λ, i.e. $\lambda \in \Gamma(\xi^*)$.

□

Remark. We do not require any regularity of the functional Λ (we do not even need to consider topologies of the vector spaces $\Gamma(\xi)$ and $\mathcal{C}^\infty(M)$). The regularity is hidden in the fact that given a smooth section $\sigma \in \Gamma(\xi)$, the resulting function $\Lambda(\sigma)$ is smooth.

6.5.2 The homomorphism bundle

Given two vector spaces V, W and elements $\lambda \in V^*, w \in W$, we can define the linear map $\lambda \otimes w \in \mathrm{Hom}(V, W)$ by $\lambda \otimes w(v) = \lambda(v)w$ for all $v \in V$.

If $(e_1, \ldots, e_r)$ is a basis of V and $(f_1, \ldots, f_s)$ is a basis of W, then $(e_i^* \otimes f_j)$ is a basis of $\mathrm{Hom}(V, W)$.

Definition 6.5.4. Let $\xi = (E,p,M)$ and $\eta = (F,q,M)$ be vector bundles over the same manifold. The homomorphism bundle $\mathrm{Hom}(\xi,\eta)$ is the vector bundle defined by $\mathrm{Hom}(\xi,\eta)_x = \mathrm{Hom}(\xi_x,\eta_x)$ for all $x \in M$, and such that for local frames $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ and $(\zeta_1,\dots,\zeta_s)$ of η over a same open set $U \subset M$, the maps $(\varepsilon^i \otimes \zeta_j)$ form a frame field.

There are several fibrewise operations that produce and/or use sections of homomorphism bundles.

- Sections $\sigma \in \Gamma(\xi)$ and $A \in \Gamma(\mathrm{Hom}(\xi,\eta))$ produce a section $A(\sigma) \in \Gamma(\eta)$ defined by $A(\sigma)(x) = A_x(\sigma(x))$.
- Sections $\lambda \in \Gamma(\xi^*)$ and $\sigma \in \Gamma(\eta)$ produce a section $\lambda \otimes \sigma \in \Gamma(\mathrm{Hom}(\xi,\eta))$ defined by $\lambda \otimes \sigma_x = \lambda_x \otimes \sigma(x)$.
- A section $A \in \Gamma(\mathrm{Hom}(\xi,\eta))$ produces a section ${}^tA \in \Gamma(\mathrm{Hom}(\eta^*,\xi^*))$ by considering the fibrewise transpose (recall that given $u \in \mathrm{Hom}(V,W)$, the transpose ${}^tu \in \mathrm{Hom}(W^*,V^*)$ is defined by ${}^tu(\lambda) = \lambda \circ u$).

Sections $\sigma \in \Gamma(\xi)$ and $A \in \Gamma(\mathrm{Hom}(\xi,\eta))$ produce a section $A(\sigma) \in \Gamma(\eta)$. So a section $A \in \Gamma(\mathrm{Hom}(\xi,\eta))$ induces a linear map $\Gamma(\xi) \to \Gamma(\eta)$, and once again $\mathcal{C}^\infty(M)$-linearity is the distinguishing feature of these maps.

Lemma 6.5.5. *Let ξ and η be vector bundles over the same base M. The map*

$$\left\{\begin{array}{ccc} \Gamma(\mathrm{Hom}(\xi,\eta)) & \to & \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(\xi),\Gamma(\eta)) \\ A & \mapsto & \sigma \mapsto A(\sigma) \end{array}\right.$$

is an isomorphism of $\mathcal{C}^\infty(M)$-modules.

Proof. The proof of Lemma 6.5.3 carries out exactly in the same way until the last formula that should be changed into $\lambda = \sum_{i=1}^r \varepsilon^i \otimes \Lambda(\psi\varepsilon_i)$. □

6.5.3 The tensor product bundle

Let R be a commutative unital ring, and $M_1,\dots,M_k,N$ be R-modules. We denote by $\mathcal{M}\mathrm{ult}(M_1,\dots,M_k;N)$ the space of multi-R-linear maps from $M_1 \times \cdots \times M_k$ to N. If there is any ambiguity on the ring R (mostly $\mathbb{R}$, $\mathbb{C}$ or $\mathcal{C}^\infty(M)$ for some manifold M), we will write $\otimes_R$ to specify the ring.

The construction of the tensor product $M_1 \otimes \cdots \otimes M_k$ produces a multilinear map $\left\{\begin{array}{ccc} M_1 \times \cdots \times M_k & \to & M_1 \otimes \cdots \otimes M_k \\ (x_1,\dots,x_k) & \mapsto & x_1 \otimes \cdots \otimes x_k \end{array}\right.$
Recall the universal property of the tensor product: if N is a R-module and $\varphi \in \mathcal{M}\mathrm{ult}_R(M_1,\dots,M_k;N)$, there is a unique map $\Phi \in \mathrm{Hom}_R(M_1 \otimes \cdots \otimes M_k, N)$ such that $\Phi(x_1 \otimes \cdots \otimes x_k) = \varphi(x_1,\dots,x_k)$ for all $(x_1,\dots,x_k) \in M_1 \times \cdots \times M_k$.

The tensor product $V^* \otimes W$

In the setting of smooth vector bundles, we will work with finite dimensional vector spaces over the fields $\mathbb{R}$ and $\mathbb{C}$. The construction of the tensor product is much simpler in the case. The easiest case is actually the tensor product $V^* \otimes W$ involving a dual space. Indeed, we have seen that a linear form $\lambda \in V^*$ and a vector $w \in W$ define a linear map $\lambda \otimes w \in \mathrm{Hom}(V, W)$. One can check that there is a unique isomorphism from $V^* \otimes W$ to $\mathrm{Hom}(V, W)$ identifying pure tensors $\lambda \otimes w \in V^* \otimes W$ with the linear map $\lambda \otimes w \in \mathrm{Hom}(V, W)$. For our intended purposes, we can treat this as a definition and consider that $V^* \otimes W = \mathrm{Hom}(V, W)$.

Recall that if $(e_i)_{1 \leq i \leq r}$ is a basis of V and $(f_j)_{1 \leq j \leq s}$ is a basis of W, then $(e^i \otimes f_j)$ is a basis of $V^* \otimes W$.

Remark. One sees quite clearly in this description of the tensor product that most elements of $V^* \otimes W$ are not pure tensors. Indeed, non trivial pure tensors in $\mathrm{Hom}(V, W)$ are exactly linear maps of rank 1.

Tensor product of dual spaces

This leads us directly to defining the tensor product $V^* \otimes W^*$ as the vector space $\mathrm{Hom}(V, W^*)$. There is a natural identification between $\mathrm{Hom}(V, W^*)$ and $\mathcal{M}\mathrm{ult}(V, W; \mathbb{K})$ as a map to a set of maps is the same as a two-variable map, i.e. the natural isomorphism is

$$\left\{ \begin{array}{ccc} \mathrm{Hom}(V, W^*) & \to & \mathcal{M}\mathrm{ult}(V, W; \mathbb{K}) \\ \varphi & \mapsto & (v, w) \mapsto \varphi(v)(w) \end{array} \right.$$

and its inverse

$$\left\{ \begin{array}{ccc} \mathcal{M}\mathrm{ult}(V, W; \mathbb{K}) & \to & \mathrm{Hom}(V, W^*) \\ \Phi & \mapsto & v \mapsto \Phi(v, \bullet) \end{array} \right. .$$

We can thus consider that $V^* \otimes W^* = \mathcal{M}\mathrm{ult}(V, W; \mathbb{K})$. Given linear forms $\lambda \in V^*$ and $\mu \in W^*$, the tensor product is $\lambda \otimes \mu \in \mathcal{M}\mathrm{ult}(V, W; \mathbb{K})$ is defined by $\lambda \otimes \mu(v, w) = \lambda(v)\mu(w)$ for all $(v, w) \in V \times W$.

More generally, we can consider the tensor product $V_1^* \otimes \cdots \otimes V_k^*$ as the space $\mathcal{M}\mathrm{ult}_{\mathbb{K}}(V_1, \ldots, V_k; \mathbb{K})$, and $\lambda_1 \otimes \cdots \otimes \lambda_k(v_1, \ldots, v_k) = \lambda_1(v_1) \cdots \lambda_k(v_k)$ for $(\lambda_1, \ldots, \lambda_k) \in V_1^* \times \cdots \times V_k^*$ and $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$.

If (λ_j^i) is a basis of V_j^*, then the family $(\lambda_1^{i_1} \otimes \cdots \otimes \lambda_k^{i_k})$ is a basis of $V_1^* \otimes \cdots \otimes V_k^*$.

General tensor products of finite dimensional vector spaces

We are now lead to describe a tensor product $V \otimes W$ and the tensor product of their biduals $V^{**} \otimes W^{**}$, so we can consider that $V \otimes W = \mathcal{M}\mathrm{ult}(V^*, W^*; \mathbb{K})$.

The tensor product of $(v,w) \in V \times W$ is defined by $v \otimes w(\lambda,\mu) = \lambda(v)\mu(w)$ for all $(\lambda,\mu) \in V^* \times W^*$.

More generally, we can consider the tensor product $V_1 \otimes \cdots \otimes V_k$ as the space $\mathcal{M}\mathrm{ult}_{\mathbb{K}}(V_1^*,\ldots,V_k^*;\mathbb{K})$, and $v_1 \otimes \cdots \otimes v_k(\lambda_1,\ldots,\lambda_k) = \lambda_1(v_1)\cdots\lambda_k(v_k)$ for $(\lambda_1,\ldots,\lambda) \in V_1^* \times \cdots \times V_k^*$.

If (e_i^j) is a basis of V_j, then the family $(e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k)$ is a basis of $V_1 \otimes \cdots \otimes V_k$.

Remark. Another approach is that the dual space $(V \otimes W)^*$ can be identified with $\mathcal{M}\mathrm{ult}(V,W;\mathbb{K})$ by the universal property of tensor products. So one can also consider $V \otimes W$ as the dual space $\mathcal{M}\mathrm{ult}(V,W;\mathbb{K})^*$, but this is not as easy to deal with as $\mathcal{M}\mathrm{ult}(V^*,W^*;\mathbb{K})$ is. Note that there is not direct natural identification between these two spaces, the most natural way of identifying them goes through the tensor product (one can also choose bases of V and W and identify the corresponding bases of $\mathcal{M}\mathrm{ult}(V,W;\mathbb{K})^*$ and $\mathcal{M}\mathrm{ult}(V^*,W^*;\mathbb{K})$, then check that this isomorphism does not depend on the choice of bases).

Associativity and commutativity of tensor products

The operation of taking tensor products is commutative, meaning that for any permutation $\sigma \in \mathfrak{S}_k$, there is a unique isomorphism from $V_1 \otimes \cdots \otimes V_k$ to $V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(k)}$ sending each pure tensor $v_1 \otimes \cdots \otimes v_k$ to $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$. This is quite straightforward from the description of the tensor product as multilinear maps.

This operations is also associative, in the sense that there a unique isomorphism from $(V_1 \otimes V_2) \otimes V_3$ to $V_1 \otimes V_2 \otimes V_3$ sending pure tensors $(v_1 \otimes v_2) \otimes v_3$ to $v_1 \otimes v_2 \otimes v_3$. This is a consequence of the isomorphism between the appropriate spaces of multilinear maps given by partial evaluations.

One example of particular interest is the fact that a tensor product of the form $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ can be described as $\mathcal{M}\mathrm{ult}(V_1,\ldots,V_k;W)$.

Tensor products of vector bundles

Definition 6.5.6. Let $\xi_1 = (E_1,p_1,M),\ldots,\xi_k = (E_k,p_k,M)$ be vector bundles over the same manifold M. The **tensor product bundle** $\xi_1 \otimes \cdots \otimes \xi_k$ is the vector bundle defined by $(\xi_1 \otimes \cdots \otimes \xi_k)_x = (\xi_1)_x \otimes \cdots \otimes (\xi_k)_x$ for all $x \in M$, and such that for local frames $(\varepsilon_1^j,\ldots,\varepsilon_{r_j}^j)$ of each ξ_j defined over a same open set $U \subset M$, the maps $(\varepsilon_{i_1}^1 \otimes \cdots \otimes \varepsilon_{i_k}^k)$ for a local frame field.

Given a finite dimensional space V, we set $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$. Similarly, given a vector bundle ξ we can define $\xi^{\otimes k}$.

There is also an operation on sections: considering $\sigma \in \Gamma(\xi)$ and $\tau \in \Gamma(\eta)$, we can define $\sigma \otimes \tau \in \Gamma(\xi \otimes \eta)$ through fibrewise tensor product. This actually leads to an isomorphism between $\Gamma(\xi\otimes\eta)$ and $\Gamma(\xi)\otimes_{\mathcal{C}^\infty(M)}\Gamma(\eta)$, but we will not attempt to prove this fact as it will not be of any interest for us.

Tensors on manifolds

Most of the vector bundles that we will be interested in will be tensor powers of the tangent bundle of a manifold, or subbundles of these.

Definition 6.5.7. Let M be a manifold, and $p,q\in\mathbb{N}$. A **tensor of type** (p,q) on M is a section of $(T^*M)^{\otimes p}\otimes TM^{\otimes q}$. We denote by $\mathcal{T}^{p,q}(M)$ the space of tensors of type (p,q) on M.

A tensor of type $(p,0)$ is called **covariant** and a tensor of type $(0,q)$ is called **contravariant**.

There is a widely accepted consensus as to how one should write a tensor in coordinates. Given $T\in\mathcal{T}^{p,q}(M)$, we always write its expression in coordinates with indices relating to the covariant part on the bottom, and indices for the contravariant part on top, i.e.

$$T=\sum_{1\leq i_1,\dots,i_p,j_1,\dots,j_q\leq d} T_{i_1,\dots,i_p}^{j_1,\dots,j_q}dx^{i_1}\otimes\cdots\otimes dx^{i_p}\otimes\partial_{j_1}\otimes\partial_{j_q}$$

For example, a vector field $X\in\mathcal{X}(M)=\mathcal{T}^{0,1}(M)$, which is a contravariant tensor, is always written locally as $X=\sum_{i=1}^d X^i\partial_i$. A differential 1-form $\omega\in\Omega^1(M)=\mathcal{T}^{1,0}(M)$, on the other hand, is a covariant tensor, and is written locally as $\omega=\sum_{i=1}^d\omega_i dx^i$.

The tensoriality lemma

Definition 6.5.8. Consider vector bundles $\xi,\xi_1,\dots,\xi_m$ over the same basis M. A multi-linear map $A:\Gamma(\xi_1)\times\cdots\times\Gamma(\xi_m)\to\Gamma(\xi)$ is called **tensorial** if it is $\mathcal{C}^\infty(M)$-multi-linear, i.e.

$$\forall(f_1,\dots,f_m)\in\mathcal{C}^\infty(M)^m\ \forall(\sigma_1,\dots,\sigma_m)\in\Gamma(\xi_1)\times\cdots\times\Gamma(\xi_m)$$
$$A(f_1\sigma_1,\dots,f_m\sigma_m)=f_1\cdots f_mA(\sigma_1,\dots,\sigma_m)$$

Theorem 6.5.9 (Tensoriality Lemma). *Consider vector bundles $\xi,\xi_1,\dots,\xi_m$ over the same basis M, and a multi-linear map*

$$A:\Gamma(\xi_1)\times\cdots\times\Gamma(\xi_m)\to\Gamma(\xi)$$

The following are equivalent:

1. *A is tensorial*
2. *There is $\alpha\in\Gamma(\xi_1^*\otimes\cdots\otimes\xi_m^*\otimes\xi)$ such that:*

$$\forall x\in M\ \forall(\sigma_i)_{1\leq i\leq m}\in\prod_{i=1}^m\Gamma(\xi_i)\quad A(\sigma_1,\dots,\sigma_m)(x)=\alpha_x(\sigma_1(x),\dots,\sigma_m(x))$$

Pull-backs and the Lie derivative

Given a vector bundle $\xi = (E, p, M)$ and a smooth map $\varphi : N \to M$, the pull-back $\varphi^*\sigma$ of a section $\sigma \in \Gamma(\xi)$ is a section of $f^*\xi$. In the case of a covariant tensor bundle $\xi = \mathcal{T}^{p,0}(M)$, there is another notion of pull-back which defines a tensor on N. We use the same notation for both.

Given a smooth map $\varphi : N \to M$ and a covariant tensor $T \in \mathcal{T}^{p,0}(M)$, one can define its pull-back $\varphi^*T \in \mathcal{T}^{p,0}(N)$ by

$$(\varphi^*T)_x(v_1, \dots, v_p) = T_{\varphi(x)}(d_x\varphi(v_1), \dots, d_x\varphi(v_p)).$$

This is well defined because $T_y(w_1, \dots, w_p)$ is just a real number for any $y \in M$ and $w_1, \dots, w_p \in T_yM$.

For a general tensor $T \in \mathcal{T}^{p,q}(M)$, then we can define the pull-back $\varphi^*T \in \mathcal{T}^{p,q}(N)$ under the additional hypothesis that φ is a local diffeomorphism. For $q = 1$, the formula is

$$(\varphi^*T)_x(v_1, \dots, v_p) = (d_x\varphi)^{-1}\left[T_{\varphi(x)}(d_x\varphi(v_1), \dots, d_x\varphi(v_p))\right]$$

Given a vector field $X \in \mathcal{X}(M)$ and a tensor $T \in \mathcal{T}^{p,q}(M)$, we can define the **Lie derivative** $\mathcal{L}_X T \in T \in \mathcal{T}^{p,q}(N)$ by

$$\mathcal{L}_X T = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_X^t\right)^* T$$

It acts as a derivation on tensors, in the sense that $\mathcal{L}_X(fT) = df(X)T + f\mathcal{L}_X T$ for any function $f \in \mathcal{C}^\infty(M)$.

Note that a tensor of type $(0,0)$ is just a function, and in this case we get $\mathcal{L}_X f = df(X)$. For $X, Y \in \mathcal{X}(M)$, we have $[X, Y] = \mathcal{L}_X Y = -\mathcal{L}_Y X$.

6.5.4 Exterior and symmetric powers of a vector bundle

The **exterior power** $\Lambda^k V^*$ is the subspace of $(V^*)^{\otimes k}$ composed of skew-symmetric forms (i.e. $u(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \varepsilon(\sigma)u(x_1, \dots, x_k)$ for $(x_1, \dots, x_k) \in V^k$ and $\sigma \in \mathfrak{S}_k$).

Given $j, k \in \mathbb{N}$, $\lambda \in \Lambda^j V^*$ and $\mu \in \Lambda^k V^*$, we define $\lambda \wedge \mu \in \Lambda^{j+k} V^*$ as:

$$\lambda \wedge \mu(x_1, \dots, x_{j+k}) = \frac{1}{j!k!} \sum_{\sigma \in \mathfrak{S}_{j+k}} \varepsilon(\sigma)\lambda(x_{\sigma(1)}, \dots, x_{\sigma(j)})\mu(x_{\sigma(j+1)}, \dots, x_{\sigma(j+k)})$$

This operation is associative and anti-commutative ($\mu \wedge \lambda = (-1)^{j(k-j)}\lambda \wedge \mu$). Given $(\lambda_1, \dots, \lambda_k) \in (V^*)^k$, the formula for $\lambda_1 \wedge \cdots \wedge \lambda_k \in \Lambda^k V^*$ is:

$$\begin{aligned}
\lambda_1 \wedge \cdots \wedge \lambda_k(x_1, \dots, x_k) &= \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\lambda_1(x_{\sigma(1)}) \cdots \lambda_k(x_{\sigma(k)}) \\
&= \det\left(\lambda_i(x_j)\right)_{1 \le i,j \le k}
\end{aligned}$$

If $(e_1, \dots, e_r)$ is a basis of V, then $(e^{i_1} \wedge \cdots \wedge e^{i_k})_{i_1 < \cdots < i_k}$ is a basis of $\Lambda^k V^*$.

Definition 6.5.10. Let $\xi = (E,p,M)$ be a vector bundle of rank r, and k an integer. The k^{th}-**exterior power** of ξ^* is the vector subbundle $\Lambda^k\xi^*$ of $(\xi^*)^{\otimes k}$ defined by $\left(\Lambda^k\xi^*\right)_x = \Lambda^k(\xi^*)_x$ for all $x \in M$.

The **symmetric power** S^kV^* is the subspace of $(V^*)^{\otimes k}$ composed of symmetric forms (i.e. $u(x_{\sigma(1)},\dots,x_{\sigma(k)}) = u(x_1,\dots,x_k)$ for $(x_1,\dots,x_k) \in V^k$ and $\sigma \in \mathfrak{S}_k$).

Given $j,k \in \mathbb{N}$, $\lambda \in S^jV^*$ and $\mu \in S^kV^*$, we define $\lambda \vee \mu \in S^{j+k}V^*$ as:

$$\lambda \vee \mu(x_1,\dots,x_{j+k}) = \frac{j!k!}{(j+k)!}\sum_{\sigma\in\mathfrak{S}_{j+k}} \lambda(x_{\sigma(1)},\dots,x_{\sigma(j)})\mu(x_{\sigma(j+1)},\dots,x_{\sigma(j+k)}).$$

This operation is associative and commutative. For $(\lambda_1,\dots,\lambda_k) \in (V^*)^k$, the product $\lambda_1 \vee \cdots \vee \lambda_k \in S^kV^*$ is given by:

$$\lambda_1 \vee \cdots \vee \lambda_k(x_1,\dots,x_k) = \sum_{\sigma\in\mathfrak{S}_k} \lambda_1(x_{\sigma(1)})\cdots\lambda_k(x_{\sigma(k)})$$

If $(e_1,\dots,e_r)$ is a basis of V, then $(e^{i_1} \vee \cdots \vee e^{i_k})_{i_1\leq\cdots\leq i_k}$ is a basis of S^kV^*.

Definition 6.5.11. Let $\xi = (E,p,M)$ be a vector bundle of rank r, and k an integer. The k^{th}-**symmetric power** of ξ^* is the vector subbundle $S^k\xi^*$ of $(\xi^*)^{\otimes k}$ defined by $\left(\Lambda^k\xi^*\right)_x = S^k(\xi^*)_x$ for all $x \in M$.

Differential forms on a manifold

A **differential** k**-form** on a manifold M is a section of Λ^kT^*M. We use the notation $\Omega^k(M) = \Gamma\left(\Lambda^kT^*M\right)$. There are four main operations on differential forms.

The **wedge product:** Given $\omega_1 \in \Omega^{k_1}(M)$ and $\omega_2 \in \Omega^{k_2}(M)$ we can define $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(M)$ fibrewise.

The **interior product:** Given $X \in \mathcal{X}(M)$ and $\omega \in \Omega^{k+1}(M)$, we define $\iota_X\omega \in \Omega^k(M)$ by:

$$\forall x \in M\ \forall (v_1,\dots,v_k) \in T_xM^k \quad (\iota_X\omega)_x(v_1,\dots,v_k) = \omega_x(X(x),v_1,\dots,v_k)$$

The **Lie derivative:** if $\omega \in \Omega^k(M) \subset \mathcal{T}^{k,0}(M)$ and $X \in \mathcal{X}(M)$, then the Lie derivative $\mathcal{L}_X\omega \in \mathcal{T}^{p,0}(M)$ is still skew-symmetric, i.e. $\mathcal{L}_X\omega \in \Omega^k(M)$, and the Lie derivative is an operation on differential forms.

The **exterior derivative:** Given $\omega \in \Omega^k(M)$, we define $d\omega \in \Omega^{k+1}(M)$. In coordinates, if $\omega = \sum_{|I|=k}\omega_I dx^I$ (where $dx^{\{i_1,\dots,i_k\}} = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ if $i_1 < \cdots < i_k$), then $d\omega = \sum_{i=1}^d \sum_{|I|=k} \partial_i\omega_I dx^i \wedge dx^I$.

There are several formulae relating these operations. The interior product and the wedge product are related by

$$\iota_X(\omega_1 \wedge \omega_2) = (\iota_X \omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge (\iota_X \omega_2)$$

We have the following rule for the exterior derivative of a wedge product.

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge \omega_2$$

Note that this formula, together with the fact that df is the usual differential for a 0-form f, and $d \circ d = 0$, are the defining features of the exterior derivative.

The exterior derivative, the Lie derivative and the interior product are related through **Cartan's magic formula**.

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega$$

Remark. The Tensoriality Lemma can be used to define the exterior derivative of a differential form, using Cartan's magic formula as a definition instead of a property. For $\omega \in \Omega^1(M)$, it gives

$$d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega(X, Y)$$

In the general case $\omega \in \Omega^p(M)$), given $X_0, \ldots, X_p \in \mathcal{X}(M)$, we find:

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p). \end{aligned}$$

One can use this forumula as a definition, since the Tensoriality Lemma shows that this expression defines $d\omega \in \Gamma\left(T^*M^{\otimes p+1}\right)$, then one can check that it is skew-symmetric, i.e. $d\omega \in \Gamma\left(\Lambda^{p+1} T^*M\right) = \Omega^{p+1}(M)$.

6.5.5 Vector bundle valued differential forms

Definition 6.5.12. Let $\xi = (E, p, M)$ be a vector bundle and $p \in \mathbb{N}$. A ξ-**valued differential** k**-form** ω is a section $\omega \in \Gamma\left(\Lambda^k T^*M \otimes \xi\right)$.

We will use the notation $\Omega^k(\xi) = \Gamma\left(\Lambda^k T^*M \otimes \xi\right)$. Note that for $\underline{\mathbb{R}}_M = (M \times \mathbb{R}, \pi_1, M)$, we get usual differential forms $\Omega^k(\underline{\mathbb{R}}_M) = \Omega^k(M)$.

The **interior product** of a vector field $X \in \mathcal{X}(M)$ and a ξ-valued differential form $\omega \in \Omega^{k+1}(\xi)$ can be still be defined as $\iota_X \omega \in \Omega^k(\xi)$ through the same formula $(\iota_X \omega)_x(v_1, \ldots, v_k) = \omega_x(X(x), v_1, \ldots, v_k)$.

The **wedge product** is trickier. In general, given $\omega_1 \in \Omega^{k_1}(\xi)$ and $\omega_2 \in$

$\Omega^{k_2}(\xi)$, there is no natural way of defining some element of $\Omega^{k_1+k_2}(\xi)$, since the definition of the wedge product involves multiplication in fibres. The most general construction possible is to start with two vector bundles ξ, η over M and forms $\alpha \in \Omega^p(\xi)$, $\beta \in \Omega^q(\eta)$, then $\alpha \wedge \beta$ can be defined as an elements of $\Omega^{p+q}(\xi \otimes \eta)$.

What makes sense is the exterior product $\alpha \wedge \beta \in \Omega^{p+q}(\xi)$ of a usual differential form $\alpha \in \Omega^p(M)$ and a ξ-valued differential form $\beta \in \Omega^q(\xi)$, the formula is the same as usual:

$$(\alpha \wedge \beta)_x(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} \varepsilon(\sigma) \alpha_x(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \beta_x(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}).$$

For $\omega \in \Omega^p(M)$ and $\sigma \in \Gamma(\xi) = \Omega^0(\xi)$, the notation $\omega \otimes \sigma \in \Omega^p(\xi)$ is more appropriate. We will mostly use it for 1-forms: given $\lambda \in \Omega^1(M) = \Gamma(T^*M)$ and $\sigma \in \Gamma(\xi)$, we define $\lambda \otimes \sigma \in \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$. The formula is:

$$\forall x \in M \ \forall v \in T_xM \quad (\lambda \otimes \sigma)_x(v) = \lambda_x(v)\sigma(x)$$

It is also possible to define pull-backs of vector bundle valued differential forms. If $\xi = (E, p, M)$ is a vector bundle, $\omega \in \Omega^k(\xi)$ a ξ-valued differential form and $\varphi : N \to M$ a smooth map, the pull-back $\varphi^*\omega \in \Omega^k(\varphi^*\xi)$ is defined by:

$$(\varphi^*\omega)_x(v_1, \ldots, v_k) = \omega_{\varphi(x)}(d_x\varphi(v_1), \ldots, d_x\varphi(v_k)) \in \xi_{\varphi(x)} = (\varphi^*\xi)_x.$$

6.5.6 The conjugate of a complex vector bundle

All the above constructions work for both complex and real vector bundles. However we only consider real manifolds, so tangent bundles, and more generally tensor bundles $\mathcal{T}^{p,q}(M)$, are real vector bundles.

When considering complex vector bundles, there are a few more operations. Complex vector spaces come in pair. Indeed, recall that a vector space V over a field $\mathbb{K}$ is not just a set, but a triple $V = (|V|, +, .)$ where $|V|$ is a set, the addition map $+ : |V| \times |V| \to |V|$ is such that $(|V|, +)$ is an abelian group, and the exterior multiplication $. : \mathbb{K} \times |V| \to |V|$ satisfies a rather long list of axioms.

In the case where $\mathbb{K} = \mathbb{C}$, we can associate to V its **conjugate space** $\overline{V}$ defined by $\overline{V} = (|V|, +, \overline{.})$, where $\lambda \overline{.} v = \overline{\lambda}.v$. One can check that it is also a complex vector space. The (complex) dual space $\left(\overline{V}\right)^*$ of the conjugate consists of maps $\lambda : |V| \to \mathbb{C}$ that are anti-linear for V (i.e. satisfy $\lambda(z.v) = \overline{z}\lambda(v)$). In particular, we can identify $\left(\overline{V}\right)^* \otimes_{\mathbb{C}} V^*$ with the space of maps $\varphi : V \times V \to \mathbb{C}$ that are sesquilinear, i.e. anti-linear with respect to the first variable and linear with respect to the second.

Given a complex vector bundle ξ, we can define its conjugate $\overline{\xi}$ as we have for all other algebraic operations. This allows us to define the vector bundle of sesquilinear forms over ξ as $(\overline{\xi})^* \otimes \xi^*$.

A complex vector space $V = (|V|, +, .)$ defines a real vector space $V_{\mathbb{R}} = (|V|, +, .|_{\mathbb{R}\times|V|})$, this process is known as the restriction of scalars. Similarly, a complex vector bundle ξ defines a real vector bundle $\xi_{\mathbb{R}}$ by fibrewise restriction of scalars.

Starting with a real vector space W, we can wonder if there is a complex vector space V such that $V_{\mathbb{R}} = W$. For this, one must find a way to define the multiplication by arbitrary complex numbers instead of just real numbers. The map $J : |V| \to |V|$ defined by $J(v) = i.v$ is a linear map of $V_{\mathbb{R}}$ which satisfies $J^2 = -\mathrm{Id}$. Starting with a real vector space $W = (|W|, +, .)$ and a linear map $J \in \mathrm{Hom}_{\mathbb{R}}(W)$ satisfying $J^2 = -\mathrm{Id}$, we can define the complex vector space $V = (|W|, +, *)$ where $(x+iy)*w = x.v + y.J(v)$. This is a complex vector space such that $V_{\mathbb{R}} = W$ (note that this is an actual equality, not just an isomorphism).

This questions of knowing whether a real vector space can be endowed with a complex structure should not be confused with extension of scalars. A real vector space V defines a complex vector space simply by looking at $\mathbb{C} \otimes_{\mathbb{R}} V$. Extension of scalars can be considered at the level of vector bundles, through the tensor product with the trivial bundle $\underline{\mathbb{C}}_M$.

Note that not all complex vector bundles (on real manifolds) can be obtained in this way. Indeed, the tangent bundle over $\mathbb{S}^2$ admits a structure of a one dimensional complex vector space (e.g. because $\mathbb{S}^2 \approx \mathbb{CP}^1$ is a complex manifold). If there were to exist a real bundle ξ such that $\underline{\mathbb{C}}_{\mathbb{S}^2} \otimes \xi$ is isomorphic to $T\mathbb{S}^2$, then it would possess a rank 1 subbundle (e.g. $\underline{\mathbb{R}}_{\mathbb{S}^2} \otimes \xi$). But the Hairy Ball Theorem implies that there is no such bundle.

6.5.7 Grassmannian bundles

Up to now, we saw that vector bundles produce more vector bundles. But they also produce fibre bundles that are not vector bundle. The main examples are Grassmannians.

Let $\xi = (E, p, M)$ be a vector bundle of rank r, and $k \in \{1, \ldots, r\}$. We wish to define a fibre bundle $\mathcal{G}_k(\xi)$ over M with fibres $\mathcal{G}_k(\xi)_x = \mathcal{G}_k(\xi_x)$ for $x \in M$. For this matter, we consider an open set $U \subset M$ and a frame field $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\xi|_U$.

Recall that the differentiable structure on $\mathcal{G}_k(\mathbb{R}^r)$ is given by that of the homogeneous space $\mathrm{GL}(r, \mathbb{R})/P_{k,r}$ where:

$$P_{k,r} = \left\{ \left(\begin{array}{cc} A & B \\ 0 & C \end{array} \right) \middle| A \in \mathrm{GL}(k, \mathbb{R}), B \in \mathcal{M}_{k,r-k}(\mathbb{R}), C \in \mathrm{GL}(r-k, \mathbb{R}) \right\}$$

For any $x \in U$, the basis $\varepsilon(x)$ of ξ_x induces an isomorphism between $\mathrm{GL}(r, \mathbb{R})$ and $\mathrm{GL}(\xi_x)$, sending $P_{k,r}$ to some group $P_\varepsilon(x) \subset \mathrm{GL}(\xi_x)$. We therefore get a diffeomorphism between the homogeneous spaces $\mathrm{GL}(r,\mathbb{R})/P_{k,r}$ and $\mathrm{GL}(\xi_x)/P_\varepsilon(x)$, hence a diffeomorphism $\theta_x^\varepsilon : \mathcal{G}_k(\mathbb{R}^r) \to \mathcal{G}_k(\xi_x)$.

Given another frame field δ, the transition map $(\theta_x^\delta)^{-1} \circ \theta_x^\varepsilon : \mathcal{G}(k,\mathbb{R}^r) \to \mathcal{G}(k,\mathbb{R})$ lifts to a diffeomorphism $\mathrm{GL}(k,\mathbb{R}^r) \to \mathrm{GL}(k,\mathbb{R}^r)$ which is the conjugation by the change-of-base matrix between $\varepsilon(x)$ and $\delta(x)$, so it depends smoothly on x. By Theorem 5.1.6, we have found a fibre bundle structure on $\mathcal{G}_k(\xi) = (\sqcup_{x\in M}\mathcal{G}_k(\xi_x), p, M, \mathcal{G}_k(\mathbb{R}^d))$.

Definition 6.5.13. Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $k \in \{1,\dots,r\}$. The fibre bundle $\mathcal{G}_k(\xi)$ constructed above with fibre $\mathcal{G}_k(\xi_x)$ over $x \in M$ is called the k^{th}-**Grassmannian bundle** of ξ.

Note that since Grassmannian spaces are compact, the total space of a Grassmannian bundle of a vector bundle over a compact manifold is compact, due to Proposition 5.1.8.

6.6 Classification of vector bundles

The results of this section will only be stated, not proved.

Two continuous maps $f,g : B \to M$ are called **homotopic** if there is a continuous map $h : [0,1]\times B \to M$ such that $h(0,\cdot) = f$ and $h(1,g)$. If f and g are smooth, it is always possible to choose a smooth h.

We say that B is **contractible** if $\mathrm{Id} : B \to B$ is homotopic to a constant.

Theorem 6.6.1. *Any vector bundle over a contractible manifold is trivialisable.*

We will see a simple proof of this fact using connections. Note that the result is also true for general fibre bundles.

Theorem 6.6.2. *Let M be a smooth manifold.*

1. *If ξ is a vector bundle of rank r over M, there are an integer N and a smooth map $f : M \to \mathcal{G}_r(\mathbb{K}^N)$ such that ξ is isomorphic to $f^*\tau_r(\mathbb{K}^N)$ where $\tau_r(\mathbb{K}^N)$ is the tautological bundle.*

2. *Given two smooth maps $f,g : M \to \mathcal{G}_r(\mathbb{K}^N)$, the vector bundles $f^*\tau_n(\mathbb{K}^N)$ and $g^*\tau_n(\mathbb{K}^N)$ are isomorphic if and only if there is an integer N' such that f and g are homotopic as maps from M to $\mathcal{G}_r(\mathbb{K}^{N'})$.*

The first point follows easily from Proposition 6.4.1: if ξ is a vector subbundle of the trivial bundle $\underline{\mathbb{K}^N}_M$, then it is isomorphic to $f^*\tau_r(\mathbb{K}^N)$ where $f : M \to \mathcal{G}_r(\mathbb{K}^N)$ is defined by $f(x) = \xi_x$.

Chapter 7

Covariant derivatives

In order to make sense of the definition of a connection, we need to understand how central the Leibniz rule is in calculus, and that it is a defining rule of differentiation.

Proposition 7.0.1. *Let $D : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ be a linear map such that $D(fg) = D(f)g + fD(g)$ for all $f, g \in \mathcal{C}^\infty(\mathbb{R})$. There is $\lambda \in \mathcal{C}^\infty(\mathbb{R})$ such that $D(f) = \lambda \dot{f}$ for all $f \in \mathcal{C}^\infty(\mathbb{R})$.*

7.1 Vector bundles over the line

7.1.1 Existence of frame fields

The study of vector bundles over an interval is made easy by the fact that they are all trivialisable.

Theorem 7.1.1. *Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ be a vector bundle of rank r. Then ξ is trivialisable.*

This is a consequence of Theorem 6.6.1, however we will give a proof of Theorem 7.1.1 as we will use it repeatedly.

The key is that if there is only one dimension, then there is only one way of "gluing" local trivialisations. This is because the intersection of two intervals is an interval.

Lemma 7.1.2. *Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ be a vector bundle of rank r. Assume that there are two subintervals $I_0, I_1 \subset I$ such that $\xi|_{I_0}$ and $\xi|_{I_1}$ are trivialisable, and $I_0 \cup I_1 = I$. Then ξ is trivialisable.*
Moreover, for any compact subinterval $J \subset I_0$ and any frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_{I_0}$, there is a frame field $(\overline{\varepsilon_1}, \dots, \overline{\varepsilon_r})$ of ξ such that $\overline{\varepsilon_i}(t) = \varepsilon_i(t)$ for all $t \in J$ and $i \in \{1, \dots, r\}$.

Proof. Notice that the second point implies the first. Up to enlarging J, we can assume that $J \cap I_1 \neq \emptyset$.

Let $(\varepsilon_1, \ldots, \varepsilon_r)$ be a frame field of $\xi|_{I_0}$, and $(\varepsilon'_1, \ldots, \varepsilon'_r)$ a frame field of $\xi|_{I_1}$. For $t \in I_0 \cap I_1$, we can decompose:

$$\varepsilon_i(t) = \sum_{j=1}^{r} A_i^j(t)\varepsilon'_j(t)$$

This defines a smooth curve $A = (A_i^j)_{1 \leq i,j \leq r} : I_0 \cap I_1 \to \mathrm{GL}(r, \mathbb{R})$. Consider a smooth curve $\overline{A} : I_1 \to \mathrm{GL}(r, \mathbb{R})$ such that $\overline{A}(t) = A(t)$ for all $t \in J \cap I_1$ (this is possible because $\mathrm{GL}(r, \mathbb{R})$ is a manifold).

Now define $(\overline{\varepsilon_1}, \ldots, \overline{\varepsilon_r})$ in the following way:

$$\overline{\varepsilon_i}(t) = \begin{cases} \varepsilon_i(t) & \text{if } t \in I_0 \setminus I_1 \\ \sum_{j=1}^{r} \overline{A}_i^j(t)\varepsilon'_j(t) & \text{if } t \in I_1 \end{cases}$$

One easily checks that $(\overline{\varepsilon_1}, \ldots, \overline{\varepsilon_r})$ is a frame field with the required property. □

Proof of Theorem 7.1.1. Using the topological properties of $\mathbb{R}$, given $t_0 \in I$, we can find a sequence $(I_i)_{i \in \mathbb{Z}}$ of open intervals such that:

- $t_0 \in I_0$.
- For all $i \in \mathbb{Z}$, I_i is a trivialising domain of ξ.
- $I_i \cap I_j = \emptyset$ whenever $|i - j| \geq 2$.
- For all $i \in \mathbb{Z}$, $I_i \cap I_{i+1}$ is an interval.
- $\bigcup_{i \in \mathbb{Z}} I_i = I$.

Using Lemma 7.1.2, we can construct a sequence of frame fields $(\varepsilon_1^k, \ldots, \varepsilon_n^k)$ over $J_k = \bigcup_{-k \leq j \leq k} I_j$ such that
e_i^{k+1} restricts to ε_i^k on J_k, which leads to a trivialization of ξ. □

If frame fields exist, they are however far from being unique. One can check that given $t_0 \in I$ and a vector basis $(e_1, \ldots, e_r)$ of ξ_{t_0}, there exists a trivializing frame $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ such that $\varepsilon_i(t_0) = e_i$ for all $i \in \{1, \ldots, r\}$. But this frame field is still far from being unique.

7.1.2 The archetypal vector bundle over the line

Since frame fields of vector bundles over the line exist but are not unique, we can ask whether there is a canonical choice. In order to understand this, let us understand the archetypal construction of a vector bundle over the line. For this, consider a manifold M, and a smooth curve $\gamma : I \to M$. Then

γ^*TM is a vector bundle over I. If M is not parallelisable, then there is really no reason for a "canonical" frame field to exist (the fact that γ^*TM is trivial is a property of the interval, not of M or γ).

In a trivial bundle $\xi = (I \times \mathbb{R}^r, \pi_1, I)$, there is a canonical way of choosing such a frame fields: picking sections that are constant maps $I \to \mathbb{R}^r$. But for a pulled back bundle γ^*TM there is no natural definition of a constant frame field. Even when M is a submanifold of $\mathbb{R}^d$, there is no reason for a section of γ^*TM with constant coordinates in $\mathbb{R}^d$ to exist.

7.1.3 Intrinsic derivatives

Definition 7.1.3. Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ a vector bundle of rank r. An **intrinsic derivative** on ξ is a linear map $\frac{D}{dt} : \Gamma(\xi) \to \Gamma(\xi)$ such that:

$$\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(I) \quad \frac{D}{dt}(f\sigma) = f\frac{D}{dt}\sigma + \dot{f}\sigma$$

A section $\sigma \in \Gamma(\xi)$ is **parallel** if $\frac{D}{dt}\sigma = 0$. A frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ is called parallel if $\varepsilon_1, \ldots, \varepsilon_r$ are parallel.

A frame field allows us to define an intrinsic derivative.

Proposition 7.1.4. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ a frame field of ξ. There is a unique intrinsic derivative on ξ for which ε is parallel.*

Proof. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i$ where $\sigma^i = \varepsilon^i(\sigma) \in \mathcal{C}^\infty(I)$. Define:

$$\frac{D}{dt} : \left\{ \begin{array}{ccc} \Gamma(\xi) & \to & \Gamma(\xi) \\ \sigma = \sum_{i=1}^r \sigma^i \varepsilon_i & \mapsto & \sum_{i=1}^r \dot{\sigma}^i \varepsilon_i \end{array} \right.$$

It is linear, and the product rule for derivation of functions shows that it is an intrinsic derivative. By definition, ε is parallel.

Reciprocally, if $\frac{D}{dt}$ is an intrinsic derivative for which ε is parallel, then:

$$\frac{D}{dt}\left(\sum_{i=1}^r \sigma^i \varepsilon_i\right) = \sum_{i=1}^r \sigma^i \underbrace{\frac{D}{dt}\varepsilon_i}_{=0} + \sum_{i=1}^r \dot{\sigma}^i \varepsilon_i = \sum_{i=1}^r \dot{\sigma}^i \varepsilon_i$$

This yields uniqueness. □

Another situation in which we can define a "canonical" intrinsic derivative is for pullbacks of tangent bundles of submanifolds of $\mathbb{R}^n$ by curves.

Proposition 7.1.5. *Let $M \subset \mathbb{R}^n$ be a submanifold, and let $\gamma : I \to M$ be a smooth curve. For $x \in M$, we let $p_x \in \mathrm{End}(\mathbb{R}^n)$ be the orthogonal projection onto T_xM for the canonical inner product.*
The map $\frac{D}{dt} : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ defined by $\frac{D}{dt}\sigma(t) = p_{\gamma(t)}(\dot{\sigma}(t))$ is an intrinsic derivative.

Proof. Firstly, since M is a smooth manifold, the linear map $p_x \in \mathrm{End}(\mathbb{R}^n)$ depends smoothly on x, so $\frac{D}{dt}\sigma$ is smooth for any $\sigma \in \Gamma(\gamma^* TM)$, and $\frac{D}{dt} : \Gamma(\gamma^* TM) \to \Gamma(\gamma^* TM)$ is well defined. It is linear because the orthogonal projections are linear.
Let $\sigma \in \Gamma(\gamma^* TM)$ and $f \in \mathcal{C}^\infty(I)$. Since $\sigma(t) \in T_{\gamma(t)}M$ for all $t \in I$, we have that $p_{\gamma(t)}(\sigma(t)) = \sigma(t)$, and it follows that:

$$\frac{D}{dt}(f\sigma)(t) = p_{\gamma(t)}(f(t)\dot{\sigma}(t) + \dot{f}(t)\sigma(t)) = f(t)\frac{D}{dt}\sigma(t) + \dot{f}(t)\sigma(t)$$

□

We now wish to show that any intrinsic derivative possesses a parallel frame field. The whole point is that seeking parallel sections amounts to solving a linear ODE.

Proposition 7.1.6. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ. For all $t_0 \in I$ and $v \in \xi_{t_0}$, there is a unique parallel section $\sigma \in \Gamma(\xi)$ such that $\sigma(t_0) = v$.*

Proof. Since ξ is trivialisable according to Theorem 7.1.1, consider a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ. Any section σ can be written as $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i$ for some functions $\sigma^1, \ldots, \sigma^r \in \mathcal{C}^\infty(I)$.

Consider $A = \left(A_i^j\right)_{1 \le i,j \le r} : I \to \mathcal{M}(r, \mathbb{R})$ such that:

$$\frac{D}{dt}\varepsilon_i = \sum_{j=1}^r A_i^j \varepsilon_j$$

Since $\frac{D}{dt}\sigma = \sum_{i=1}^r \sigma^i \frac{D}{dt}\varepsilon_i + \dot{\sigma}_i \varepsilon_i$, we find:

$$\frac{D}{dt}\sigma = 0 \iff \forall j \in \{1, \ldots, r\}\ \dot{\sigma}^j + \sum_{i=1}^r A_i^j \sigma^i = 0$$

Existence and uniqueness follow from the Cauchy-Lipschitz Theorem. □

The existence of parallel sections allows us to define linear maps between fibres (i.e. we connect fibres with each other).

Definition 7.1.7. Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ. For $t_0, t_1 \in I$, the map $\|_{t_0}^{t_1} : \xi_{t_0} \to \xi_{t_1}$ defined by $\|_{t_0}^{t_1} v = \sigma(t_1)$, where $\sigma \in \Gamma(\xi)$ is parallel and $\sigma(t_0) = v$, is called the **parallel transport**.

The parallel transport has a semi-group type property (which is not surprising since it is more or less defined as the flow of an ODE).

Proposition 7.1.8. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ.*

- *For $t_0, t_1 \in I$, the parallel transport $\|_{t_0}^{t_1} : \xi_{t_0} \to \xi_{t_1}$ is an isomorphism, with inverse $\left(\|_{t_0}^{t_1}\right)^{-1} = \|_{t_1}^{t_0}$.*
- *For $t_0, t_1, t_2 \in I$ we have $\|_{t_1}^{t_2} \circ \|_{t_0}^{t_1} = \|_{t_0}^{t_2}$.*

Proof. Linearity of the parallel transport is because the space of parallel vector fields is a vector space. Everything else is a consequence of the uniqueness in Proposition 7.1.6. □

This can be used to show the existence of parallel frame fields

Proposition 7.1.9. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ.*
Given $t_0 \in I$ and a basis $(e_1, \ldots, e_r)$ of ξ_{t_0}, there is a unique parallel frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ such that $(\varepsilon_1(t_0), \ldots, \varepsilon_r(t_0)) = (e_1, \ldots, e_r)$.

Proof. Uniqueness is a consequence of the uniqueness in Proposition 7.1.6. For the existence, we only need to check that if $e_1, \ldots, \varepsilon_r \in \Gamma(\xi)$ are parallel and $(e_1(t_0), \ldots, e_r(t_0))$ is linearly independent, then $(e_1(t), \ldots, e_r(t))$ is linearly independent for all $t \in I$. This is true because the parallel transport $\|_{t_0}^{t}$ is an isomorphism (Proposition 7.1.8). □

The data of a parallel frame field allows for easy computations of everything that involves the intrinsic derivative.

Lemma 7.1.10. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r, $\frac{D}{dt}$ an intrinsic derivative on ξ and $(\varepsilon_1, \ldots, \varepsilon_r)$ a parallel frame field.*

- *Given $\sigma_1, \ldots, \sigma_r \in \mathcal{C}^\infty(I)$, if $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i \in \Gamma(\xi)$, then $\frac{D}{dt}\sigma = \sum_{i=1}^r \dot{\sigma}^i \varepsilon_i$.*
- *Given $t_0, t_1 \in I$ and $v = \sum_{i=1}^r v^i \varepsilon_i(t_0) \in \xi_{t_0}$, we find $\|_{t_0}^{t_1} v = \sum_{i=1}^r v^i \varepsilon_i(t_0)$.*

Proof. The first point is a consequence of the Leibniz rule and the fact that $\frac{D}{dt}\varepsilon_i = 0$. The second point is a consequence of the linearity of the parallel transport, and the fact that $\|_{t_0}^{t_1}\varepsilon_i(t_0) = \varepsilon_i(t_1)$. □

Luckily for us, in many situations we will not need to know an explicit parallel frame field, but it will be enough to know that they exist.

7.2 Koszul connections

Given a function $f \in \mathcal{C}^\infty(M)$, a point $x \in M$ and a tangent vector $v \in T_xM$, we can define $d_xf(v) \in \mathbb{R}$. If we replace f with a section $\sigma \in \Gamma(\xi)$ of a vector bundle ξ, then $d_x\sigma(v) \in T_xE$. The whole point of vector bundles (and more generally fibre bundles) is that we do not want to consider the manifold structure of the total space, but rather work either in a given fibre, or on the base. More simply put, TE is a vector bundle over a vector bundle, so it is meant to mess with your head.

To define connections on vector bundles with arbitrary basis, we first notice that differentiating a function $f \in \mathcal{C}^\infty(M)$ does not yield another function, but a 1-form $df \in \Gamma(T^*M) = \Omega^1(M)$. For sections of bundles, we wish to define a differential $\nabla_x\sigma(v) \in \xi_x$ for $v \in T_xM$, hence a linear map $\nabla_x\sigma : T_xM \to \xi_x$, i.e. a ξ-valued 1-form $\nabla\sigma \in \Gamma(\mathrm{Hom}(TM,\xi)) = \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$.

Definition 7.2.1. Let $\xi = (E,p,M)$ be a vector bundle. A **connection** on ξ is a linear map $\nabla : \Gamma(\xi) \to \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$ satisfying the **Leibniz rule**:

$$\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(M) \quad \nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

Remark. By definition, $(df \otimes \sigma)_x(v) = d_xf(v)\sigma(x)$ for $x \in M$ and $v \in T_xM$. We will write $\nabla_x\sigma(v)$, however the notation $\nabla_v\sigma$ is more frequent in the literature.

We will only use the notation with the vector as a subscript for vector fields: given $X \in \mathcal{X}(M)$ and $\sigma \in \Gamma(\xi)$, we will write $\nabla_X\sigma = \iota_X(\nabla\sigma) = \nabla\sigma(X)$. This defines a map $\nabla_X : \Gamma(\xi) \to \Gamma(\xi)$ which is $\mathbb{R}$-linear but NOT $\mathcal{C}^\infty(M)$-linear, instead it satisfies the Leibniz rule $\nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma$. It is quite common in the literature to see a connection defined as a $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \Gamma(\xi) \to \Gamma(xi)$ which is $\mathcal{C}^\infty(M)$-linear in the first variable and satisfies the Leibniz rule in the second variable. A simple use of the Tensoriality Lemma shows that this definition is equivalent to 7.2.1.

On a trivial bundle $\underline{V}_M$, sections can be identified with smooth functions $f \in \mathcal{C}^\infty(M,V)$. One can check that $D_xf(v) = d_xf(v)$ then defines a connection D on $\underline{V}_M$, called the trivial connection. Let us see how to treat this in terms of frame fields.

Proposition 7.2.2. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $(\varepsilon_1,\ldots,\varepsilon_r)$ a frame field of ξ. There is a unique connection D on ξ, called the* ***trivial connection****, such that:*

$$\forall i \in \{1,\ldots,r\} \quad D\varepsilon_i = 0$$

Proof. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i\varepsilon_i$. Define $D : \Gamma(\xi) \to \Omega^1(\xi)$ by:

$$D\sigma = \sum_{i=1}^r d\sigma^i \otimes \varepsilon_i$$

It is linear and satisfies the Leibniz rule for connections because of the usual Leibniz rule for functions. We also have $D\varepsilon_i = 0$ for all i.
If ∇ is another connection with the same property, then the Leibniz rule ensures that $\nabla = D$. □

Definition 7.2.3. Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on ξ. A section $\sigma \in \Gamma(\xi)$ is called **parallel** if $\nabla\sigma = 0$.

Proposition 7.2.4. *Let $M \subset \mathbb{R}^n$ be a submanifold. For $x \in M$, we let $p_x \in \mathrm{End}(\mathbb{R}^n)$ be the orthogonal projection onto T_xM for the canonical inner product. The map $\nabla : \mathcal{X}(M) \to \Omega^1(TM) = \mathcal{T}^{1,1}(M) = \Gamma(\mathrm{End}(TM))$ defined by $\nabla_x\sigma(v) = p_x(d_x\sigma(v))$ is a connection.*

The proof is very similar to 7.1.5.

If $M = I \subset \mathbb{R}$ is an interval, then connections on vector bundles over I are essentially the same as intrinsic derivatives.

Recall that $\frac{d}{dt} \in \Gamma(TI)$ is defined by $\frac{d}{dt} \cdot f = \dot{f}$ for every $f \in \mathcal{C}^\infty(I)$, and $dt \in \Gamma(T^*I)$ satisfies $dt\left(\frac{d}{dt}\right) = 1$.

Proposition 7.2.5. *Set $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ a vector bundle. If $\frac{D}{dt}$ is an intrinsic derivative on ξ, then $\nabla : \sigma \mapsto dt \otimes \frac{D}{dt}\sigma$ is a connection. Reciprocally, if ∇ is a connection on ξ, then $\frac{D}{dt} : \sigma \mapsto \nabla_{\frac{d}{dt}}\sigma$ is an intrinsic derivative.*

Remark. These maps are mutually reciprocal bijections.

Proof. One just has to check that the Leibniz rules are equivalent. Note that for $f \in \mathcal{C}^\infty(I)$, we have $df = \dot{f}\,dt$ and $df\left(\frac{d}{dt}\right) = \dot{f}$.
Given an intrinsic derivative $\frac{D}{dt}$, we check:

$$\nabla(f\sigma) = dt \otimes \frac{D}{dt}(f\sigma) = dt \otimes (\dot{f}\sigma) + dt \otimes f\frac{D}{dt}\sigma = dt \otimes \nabla\sigma + f\nabla\sigma$$

Given a connection ∇, we check:

$$\frac{D}{dt}(f\sigma) = \nabla_{\frac{d}{dt}}(f\sigma) = df \otimes \sigma\left(\frac{d}{dt}\right) + f\nabla_{\frac{d}{dt}}\sigma = \dot{f}\sigma + f\frac{D}{dt}\sigma$$

□

Proposition 7.2.6. *Let ξ, ξ' be vector bundles with the same base M, and ∇, ∇' connections on ξ, ξ'.*
There are unique connections ∇^*, $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ on ξ^*, $\xi \oplus \xi'$ and $\xi \otimes \xi'$ such that:

$$\begin{aligned}
&\forall \lambda \in \Gamma(\xi^*)\ \forall \sigma \in \Gamma(\xi)\ \forall \sigma' \in \Gamma(\xi') \\
&\nabla^*\lambda(\sigma) = d(\lambda(\sigma)) - \lambda(\nabla\sigma) \\
&(\nabla \oplus \nabla')(\sigma + \sigma') = \nabla\sigma + \nabla'\sigma' \\
&(\nabla \otimes \nabla')(\sigma \otimes \sigma') = \nabla\sigma \otimes \sigma' + \sigma \otimes \nabla'\sigma'
\end{aligned}$$

More generally, ∇ induces a connection, still denoted by ∇, on the vector bundles $(\xi^*)^{\otimes p} \otimes \xi^{\otimes q}$. For $\omega \in \mathcal{T}^{p,0}(M)$ a covariant tensor, we get:

$$\nabla\omega(X_1,\dots,X_p) = d(\omega(X_1,\dots,X_p)) - \sum_{i=1}^{p} \omega(X_1,\dots,\nabla X_i,\dots,X_p)$$

In other words, we can differentiate $\omega(X_1,\dots,X_p)$ using a Leibniz rule:

$$d(\omega(X_1,\dots,X_p)) = \nabla\omega(X_1,\dots,X_p) + \sum_{i=1}^{p} \omega(X_1,\dots,\nabla X_i,\dots,X_p)$$

Note that this formula also preserves skew-symmetry, so a connection ∇ on TM also induces a connection on $\Lambda^k T^*M$.

For $R \in \mathcal{T}^{p,1}(M)$ a type-$(p,1)$ tensor, we get:

$$\nabla R(X_1,\dots,X_p) = \nabla(R(X_1,\dots,X_p)) - \sum_{i=1}^{p} R(X_1,\dots,\nabla X_i,\dots,X_p)$$

Proposition 7.2.7. *Let $\xi = (E,p,M)$ be a vector bundle. The set of connections on ξ has the structure of an affine space, with underlying vector space $\Gamma(\xi^* \otimes T^*M \otimes \xi) = \Gamma(T^*M \otimes \mathrm{End}(\xi)) = \Omega^1(\mathrm{End}(\xi))$.*

Proof. Simply notice that if ∇ and ∇' are connections on ξ, then $\nabla - \nabla'$ is tensorial. $\square$

This idea can be used to show the existence of connections.

Proposition 7.2.8. *Every vector bundle has a connection.*

Proof. Let $\xi = (E,p,M)$ be a vector bundle, and consider a locally finite open cover $M = \bigcup_{U\in\mathcal{U}} U$ such that every restriction $\xi|_U$ is trivialisable. Consider a connection ∇^U on $\xi|_U$ (which exists because of Proposition 7.2.2).

We let $(\varphi_U)_{U\in\mathcal{U}}$ be a partition of unity associated to $\mathcal{U}$. For $\sigma \in \Gamma(\xi)$ and $x \in M$, we set:

$$\nabla\sigma(x) = \sum_{x\in U} \varphi_U(x)\nabla^U(\sigma|_U)(x)$$

Since $\mathcal{U}$ is locally finite, $V = \bigcap_{x\in U} U$ is a neighbourhood of x, and the indexes in the sum for $\nabla\sigma(y)$ for $y \in V$ are the same as for x, which shows that $\nabla\sigma$ is a smooth map, hence an element of $\Gamma(T^*M \otimes \xi)$.

For $f \in \mathcal{C}^\infty(M)$, we find:

$$\begin{aligned}
\nabla(f\sigma) &= \sum_{U\in\mathcal{U}} \varphi_U \nabla^U (f\sigma|_U) \\
&= \sum_{U\in\mathcal{U}} \varphi_U \left(df\otimes\sigma + f\nabla^U(\sigma|_U)\right) \\
&= \left(\sum_{U\in\mathcal{U}} \varphi_U\right) df\otimes\sigma + f\sum_{U\in\mathcal{U}} \varphi_U\nabla^U(\sigma|_U) \\
&= df\otimes\sigma + f\nabla\sigma
\end{aligned}$$

□

7.2.1 Local description of a connection

Even though the definition of a connection is through global sections, it still allows us to differentiate local sections.

Lemma 7.2.9. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. Consider $\sigma,\sigma' \in \Gamma(\xi)$. Let $U \subset M$ be an open set such that $\sigma(x) = \sigma'(x)$ for all $x \in U$. Then $\nabla_x\sigma = \nabla_x\sigma'$ for all $x \in U$.*

Proof. Fix $x \in U$, and consider a plateau function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside of U. Since $\varphi(x) = 1$ and $d_x\varphi = 0$, we find:

$$\begin{aligned}
\nabla_x(\varphi\sigma) &= d_x\varphi\otimes\sigma(x) + \varphi(x)\nabla_x\sigma \\
&= \nabla_x\sigma
\end{aligned}$$

As $\varphi\sigma = \varphi\sigma'$, it follows that $\nabla_x\sigma = \nabla_x\sigma'$ □

An immediate consequence is that $\nabla_x\sigma$ only depends on the 1-jet of σ at x.

Proposition 7.2.10. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. For all $\sigma \in \Gamma(\xi)$, $x \in M$ and $v \in T_xM$, the value of $\nabla_x\sigma(v)$ only depends on $\sigma(x) \in \xi_x$ and $d_x\sigma(v) \in T_{\sigma(x)}E$.*

Proof. If $\sigma' \in \Gamma(\xi)$ satisfies $\sigma'(x) = \sigma(x)$ then according to Lemma 6.2.5, there are sections $s_1,\dots,s_r \in \Gamma(\xi)$ and functions $f_1,\dots,f_r \in \mathcal{C}^\infty(M)$ such that $\sigma' = \sigma + \sum f_i s_i$ on a neighbourhood of x and $f_i(x) = 0$.
If $d_x\sigma'(v) = d_x\sigma(v)$, then we also get $d_xf_i(v) = 0$, and Lemma 7.2.9 yields:

$$\begin{aligned}
\nabla_x\sigma'(v) &= \nabla_x\left(\sigma + \sum f_i s_i\right)(v) \\
&= \nabla_x\sigma(v) + \sum\left(d_xf_i(v)s_i(x) + f_i(x)\nabla_x s_i(v)\right) \\
&= \nabla_x\sigma(v)
\end{aligned}$$

□

Proposition 7.2.11. *Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on ξ. For any open set $U \subset M$, there is a unique connection $\nabla|_U$ on $\xi|_U$ such that:*

$$\forall \sigma \in \Gamma(\xi) \ \ (\nabla|_U)\sigma|_U = (\nabla\sigma)|_U.$$

Proof. Lemma 7.2.9 and a tiny bit of thinking guarantee the uniqueness. To prove the existence, consider $\sigma \in \Gamma(\xi|_U)$.
For $x \in x \in U$, consider a plateau function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside of U. This allows us to define $\varphi\sigma \in \Gamma(\xi)$, and set $(\nabla|_U)_x\sigma = \nabla_x(\varphi\sigma)$. Following Lemma 7.2.9, $(\nabla|_U)_x\sigma(x)$ does not depend on the choice of φ. Since the same function φ works for any point in a neighbourhood of x, we do obtain a smooth section of $T^*U \otimes \xi|_U$. We have defined a map $\nabla|_U : \Gamma(TU) \to \Gamma(T^*U \otimes \xi|_U)$. Its linearity and the Leibniz rule follow from the same properties for ∇. □

Most of the time, we will use the notation ∇ for $\nabla|_U$ (so Proposition 7.2.11 means that $\nabla\sigma$ can be defined even if σ is only defined on an open subet of M).

Reciprocally, knowing the local expressions of a connection is enough to recover the whole connection.

Lemma 7.2.12. *Let $\xi = (E, p, M)$ be a vector bundle. Consider an open cover $M = \bigcup_{i\in I} U_i$ of M and connections ∇^i on $\xi|_{U_i}$. If*

$$\forall i, j \in I \quad \nabla^i\Big|_{U_i \cap U_j} = \nabla^j\Big|_{U_i \cap U_j}$$

then there is a unique connection ∇ on ξ such that

$$\forall i \in I \ \ \nabla|_{U_i} = \nabla^i$$

Remark. Readers who are familiar with algebraic geometry will recognize that Proposition 7.2.11 shows that connections on open subsets of M form a pre-sheaf, and Lemma 7.2.12 shows that it is actually a sheaf.

Proof. Uniqueness is a consequence of Lemma 7.2.9. For the existence, simply set $\nabla_x\sigma = \nabla^i_x\left(\sigma|_{U_i}\right)$ whenever $x \in U_i$. Because of the assumption on the restriction to intersections, it does not depend on the choice of U_i, and one easily checks that it defines a connection.

□

The restriction of a connection is what allows us to study a connection from a local point of view, i.e. in coordinates.

Consider a trivializing domain $U \subset M$, and a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_U$.

According to Proposition 7.2.2, there is a unique connection D on $\xi|_U$ for which $\varepsilon_1, \dots, \varepsilon_r$ are parallel. According to Proposition 7.2.7, $\nabla - D$ is a represented by a section $A \in \Gamma(T^*U \otimes \mathrm{End}(\xi|_U)) = \Omega^1(\mathrm{End}(\xi|_U))$, called the **connection form** of (U,Φ) (or the **connection 1-form**). We find:

$$\nabla_x\sigma(v) = D_x\sigma(v) + A_x(v)(\sigma(x)).$$

Every section $\sigma \in \Gamma(\xi|_U)$ decomposes as $\sigma = \sum_{\alpha=1}^r \sigma^\alpha \varepsilon_\alpha$.

Now consider also a coordinate system $(x^1,\dots,x^d)$ on U. We let $\partial_i = \frac{\partial}{\partial x^i}$ be the fundamental vector fields, and dx^i the differentials of the functions x^i.

$$\begin{aligned}\nabla\sigma &= \sum_{\alpha=1}^r \Big[D(\sigma^\alpha\varepsilon_\alpha) + A(\sigma^\alpha\varepsilon_\alpha)\Big]\\ &= \sum_{\alpha=1}^r \Big[d\sigma^\alpha \otimes \varepsilon_\alpha + \sigma^\alpha A(\varepsilon_\alpha)\Big]\end{aligned}$$

Since $D\varepsilon_\alpha = 0$, we find:

$$\nabla\varepsilon_\alpha = A(\varepsilon_\alpha) = \sum_{\beta=1}^r A_{i,\alpha}^\beta dx^i \otimes \varepsilon_\beta$$

For $v = \sum v^i\partial_i$ and $\sigma = \sum \sigma^\alpha\varepsilon_\alpha$, we find:

$$\nabla_x\sigma(v) = \underbrace{\sum_{\substack{1\le i\le d\\ 1\le\alpha\le r}} v^i\partial_i\sigma^\alpha(x)\varepsilon_\alpha(x)}_{=D_x\sigma(v)} + \underbrace{\sum_{\substack{1\le i\le d\\ 1\le\alpha,\beta\le r}} A_{i,\alpha}^\beta(x)v^i\sigma^\alpha(x)\varepsilon_\beta(x)}_{=A_x(v)(\sigma(x))}$$

When calculations start to involve sums on many indexes, it is more practical to use Einstein's convention: we do not write the sum symbol $\sum$, but whenever an index is repeated once as a subscript and once as a superscript, we implicitely consider the sum over all possible values of this index. Here, we find:

$$\nabla_x\sigma(v) = v^i\partial_i\sigma^\alpha\varepsilon_\alpha + A_{i,\alpha}^\beta v^i\sigma^\alpha\varepsilon_\beta$$

There will always be a warning before using Einstein's convention.

One way of interpreting Lemma 7.2.12 is that it is possible to define a connection by using local coordinates, as long as the expression is invariant under a change of coordinates. For this, we need to know how the connection form behaves under a change of coordinates. On an arbitrary vector bundle, we need to choose local coordinates on the base manifold and a local frame field. Let us focus on the frame field, and see how the connection form changes.

Lemma 7.2.13. *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ and $\eta = (\eta_1, \dots, \eta_r)$ be two frame fields of a vector bundle ξ, and ∇ a connection on ξ. Let $\tau \in \Gamma(\mathrm{End}(\xi))$ be defined by $\tau(\varepsilon) = \eta$. Let D be the trivial connection defined by ε, and $A = \nabla - D \in \Omega^1(\xi)$ the connection form of ∇ in the frame field ε. The connection form B of ∇ is the frame field η is given by:*

$$B = \tau^{-1} D\tau + \tau^{-1} A\tau.$$

Remarks.

- We still denote by D the connection induced on $\mathrm{End}(\xi)$ in order to define $D\tau \in \Omega^1(\mathrm{End}(\xi))$.
- In coordinates, using Einstein's convention and letting $\eta_\alpha = \tau_\alpha^\beta$, the formula is:
$$B_\alpha^\beta = \left(\tau^{-1}\right)_\gamma^\beta d\tau_\alpha^\gamma + \left(\tau^{-1}\right)_\delta^\beta A_\gamma^\delta \tau_\alpha^\gamma.$$

Proof. We must compute $\nabla\eta_\alpha = B_\alpha^\beta \eta_\beta$, which mostly consists in renaming indices.

$$\begin{aligned}
\nabla\eta_\alpha &= \nabla\left(\tau_\alpha^\beta \varepsilon_\beta\right) \\
&= d\tau_\alpha^\beta \otimes \varepsilon_\beta + \tau_\alpha^\beta \nabla\varepsilon_\beta \\
&= d\tau_\beta^\alpha \otimes \varepsilon_\beta + \tau_\alpha^\beta A_\beta^\gamma \otimes \varepsilon_\gamma \\
&= \left(d\tau_\beta^\alpha + \tau_\alpha^\gamma A_\gamma^\beta\right) \otimes \varepsilon_\beta \\
&= \left(d\tau_\beta^\alpha + \tau_\alpha^\gamma A_\gamma^\beta\right) \otimes \left(\left(\tau^{-1}\right)_\beta^\delta \eta_\delta\right) \\
&= \left(\left(\tau^{-1}\right)_\gamma^\beta d\tau_\alpha^\gamma + \left(\tau^{-1}\right)_\delta^\beta A_\gamma^\delta \tau_\alpha^\gamma\right) \otimes \eta_\beta
\end{aligned}$$

□

Now let us look at the particular case of a connection on the tangent bundle, where the frame field is made of the coordinate vector fields.

Lemma 7.2.14. *Let M be a manifold, and ∇ a connection on TM. Consider two local coordinate systems $(x^1, \dots, x^d)$ and $(y^1, \dots, y^d)$ defined on the same open set $U \subset M$.*

The components of the connection form $A_{i,j}^k$ (resp. $\overline{A}_{i,j}^k$) of ∇ with respect to the coordinates $(x^1, \dots, x^d)$ (resp. $(y^1, \dots, y^d)$) are related by:

$$\overline{A}_{i,j}^k = \frac{\partial^2 x^p}{\partial y^j \partial y^i} \frac{\partial y^k}{\partial x^p} + A_{p,q}^r \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \frac{\partial y^k}{\partial x^r}$$

Remark. We use Einstein's convention.

Proof. The coefficients $A^k_{i,j}$ are defined by:

$$\nabla\frac{\partial}{\partial x^i} = A^k_{i,j}dx^j \otimes \frac{\partial}{\partial x^k}$$

Since $\frac{\partial}{\partial y^i} = \frac{\partial x^p}{\partial y^i}\frac{\partial}{\partial x^p}$ and $\frac{\partial}{\partial x^p} = \frac{\partial y^k}{\partial x^p}\frac{\partial}{\partial y^k}$, we find:

$$\begin{aligned}
\nabla\frac{\partial}{\partial y^i} &= \frac{\partial^2 x^p}{\partial y^j \partial y^i}dy^j \otimes \frac{\partial}{\partial x^p} + \frac{\partial x^p}{\partial y^i}\nabla\frac{\partial}{\partial x^p} \\
&= \frac{\partial^2 x^p}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^p}dy^j \otimes \frac{\partial}{\partial y^k} + \frac{\partial x^p}{\partial y^i}A^r_{p,q}dx^q \otimes \frac{\partial}{\partial x^r} \\
&= \frac{\partial^2 x^p}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^p}dy^j \otimes \frac{\partial}{\partial y^k} + \frac{\partial x^p}{\partial y^i}A^r_{p,q}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r}dy^j \otimes \frac{\partial}{\partial y^k}
\end{aligned}$$

The formula follows from the identification of each term with the definition of $\overline{A}^k_{i,j}$:

$$\nabla\frac{\partial}{\partial y^i} = \overline{A}^k_{i,j}dy^j \otimes \frac{\partial}{\partial y^k}$$

□

7.2.2 Connections and curves

We have seen that intrinsic derivatives on vector bundles over an interval are related to choices of trivializations of these bundles. However, connections on higher dimensional manifolds do not lead to trivializations. One reason is that vector bundles need to be trivializable, but there are also local obstructions, which we will study later.

However, connections can be studied by considering curves on the base.

Definition 7.2.15. Let $\xi = (E, p, M)$ be a vector bundle. Given a smooth curve $c : I \to \mathbb{R}$, a **ξ-valued vector field along** c is a section of $c^*\xi$, i.e. a smooth function $\sigma : I \to E$ such that:

$$\forall t \in I \quad \sigma(t) \in \xi_{c(t)}$$

Note that every $\sigma \in \Gamma(\xi)$ defines a ξ-valued vector field $\sigma \circ c$ along c. However, not all ξ-valued vector fields along c can be obtained in this way (the curve c is not necessarily injective).

A vector field along c (without precising the vector bundle) is a TM-valued vector field along c.

Proposition 7.2.16. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on ξ, and $c : I \to M$ a smooth curve.*
There is a unique intrinsic derivative $\frac{D}{dt}$ on $c^*\xi$, called the ***intrinsic derivative along*** *c, such that:*

$$\forall \sigma \in \Gamma(\xi)\ \forall t \in I \quad \frac{D}{dt}(\sigma \circ c)(t) = \nabla_{c(t)}\sigma(\dot{c}(t))$$

Remark. For $\sigma \in \Gamma(\xi)$, we will usually write $\frac{D}{dt}\sigma$ instead of $\frac{D}{dt}(\sigma \circ c)$.

Proof. Let us start with the case where ξ is trivializable. Consider a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ, the associated trivial connection D, and the connection form $A \in \Gamma(T^*M \otimes \mathrm{End}(\xi))$.

If $\frac{D}{dt}$ is an intrinsic derivative on $c^*\xi$ satisfying the requirement, then:

$$\forall \alpha \in \{1, \ldots, r\}\ \forall t \in I \quad \frac{D}{dt}(\varepsilon_\alpha \circ c)(t) = \nabla_{c(t)}\varepsilon_\alpha(\dot{c}(t)) = A_{c(t)}(\dot{c}(t))(\varepsilon_\alpha(c(t)))$$

Since any ξ-valued vector field along c decomposes as $\sigma = \sum_{\alpha=1}^{r} \sigma^\alpha \varepsilon_\alpha \circ c$ for some functions $\sigma^\alpha \in \mathcal{C}^\infty(I)$, we get:

$$\frac{D}{dt}\sigma = \sum_{\alpha=1}^{r} \Big(\dot{\sigma}^\alpha \varepsilon_\alpha + \sigma^\alpha A(\dot{c})(\varepsilon_\alpha)\Big)$$

This formula not only guarantees uniqueness, but can also be used to prove existence.

In order to move on to the general case, we use the fact that for every $t \in I$, we can choose an open interval $I_t \subset I$ containing I and a trivializing domain $U_t \subset M$ such that $c(I_t) \subset U_t$. The previous discussion shows that the intrinsic derivative along $c|_{I_t}$ is uniquely defined. Because of the uniqueness, they coincide on $I_t \cap I_s$ for $t, s \in I$, so Lemma 7.2.12 allows us to define a unique intrinsic derivative with the same property for the whole curve c (actually Lemma 7.2.12 only concerns connections on $c^*\xi$, but the relationship with intrinsic derivatives is given by Proposition 7.2.5). □

The proof actually gave us the local expression for $\frac{D}{dt}$, given a local frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ:

$$\frac{D}{dt}\sigma = \sum_{\alpha=1}^{r} \Big(\dot{\sigma}^\alpha \varepsilon_\alpha + \sigma^\alpha A(\dot{c})(\varepsilon_\alpha)\Big)$$

If c has values inside a chart domain of M, and $(x^1, \ldots, x^d)$ are local coordinates, then we decompose $\dot{c}(t) = \sum_{i=1}^{d} \dot{c}^i(t)\partial_i$. We now find:

$$A(\dot{c})(\varepsilon_\alpha) = \sum_{\substack{1 \le i \le d \\ 1 \le \beta \le r}} \Big(A_{i,\alpha}^{\beta} \circ c\Big)\dot{c}^i \varepsilon_\beta$$

Which leads to:

$$\frac{D}{dt}\sigma = \sum_{1\le\alpha\le r} \dot{\sigma}^\alpha \varepsilon_\alpha + \sum_{\substack{1\le i\le d\\ 1\le\alpha,\beta\le r}} \left(A^\beta_{i,\alpha}\circ c\right)\sigma^\alpha \dot{c}^i \varepsilon_\beta$$

Example 7.2.17. With the same notations for local coordinates and frame field, through any point $x \in U$ we can consider the curves obtained by varying one of the coordinates, i.e. $t \mapsto (x^1(x),\dots,x^{i-1}(x),t,x^{i+1}(x),\dots,x^d(x))$. We will denote by $\frac{D}{\partial x^i}$ the intrinsic derivative along this curve. Note that for $\sigma \in \Gamma(\xi)$, we have $\frac{D}{\partial x^i}\sigma = \nabla_{\partial_i}\sigma$.

Since a connection defines an intrinsic derivative for any curve in M, it also defines a parallel transport. Since $(c^*\xi)_t = \xi_{c(t)}$, it defines linear isomorphisms between different fibres of ξ.

Definition 7.2.18. Let $\xi = (E,p,M)$ be a vector bundle, ∇ a connection on ξ, and $c: I \to M$ a smooth curve. Given $t_0,t_1 \in I$, the parallel transport $\|_{t_0}^{t_1} : \xi_{c(t_0)} \to \xi_{c(t_1)}$ for the intrinsic derivative along c is called the **parallel transport along** c.

Note that because $\|_{t_1}^{t_2} \circ \|_{t_0}^{t_1} = \|_{t_0}^{t_2}$, we can define the parallel transport along any **piecewise smooth** curve in a consistent way, so that we still have the same property for composition (we use the convention that piecewise smooth curves are continuous).

Parallel transport can be used to prove Theorem 6.6.1 (vector bundles over a contractible manifold are trivial). Indedd, if $f : [0,1]\times M \to M$ is a smooth map such that $f(0,\cdot)$ is a constant $x_0 \in M$ and $f(1,\cdot) = \mathrm{Id}$, then the parallel transport along curves $t \mapsto f(t,x)$ gives an isomorphism between the trivial bundle $\underline{\xi_{x_0}}_M$ and ξ.

Note that the parallel transport usually depends heavily on the choice of a curve. In particular, if $c(t_0) = c(t_1)$, then there is no reason that $\|_{t_0}^{t_1}$ should be the identity map.

Definition 7.2.19. Let $\xi = (E,p,M)$ be a vector bundle, ∇ a connection on ξ, and $x \in M$. The **holonomy group** of ∇ at x is the subgroup $\mathrm{Hol}_x \subset \mathrm{GL}(\xi_x)$ of parallel transports $\|_0^1$ along piecewise smooth curves $c : [0,1] \to M$ such that $c(0) = c(1) = x$.
The **restricted holonomy group** is the subgroup $\mathrm{Hol}_x^\circ \subset \mathrm{Hol}_x$ of parallel transporst along null homotopic piecewise smooth curves.

Note that if M is connected, then Hol_x does not really depend on x, as for $x,y \in M$ there is a linear isomorphism $\varphi : \xi_x \to \xi_y$ such that $\mathrm{Hol}_x = \varphi^{-1}\mathrm{Hol}_y\varphi$ (simply take φ to be the parallel transport along any curve from x to y).

Not only can the parallel transport along curves give many informations about a connection, it actually allows to retrieve the whole connection.

Proposition 7.2.20. *Let $\xi = (E,p,M)$ be a vector bundle and ∇ a connection on ξ. Consider $x \in M$, $v \in T_xM$, and a smooth curve $c :]-\varepsilon,\varepsilon[\to M$ such that $c(0) = x$ and $\dot{c}(0) = v$. For all $\sigma \in \Gamma(\xi)$, we get:*

$$\nabla_x\sigma(v) = \left.\frac{d}{dt}\right|_{t=0} \|_t^0 \sigma(c(t))$$

Proof. Consider a parallel frame $(\varepsilon_1,\dots,\varepsilon_r)$ along c so we can decompose $\sigma(c(t)) = \sum_{\alpha=1}^r \sigma^\alpha(t)\varepsilon_\alpha(t)$. We have that:

$$\nabla_x\sigma(v) = \frac{D}{dt}(\sigma \circ c)(0) = \sum_{\alpha=1}^r \dot{\sigma}^\alpha(0)\varepsilon_\alpha(0)$$

We also have:

$$\|_t^0 \sigma(c(t)) = \sum_{\alpha=1}^r \sigma^\alpha(t)\varepsilon_\alpha(0)$$

Differentiating this expression at $t = 0$ offers the conclusion. □

7.2.3 Pulling back connections

Recall that the pulled back bundle $f^*\xi$ of a vector bundle $\xi = (E,p,M)$ by a smooth map $f : N \to M$ is defined by $f^*\xi = (f^*E, N, f^*p)$ where

$$f^*E = \{(x,v) \in N \times E | p(v) = f(x)\}$$

and $f^*p(x,v) = x$.

In other words, the fibre above $x \in N$ is the fibre above $f(x) \in M$:

$$(f^*\xi)_x = \xi_{f(x)}$$

Note that pulling back vector bundles is compatible with algebraic operations on vector bundles, i.e. $f^*(\xi \oplus \xi') = (f^*\xi) \oplus (f^*\xi')$, $f^*(\xi \oplus \xi') = (f^*\xi) \oplus (f^*\xi')$ and $f^*(\xi^*) = (f^*\xi)^*$.

Any section $\sigma \in \Gamma(\xi)$ can be pulled back to a section $f^*\sigma = \sigma \circ f \in \Gamma(f^*\xi)$. Not all sections of $f^*\xi$ can be obtained in this way (indeed, f could be constant), but sections can all be recovered from these.

Lemma 7.2.21. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, $f : N \to M$ a smooth map, and $\sigma \in \Gamma(f^*\xi)$. For every $x \in N$, there are sections $s_1,\dots,s_r \in \Gamma(\xi)$ and functions $u^1,\dots,u^r \in \mathcal{C}^\infty(N)$ such that $\sum_{i=1}^r u^i f^* s_i$ is equal to σ on a neighbourhood of x.*

Proof. Consider a trivialising domain $U \subset M$ of ξ that contains $f(x)$, and a frame field $(\varepsilon_1,\dots,\varepsilon_r)$ of $\xi|_U$. Then $(f^*\varepsilon_1,\dots,f^*\varepsilon_r)$ is a frame field of $f^*\xi|_{f^{-1}(U)}$. Write $\sigma|_U = \sum_{i=1}^r \sigma^i f^*\varepsilon_i$, and consider a plateau function φ on M such that $\varphi = 1$ on a neighbourhood of $f(x)$ and $\varphi = 0$ outside of U. Then $s_i = \varphi\varepsilon_i$ and $u^i = (\varphi \circ f)\sigma^i$ are the required functions and sections. □

In order to define a connection on the pulled back bundle, notice that we can also pull back bundle-valued differential forms: for $\omega \in \Omega^1(\xi)$, define $f^*\omega \in \Omega^1(f^*\xi)$ by:

$$\forall x \in N \ \forall v \in T_x N \quad (f^*\omega)_x(v) = \omega_{f(x)}(d_x f(v)) \in \xi_{f(x)} = (f^*\xi)_x$$

Proposition 7.2.22. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, ∇ a connection on M, and $f : N \to M$ a smooth map. There is a unique connection $f^*\nabla$ on $f^*\xi$ such that:*

$$\forall \sigma \in \Gamma(\xi) \quad (f^*\nabla)(f^*\sigma) = f^*(\nabla\sigma)$$

Proof. Let us start with uniqueness: let ∇^1, ∇^2 be connections on $f^*\xi$ such that $\nabla^1 f^*\sigma = \nabla^2 f^*\sigma = f^*(\nabla\sigma)$ for every $\sigma \in \Gamma(\xi)$.

Let $\sigma \in \Gamma(f^*\xi)$ and $x \in N$. According to Lemma 7.2.21, there are functions $u^1, \ldots, u^r \in \mathcal{C}^\infty(N)$ and sections $s_1, \ldots, s_r \in \Gamma(\xi)$ such that $\sum_{i=1}^r u^i f^* s_i$ is equal to σ on a neighbourhood of x. From Lemma 7.2.9 we know that $\nabla^j_x \sigma = \nabla^j_x \left(\sum_{i=1}^r u^i f^* s_i\right)$ for $j = 1, 2$.

$$\begin{aligned}
\nabla^j_x \sigma &= \nabla^j_x \left(\sum_{i=1}^r u^i f^* s_i\right) \\
&= \sum_{i=1}^r \left(d_x u^i \otimes f^* s_i(x) + u^i(x) \nabla^j_x f^* s_i\right) \\
&= \sum_{i=1}^r \left(d_x u^i \otimes f^* s_i(x) + u^i(x) (f^*\nabla s_i)_x\right)
\end{aligned}$$

Since the last line does not depend on j, we find that $\nabla^1 = \nabla^2$. In order to show the existence, we simply need to check that the expression $\sum_{i=1}^r \left(du^i \otimes f^* s_i + u^i (f^*\nabla s_i)_x\right)$ defines a connection on $f^*\xi$. $\square$

Let us compute the components of the connection form of the pulled back connection. Consider an open set $U \subset M$ and a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of $\xi|_U$. Then $(f^*\varepsilon_1, \ldots, f^*\varepsilon_r)$ is a frame field of $f^*(\xi|_U) = (f^*\xi)|_{f^{-1}(U)}$.

Consider also coordinates $(x^1, \ldots, x^d)$ on U and coordinates $(y^1, \ldots, y^{d'})$ on an open subset of $f^{-1}(U)$.

For $\omega \in \Gamma(T^*M \otimes \xi)$, we write locally $\omega = \sum\limits_{\substack{1 \le i \le d \\ 1 \le \alpha \le r}} \omega_i^\alpha dx^i \otimes \varepsilon_\alpha$. Then $f^*\omega$ is locally given by:

$$f^*\omega = \sum_{\substack{1 \le j \le d' \\ 1 \le i \le d \\ 1 \le \alpha \le r}} \frac{\partial f^i}{\partial y^j} \omega_i^\alpha dy^j \otimes (f^*\varepsilon_\alpha)$$

Since $(f^*\nabla)(f^*\varepsilon_\alpha) = f^*(\nabla\varepsilon_\alpha)$, we find the components of the connection form f^*A of $f^*\nabla$.

$$(f^*A)^\beta_{j,\alpha}(y) = \sum_{i=1}^{d} A^\beta_{i,\alpha}(f(y))\frac{\partial f^i}{\partial y^j}(y)$$

Note that the formula justifies the notation f^*A, as it is also the formula for the pull-back of A.

The pulled back connection can also be described in terms of parallel transport.

Proposition 7.2.23. *Let $\xi = (E,p,M)$ be a vector bundle, ∇ a connection on M, $f: N \to M$ a smooth map, and $f^*\nabla$ the pulled back connection on $f^*\xi$. For every smooth curve $c: I \to N$ and $t_0, t_1 \in I$, the parallel transport along c from t_0 to t_1*

$$\|_{t_0}^{t_1} : (f^*\xi)_{c(t_0)} = \xi_{f(c(t_0))} \to (f^*\xi)_{c(t_1)} = \xi_{f(c(t_1))}$$

is equal to the parallel transport along $f \circ c$ from t_0 to t_1.

Proof. If σ is a ξ-valued vector field along $f \circ c$, then $f^*\sigma$ is a $f^*\xi$-valued vector field along c. By working locally, we can see that $\frac{D}{dt}(f^*\sigma) = f^*\left(\frac{D}{dt}\sigma\right)$, so σ is parallel along $f \circ c$ if and only if $f^*\sigma$ is parallel along c. $\square$

We will mostly deal with pulled back connections in two situations: constant maps and the inclusion of a submanifold.

Pulled back connection by a constant map

Assume that the map $f : N \to M$ is constant, say $f(x) = y_0$ for all $x \in N$. Then the vector bundle $f^*\xi$ is trivialisable: the fibre over $x \in N$ is always the same vector space ξ_{y_0}. Given a vector basis $(e_1, \dots, e_r)$ of ξ_{y_0}, the sections $(\varepsilon_1, \dots, \varepsilon_r)$ of $f^*\xi$ given by $\varepsilon_\alpha(x) = e_\alpha$ for all $x \in N$ form a frame field of $f^*\xi$.

The connection $f^*\nabla$ is equal to the trivial connection D associated to $(\varepsilon_1, \dots, \varepsilon_r)$. In other words, sections of $f^*\xi$ can be identified with smooth maps from N to the vector space ξ_{y_0}, the connection $f^*\nabla$ is then mapped to the usual differential.

Note that sections of the form $f^*\sigma$ correspond to constant maps $N \to \xi_{y_0}$.

Restriction of a connection to a submanifold

Let $\xi = (E,p,M)$ be a vector bundle equipped with a connection ∇, and consider an immersed submanifold $N \subset M$. Recall that by calling $N \subset M$ an immersed submanifold, we have implicitely chosen a differentiable structure on N for which the inclusion $i : N \to M$ is an immersion.

The pulled back bundle $i^*\xi$ is the vector bundle over N whose fibres are given by $(i^*\xi)_x = \xi_x$ for $x \in N$, i.e. $i^*\xi$ should be seen as the **restriction** $\xi|_N$ of ξ to N.

The connection $i^*\nabla$ should also be considered as the restriction of ∇ to $\xi|_N$. This means that given a section $\sigma \in \Gamma(i^*\xi)$, which should be seen as a section of ξ which is only defined on the submanifold N, we can define $\nabla_v \sigma \in \xi_x$ for $x \in N$ and $v \in T_x N$ (but not necessarily for a general $v \in T_x M$). This is exactly the same as for smooth real valued functions defined on submanifolds: a function defined on N can be differentiated along a direction tangent to N.

Note that sections of the form $i^*\sigma$ for some $\sigma \in \Gamma(\xi)$ are sections defined over N that can be extended to sections defined over M.

7.3 Tensorial invariants of a connection

We will now see how the non triviality of a connection can be encoded in two fields. A first candidate would be the connection form. However, it depends on the choice of a trivialisation, and its value at one point cannot help in the definition of an invariant.

Proposition 7.3.1. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, ∇ a connection on ξ and $x \in M$. There is a neighbourhood $U \subset M$ and a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_U$ such that $\nabla \varepsilon_\alpha$ vanishes at x for all $\alpha \in \{1, \dots, r\}$.*

Remark. In local coordinates $(x^1, \dots, x^d)$ around $x \in M$, this means that $A_{i,\alpha}^\beta(x) = 0$ for all $i \in \{1, \dots, d\}$ and $\alpha, \beta \in \{1, \dots, r\}$.

Proof. Start with a local frame field $(\overline{\varepsilon}_1, \dots, \overline{\varepsilon}_n)$ on some neighbourhood of x, and decompose $\nabla_x \overline{\varepsilon}_\alpha = \sum_{\beta=1}^r \omega_\alpha^\beta \otimes \overline{\varepsilon}_\beta(x)$ where $\omega_\alpha^\beta \in T_x^* M$ (given local coordinates $(x^1, \dots, x^d)$, the formula is $\omega_\alpha^\beta = \sum_{i=1}^d A_{i,\alpha}^\beta dx^i$).

Choose functions $f_\alpha^\beta \in \mathcal{C}^\infty(M)$ such that $f_\alpha^\beta(x) = 0$ and $d_x f_\alpha^\beta = -\omega_\alpha^\beta$ (this is just a punctual condition at x, not a differential equation). Let $\varepsilon_\alpha = \overline{\varepsilon}_\alpha + \sum_{\beta=1}^r f_\alpha^\beta \overline{\varepsilon}_\beta$.

Since $\varepsilon_\alpha(x) = \overline{\varepsilon}_\alpha(x)$, we find that $(\varepsilon_1, \dots, \varepsilon_n)$ is a local frame field around x.

$$\begin{aligned}
\nabla_x \varepsilon_\alpha &= \nabla_x \left(\overline{\varepsilon}_\alpha + \sum_{\beta=1}^r f_\alpha^\beta \overline{\varepsilon}_\beta \right) \\
&= \sum_{\beta=1}^r \underbrace{\left(\omega_\alpha^\beta + d_x f_\alpha^\beta \right)}_{=0} \otimes \overline{\varepsilon}_\beta(x) + \sum_{\beta=1}^r \underbrace{f_\alpha^\beta(x)}_{=0} \nabla_x \overline{\varepsilon}_\beta \\
&= 0
\end{aligned}$$

$\square$

Because of this, we have two options. The first one is to consider derivatives of the connection form, this leads to the notion of **curvature**. The second one is to only consider some specific trivialising frames, which we can do on the tangent bundle of a manifold, this will lead to the notion of **torsion**.

Even though curvature and torsion are two very different types of invariants, they share a common interpretation as the failure of generalisations of the Schwarz Lemma for second order covariant derivatives. Curvature is the failure of the Schwarz Lemma when considering second order covariant derivatives of sections of a vector bundle, and torsion is the failure of the Schwarz Lemma for second order derivatives of functions on a manifold (we will see how a connection on the tangent bundle allows us to define second order differentials of functions).

7.3.1 The curvature of a connection

When dealing with connections on vector bundles over the line, we saw how useful it was to be able to consider parallel frame fields. So it is natural to ask whether these exist in higher dimensions.

Definition 7.3.2. Let $\xi = (E, p, M)$ be a vector bundle of rank r and ∇ a connection on M. We say that ∇ is **locally trivial** if every $x \in M$ has an open neighbourhood $U \subset M$ on which there is a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ such that $\nabla|_U$ is equal to the corresponding trivial connection (i.e. $\forall \alpha \in \{1, \ldots, r\}$ $\nabla \varepsilon_\alpha = 0$).

In order to know whether a connection is locally trivial, a starting point would be to ask if given a point $x \in M$ and a vector $v \in \xi_x$, we can find a parallel section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$. First, we notice that uniqueness still holds.

Proposition 7.3.3. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a connection on ξ. Let $\sigma \in \Gamma(\xi)$ be parallel. If σ vanishes a some point, and if M is connected, then σ vanishes identically on M.*

Proof. Consider $x \in M$ such that $\sigma(x) = 0$, and $y \in M$. Since M is connected, we can consider a smooth path c joining x and y. Then $\sigma \circ c$ is a parallel section of $c^*\xi$ that vanishes at some point, so the uniqueness in Proposition 7.1.6 guarantees that $\sigma \circ c = 0$, therefore $\sigma(y) = 0$. □

This tells us that non trivial parallel sections should not always exist, because nowhere vanishing sections do not always exist (e.g. the tangent bundle of $\mathbb{S}^2$). However, one should not expect local existence to hold either, since the equation $\nabla \sigma = 0$ is a partial differential equation, and a generic partial differential equation tends to not have solutions. What stands out is that the obstruction to the existence of parallel section is encoded in a field, called the curvature.

Lemma 7.3.4. *Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on M. The map $F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(\xi) \to \Gamma(\xi)$ defined by*

$$F(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma$$

is tensorial, and skew-symmetric in the first two variables.

Proof. Skew-symmetry is straightforward. For tensoriality with respect to σ, we consider $f \in \mathcal{C}^\infty(M)$, and we first compute $\nabla_X \nabla_Y (f\sigma)$:

$$\begin{aligned}\nabla_X \nabla_Y (f\sigma) &= \nabla_X \Big(df(Y)\sigma + f\nabla_Y\sigma \Big) \\ &= d\Big[df(Y)\Big](X)\sigma + df(Y)\nabla_X\sigma + df(X)\nabla_Y\sigma + f\nabla_X\nabla_Y\sigma\end{aligned}$$

Similarly, we get:

$$\nabla_Y \nabla_X (f\sigma) = d\Big[df(X)\Big](Y)\sigma + df(X)\nabla_Y\sigma + df(Y)\nabla_X\sigma + f\nabla_Y\nabla_X\sigma$$

Since $d\Big[df(Y)\Big](X) - d\Big[df(X)\Big](Y) = df\Big([X,Y]\Big)$, we find:

$$\nabla_X \nabla_Y (f\sigma) - \nabla_Y \nabla_X (f\sigma) = df\Big([X,Y]\Big)\sigma + f\Big(\nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma\Big)$$

This simplifies to:

$$\begin{aligned}F(X,Y)(f\sigma) &= -\nabla_{[X,Y]}(f\sigma) + df\Big([X,Y]\Big)\sigma + f\Big(F(X,Y)\sigma + \nabla_{[X,Y]}\sigma\Big) \\ &= fF(X,Y)\sigma\end{aligned}$$

Let us now prove tensoriality with respect to X. First, recall that:

$$[fX, Y] = f[X,Y] - df(Y)X$$

It follows that:

$$\nabla_{[fX,Y]}\sigma = f\nabla_{[X,Y]}\sigma - df(Y)\nabla_X\sigma$$

We also find:

$$\begin{aligned}\nabla_{fX}\nabla_Y\sigma - \nabla_Y\nabla_{fX}\sigma &= f\nabla_X\nabla_Y\sigma - \nabla_Y\Big(f\nabla_X\sigma\Big) \\ &= f\Big(\nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma\Big) - df(Y)\nabla_X\sigma(X)\end{aligned}$$

Combining with the previous computation, we get:

$$F(fX,Y)\sigma = fF(X,Y)\sigma$$

Tensoriality with respect to Y follows from the skew-symmetry. $\square$

Definition 7.3.5. Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on M. The **curvature** of ∇ is the field $F \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(\xi)) = \Omega^2(\mathrm{End}(\xi))$ such that, for all $X, Y \in \mathcal{X}(M)$ and $\sigma \in \Gamma(\xi)$, we have:

$$F(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma$$

We say that ∇ is **flat** if F vanishes identically on M.

Let us describe the curvature tensor in coordinates. Let $(x^1, \dots, x^d)$ be a local coordinate system on M, and $(\varepsilon_1, \dots, \varepsilon_r)$ a local frame field of ξ.
For $X = \sum_{i=1}^d X^i \partial_i$, $Y = \sum_{i=1}^d Y^i \partial_i \in \mathcal{X}(M)$ and $\sigma = \sum_{\alpha=1}^r \sigma^\alpha \varepsilon_\alpha$, we get:

$$\begin{aligned}
F(X,Y)\sigma &= \sum_{\substack{1 \le i,j \le d \\ 1 \le \alpha \le r}} X^i Y^j \sigma^\alpha F(\partial_i, \partial_j)\varepsilon_\alpha \\
&= \sum_{\substack{1 \le i<j \le d \\ 1 \le \alpha \le r}} \left(X^i Y^j - X^j Y^i\right) \sigma^\alpha F(\partial_i, \partial_j)\varepsilon_\alpha
\end{aligned}$$

We can compute $F(\partial_i, \partial_j)\varepsilon_\alpha$ by using the components of the connection form:

$$\begin{aligned}
F(\partial_i, \partial_j)\varepsilon_\alpha &= \nabla_{\partial_i}\nabla_{\partial_j}\varepsilon_\alpha - \nabla_{\partial_j}\nabla_{\partial_i}\varepsilon_\alpha - \underbrace{\nabla_{[\partial_i,\partial_j]}\varepsilon_\alpha}_{=0} \\
&= \nabla_{\partial_i}\left(\sum_{\beta=1}^r A^\beta_{j,\alpha}\varepsilon_\beta\right) - \nabla_{\partial_j}\left(\sum_{\beta=1}^r A^\beta_{i,\alpha}\varepsilon_\beta\right) \\
&= \sum_{1 \le \beta \le r} \left(\partial_i A^\beta_{j,\alpha} - \partial_j A^\beta_{i,\alpha}\right)\varepsilon_\beta + \sum_{1 \le \beta,\gamma \le r} \left(A^\beta_{j,\alpha} A^\gamma_{i,\beta} - A^\beta_{i,\alpha} A^\gamma_{j,\beta}\right)\varepsilon_\gamma
\end{aligned}$$

So we find $F(\partial_i, \partial_j)\varepsilon_\alpha = \sum_{\beta=1}^r F^\beta_{i,j,\alpha}\varepsilon_\beta$ where:

$$F^\beta_{i,j,\alpha} = \partial_i A^\beta_{j,\alpha} - \partial_j A^\beta_{i,\alpha} + \sum_{\gamma=1}^r \left(A^\gamma_{j,\alpha} A^\beta_{i,\gamma} - A^\gamma_{i,\alpha} A^\beta_{j,\gamma}\right)$$

Reformulation: This formula can be written as $F = dA + \frac{1}{2}[A, A]$ where dA is the exterior differential of the connection 1-form A (it only makes sense for a trivial bundle, but A already depends on the choice of a trivialization), and $[A, B] \in \Omega^2(\mathrm{End}(\xi))$ is defined for $A, B \in \Omega^1(\mathrm{End}(\xi))$ by $[A,B](u,v) = [A(u), B(v)] - [A(v), B(u)]$.

Proposition 7.3.6. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on ξ, and F its curvature.*
Given a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$ is open, we denote by:

- $\frac{D}{\partial t}$ *the intrinsic derivatives along the curves* $t \mapsto f(t,s_0)$ *for fixed* s_0.
- $\frac{D}{\partial s}$ *the intrinsic derivatives along the curves* $s \mapsto f(t_0,s)$ *for fixed* t_0.

Then for all $\sigma \in \Gamma(f^*\xi)$, *we have that:*

$$\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma = F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma$$

Remark. The result can be abbreviated as $\left[\frac{D}{\partial t}, \frac{D}{\partial s}\right] = F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)$.

Proof. Since both sides of the equation can be computed locally, consider a local coordinate system $(x^1,\dots,x^d)$ on M, and a local frame field $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ.

Write $f^i(t,s) = x^i(f(t,s))$ for $i \in \{1,\dots,d\}$ and $(t,s) \in U$, and decompose:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial t}\partial_i \quad ; \quad \frac{\partial f}{\partial s} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial s}\partial_i$$

We also decompose $\sigma = \sum_{\alpha=1}^{r} \sigma^\alpha \varepsilon_\alpha$ (here ε_α stands for an abbreviation of $\varepsilon_\alpha \circ f$).

First, we compute $\frac{D}{\partial t}\varepsilon_\alpha$:

$$\frac{D}{\partial t}\varepsilon_\alpha = \nabla_{\frac{\partial f}{\partial t}}\varepsilon_\alpha = A\left(\frac{\partial f}{\partial t}, \varepsilon_\alpha\right) = \sum_{i=1}^{d} \frac{\partial f^i}{\partial t} A(\partial_i, \varepsilon_\alpha) = \sum_{\substack{1\le i\le d\\1\le\beta\le r}} \frac{\partial f^i}{\partial t} A^\beta_{i,\alpha}\varepsilon_\beta$$

Similarly, we find:

$$\frac{D}{\partial s}\varepsilon_\alpha = \sum_{\substack{1\le i\le d\\1\le\beta\le r}} \frac{\partial f^i}{\partial s} A^\beta_{i,\alpha}\varepsilon_\beta$$

We can compute $\frac{D}{\partial t}\sigma$ and $\frac{D}{\partial s}\sigma$:

$$\begin{aligned}\frac{D}{\partial t}\sigma &= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t}\varepsilon_\alpha + \sigma^\alpha \frac{D}{\partial t}\varepsilon_\alpha\right)\\ &= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t}\varepsilon_\alpha + \sum_{\substack{1\le i\le d\\1\le\beta\le r}} \sigma^\alpha \frac{\partial f^i}{\partial t} A^\beta_{i,\alpha}\varepsilon_\beta\right)\\ &= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t} + \sum_{\substack{1\le i\le d\\1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial t} A^\alpha_{i,\beta}\right)\varepsilon_\alpha\end{aligned}$$

In slightly simpler terms, $\frac{D}{\partial t}\sigma = \sum_{\alpha=1}^{r} \left(\frac{D}{\partial t}\sigma\right)^{\alpha} \varepsilon_\alpha$ where:

$$\left(\frac{D}{\partial t}\sigma\right)^{\alpha} = \frac{\partial \sigma^\alpha}{\partial t} + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial t} A^{\alpha}_{i,\beta}$$

Similarly, we find:

$$\left(\frac{D}{\partial s}\sigma\right)^{\alpha} = \frac{\partial \sigma^\alpha}{\partial s} + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial s} A^{\alpha}_{i,\beta}$$

We now have everything in place to (reluctantly) compute $\frac{D}{\partial t}\frac{D}{\partial s}\sigma$:

$$\begin{aligned}\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma\right)^{\alpha} = \frac{\partial^2 \sigma^\alpha}{\partial t \partial s} &+ \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \left(\frac{\partial \sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s} + \sigma^\beta \frac{\partial^2 f^i}{\partial t \partial s}\right) A^{\alpha}_{i,\beta} \\ &+ \sum_{\substack{1\le i,j\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t} \partial_j A^{\beta}_{i,\alpha} \\ &+ \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \left(\frac{\partial \sigma^\beta}{\partial s} + \sum_{\substack{1\le j\le d\\ 1\le\gamma\le r}} \sigma^\gamma \frac{\partial f^j}{\partial s} A^{\beta}_{j,\gamma}\right) \frac{\partial f^i}{\partial t} A^{\alpha}_{i,\beta}\end{aligned}$$

It becomes slightly tidier when using Einstein's convention:

$$\begin{aligned}\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma\right)^{\alpha} = \frac{\partial^2 \sigma^\alpha}{\partial t \partial s} &+ \left(\frac{\partial \sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s} + \sigma^\beta \frac{\partial^2 f^i}{\partial t \partial s}\right) A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t} \partial_j A^{\beta}_{i,\alpha} \\ &+ \frac{\partial \sigma^\beta}{\partial s}\frac{\partial f^i}{\partial t} A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^i}{\partial s} A^{\gamma}_{i,\beta} \frac{\partial f^j}{\partial t} A^{\alpha}_{j,\gamma}\end{aligned}$$

We have developed the last term and swapped the indexes β and γ as well as i and j in the very last sum, so that it will be easier to see what cancels out with $\frac{D}{\partial s}\frac{D}{\partial t}\sigma$.

For that purpose, we label the indexes differently for $\frac{D}{\partial s}\frac{D}{\partial t}\sigma$: whenever there is a sum over both i and j, we swap them. Other sums keep the same labelling.

$$\begin{aligned}\left(\frac{D}{\partial s}\frac{D}{\partial t}\sigma\right)^{\alpha} = \frac{\partial^2 \sigma^\alpha}{\partial s \partial t} &+ \left(\frac{\partial \sigma^\beta}{\partial s}\frac{\partial f^i}{\partial t} + \sigma^\beta \frac{\partial^2 f^i}{\partial s \partial t}\right) A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^j}{\partial t}\frac{\partial f^i}{\partial s} \partial_i A^{\beta}_{j,\alpha} \\ &+ \frac{\partial \sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s} A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^j}{\partial t} A^{\gamma}_{j,\beta} \frac{\partial f^i}{\partial s} A^{\alpha}_{i,\gamma}\end{aligned}$$

Out of the six terms in each sum, four will cancel out, either directly or by permuting the order of partial derivatives with respect to s and t thanks to the Schwarz Lemma. We are left with:

$$\begin{aligned}
\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma\right)^{\alpha} &= \sigma^{\beta}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\left(\partial_j A^{\beta}_{i,\alpha} - \partial_i A^{\beta}_{j,\alpha} + A^{\gamma}_{i,\beta}A^{\alpha}_{j,\gamma} - A^{\gamma}_{j,\beta}A^{\beta}_{i,\gamma}\right) \\
&= \sigma^{\beta}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}F^{\beta}_{j,i,\alpha} \\
&= \left(F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma\right)^{\alpha}
\end{aligned}$$

□

Remark. This kind of calculation can seem a daunting task when first approaching the subject, but with time and practice they become rather easy. Clever labelling of indices is the key (e.g. note that we used Greek letters for coordinates in the fibres and Roman letters for coordinates on the base manifold).

Note that the computation can be made much simpler if f is assumed to be an immersion. In this case, we can find local coordinates $(x^1, \dots, x^d)$ on M such that $x^1(f(t,s)) = t$ and $x^2(f(t,s)) = s$, as well as $x^i(f(t,s)) = 0$ for $i > 2$. It follows that for $\sigma \in \Gamma(\xi)$, we have $\frac{D}{\partial t}\sigma = \nabla_{\partial_1}\sigma$ and $\frac{D}{\partial s}\sigma = \nabla_{\partial_2}\sigma$, so using the fact that $[\partial_1, \partial_2] = 0$ we find $\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma = F(\partial_1, \partial_2)\sigma = F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma$. A double use of the Leibniz rule allows to replace $\sigma \in \Gamma(\xi)$ with $\sigma \in \Gamma(f^*\xi)$ by decomposing σ in a local trivializing frame of ξ (which amounts to showing that $\left[\frac{D}{\partial t}, \frac{D}{\partial s}\right]$ is tensorial on $\Gamma(f^*\xi)$).

Proposition 7.3.6 can also be retrieved abstractly (i.e. without any computation) from a more general result on pull-backs.

Proposition 7.3.7. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on ξ and $\varphi : N \to M$ a smooth map. The curvature tensor $F^{\varphi^*\nabla} \in \Omega^2(\mathrm{End}(\varphi^*\xi))$ of the pull-back connection $\varphi^*\nabla$ is related to the curvature $F^{\nabla} \in \Omega^2(\xi)$ of ∇ by:*

$$F^{\varphi^*\nabla} = \varphi^* F^{\nabla}.$$

Remark. We use the identification $\mathrm{End}(\varphi^*\xi) = \varphi^*\mathrm{End}(\xi)$.

Proof. Locally, the connection form of $\varphi^*\nabla$ is φ^*A where A is the connection form of ∇. So the result is just a consequence of the commutation of the pull-back with the exterior derivative and the exterior product. □

Theorem 7.3.8. *Let $\xi = (E, p, M)$ be a vector bundle. A connection ∇ on ξ is flat if and only if it is locally trivial.*

Lemma 7.3.9. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a flat connection on M. Then for any $x \in M$ and $v \in \xi_x$, there is a local parallel section σ around x such that $\sigma(x) = v$.*

Proof. To understand the idea, let us start with the two-dimensional case. Consider a local parametrization $f : \mathbb{R}^2 \to M$ such that $f(0,0) = x$. Denote by $\frac{D}{\partial t}$ (resp. $\frac{D}{\partial s}$) the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0 (resp. $t \mapsto f(t_0, s)$ for fixed t_0).

Define σ in two steps:

- First define $\sigma(f(t,0))$ such that $\sigma(x) = v$ and $\frac{D}{\partial t}\sigma((f(t,0)) = 0$ for all $t \in \mathbb{R}$ (recall that from Proposition 7.1.6, there is a unique way of doing so).
- Then define $\sigma(f(t,s))$ with the value at $s = 0$ given by the first step, and so that $\frac{D}{\partial s}\sigma(f(t,s)) = 0$ for all $(t,s) \in \mathbb{R}^2$.

Smoothness of σ comes from the Cauchy-Lipschitz Theorem and the fact that $\frac{D}{\partial t}\sigma = 0$ and $\frac{D}{\partial s}\sigma = 0$ are ordinary differential equations (for exemple, in coordinates, $\frac{D}{\partial s}\sigma = 0$ writes as $\frac{\partial \sigma^\alpha}{\partial s} + A^\alpha_{2,\beta}\sigma^\beta = 0$).

We have that $\frac{D}{\partial s}\sigma = 0$, and according to Proposition 7.3.6:

$$\frac{D}{\partial s}\frac{D}{\partial t}\sigma = \frac{D}{\partial t}\frac{D}{\partial s}\sigma + F(\partial_t, \partial_s)\sigma = 0$$

For all $t_0 \in \mathbb{R}$, the ξ-valued vector field $s \mapsto \frac{D}{\partial t}\sigma(f(t_0, s))$ along the curve $s \mapsto f(t_0, s)$ is parallel, and vanishes at $s = 0$, so it must vanish identically because of the uniqueness in Proposition 7.1.6.

We now know that $\frac{D}{\partial t}\sigma = \frac{D}{\partial s}\sigma = 0$, hence $\nabla_{\partial_t}\sigma = \nabla_{\partial_s}\sigma = 0$. Tensoriality with respect to the vector field shows that $\nabla\sigma = 0$.

To proceed in arbitrary dimension, we use an induction process. For $d \geq 0$, let $\mathcal{A}(d)$ be the assertion: "Lemma 7.3.9 is true for d-dimensional manifolds". We have proved $\mathcal{A}(2)$, but note that $\mathcal{A}(0)$ and $\mathcal{A}(1)$ trivially hold.

Assume that $\mathcal{A}(d)$ is true. Consider that M has dimension $d+1$, and let $f : \mathbb{R}^{d+1} \to M$ be a local parametrization such that $f(0) = x$. Let $\frac{D}{\partial x_i}$ be the intrinsic derivative along the curves $t \mapsto f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{d+1})$ for fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}) \in \mathbb{R}^d$.

Using the assumption $\mathcal{A}(d)$ applied to the manifold $N = f\left(\mathbb{R}^d \times \{0\}\right)$, and the pulled-back connection (which is still flat because of Proposition 7.3.6), we can define σ on N such that $\sigma(x) = v$ and $\frac{D}{\partial x_i}\sigma(y) = 0$ for all $y \in N$ and $i \in \{1, \ldots, d\}$.

We now defined σ on $f\left(\mathbb{R}^{d+1}\right)$ with given value on N and such that $\frac{D}{\partial x_{d+1}}\sigma = 0$.

Once again, the theory of ordinary diffential equations guarantees the smoothness of σ.

Just as in the 2-dimensional case, we have that $\frac{D}{\partial x_i}\sigma$ is parallel along all curves $t \mapsto f(x_1,\dots,x_d,t)$ for $(x_1,\dots,x_d) \in \mathbb{R}^d$, and vanishes at $t = 0$, so it must be identically 0.

Tensoriality of ∇ with respect to the vector field once again leads to $\nabla\sigma = 0$. □

Remark. We just solved a partial differential equation. The method we used applies to a family of equations called transport equations, or conservation laws.

Proof of Theorem 7.3.8. Notice that if $\sigma \in \Gamma(\xi)$ is parallel, the definition of the curvature tensor directly leads to $F(X,Y)\sigma = 0$ for all $X,Y \in \mathcal{X}(M)$. Added to the tensoriality of the curvature, this shows that locally trivial connections are flat.

We now assume that ∇ is flat. Consider $x \in M$ and a basis $(e_1,\dots,e_r)$ of ξ_x. According to Lemma 7.3.9, we can define local parallel sections $\varepsilon_1,\dots,\varepsilon_r$ of ξ such that $\varepsilon_\alpha(x) = e_\alpha$ for $\alpha \in \{1,\dots,r\}$ (note that they can be defined on the same neighbourhood of x because a finite intersection of open sets is open).

According to the uniqueness in Proposition 7.2.2, we only need to check that $(\varepsilon_1,\dots,\varepsilon_r)$ is a frame field on a neighbourhood of x. For this consider a local frame field $(\delta_1,\dots,\delta_r)$, and write $\varepsilon_\alpha = \sum_{\beta=1}^r A_\alpha^\beta \delta_\beta$. Since the matrix $(A_\alpha^\beta(x))_{1\le\alpha,\beta\le r}$ is invertible and $\mathrm{GL}(r,\mathbb{R})$ is open in $\mathcal{M}(r,\mathbb{R})$, we have that $(A_\alpha^\beta(y))_{1\le\alpha,\beta\le r} \in \mathrm{GL}(r,\mathbb{R})$ for y sufficiently close to x, so $(\varepsilon_1(y),\dots,\varepsilon_r(y))$ is a vector basis of ξ_y. □

7.3.2 Flat connections and representations of the fundamental group

The parallel transport of a flat connection only depends on the homotopy class.

Proposition 7.3.10. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a flat connection on ξ. Let $x,y \in M$, and consider smooth curves $c_1,c_2 : [0,1] \to M$ such that $c_0(0) = c_1(0) = x$ and $c_0(1) = c_1(1) = y$. If c_1 and c_2 are homotopic, then their parallel transports from ξ_x to ξ_y are equal.*

Proof. Since c_1 and c_2 are homotopic, there is a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$ is open and contains $[0,1]^2$, such that $f(\cdot,0) = c_0$, $f(\cdot,1) = c_1$, $f(0,s) = x$ and $f(1,s) = y$ for all s. We use the notations of Proposition 7.3.6.

Consider $v \in \xi_x$, and define $\sigma \in \Gamma(f^*\xi)$ in the following way:

- First define $\sigma(t,0)$ that is parallel along c_0 such that $\sigma(0,0) = v$.

- Then define $\sigma(t,s)$ that is parallel along the curves $f(t,\cdot)$ for fixed t, with the value $\sigma(t,0)$ given in the first step.

Smoothness of σ comes from the Cauchy-Lipschitz Theorem and the fact that $\frac{D}{\partial t}\sigma = 0$ and $\frac{D}{\partial s}\sigma = 0$ are ordinary differential equations (for exemple, in coordinates, $\frac{D}{\partial s}\sigma = 0$ writes as $\frac{\partial \sigma^\alpha}{\partial s} + A^\alpha_{2,\beta}\sigma^\beta = 0$).

By definition, we have that $\frac{D}{\partial s}\sigma = 0$ on U and $\frac{D}{\partial t}\sigma(t,0) = 0$. According to Proposition 7.3.6, we find:

$$\frac{D}{\partial s}\frac{D}{\partial t}\sigma = \frac{D}{\partial t}\frac{D}{\partial s}\sigma + F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma = 0$$

It follows that $\frac{D}{\partial t}\sigma$ is parallel along the curves $f(t,\cdot)$. Since it vanishes at $s = 0$, it must be identically 0, i.e. $\frac{D}{\partial t}\sigma = 0$ on U. Therefore $\sigma(\cdot, 1)$ is parallel along c_1, so the parallel transport of v along c_1 is equal to $\sigma(1,1)$. By definition of σ, the parallel transport of v along c_0 is equal to $\sigma(1,0)$.

Since $\sigma(1,s) \in T_yM$ and $\frac{D}{\partial s}\sigma = 0$, we find that $\sigma(1,\cdot)$ is constant, hence $\sigma(1,0) = \sigma(1,1)$.

□

A first consequence is that flat connections on vector bundles over simply connected manifolds only exist on trivialisable vector bundles.

Corollary 7.3.11. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a flat connection on ξ. If M is simply connected, then ξ is trivialisable.*

Proof. For $x,y \in M$, we let $\|_x^y : \xi_x \to \xi_y$ be the parallel transport along any curve joining x and y (according to Proposition 7.3.10, it does not depend on the choice of such a curve because M is simply connected). Fix some $o \in M$, and let $(e_1,\dots,e_r)$ be a vector basis of ξ_o. For $x \in M$, we write $\varepsilon_i(x) = \|_o^x(e_i)$, so that $(\varepsilon_1(x),\dots,\varepsilon_r(x))$ is a vector basis of ξ_x.

We need to check that the sections ε_i are smooth (so that $(\varepsilon_1,\dots,\varepsilon_r)$ is a frame field of M, and ξ is trivialisable). The same arguments as in the proofs of Proposition 7.3.10, Theorem 7.3.8 and Lemma 7.3.9, which rely on locally choosing curves that depend smoothly on x, apply. □

A straightforward consequence of Corollary 7.3.11 is that $T\mathbb{S}^2$ admits no flat connection.

We now wish to discuss the relationship between flat connections on vector bundles over a manifold M and linear representations of the fundamental group $\pi_1(M)$, i.e. group homomorphisms $\rho : \pi_1(M) \to \mathrm{GL}(r,\mathbb{R})$ (or $\mathrm{GL}(r,\mathbb{C})$ for complex vector bundles).

Start with a vector bundle $\xi = (E,p,M)$ and a flat connection ∇ on ξ. Fix some $x \in M$. For $\gamma \in \pi_1(M)$, we set $\rho(\gamma) \in \mathrm{GL}(\xi_x)$ to be the parallel transport along a closed curve based at x representing γ (according to

Proposition 7.3.10, it only depends on the homotopy class γ). Because of the semi-group property of parallel transport, this gives a group representation $\rho : \pi_1(M) \to \mathrm{GL}(\xi_x)$, called the **holonomy representation**. Note it is well defined up to conjugacy: if we choose a different point $y \in M$, then the representations are conjugated by the parallel transport along any curve from x to y.

Any representation of $\rho : \pi_1(M) \to \mathrm{GL}(V)$ can be obtained in this way. Consider the action of $\pi_1(M)$ on $\widetilde{M} \times V$ defined by $\gamma.(x,v) = (\gamma.x, \rho(\gamma)v)$. It is free and properly discontinuous (because $\pi_1(M) \curvearrowright \widetilde{M}$ is). We can consider the quotient manifold $E = (\widetilde{M} \times V)/\pi_1(M)$, and the map $p : E \to M$ defined by $p((x,v) \bmod \pi_1(M)) = \pi(x)$ (where $\pi : \widetilde{M} \to M$ is the universal covering map). One can check that $\xi = (E, p, M)$ is a vector bundle. Since the canonical flat connection on the trivial bundle $\widetilde{M} \times V$ is invariant under $\pi_1(M)$, it descends to a flat connection on ξ, whose holonomy representation is ρ.

7.3.3 Torsion of a connection on a manifold

So far, we have worked on arbitrary tangent bundles, and done most computations using local trivializing frames. In the specific case of the tangent bundle of a manifold, it is often convenient to choose frame fields coming from local coordinates, i.e. $(\partial_1, \ldots, \partial_d)$.

For example, given a flat connection on TM, one can ask if there are local coordinates for which the associated frame field is parallel. Once again, the answer is no, and the obstruction is encoded in a tensor.

In terms of calculation rules, a connection ∇ on the tangent bundle TM of M allows to compare $\nabla_X Y$ and $\nabla_Y X$ for vector fields $X, Y \in \mathcal{X}(M)$. Let us calculate the difference for the trivial connection.

If $(x^1, \ldots, x^d)$ are local coordinates and D is the trivial connection associated to the frame field $(\partial_1, \ldots, \partial_d)$ of TM, then for $X = \sum_{i=1}^n X^i \partial_i$ and $Y = \sum_{i=1}^n Y^i \partial_i$, the formula for D is:

$$D_X Y = \sum_{1 \le i,j \le d} X^j \partial_j Y^i \partial_i$$

We get:

$$D_X Y - D_Y X = \sum_{1 \le i,j \le d} \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right) \partial_i = [X, Y]$$

This formula, $D_X Y - D_Y X = [X, Y]$, does not hold for an arbitrary connection on TM, but the failure of this formula defines a tensor, called the torsion.

Lemma 7.3.12. *Let M be a manifold, and ∇ a connection on TM. The map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by:*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is tensorial and skew-symmetric.

Proof. Skew-symmetry is just a consequence of the skew-symmetry of the Lie bracket. Consider $f \in \mathcal{C}^\infty(M)$. Recall that:

$$[fX, Y] = f[X, Y] - (Y \cdot f)X$$

We get:

$$\begin{aligned} T(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - (Y \cdot f)X - \big(f[X, Y] - (Y \cdot f)X\big) \\ &= f\nabla_X Y - f\nabla_Y X - f[X, Y] \\ &= fT(X, Y) \end{aligned}$$

Skew-symmetry implies tensoriality with respect to Y. □

Definition 7.3.13. Let M be a manifold, and ∇ a connection on TM. The **torsion** of ∇ is the tensor $T \in \Gamma\big(\Lambda^2 T^*M \otimes TM\big) = \Omega^2(TM)$ such that, for all $X, Y \in \mathcal{X}(M)$, we have:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

We say that ∇ is **torsion free** if T vanishes identically.

The absence of torsion should be seen as a much weaker condition that flatness. One of the many reasons is the general existence (and abundance) of torsion free connections.

Proposition 7.3.14. *Let M be a manifold, ∇ a connection on TM, and T its torsion. The connection $\nabla' = \nabla - \frac{1}{2}T$ on TM is torsion free.*

Remark. Since $T \in \Omega^2(TM) \subset \Omega^1(\mathrm{End}(TM))$, this expression does define a connection ∇' thanks to Proposition 7.2.7. The formula is:

$$\forall X, Y \in \mathcal{X}(M) \quad \nabla'_X Y = \nabla_X Y - \frac{1}{2}T(X, Y)$$

Proof. This is a simple computation, using the skew symmetry of T.

$$\begin{aligned} \nabla'_X Y - \nabla'_Y X &= \nabla_X Y - \frac{1}{2}T(X, Y) - \nabla_Y X + \frac{1}{2}T(Y, X) \\ &= \nabla_X Y - \nabla_Y X - T(X, Y) \\ &= [X, Y] \end{aligned}$$

□

Let us have a look at the expression of the torsion in local coordinates $(x^1,\dots,x^d)$. For $X=\sum_{i=1}^n X^i\partial_i$ and $Y=\sum_{i=1}^n Y^i\partial_i$, we get:

$$\begin{aligned}T(X,Y) &= \sum_{1\le i,j\le d} X^iY^jT(\partial_i,\partial_j)\\ &= \sum_{1\le i<j\le d}\left(X^iY^j-X^jY^i\right)T(\partial_i,\partial_j)\end{aligned}$$

We can compute $T(\partial_i,\partial_j)$ by using the connection form in the frame field $(\partial_1,\dots,\partial_d)$ (i.e. $\nabla_{\partial_i}\partial_j=\sum_{k=1}^d A_{i,j}^k\partial_k$):

$$\begin{aligned}T(\partial_i,\partial_j) &= \nabla_{\partial_i}\partial_j-\nabla_{\partial_j}\partial_i-\underbrace{[\partial_i,\partial_j]}_{=0}\\ &= \sum_{k=1}^d\left(A_{i,j}^k-A_{j,i}^k\right)\partial_k\end{aligned}$$

So we find $T(\partial_i,\partial_j)=\sum_{k=1}^d T_{i,j}^k\partial_k$ where:

$$T_{i,j}^k=A_{i,j}^k-A_{j,i}^k$$

Lemma 7.3.15. *Let M be a manifold, and ∇ a torsion free connection on TM. For every $x\in M$, there is a local coordinate system $(x^1,\dots,x^d)$ of M around x such that $A_{i,j}^k(x)=0$ for all $i,j,k\in\{1,\dots,d\}$.*

Proof. Consider a local coordinate system $(y^1,\dots,y^d)$ on an open set $V\subset M$ centered at x (i.e. $y^1(x)=\cdots=y^d(x)=0$). Write $\overline{A}_{i,j}^k$ the components of the connection form in these coordinates. Consider functions f_i^k on M for $i,k\in\{1,\dots,d\}$ such that $f_i^k(x)=0$ and $d_xf_i^k=\sum_{j=1}^d\overline{A}_{i,j}^k(x)d_xy^j$.

Since ∇ is torsion free, the derivatives of the functions f_i^k satisfy the following formula:

$$\frac{\partial f_i^k}{\partial y^j}=\overline{A}_{i,j}^k=\overline{A}_{j,i}^k=\frac{\partial f_j^k}{\partial y^i}$$

This means that the differential 1-forms $\alpha^1,\dots,\alpha^d\in\Omega^1(U)$ defined by $\alpha^k=\sum_{i=1}^d f_i^kdy^i$ are closed. Let $U\subset V$ be an open set diffeomorphic to $\mathbb{R}^d$ that contains x. Poincaré's Lemma gives the existence of functions $z^1,\dots,z^d$ on U such that $dz^k=\alpha^k$.

Note that $\alpha_x^k=0$, hence $d_xz^k=0$. By setting $x^i=y^i+z^i$, the Local Inverse Function Theorem ensures that up to shrinking U, the map $(x^1,\dots,x^d)$ is a coordinate system on U.

Note that $\frac{\partial x^i}{\partial y^j}(x) = \frac{\partial y^i}{\partial x^j}(x) = \delta_i^j$. It follows from Lemma 7.2.14 that:

$$\overline{A}_{i,j}^k(x) = \frac{\partial^2 x^k}{\partial y^j \partial y^i}(x) + A_{i,j}^k(x) = \frac{\partial f_i^k}{\partial y^j}(x) + A_{i,j}^k(x) = \overline{A}_{i,j}^k(x) + A_{i,j}^k(x)$$

It follows that $A_{i,j}^k(x) = 0$. □

The torsion being a point-wise invariant, we can turn this into a point-wise result.

Proposition 7.3.16. *Let M be a manifold, ∇ a connection on TM and T its torsion. Given $x \in M$, the following are equivalent:*

1. *The torsion vanishes at x: $\forall u, v \in T_xM\ T_x(u,v) = 0$.*

2. *There is a local coordinate system $(x^1, \dots, x^d)$ of M around x such that $A_{i,j}^k(x) = 0$ for all $i, j, k \in \{1, \dots, d\}$.*

Proof. The formula given for the components of the torsion in coordinates show that *2.⇒1.*

If $T_x = 0$, we consider the torsion free connection ∇' given by Proposition 7.3.14. According to Lemma 7.3.15, there is a local coordinate system $(x^1, \dots, x^d)$ of M around x such that $A'^k_{i,j}(x) = 0$ for all $i, j, k \in \{1, \dots, d\}$, where the $A'^k_{i,j}$ are the components of the connection form of ∇'. The definition of ∇' gives the following relationship between the connection forms A of ∇ and that of ∇':

$$A'^k_{i,j}(x) = A_{i,j}^k(x) - \frac{1}{2} T_{i,j}^k(x)$$

It follows that $A_{i,j}^k(x) = 0$. □

Proposition 7.3.17. *Let M be a manifold, ∇ a connection on TM, and T its torsion.*

Given a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$, we define by:

- *$\frac{D}{\partial t}$ the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0.*

- *$\frac{D}{\partial s}$ the intrinsic derivatives along the curves $s \mapsto f(t_0, s)$ for fixed t_0.*

We have that:

$$\frac{D}{\partial t}\frac{\partial f}{\partial s} - \frac{D}{\partial s}\frac{\partial f}{\partial t} = T\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)$$

Proof. Consider local coordinates $(x^1, \dots, x^d)$ on M. Write $f^i(t,s) = x^i(f(t,s))$ for $i \in \{1, \dots, d\}$ and $(t,s) \in U$, and decompose:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial t} \partial_i \quad ; \quad \frac{\partial f}{\partial s} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial s} \partial_i$$

The calculations made in the proof of Proposition 7.3.6 (replacing ε_α with ∂_i) translate as:

$$\frac{D}{\partial t} \partial_i = \sum_{1 \le j,k \le d} \frac{\partial f^j}{\partial t} A^k_{j,i} \partial_k \quad ; \quad \frac{D}{\partial s} \partial_i = \sum_{1 \le j,k \le d} \frac{\partial f^j}{\partial s} A^k_{j,i} \partial_k$$

We find:

$$\begin{aligned}
\frac{D}{\partial t} \frac{\partial f}{\partial s} &= \frac{D}{\partial t} \left(\sum_{i=1}^{d} \frac{\partial f^i}{\partial t} \partial_i \right) \\
&= \sum_{i=1}^{d} \left(\frac{\partial^2 f^i}{\partial t \partial s} \partial_i + \frac{\partial f^i}{\partial s} \frac{D}{\partial t} \partial_i \right) \\
&= \sum_{i=1}^{d} \left(\frac{\partial^2 f^i}{\partial t \partial s} \partial_i + \frac{\partial f^i}{\partial s} \sum_{1 \le j,k \le d} \frac{\partial f^j}{\partial t} A^k_{j,i} \partial_k \right) \\
&= \sum_{i=1}^{d} \frac{\partial^2 f^i}{\partial t \partial s} \partial_i + \sum_{1 \le i,j,k \le d} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t} A^k_{j,i} \partial_k
\end{aligned}$$

We swap i and j in the second sum for the expression of $\frac{D}{\partial s} \frac{\partial f}{\partial t}$:

$$\frac{D}{\partial s} \frac{\partial f}{\partial t} = \sum_{i=1}^{d} \frac{\partial^2 f^i}{\partial s \partial t} \partial_i + \sum_{1 \le i,j,k \le d} \frac{\partial f^j}{\partial t} \frac{\partial f^i}{\partial s} A^k_{i,j} \partial_k$$

Combining the two, we get:

$$\begin{aligned}
\frac{D}{\partial t} \frac{\partial f}{\partial s} - \frac{D}{\partial s} \frac{\partial f}{\partial t} &= \sum_{1 \le i,j,k \le d} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t} \left(A^k_{j,i} - A^k_{i,j} \right) \partial_k \\
&= \sum_{1 \le i,j,k \le d} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t} T^k_{j,i} \partial_k \\
&= T\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right)
\end{aligned}$$

□

Theorem 7.3.18. *Let M be a manifold and ∇ a connection on M. The following are equivalent:*

1. *∇ is flat and torsion free.*
2. *Around every point in M, there is a local coordinate system $(x^1, \dots, x^d)$ such that $\nabla \partial_i = 0$ for all $i \in \{1, \dots, d\}$.*

Proof. We have seen that the trivial connection associated to a coordinate system is torsion free and flat, so *2.⇒1.*

To show *1.⇒2.*, we fix $x \in M$ and use Theorem 7.3.8 that provides a local trivializing frame $(\varepsilon_1, \dots, \varepsilon_d)$ of TM around x such that $\nabla \varepsilon_i = 0$ for all $i \in \{1, \dots, d\}$.

Consider the flows $\varphi^t_{\varepsilon_i}$ of the vector fields ε_i, and define the map $f : U \to M$ on a small neighbourhood $U \subset \mathbb{R}^d$ of 0 by:

$$f(x^1, \dots, x^d) = \varphi^{x^1}_{\varepsilon_1} \circ \dots \circ \varphi^{x^d}_{\varepsilon_d}(x)$$

Since $d_0 f(v^1, \dots, v^d) = v^1 \varepsilon_1(x) + \dots + v^d \varepsilon_d(x)$, the Inverse Function Theorem states that f is a diffeomorphism from a neighbourhood of 0 in $\mathbb{R}^d$ to a neighbourhood of x in M. Consider the local coordinates $(x^1, \dots, x^d)$ on M defined by $f^{-1}(y) = (x^1(y), \dots, x^d(y))$.

For $i, j \in \{1, \dots, d\}$, the vector fields ε_i and ε_j commute:

$$[\varepsilon_i, \varepsilon_j] = \nabla_{\varepsilon_i} \varepsilon_j - \nabla_{\varepsilon_j} \varepsilon_i + T(\varepsilon_i, \varepsilon_j) = 0$$

It follows that the flows also commute, and therefore $\partial_i = \varepsilon_i$ is parallel. □

Remark. If the flows do not commute, there is no reason for ∂_i to be equal to ε_i.

7.3.4 Torsion and symmetry of the Hessian

Consider a manifold M. Recall that a connection ∇ on TM also defines a connection ∇^* on the cotangent bundle T^*M (the data of one or the other are equivalent). This allows us to define a second order differential for smooth functions of M.

Definition 7.3.19. Let M be a manifold, ∇^* a connection on T^*M and $f \in \mathcal{C}^\infty(M)$. The **Hessian** of f with respect to ∇^* is the tensor $\mathrm{Hess}(f) = \nabla^* df \in \Omega^1(T^*M) = \Gamma(T^*M \otimes T^*M)$.

Let us try to understand this definition. For $x \in M$ and $u \in T_xM$, we can define $\nabla^*_x(df)(u) \in T^*_xM$, and apply this linear form to some $v \in T_xM$ to get $\mathrm{Hess}(f)(u, v) = \Big(\nabla^*_x df(u)\Big)(v)$.

If ∇^* is the dual connection of ∇, then for $X, Y \in \mathcal{X}(M)$, we find:

$$\mathrm{Hess}(f)(X,Y) = d\Big(df(X)\Big)(Y) - df\Big(\nabla_Y X\Big)$$

This formula can be understood as a product rule for differentiating the function $df(X)$: it should be the sum of a term that differentiates df (the Hessian $\mathrm{Hess}(f)(X,Y)$) and a term that differentiates X (the second term $df\left(\nabla_Y X\right)$).

Proposition 7.3.20. *Let M be a manifold, ∇ a connection on TM and ∇^* its dual connection on T^*M. The following assertions are equivalent:*

1. *The connection ∇ is torsion free.*
2. *For all $f \in \mathcal{C}^\infty(M)$, the Hessian of f with respect to ∇^* is symmetric.*

Proof. For $X, Y \in \mathcal{X}(M)$, we have:

$$\mathrm{Hess}(f)(X,Y) = d\Big(df(X)\Big)(Y) - df\Big(\nabla_Y X\Big)$$

This leads to:

$$\mathrm{Hess}(f)(X,Y) - \mathrm{Hess}(f)(Y,X) = df\Big(T(X,Y)\Big)$$

The abundance of smooth functions allows to prove the equivalence. □

7.3.5 Bianchi identities

Most torsion free connections are not flat. However, vanishing of the torsion adds a type of symmetry on the curvature tensor.

Proposition 7.3.21 (First Bianchi identity)**.** *Let M be a manifold and ∇ a torsion free connection on TM. For $x \in M$ and $u, v, w \in T_xM$, we have that:*

$$F_x(u,v)w + F_x(v,w)u + F_x(w,u)v = 0$$

Remark. The first Bianchi identity is also called the algebraic Bianchi identity.

Proof. Consider a smooth map $f : \left\{ \begin{array}{ccc} \mathbb{R}^3 & \to & M \\ (t,s,r) & \mapsto & f(t,s,r) \end{array} \right.$ such that

$$f(0,0,0) = x,\ \frac{\partial f}{\partial t}(0,0,0) = u,\ \frac{\partial f}{\partial s}(0,0,0) = v,\ \frac{\partial f}{\partial r}(0,0,0) = w.$$

This is always possible, e.g. by choosing a diffeomorphism $\varphi : \mathbb{R}^d \to U$ where $U \subset M$ is open, and $\varphi(0) = x$, then setting:

$$f(t,s,r) = \varphi\left(td_x\varphi^{-1}(u) + sd_x\varphi^{-1}(v) + rd_x\varphi^{-1}(w)\right)$$

According to Proposition 7.3.6, we have:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial r} - \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial r}$$

Since ∇ is torsion free, we can change the second term by means of Proposition 7.3.17:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial r} - \frac{D}{\partial s}\frac{D}{\partial r}\frac{\partial f}{\partial t}$$

Cyclically permuting the indexes and summing, each term on the right hand side appears twice, once with a positive sign and once with a negative sign, so they cancel each other out. In the end, we find:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} + F\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\frac{\partial f}{\partial t} + F\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s} = 0$$

Evaluation at $(0,0,0)$ offers the conclusion. □

Recall that a connection ∇ on a vector bundle ξ defines connections, still denoted by ∇, on the dual bundle ξ^* and all tensor products (Proposition 7.2.6).

In particular, if ∇ is a connection on TM, then the curvature tensor $F \in \Omega^2(\mathrm{End}(TM))$ is a section of $\Lambda^2 T^*M \otimes \mathrm{End}(TM)$, so we can define $\nabla F \in \Omega^1(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$.

Let us review the notation: given $x \in M$ and $u \in T_xM$, we can define $\nabla_x F(u)$ which is the same type of tensor as F_x, i.e. $\nabla_x F(u) \in \Lambda^2 T_x^*M \otimes \mathrm{End}(T_xM)$, so we use the same notation $\nabla_x F(u)(v,w) \in \mathrm{End}(T_xM)$ for $v,w \in T_xM$.

The very definition of the associated connection on $\Lambda^2 T^*M \otimes \mathrm{End}(TM)$ allows us to use of product rule for differentiating. Namely, given vector fields $X,Y,Z,W \in \mathcal{X}(M)$, if we want to differentiate the vector field $F(X,Y)Z \in \mathcal{X}(M)$ in the direction W, i.e. calculate

$$\nabla_W\left(R(X,Y)Z\right)$$

then the result is a sum of terms where we differentiate each term of the expression $F(X,Y)Z$ at a time, i.e. consider $\nabla_W X$, $\nabla_W Y$, $\nabla_W Z$ as well as $\nabla_W F$:

$$\begin{aligned}\nabla_W\left(F(X,Y)Z\right) = \left(\nabla_W F\right)&(X,Y)Z\\ &+ F(\nabla_W X, Y)Z + F(X, \nabla_W Y)Z\\ &\quad + F(X,Y)[\nabla_W Z]\end{aligned}$$

Proposition 7.3.22 (Second Bianchi identity). *Let M be a manifold and ∇ a torsion free connection on TM. For $x \in M$ and $u, v, w \in T_xM$, we have that:*

$$\Big[\nabla_x F(u)\Big](v,w) + \Big[\nabla_x F(v)\Big](w,u) + \Big[\nabla_x F(w)\Big](u,v) = 0$$

Remarks.

- The second Bianchi identity is also called the differential Bianchi identity.
- This is an equation in $\mathrm{End}(T_xM)$. Written in terms of vector fields $X, Y, Z, W \in \mathcal{X}(M)$, the equation is:

$$\Big(\nabla_W F\Big)(X,Y)Z + \Big(\nabla_X F\Big)(Y,W)Z + \Big(\nabla_Y F\Big)(W,X)Z = 0$$

Proof. Consider once again a smooth map $f : \left\{ \begin{array}{ccc} \mathbb{R}^3 & \to & M \\ (t,s,r) & \mapsto & f(t,s,r) \end{array} \right.$ such that $f(0,0,0) = x$, $\frac{\partial f}{\partial t}(0,0,0) = u$, $\frac{\partial f}{\partial s}(0,0,0) = v$ and $\frac{\partial f}{\partial r}(0,0,0) = w$.

Let $z \in T_xM$, and consider a section $\sigma \in \Gamma(f^*TM)$ (i.e. a smooth map $\sigma : \mathbb{R}^3 \to TM$ such that $\sigma(t,s,r) \in T_{f(t,s,r)}M$ for all $(t,s,r) \in \mathbb{R}^3$) such that $\sigma(0,0,0) = z$ (existence of σ is guaranteed by Lemma 6.2.4).

$$\begin{aligned}
\left(\frac{D}{\partial t}R\right)\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\sigma &= \frac{D}{\partial t}\left(R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\sigma\right) - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\sigma \\
&\qquad - R\left(\frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial r}\right)\sigma - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\frac{D}{\partial t}\sigma \\
&= \frac{D}{\partial t}\left[\frac{D}{\partial s}, \frac{D}{\partial r}\right]\sigma - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\sigma \\
&\qquad - R\left(\frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial r}\right)\sigma - \left[\frac{D}{\partial s}, \frac{D}{\partial r}\right]\frac{D}{\partial t}\sigma \\
&= \left[\frac{D}{\partial t}, \left[\frac{D}{\partial s}, \frac{D}{\partial r}\right]\right]\sigma - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\sigma \\
&\qquad + R\left(\frac{D}{\partial r}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s},\right)\sigma
\end{aligned}$$

Cyclically permuting the indexes and summing, each term involving R on the right hand side appears twice, once with a positive sign and once with a negative sign, so they cancel each other out. The other terms sum up to zero because of the Jacobi identity (for linear endomorphisms of the vector space $\Gamma(f^*TM)$). Evaluating at $(0,0,0)$ offers the conclusion.

□

The second Bianchi identity is related to the fact that the curvature, when seen as a 2-form with values in $\mathrm{End}(TM)$, is closed. This is actually true for connections on general vector bundles (so torsion free is not necessary). In order to make sense of this, we need to work with differential forms with values in a vector bundle and consider the exterior covariant derivative.

Proposition 7.3.23. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a connection on ξ. For any integer $k \in \mathbb{N}$, there is a unique linear map $d^\nabla : \Omega^k(\xi) \to \Omega^{k+1}(\xi)$, called the* ***exterior covariant derivative****, such that*

$$d^\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^k \alpha \wedge \nabla\sigma$$

for all $\alpha \in \Omega^k(M)$ and $\sigma \in \Gamma(\xi)$.

Proof. In a local frame field $(\varepsilon_1, \ldots, \varepsilon_r)$, any $\omega \in \Omega^k(\xi)$ can be decomposed as $\omega = \omega^\alpha \otimes \varepsilon_\alpha$ where $\omega^1, \ldots, \omega^r \in \Omega^k(M)$. Then we must have:

$$d^\nabla \omega = d\omega^\alpha \otimes \varepsilon_\alpha + (-1)^k \omega^\alpha \otimes \nabla\varepsilon_\alpha.$$

This formula proves both existence and uniqueness. □

Note that if A is the connection form, then the local expression can also be written as

$$d^\nabla \omega = d\omega + A \wedge \omega$$

where $d\omega$ stands for $d^D\omega$ where D is the trivial connection associated to the chosen frame field.

For $k = 0$, i.e. $\omega \in \Gamma(\xi)$, we simply have $d^\nabla\omega = \nabla\omega$. For a ξ-valued differential 1-form $\omega \in \Omega^1(\xi)$ and $X, Y \in \mathcal{X}(M)$, there is a covariant magic formula:

$$d^\nabla\omega(X,Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X,Y]).$$

More generally, for $\omega \in \Omega^k(\xi)$ and $X_0, \ldots, X_k \in \mathcal{X}(M)$ we have:

$$\begin{aligned} d^\nabla\omega(X_0,\ldots,X_k) = &\sum_{0\le i\le k} (-1)^i \nabla_{X_i}\left(\omega(X_0,\ldots,\widehat{X_i},\ldots,X_k)\right) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k). \end{aligned}$$

Proposition 7.3.24. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on M, and F its curvature.*

1. $\forall X, Y \in \mathcal{X}(M)\ \forall \sigma \in \Gamma(\xi) \quad F(X,Y)\sigma = d^\nabla\left(d^\nabla\sigma\right)(X,Y)$

2. $d^\nabla F = 0$

Remarks.

- We will see later that 2. is equivalent to the second Bianchi identity.
- The exterior covariant derivative $d^\nabla F$ in 2. is the exterior covariant derivative for $\mathrm{End}(\xi)$-valued differential forms, defined by the connection on $\mathrm{End}(\xi)$ induced by ∇ (i.e. $\forall u \in \Gamma(\mathrm{End}(\xi)) \forall \sigma \in \Gamma(\xi)$ $(\nabla u)\sigma = \nabla(u(\sigma)) - u(\nabla\sigma)$).
- A flat vector bundle defines a cohomology complex $\left(\Omega^\bullet(\xi), d^\nabla\right)$. This allows for the construction of topological invariants associated to flat vector bundles (or equivalently to linear representations of $\pi_1 M$).

Proof.

1. Just apply the formulas given above for d^∇ on ξ-valued 0-forms (i.e. sections of ξ) and 1-forms.
2. Start by noticing that for $\omega \in \Omega^p(\mathrm{End}(\xi))$ and $\sigma \in \Gamma(\xi)$, we have that:

$$\left(d^\nabla \omega\right)(\sigma) = d^\nabla(\omega(\sigma)) - \omega\left(d^\nabla \sigma\right)$$

Applied to $F \in \Omega^2(\mathrm{End}(\xi))$, we find:

$$\begin{aligned}\left(d^\nabla F\right)(\sigma) &= d^\nabla(F(\sigma)) - F\left(d^\nabla\sigma\right)\\ &= d^\nabla\left(d^\nabla \circ d^\nabla(\sigma)\right) - d^\nabla \circ d^\nabla\left(d^\nabla \sigma\right)\\ &= 0\end{aligned}$$

□

Let us now see how Proposition 7.3.24 gives another proof of Proposition 7.3.22.

Consider $X, Y, Z \in \mathcal{X}(M)$ such that $X(x) = u$, $Y(x) = v$ and $Z(x) = w$. Note that up to working locally, we can choose X, Y, Z that pairwise commute (say by choosing them to be constant in some local coordinates).

To make the expressions lighter, we fix $W \in \mathcal{X}(M)$, and set:

$$\mathcal{B}(X,Y,Z)W = \nabla_X F(Y,Z)W + \nabla_Y F(Z,X)W + \nabla_Z F(X,Y)W$$

So we wish to show that $\mathcal{B}(X,Y,Z)W = 0$.

By definition of the exterior covariant derivative, the expression of the $\mathrm{End}(TM)$-valued differential 3-form $d^\nabla F$ is:

$$d^\nabla F(X,Y,Z) = \nabla_X\left(F(Y,Z)\right) + \nabla_Y\left(F(Z,X)\right) + \nabla_Z\left(F(X,Y)\right)$$

Now $\nabla_X\left(F(Y,Z)\right)$ is a section of $\mathrm{End}(TM)$. Its definition is:

$$\begin{aligned}\nabla_X\Big(F(Y,Z)\Big)W &= \nabla_X\Big(F(Y,Z)W\Big) - F(Y,Z)[\nabla_X W] \\ &= \Big(\nabla_X F\Big)(Y,Z)W + F(\nabla_X Y,Z)W + F(Y,\nabla_X Z)W\end{aligned}$$

Since $d^\nabla F = 0$, we find:

$$\begin{aligned}-\mathcal{B}(X,Y,Z)W = F(\nabla_X Y,Z)W &+ F(Y,\nabla_X Z)W \\ &+ F(\nabla_Y Z,X)W + F(Z,\nabla_Y X)W \\ &+ F(\nabla_Z X,Y)W + F(X,\nabla_Z Y)W\end{aligned}$$

Since ∇ is torsion free, we can simplify this expression:

$$-\mathcal{B}(X,Y,Z)W = F([X,Y],Z) + F([Z,X],Y) + F([Y,Z],X)$$

Since we have chosen X,Y,Z that pairwise commute, we get $\mathcal{B}(X,Y,Z)W = 0$.

7.4 Geodesics of a connection

7.4.1 The geodesic equation

Definition 7.4.1. Let M be a manifold and ∇ a connection on TM. A **geodesic** of ∇ is a smooth curve $c : I \to M$ such that $\frac{D}{dt}\dot{c} = 0$.

Remarks.

- This notion is only well defined for connections on the tangent bundle TM.

- In local coordinates $(x^1,\dots,x^d)$, the equation becomes:

  $$\forall k \in \{1,\dots,d\}\ \ \ddot{c}^k(t) + \sum_{1\leq i,j\leq d} A^k_{i,j}(c(t))\dot{c}^i(t)\dot{c}^j(t) = 0$$

  It is a second order ordinary differential equation, non linear except in some exceptional situations.

- The equation depends on the parameterization of the curve, not only on its image in M. For $M = \mathbb{R}^d$ and the trivial connection, geodesics are solution to $\ddot{c} = 0$, i.e. affinely parameterized straight lines.

7.4.2 The exponential map of a connection

According to the theory of second order ordinary differential equations, a point $x \in M$ and a tangent vector $v \in T_xM$ define a unique maximal solution $c_v : I_v \to M$ such that $c_v(0) = x$ and $\dot{c}_v(0) = v$.

Because of uniqueness, we have that $I_{sv} = s^{-1}I_v$ for $s \in \mathbb{R}^*$, and $c_{sv}(t) = c_v(st)$.

Definition 7.4.2. Let M be a manifold and ∇ a connection on TM. The **exponential map** of ∇ is the map

$$\exp : \begin{cases} \{v \in TM \,|\, 1 \in I_v\} & \to & M \\ v & \mapsto & c_v(1) \end{cases}$$

We say that ∇ is **complete** if exp is defined on all of TM.

For $x \in M$ we set $\exp_x = \exp|_{T_xM}$.

According to the theory of second order ordinary differential equations, the domain on which exp is defined is an open subset of TM (containing the zero section), and exp is smooth.

Proposition 7.4.3. *Let M be a manifold and ∇ a connection on TM. For all $x \in M$, we have that $d_0 \exp_x = \mathrm{Id}_{T_xM}$.*

Remarks.

- Since $\exp_x$ is defined on an open set of the vector space T_xM, its differential $d_0 \exp_x$ is defined on $T_0(T_xM) = T_xM$.
- According to the Local Inverse Function Theorem, $\exp_x$ defines a diffeomorphism from a neighbourhood of 0 in T_xM to a neighbourhood of x in M, hence local coordinates around x.

Proof. Since $\exp_x(tv) = c_{tv}(1) = c_v(t)$, we get:

$$d_0 \exp_x(v) = \left.\frac{d}{dt}\right|_{t=0} \exp_x(tv) = \dot{c}_v(0) = v$$

□

For the trivial connection on $\mathbb{R}^d$, the exponential map $\exp : T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is simply $\exp(x, v) = x + v$.

Proposition 7.4.4. *Let M be a manifold and ∇ a connection on TM. For all $x_0 \in M$, there exist an open neighbourhood $U \subset M$ of x_0, and a smooth map $\varphi : U \times U \to TM$ such that:*

- $\forall x, y \in U \ \varphi(x, y) \in T_xM$
- $\exp_x(\varphi(x, y)) = y$

Proof. Since the result is local, we may assume that M is an open subset of $\mathbb{R}^d$. Consider the map

$$F : \begin{cases} M \times M \times \mathbb{R}^d & \to & \mathbb{R}^d \\ (x,y,v) & \mapsto & \exp_x(v) - y \end{cases}$$

The partial differential with respect to v at $(x_0, x_0, 0)$ is the identity map of $\mathbb{R}^d$, so there is an open set $U \subset M$ containing x_0 and a smooth map $\varphi : U \times U \to \mathbb{R}^d$ such that $F(x,y,\varphi(x,y)) = 0$, i.e. $\exp_x(\varphi(x,y)) = y$. □

7.4.3 Jacobi fields

We now wish to compute the differential of the exponential map at an arbitrary point. Since there is no explicit formula for the exponential map itself, we cannot expect an explicit formula for its differential. However, it can also be defined as the solution of a second order ordinary differential equation, which is linear (this is actually a general fact for the differential of the flow of a vector field).

Definition 7.4.5. Let M be a manifold, ∇ a connection on TM and $c : I \to M$ be a smooth curve. A **Jacobi field along** c is a vector field J along c such that:

$$\frac{D^2}{dt^2} J + F(J, \dot{c})\dot{c} = 0$$

where F is the curvature of ∇.

Remarks.

- By vector field along c, we mean TM-valued vector field along c.
- This definition makes sense for any curve c, however it will only be useful when c is a geodesic.

Since Jacobi fields are defined by a linear differential equation, solutions exist and are given by initial data consisting of the value and the derivative at a point.

Proposition 7.4.6. *Let M be a manifold, ∇ a connection on TM and $c : I \to M$ be a smooth curve. For every $t_0 \in I$ and $J_0, \dot{J}_0 \in T_{c(t_0)}M$, there is a unique Jacobi field J along c such that $J(t_0) = J_0$ and $\frac{D}{dt}J(t_0) = \dot{J}_0$.*

Proof. Let $(\varepsilon_1, \ldots, \varepsilon_d)$ be a parallel frame along c. Decompose:

$$F(\varepsilon_i, \dot{c})\dot{c} = \sum_{j=1}^{d} f_i^j \varepsilon_j$$

for some functions $f_i^j \in \mathcal{C}^\infty(I)$. Also decompose $J_0 = \sum_{i=1}^d J_0^i \varepsilon_i(t_0)$ and $\dot{J}_0 = \sum_{i=1}^d \dot{J}_0^i \varepsilon_i(t_0)$.

Then $J = \sum_{i=1}^d J^i \varepsilon_i$ is a Jacobi field if and only if:

$$\forall i \in \{1,\dots,d\} \quad \ddot{J}^i + \sum_{j=1}^d f_j^i J^j = 0$$

According to the Cauchy-Lipschitz Theorem for linear equations, there is a unique solution $(J^i)_{1\le i\le d} \in \mathcal{C}^\infty(I)^d$ such that $J^i(t_0) = J_0^i$ and $\dot{J}^i(t_0) = \dot{J}_0^i$ for all $i \in \{1,\dots,d\}$, i.e. a unique Jacobi field $J \in \Gamma(c^*TM)$ such that $J(t_0) = J_0$ and $\frac{D}{dt}J(t_0) = \dot{J}_0$. □

Let us now see how Jacobi fields are related to variations of geodesics.

Lemma 7.4.7. *Let M be a manifold and ∇ a torsion free connection on TM. Let $f : U \to M$ be a smooth map where $U \subset \mathbb{R}^2$. Assume that for all s_0, the curve $t \mapsto f(t,s_0)$ is a geodesic. Then for all s_0, the vector field $t \mapsto \frac{\partial f}{\partial s}(t,s_0)$ along the geodesic $t \mapsto f(t,s_0)$ is a Jacobi field.*

Proof. As usual we define:

- $\frac{D}{\partial t}$ the intrinsic derivatives along the curves $t \mapsto f(t,s_0)$ for fixed s_0.
- $\frac{D}{\partial s}$ the intrinsic derivatives along the curves $s \mapsto f(t_0,s)$ for fixed t_0.

Since ∇ is torsion free, Proposition 7.3.17 affirms that $\frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}$. Now, we use Proposition 7.3.6 and the fact that $f(\cdot,s)$ is a geodesic (which translates as $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$), to calculate:

$$\begin{aligned}\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} &= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} \\ &= \frac{D}{\partial s}\underbrace{\frac{D}{\partial t}\frac{\partial f}{\partial t}}_{=0} + F\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} \\ &= F\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\end{aligned}$$

□

Let us see how to use this to compute the differential of the exponential map.

Proposition 7.4.8. *Let M be a manifold and ∇ a torsion free connection on TM. Let $x \in M$ and $v, w \in T_xM$. If $\exp_x$ is defined at v, then:*

$$d_v \exp_x(w) = J(1)$$

where J is the Jacobi field along the geodesic c_v such that $J(0) = 0$ and $\frac{D}{dt}J(0) = w$.

Proof. Consider the map

$$f : \begin{cases} U & \to & M \\ (t,s) & \mapsto & \exp_x\big(t(v+sw)\big) \end{cases}$$

defined on an open set $U \subset \mathbb{R}^2$ which contains $[0,1]\times\{0\}$. Since $f(1,s) = \exp_x(v+sw)$, we wish to compute $d_v \exp_x(w) = \frac{\partial f}{\partial s}(1,0)$. In order to do so, we consider the vector field $t \mapsto \frac{\partial f}{\partial s}(t,0)$ along the curve $f(\cdot,0) = c_v$.

According to Lemma 7.4.7, it is a Jacobi field. Let us calculate its intial data. First, we have that $f(0,s) = x$ for all s, so $\frac{\partial f}{\partial s}(0,0) = 0$.

To get $\frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0)$, we notice that $s \mapsto f(0,s)$ is a constant curve, so the intrinsic derivative $\frac{D}{\partial s}$ along this curve is just the usual derivative of functions to T_xM. Since $\frac{\partial f}{\partial t}(0,s) = \dot{c}_{v+sw}(0) = v+sw$, we get $\frac{D}{\partial s}\frac{\partial f}{\partial t}(0,0) = w$, hence $\frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0) = w$. So $t \mapsto \frac{\partial f}{\partial s}(t,0)$ is the Jacobi field J defined above. □

7.5 Ehresmann connections

7.5.1 Vertical and horizontal bundles

So far we have only considered connections on vector bundles. It is possible to define a notion of connection on general fibre bundles. They always provide a way of differentiating sections, but do not always produce sections of the same bundle (there is no reason for this to happen if fibres are not vector spaces).

Definition 7.5.1. Let $\xi = (E,p,M,F)$ be a fibre bundle. Denote by $\varphi_E : TE \to E$ and $\pi_M : TM \to M$ the canonical projections. The **vertical bundle** of ξ is the vector sub-bundle V of TE defined by $V_z = \ker d_z p = T_z \xi_{p(z)}$ for all $z \in E$.
A **horizontal bundle**, also called an **Ehresmann connection**, is a vector sub-bundle H of TE such that $H \oplus V = TE$.

Remark. The vector bundles V and H are vector bundles above the total space E of ξ.

7.5.2 Covariant derivatives and Ehresmann connections

Let us see how a connection ∇ on a vector bundle $\xi = (E,p,M)$ defines an Ehresmann connection H^∇. First, consider $z \in E$, and let $x = p(z) \in M$. Define H^∇_z to be the space of derivatives of parallel ξ-valued vector fields along curves passing though x with value z:

$$H^\nabla_z = \left\{ \dot\sigma(0) \,\middle|\, c \in \mathcal{C}^\infty(\mathbb{R},M),\ c(0)=x,\ \sigma \in \Gamma(c^*\xi),\ \sigma(0)=z,\ \frac{D}{dt}\sigma = 0 \right\}$$

Consider local coordinates $(x^1,\dots,x^d)$ on a neighbourhood U of x and a local frame field $(\varepsilon_1,\dots,\varepsilon_r)$ for $\xi|_U$. This defines a local coordinate system $(x^1,\dots,x^d,v^1,\dots,v^r)$ on $p^{-1}(U)$.

A vector W in H^∇_z is the derivative at 0 of a curve $\gamma(t)$ in E whose coordinates $(c^1(t),\dots,c^d(t),\sigma^1(t),\dots,\sigma^r(t))$ satisfy $\dot\sigma^\alpha\varepsilon_\alpha + A^\beta_{i,\alpha}\circ c\sigma^\alpha\dot c^i\varepsilon_\beta = 0$. This means that $W = X^1\frac{\partial}{\partial x^1}+\dots+X^d\frac{\partial}{\partial x^d}+V^1\frac{\partial}{\partial v^1}+\dots+V^r\frac{\partial}{\partial v^r}$ satisfies the equations:

$$V^\alpha + \sum_{i,\beta} A^\alpha_{i,\beta}(x)v^\beta X^i = 0$$

Reciprocally, if $W\in T_zE$ satisfies this equation, then $W\in H^\nabla_z$. This shows that H^∇ is a sub-bundle of TE, and that $H\oplus V = TE$ (V is spanned by $\frac{\partial}{\partial v^1},\dots,\frac{\partial}{\partial v^r}$). Therefore H^∇ is an Ehresmann connection.

However all the properties of an affine connection are not encoded in an Ehresmann connection, so we need to impose a condition on Ehresmann connection that is linked to the structural group of vector bundles.

Definition 7.5.2. Let $\xi = (E,p,M)$ be a vector bundle. An Ehresmann connection H on ξ is called **linear** if $d_v m_\lambda(H_v) = H_{\lambda v}$ for all $v\in E$ and $\lambda\in\mathbb{R}$ (where $m_\lambda : E\to E$ is fibrewise multiplication by λ).

Most of the litterature adds a condition of invariance by sum, however it is implied by the scaling invariance.

Theorem 7.5.3. *Let $\xi = (E,p,M)$ be a vector bundle. The map $\nabla\mapsto H^\nabla$ is a bijection from the set of affine connections to the set of linear Ehresmann connections.*

Proof. Linearity of H^∇ comes from the linearity of $\frac{D}{dt}$. A ξ-valued vector field σ along a curve c is parallel if and only if $\frac{d}{dt}\sigma\in H^\nabla$. It follows that H^∇ determines the parallel transport, hence the injectivity.

To prove the surjectivity, let H be a linear Ehresmann connection. Consider $P\in\Gamma(\mathrm{End}(TE))$ the projection from TE to V parallel to H. For $v\in E$, the vertical space $V_v = T_v\xi_{p(v)}$ is the tangent space of a vector space, so it identifies to $\xi_{p(v)}$. For $\sigma\in\Gamma(\xi)$, define $\nabla_x\sigma(v) = P_v(d_x\sigma(v))$. The Leibniz rule is a consequence of linearity of P, which follows from that of H (V is also linear). □

7.5.3 Curvature of Ehresmann connections

Note that an Ehresmann connection is always characterized by a field of projections $P\in\Gamma(\mathrm{End}(TE))$ on the vertical sub-bundle. Alternatively, one can see P as a TE valued 1-form, i.e. $P\in\Omega^1(TE)$.

A notion of curvature can be defined for Ehresmann connections, as a TE-valued 2-form on E. For $X,Y\in\Gamma(TE)$, define $F(X,Y) = P([X-P(X),Y-P(Y)])$. One can check that it is tensorial, so it defines $F\in\Omega^2(TE)$.

Notice that F is exactly the obstruction for H to be integrable in the Frobenius Theorem, so flat connections are exactly integrable ones.

For linear Ehresmann connections, this curvature is linked to the previously defined one by identifying TM with H.

7.5.4 Principal connections

For a fibre bundle with a reduction of the structure group, one can always define Ehresmann connections that are invariant under this restriction. For example, there is a notion of invariant Ehresmann connections on a H-principal bundle $\xi = (E, p, M, H)$, i.e. invariant under the action of H on E. Such a connection is also characterized by a simpler object, namely a **principal connection**, i.e. a $\mathfrak{h}$-valued 1-form $A \in \Omega^1(\underline{\mathfrak{h}}_E)$ that is equivariant for the actions of H (right-action on E, and Ad on $\mathfrak{h}$), and defines an isomorphism between the vertical bundle and $\mathfrak{h}$ (so the horizontal bundle is the kernel of A). The curvature is then defined by $F = dA + \frac{1}{2}[A, A]$ where $[A, B] \in \Omega^2(\underline{\mathfrak{h}}_E)$ is defined for $A, B \in \Omega^1(\underline{\mathfrak{h}}_E)$ by $[A, B]_x(u, v) = [A_x(u), B_x(v)] - [A_x(v), B_x(u)]$.

Part III

Riemannian geometry

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Chapter 8

Pseudo-Riemannian manifolds

8.1 Metrics on vector bundles

8.1.1 Euclidean metrics

Definition 8.1.1. Let $\xi = (E, p, M)$ be a real vector bundle of rank r. A **Euclidean metric** on ξ is a section $h \in \Gamma(S^2\xi^*)$ such that for all $x \in M$, the map $h_x : \xi_x \times \xi_x \to \mathbb{R}$ is an inner product.

Proposition 8.1.2. *Euclidean metrics exist on every real vector bundle.*

Proof. Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and consider an open cover $\mathcal{U}$ of M such that $\xi|_U$ is trivialisable for every $U \in \mathcal{U}$. Consider a partition of unity $(\varphi_U)_{U\in\mathcal{U}}$ subordinate to $\mathcal{U}$, and for each $U \in \mathcal{U}$ let $\varepsilon^U = (\varepsilon_1^U, \dots, \varepsilon_r^U)$ be a frame field of $\xi|_U$. Now $h = \sum_{U\in\mathcal{U}} \varphi_U \sum_{\alpha=1}^r (e_\alpha^U)^* \otimes (e_\alpha^U)^*$ is a Euclidean metric on ξ. □

Definition 8.1.3. Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. An **orthonormal frame field** is a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ such that for all $x \in M$, the basis $(\varepsilon_1(x), \dots, \varepsilon_r(x))$ of ξ_x is orthonormal for h_x.

Proposition 8.1.4. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. If ξ possesses a frame field, then it also possesses an orthonormal frame field.*

Proof. Consider an arbitrary frame field, and apply the Gram-Schmidt process on every fibre. Since the operations involved in this process are algebraic, they are smooth and the result is an orthonormal frame field. □

Proposition 8.1.5. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. If η is a vector subbundle of ξ, then $\eta^\perp$ defined by $(\eta^\perp)_x = (\eta_x)^\perp$ for every $x \in M$ is a vector subbundle of ξ, supplementary to η.*

Proof. Consider a local frame field $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ such that $(\varepsilon_1,\dots,\varepsilon_k)$ is a frame field of η. Applying the Gram-Schmidt process, we find an orthonormal frame field $(\delta_1,\dots,\delta_r)$ such that $(\delta_1,\dots,\delta_k)$ is still a frame field of η. It follows that $(\delta_{k+1},\dots,\delta_r)$ is a frame field of $\eta^\perp$. □

Proposition 8.1.6. *Let $\xi = (E,p,M)$ be a real vector bundle of rank r. The data of a Euclidean metric on ξ is equivalent to the data of a reduction of the structural group of ξ to* $\mathrm{O}(r,\mathbb{R})$*.*

Proof. Given a Euclidean metric on ξ, the trivialisations given by local orthonormal frame fields form a reduction of the structural group to $\mathrm{O}(r,\mathbb{R})$.

Given a reduction of the structural group to $\mathrm{O}(r,\mathbb{R})$, the inner product computed in a trivialisation belonging to this reduction of the structural group does not depend on said trivialisation, so it defines a Euclidean metric on ξ. □

Proposition 8.1.7. *Let $\xi = (E,p,M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. For $x \in M$, we let $U\xi_x = \{v \in \xi_x \,|\, h_x(v,v) = 1\}$ and $UE = \bigcup_{x\in M} U\xi_x$. Then $(UE, p|_{UE}, M, \mathbb{S}^{r-1})$ is a fibre subbundle of ξ.*

Proof. Let $x \in M$, and let $(\varepsilon_1,\dots,\varepsilon_r)$ be a local orthonormal frame of ξ defined on some open set $U \subset M$ containing x. For $y \in U$, define

$$\theta_y : \begin{cases} \mathbb{R}^r & \to & \xi_y \\ (v^1,\dots,v^r) & \mapsto & v^1\varepsilon_1(y) + \cdots + v^r\varepsilon_r(y) \end{cases}$$

It is a local trivialisation of ξ sending $\mathbb{S}^{r-1} \subset \mathbb{R}^r$ diffeomorphically to $U\xi_y$ for all $y \in U$. □

Definition 8.1.8. Let $\xi = (E,p,M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. The fibre bundle $U\xi = (UE, p|_{UE}, M, \mathbb{S}^{r-1})$ is called the **unit bundle**.

An inner product $\langle\cdot|\cdot\rangle$ on a finite dimension real vector space V induces an inner product on V^*, using the isomorphism between V and V^* obtained by sending $v \in V$ to $(w \mapsto \langle v|w\rangle) \in V^*$.

Inner products on two vector finite dimensional vector spaces $(V,\langle\cdot|\cdot\rangle)$ and $(W,(\cdot,\cdot))$ also induce a unique inner product $<< \cdot\|\cdot >>$ on $V \otimes W$ satisfying $<< v_1 \otimes w_1 \| v_2 \otimes w_2 >>= \langle v_1|v_2\rangle (w_1,w_2)$ for all pure tensors.

Put together, we find inner products on all tensor powers $(V^*)^{\otimes p} \otimes V^{\otimes q}$. The identification of $V^* \otimes V$ with $\mathrm{End}(V)$ gives the usual product $\langle f|g\rangle = \mathrm{Tr}(f^t g)$.

Similarly, a Euclidean metric h on a vector bundle ξ induces Euclidean metrics on all tensor powers $(\xi^*)^{\otimes p} \otimes \xi^{\otimes q}$.

A Euclidean metric h on a vector bundle $\xi = (E,p,M)$ allows us to define semi-norms on $\Gamma(\xi)$ (or a norm when M is compact): given a compact subset $K \subset M$, define $\|\sigma\|_{\infty,K} = \sup_{x\in K}\sqrt{h_x(\sigma(x),\sigma(x))}$. The topology associated to this family of semi-norms is the compact-open topology (it does not depend on the Euclidean metric h).

8.1.2 Pseudo-Euclidean metrics

Definition 8.1.9. Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $p,q \in \mathbb{N}$ be such that $p+q=r$. A **pseudo-Euclidean metric of signature** (p,q) on ξ is a section $h \in \Gamma(S^2\xi^*)$ such that h_x has signature (p,q) for every $x \in M$.

Note that given a fixed signature, a pseudo-Euclidean metrics does not always exist.

A big difference with positive definite metrics is that the restriction of a pseudo-Euclidean metric to a vector subbundle is not necessarily a pseudo-Euclidean metric. However, if it is, then we can still define the orthogonal complement.

Proposition 8.1.10. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and h a pseudo-Riemannian metric on ξ. If η is a vector subbundle of ξ, then $\eta^\perp$ defined by $(\eta^\perp)_x = (\eta_x)^\perp$ for every $x \in M$ is a vector subbundle of ξ.*

If moreover the restriction of h_x to $\eta_x \times \eta_x$ is non degenerate for all $x \in M$, then $\eta^\perp$ is supplementary to η.

Proof. Use the corresponding statement for quadratic vector spaces and copy the proof of Proposition 8.1.5. $\square$

Let us recall some basic facts on quadratic forms over real vector spaces.

Definition 8.1.11. Let V, V' be vector spaces, and $\varphi : V \times V \to \mathbb{R}$, $\varphi' : V' \times V' \to \mathbb{R}$ be bilinear forms. We say that φ and φ' are equivalent if there is a linear isomorphism $f : V \to V'$ such that $\varphi'(f(x), f(y)) = \varphi(x,y)$ for all $x,y \in V$.

Given integers $p,q,r \in \mathbb{N}$, we let $\langle\cdot|\cdot\rangle_{p,q,r}$ be the bilinear form on $\mathbb{R}^{p+q+r}$ defined by $\langle x|y\rangle_{p,q,r} = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$ for $x,y \in \mathbb{R}^{p+q+r}$.

Theorem 8.1.12 (Sylvester's inertia law)**.** *Let V be a finite dimensional real vector space, and $\varphi : V \times V \to \mathbb{R}$ a symmetric bilinear form. There is a unique triple (p,q,r), called the **signature** of φ, such that φ is equivalent to $\langle\cdot|\cdot\rangle_{p,q,r}$.*

Lemma 8.1.13. *Let V be a finite dimensional real vector space, and φ a symmetric bilinear form on V of signature (p,q,r). Then φ is non degenerate if and only if $r=0$.*

For non degenerate forms, we will call (p,q) the signature.

Definition 8.1.14. Let V be a finite dimensional vector space, and φ a symmetric bilinear form on V. A linear map $u \in \mathrm{End}(V)$ is called φ-self adjoint if $\varphi(x,u(y)) = \varphi(u(x),y)$ for all $x,y \in V$.

Proposition 8.1.15. *Let V be a finite dimensional vector space, and φ a non degenerate symmetric bilinear form on V. If B is a symmetric bilinear form on V, then there is a unique φ-self adjoint operator $b \in \mathrm{End}(V)$ such that $B(x,y) = \varphi(x, b(y))$ for all $x, y \in V$.*

Definition 8.1.16. Let V be a finite dimensional vector space, and φ a non degenerate symmetric bilinear form on V. If B is a symmetric bilinear form on V, then the **trace of B with respect to φ** is $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}(b)$ where $b \in \mathrm{End}(V)$ is the φ-self adjoint operator such that $B(x,y) = \varphi(x, b(y))$ for all $x, y \in V$.

Recall that the matrix of a bilinear form φ in a basis $e = (e_1, \dots, e_d)$ is $(\varphi(e_i, e_j))_{1 \le i,j \le d}$.

Proposition 8.1.17. *Let V be a finite dimensional vector space, and φ, B symmetric bilinear forms on V. Assume that φ is non degenerate. Let $e = (e_1, \dots, e_d)$ be a basis of V, on consider the matrices P and Q respectively of φ and B in e. Then $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}\left(QP^{-1}\right)$.*
In particular, if φ is positive definite and e is φ-orthonormal, then $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}(Q)$.

8.1.3 Hermitian metrics on complex vector bundles

Definition 8.1.18. Let $\xi = (E, p, M)$ be a complex vector space of rank r. A **Hermitian metric** on ξ is a section $h \in \Gamma(\overline{V}^* \otimes V^*)$ such that h_x is a Hermitian inner product on ξ_x for every $x \in M$.

Proposition 8.1.19. *Every complex vector bundle possesses a Hermitian metric.*

Proof. Let $\xi = (E, p, M)$ be a complex vector bundle of rank r, and consider an open cover $\mathcal{U}$ of M such that $\xi|_U$ is trivialisable for every $U \in \mathcal{U}$. Consider a partition of unity $(\varphi_U)_{U \in \mathcal{U}}$ subordinate to $\mathcal{U}$, and for each $U \in \mathcal{U}$ let $\varepsilon^U = (\varepsilon_1^U, \dots, \varepsilon_r^U)$ be a frame field of $\xi|_U$. Now $h = \sum_{U \in \mathcal{U}} \varphi_U \sum_{\alpha=1}^r \overline{(e_\alpha^U)^*} \otimes (e_\alpha^U)^*$ is a Hermitian metric on ξ. □

8.2 Metrics on manifolds

8.2.1 Pseudo-Riemannian metrics

Definition 8.2.1. Let M be a manifold of dimension d.

A **Riemannian metric** on M is a Euclidean metric on TM, i.e. a section $g \in \Gamma(S^2 T^* M)$ such that for every $x \in M$, g_x is an inner product $T_x M$.

A **pseudo-Riemannian metric** of signature (p,q) (with $p + q = d$) is a pseudo-Euclidean metric of signature (p,q) on TM, i.e. a section $g \in$

$\Gamma(S^2T^*M)$ such that for every $x \in M$, g_x is non-degenerate and has signature (p,q).

A **Lorentzian metric** is a pseudo-Riemannian metric of signature $(d-1,1)$.

A **(pseudo-)Riemannian manifold** is a pair (M,g) where M is a manifold and g is a (pseudo-)Riemannian metric on M.

Notation: Given a pseudo-Riemannian metric g, we write $g_x(v,w) = \langle v|w\rangle_x = \langle v|w\rangle$ for $x \in M$ and $v,w \in T_xM$. If g is Riemannian, we write $\|v\| = \|v\|_x = \sqrt{\langle v|v\rangle_x}$.

Proposition 8.2.2. *Every manifold has a Riemannian metric.*

This is a consequence of the existence of Euclidean metrics on vector bundles (Proposition 8.1.2).

It does not hold for Lorentzian metrics: the sphere $\mathbb{S}^2$ has no Lorentzian metric (it is a consequence of the hairy ball theorem).

Before we move on any further with pseudo-Riemannian manifolds, let us start with the most basic example. On $\mathbb{R}^{p+q}$, we consider the bilinear form:

$$\langle x|y\rangle_{p,q} = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$$

Definition 8.2.3. The **pseudo-Euclidean space** of signature (p,q) is the pseudo-Riemannian manifold $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g)$ where $g_x = \langle\cdot|\cdot\rangle_{p,q}$ for all $x \in \mathbb{R}^{p+q}$.

For $q = 0$, $\mathbb{E}^p = \mathbb{R}^{p,0}$ is called the **Euclidean space**.

For $q = 1$, $\mathbb{M}^n = \mathbb{R}^{n-1,1}$ is called the **Minkowski space**.

Another elementary way of producing pseudo-Riemannian manifolds is through products. If (M,g) and (M,g') are pseudo-Riemannian manifolds, then we can define the **product pseudo-Riemannian manifold** $(M \times M', g \oplus g')$ where the metric is defined by:

$$(g \oplus g')_{(x,x')}((v,v'),(w,w')) = g_x(v,w) + g'_{x'}(v',w')$$

Note that if (M,g) and (M',g') are Riemannian, then so is the product. This construction generalizes to the product of a finite number of pseudo-Riemannian manifolds.

8.2.2 Local expression of a pseudo-Riemannian metric

Let (M,g) be a pseudo-Riemannian manifold of dimension d. Given local coordinates $(x^1,\ldots,x^d)$, for $u = \sum_{i=1}^d u^i \partial_i$ and $v = \sum_{i=1}^d v^i \partial_i$, we find $g_x(u,v) = \sum_{1\le i,j\le d} g_{i,j}(x)u^i v^j$, where $g_{i,j}(x) = g_x(\partial_i, \partial_j)$. We write:

$$g = \sum_{1\le i,j\le d} g_{i,j}(x)dx^i dx^j$$

Here we use the notation $dx^i dx^j = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$.

It is quite frequent to see the notation ds^2 for a pseudo-Riemannian metric, especially in coordinates:

$$ds^2 = \sum_{1\leq i,j\leq d} g_{i,j}(x) dx^i dx^j$$

Given another coordinate system $(y^1, \dots, y^d)$, if we write $g'_{i,j}$ the metric in these coordinates, i.e. $g'_{i,j}(x) = g_x\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$, the formula for the coordinate change is:

$$g'_{i,j} = \sum_{1\leq k,l\leq d} \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{k,l}$$

8.2.3 Isometric maps

Definition 8.2.4. Let (M,g) and (M',g') be (pseudo-)Riemannian manifolds. A smooth map $f : M \to M'$ is an **isometric immersion** if $f^*g' = g$.

It is called an **isometry** if it is also a diffeomorphism.

A **local isometry** is a local diffeomorphism which is an isometric immersion.

A (pseudo-)**Riemannian covering** is a map which is both a local isometry and a covering map.

Note that an isometric immersion is indeed an immersion, since the equality $d_x f(v) = 0$ implies $g_x(v,v) = 0$. In particular, its existence implies that $\dim M' \geq \dim M$.

Definition 8.2.5. Let (M,g) be a pseudo-Riemannian manifold. The **isometry group** of (M,g) is:

$$\mathrm{Isom}(M,g) = \{f \in \mathrm{Diff}(M) \,|\, f^*g = g\}$$

The isometry group is a subgroup of $\mathrm{Diff}(M)$, and we will discuss its topology later. It is important however to understand that even though most of the examples that we will work on have many isometries, a typical pseudo-Riemannian manifold (i.e. a generic metric for an appropriate topology on the set of metrics) has no non trivial isometry. Indeed, the equation $f^*g = g$ where the map f is the unknown is an overdetermined partial differential equation.

Example 8.2.6. Given $\gamma \in \mathrm{O}(p,q)$ and $v \in \mathbb{R}^{p+q}$, the affine map $x \mapsto \gamma x + v$ is an isometry of $\mathbb{R}^{p,q}$. This means that the group $\mathrm{O}(p,q) \ltimes \mathbb{R}^{p+q}$ of affine transformations whose linear part is in $\mathrm{O}(p,q)$ is a subgroup of $\mathrm{Isom}(\mathbb{R}^{p,q})$.

Proposition 8.2.7. $\mathrm{Isom}(\mathbb{R}^{p,q}) = \mathrm{O}(p,q) \ltimes \mathbb{R}^{p+q}$

Proof. Let $f \in \mathrm{Isom}(\mathbb{R}^{p,q})$. This means that for all $x, u, v \in \mathbb{R}^{p+q}$, we have:

$$\langle d_x f(u) | d_x f(v) \rangle_{p,q} = \langle u | v \rangle_{p,q} \tag{8.1}$$

Let us differentiate this expression with respect to x at some vector w.

$$\left\langle d_x^2 f(u,w) \middle| d_x f(v) \right\rangle_{p,q} + \left\langle d_x f(u) \middle| d_x^2 f(v,w) \right\rangle_{p,q} = 0 \tag{8.2}$$

The same formula remains true when switching u and w.

$$\left\langle d_x^2 f(w,u) \middle| d_x f(v) \right\rangle_{p,q} + \left\langle d_x f(w) \middle| d_x^2 f(v,u) \right\rangle_{p,q} = 0 \tag{8.3}$$

Since f is smooth, $d^2 f_x$ is symmetric, and subtracting (8.3) from (8.2) yields:

$$\left\langle d_x f(u) \middle| d_x^2 f(w,u) \right\rangle_{p,q} = \left\langle d_x f(w) \middle| d_x^2 f(v,u) \right\rangle_{p,q} \tag{8.4}$$

Now a cyclic permutation of u, v, w in (8.2) gives:

$$\left\langle d_x^2 f(v,u) \middle| d_x f(w) \right\rangle_{p,q} + \left\langle d_x f(v) \middle| d_x^2 f(w,u) \right\rangle_{p,q} = 0 \tag{8.5}$$

Combining (8.4) and (8.5), we find:

$$\left\langle d_x f(u) \middle| d_x^2 f(v,w) \right\rangle_{p,q} = 0$$

Since f is a diffeomorphism, we find that $d_x^2 f = 0$, i.e. f is affine. Now (8.1) shows that the linear part of f is in $\mathrm{O}(p,q)$. □

8.3 Volume and angles

8.3.1 Pseudo-Riemannian volume

If g is a Riemannian metric, the **Riemannian volume** is the unique Borel measure Vol^g on M such that, for any chart (U, φ) and continuous function f with support in U,

$$\int_U f \, d\,\mathrm{Vol}^g = \int_{\varphi(U)} f \circ \varphi^{-1} \sqrt{\det(g \circ \varphi^{-1})} d\lambda$$

where λ denotes the Lebesgue measure on $\mathbb{R}^d$ et $g(x)$ est the matrix $(g_{i,j}(x))_{1 \leq i,j \leq d}$.

If moreover M is oriented, the **Riemannian volume form** is the volume form $\mathrm{vol}^g \in \Gamma(\Lambda^d T^* M)$ defined in oriented coordinates by:

$$\mathrm{vol}_x^g = \sqrt{\det(g(x))} dx^1 \wedge \cdots \wedge dx^d$$

The volume can also defined for pseudo-Riemannian metrics by considering the absolute value of the determinant.

8.3.2 Conformal metrics

If (M,g) is a pseudo-Riemannian manifold and $\phi \in \mathcal{C}^\infty(M,]0,+\infty[)$, then ϕg is also a pseudo-Riemannian metric on M, with the same signature as g.

Definition 8.3.1. Let M be a manifold. Two pseudo-Riemannian metrics g and g' are called **conformal** if there is $\phi \in \mathcal{C}^\infty(M,]0,+\infty[)$ such that $g' = \phi g$.

The **conformal class** of a pseudo-Riemannian metric is the set $[g] \subset \Gamma\left(S^2T^*M\right)$ of pseudo-Riemannian metrics that are conformal to g.

Note that two conformal metrics have the same signature. Their respective pseudo-Riemannian volumes are related by $\mathrm{dvol}_{g'} = \phi^{\frac{\dim M}{2}}\,\mathrm{dvol}_g$.

In order to understand the geometric meaning of conformal Riemannian metrics, we have to define a notion of angles in Riemannian geometry.

Definition 8.3.2. Let (M,g) be a Riemannian manifold. Let $x \in M$, and $v,w \in T_xM \setminus \{0\}$. The **Riemannian angle** between v and w is the angle $\sphericalangle_x(v,w) \in [0,\pi]$ defined by:

$$\cos \sphericalangle_x(v,w) = \frac{\langle v|w\rangle_x}{\|v\|_x\,\|w\|_x}$$

One can easily prove that two Riemannian metrics on a given manifold are conformal if and only if they define the same Riemannian angles.

For a pseudo-Riemannian manifold (M,g) of signature (p,q) with $pq \neq 0$, the situation is different. Here the conformal class is characterized by the isotropic cone: given another pseudo-Riemannian metric h on M, we find:

$$h \in [g] \quad \Longleftrightarrow \quad \forall x \in M\ \{v \in T_xM \,|\, g_x(v,v) = 0\} = \{v \in T_xM \,|\, h_x(v,v) = 0\}$$

8.4 Examples of pseudo-Riemannian manifolds

8.4.1 Pseudo-Riemannian quotients

Just as examples of covering maps can be constructed from group actions, examples of pseudo-Riemannian coverings can be constructed from isometric actions.

Theorem 8.4.1. *Let $(\widetilde{M},\widetilde{g})$ be a pseudo-Riemannian manifold, and Γ be a subgroup of $\mathrm{Isom}(\widetilde{M},\widetilde{g})$ such that the action $\Gamma \curvearrowright \widetilde{M}$ is free and properly discontinuous. There is a unique pseudo-Riemannian metric g on $M = \widetilde{M}/\Gamma$ for which the projection $\widetilde{M} \to M$ is a pseudo-Riemannian covering.*

Proof. Let $\pi : \widetilde{M} \to M$ be the projection. Let $x \in M$, and consider a local inverse f of π. Set $g_x = (f^*\widetilde{g})_x$. Since two local inverses differ by an element of Γ, the metric does not depend on the choice of f, and defines a smooth Riemannian metric on M. □

Examples 8.4.2.

- For any $\lambda > 0$, we can define the **circle** $\mathbb{E}^1/\lambda\mathbb{Z}$ as the quotient of $\mathbb{E}^1$ by the group generated by the translation $x \mapsto x + \lambda$.
- The **flat torus** $\mathbb{E}^2/\mathbb{Z}^2$. More generally we can define $\mathbb{E}^n/\Lambda$ where Λ is the subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ generated by n linearly independent translations.
- The **Clifton-Pohl torus**, which is the quotient of $(\mathbb{R}^2\setminus\{0\}, \frac{2dxdy}{x^2+y^2})$ by the map $(x,y) \mapsto (2x, 2y)$. It is a Lorentzian manifold often used to point out the differences between Riemannian and Lorentzian geometries.

8.4.2 Pseudo-Riemannian submanifolds

Definition 8.4.3. Let (M,g) be a pseudo-Riemannian manifold. An immersed submanifold $N \subset M$ is called a **pseudo-Riemannian submanifold** if there are integers p', q' with $p' + q' = \dim N$ such that the restriction of g to $T_xN \times T_xN$ is non degenerate and has signature (p', q') for all $x \in N$.

It is a **Riemannian submanifold** if $q' = 0$.

If $N \subset M$ is a pseudo-Riemannian submanifold, the **induced metric**, or **restricted metric**, or the **first fundamental form** is the metric defined on N by restriction of g to tangent spaces.

Note that every submanifold of a Riemannian manifold is a Riemannian submanifold. In particular, every submanifold of $\mathbb{R}^d$ inherits a Riemannian metric in this way. It is the case for spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The induced metric g_{sph} is called the **round metric**, or standard metric on $\mathbb{S}^n$.

If $N \subset M$ is a pseudo-Riemannian embedded submanifold of (M,g) and $f \in \mathrm{Isom}(M,g)$ preserves M, then the restriction of f to N is an isometry. However N can have many more isometries.

Applied to the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we find that $\mathrm{O}(n+1) \subset \mathrm{Isom}(\mathbb{S}^n)$. We will see later that this is an equality.

Submanifolds of a pseudo-Riemannian manifold of arbitrary signature are not always pseudo-Riemannian submanifolds, since the restriction of the metric can be degenerate. If (M,g) is a pseudo-Riemannian manifold of signature (p,q), then a pseudo-Riemannian submanifold of (M,g) can have any signature (p', q') with $p' \leq p$ and $q' \leq q$.

We will now look at a Riemannian submanifold of the Minkowski space $\mathbb{M}^{n+1}$. Consider one sheet of the one-sheeted hyperboloid:

$$\mathcal{H}^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \langle x|x\rangle_{n,1} = -1 \ \&\ x_{n+1} > 0\right\}$$

For $x \in \mathcal{H}^n$, the tangent space is:

$$T_x\mathcal{H}^n = \left\{v \in \mathbb{R}^{n+1} \,\middle|\, \langle x|v\rangle_{n,1} = 0\right\}$$

Since $\langle x|x\rangle_{n,1} = -1$, its orthogonal complement is definite positive, i.e. $\mathcal{H}^n$ is a Riemannian submanifold of $\mathbb{M}^{n+1}$. This Riemannian manifold $\mathbb{H}^n = (\mathcal{H}^n, g_{\mathcal{H}^n})$ is called the **real hyperbolic space**.

By the same considerations as for the sphere, we find that $O_+(n,1) \subset \mathrm{Isom}(\mathbb{H}^n)$, where $O_+(n,1)$ is the subgroup of $O(n,1)$ preserving each sheet of the two-sheeted hyperboloid $\mathcal{H}^n$ (it has index two in $O(n,1)$, but it is not $SO(n,1)$).

8.4.3 Riemannian manifolds of dimension 1

Theorem 8.4.4. *Let (M,g) be a connected Riemannian manifold of dimension 1. Then (M,g) is isometric to an interval of $\mathbb{E}^1$ or to a circle $\mathbb{E}^1/\lambda\mathbb{Z}$.*

Proof. Consider two isometric maps $\varphi : I \to M$ and $\psi : J \to M$. Now consider the set

$$X = \left\{ t \in I \cap J \,\middle|\, \varphi(t) = \psi(t) \text{ \& } \dot\varphi(t) = \dot\psi(t) \right\}$$

It is a closed subset of $I \cap J$. Now let $t_0 \in X$. Since φ is an immersion and $\dim M = 1$, it is a local diffeomorphism. Considering a local inverse φ^{-1} near $\varphi(t_0)$, we see that $f = \varphi^{-1} \circ \psi$, which is defined on an open interval K containing t_0, satisfies $|\dot f(t)| = 1$ for all $t \in K$. The fact that $t_0 \in X$ implies that $f(t_0) = t_0$ and $\dot f(t_0) = 1$. It follows that $f = \mathrm{Id}_K$, i.e. $\varphi = \psi$ on K. This argument shows that the set X is open in $I \cap J$.

Let $x_0 \in M$, and $v_0 \in T_{x_0}M$ such that $g_{x_0}(v_0, v_0) = 1$ (there are exactly two such vectors). Consider the set E of pairs (I,φ) where $I \subset \mathbb{R}$ is an open interval containing 0 and $\varphi : I \to M$ is an isometric map such that $\varphi(0) = x_0$ and $\dot\varphi(0) = v_0$.

By the above discussion, if $(I,\varphi), (J,\psi) \in E$ then φ and ψ coincide on $I \cap J$. This allows us to define a maximal element of E: set $I_M = \bigcup_{(I,\varphi)\in E} I$, and define $\varphi_M : \mathcal{I} \to M$ by $\varphi_M(t) = \varphi(t)$ if $(I,\varphi) \in E$ and $t \in I$.

First, let us check that E is not empty. For this, start with a curve $\gamma : J \to M$ such that $\gamma(0) = x_0$ and $\dot\gamma(0) = v_0$. Up to shrinking J, we can assume that γ is an immersion. Then the function $\lambda : I \to \mathbb{R}$ defined by $\lambda(t) = \int_0^t \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}$ is increasing, so it is a diffeomorphism onto its image $I \subset \mathbb{R}$. Now set $\varphi = \gamma \circ \lambda^{-1}$, so that $(I, \varphi) \in E$.

Since E is not empty, we find that $(I_M, \varphi_M) \in E$. Now notice that since φ_M is an immersion, its image $\varphi_M(I_M)$ is open in M. Its complement is the union of such open sets obtained by starting at a different base point. It follows that $\varphi_M(I_M)$ is closed. Since M is assumed to be connected, we see that φ_M is onto.

If φ_M is injective, then it is an isometry between an interval of $\mathcal{E}^1$ to (M,g). Now assume that φ_M is not injective. We wish to show that it is periodic. Up to changing x_0 and v_0, we can assume that there is $T > 0$ such that $\varphi_M(T) = x_0$, and that T is the smallest positive real number with this property. First assume that $\dot{\varphi}_M(T) = v_0$ (we will see that it is always the case). Then $(I_M - T, \varphi_M(\cdot + T)) \in E$, and it follows from the maximality of I_M that I_M is stable by translation by T, therefore $I_M = \mathbb{R}$, and that $\varphi_M(\cdot + T) = \varphi_M$. It follows that there is a map $\varphi : \mathbb{E}^1/T\mathbb{Z} \to M$ such that $\varphi_M = \varphi \circ \pi$ where $\pi : \mathbb{E}^1 \to \mathbb{E}^1/T\mathbb{Z}$ is the canonical projection. Since T was chosen to be minimal, the map φ is injective, and it is an isometry between the circle $\mathbb{E}^1/T\mathbb{Z}$ and (M, g).

Now let us see why we must have $\dot{\varphi}_M(T) = v_0$. If not, then $\dot{\varphi}_M(T) = -v_0$. The same type of argument as above shows that $I_M = \mathbb{R}$ and that $\varphi_M(T - t) = \varphi_M(t)$ for all $t \in \mathbb{R}$. This implies that $\dot{\varphi}_M\left(\frac{T}{2}\right) = -\dot{\varphi}_M\left(\frac{T}{2}\right)$, i.e. $\dot{\varphi}_M\left(\frac{T}{2}\right) = 0$. This is a contradiction with the fact that φ_M is isometric. $\square$

This implies that the isometries of a submanifold are not all restrictions of isometries, since one dimensional submanifolds have isometries but need not be preserved by any isometry of the ambient space.

8.4.4 Conformal models of the real hyperbolic space

The Poincaré **ball model** of the real hyperbolic space $\mathbb{H}^n$ is $(\mathbb{B}^n, g_{hyp})$ where

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$$

and

$$g_{hyp} = \frac{4}{\left(1 - \|x\|^2\right)^2}\left(dx_1^2 + \cdots + dx_n^2\right)$$

It is a pseudo-Riemannian manifold conformal to the unit ball in the Euclidean space. Its volume element is $\mathrm{dvol}_{hyp} = \frac{2^n}{\left(1-\|x\|^2\right)^n} dx_1 \cdots dx_n$.

Consider the following map f from the hyperboloid model to the ball model of $\mathbb{H}^n$: set $p = (0, \dots, 0, -1) \in \mathbb{M}^{n+1}$, and embed $\mathbb{R}^n$ into $\mathbb{M}^{n+1}$ as the hyperplane $x_{n+1} = 0$. For $x \in \mathcal{H}^n$ we let $f(x)$ be the intersection of the line from x to p with $\mathbb{R}^n$.

Precisely, we have:

$$f(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}}\right)$$

The map f is an isometry.

The Lobachevsky **upper half-space model** of the real hyperbolic space $\mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$ is $(\mathbb{R}^n_+, ds^2_{hyp})$ where

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$$

and

$$ds^2_{hyp} = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

It is conformal to a half Euclidean space. The volume element is $\mathrm{dvol}_{hyp} = \frac{dx_1 \cdots dx_n}{x_n^n}$.

8.4.5 Invariant metrics on Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A pseudo-Riemannian g on G is **left invariant** if $\{L_x | x \in G\} \subset \mathrm{Isom}(G, g)$, i.e.

$$\forall x \in G \ L_x^* g = g$$

It is **right invariant** if $\{R_x | x \in G\} \subset \mathrm{Isom}(G, g)$, i.e.

$$\forall x \in G \ R_x^* g = g$$

It is **bi-invariant** if it is both left and right invariant.

Any bilinear symmetric non degenerate form $b : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a unique left invariant pseudo-Riemannian metric g on G such that $g_e = b$.

The metric g is bi-invariant if and only if b is Ad-invariant, i.e.

$$b(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = b(X, Y)$$

for all $g \in G$ and $X, Y \in \mathfrak{g}$.

8.4.6 The space of ellipsoids

For $n \geq 2$, we consider the set

$$\mathcal{E}_n = \left\{ x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x, \, x > 0, \, \det x = 1 \right\}$$

It is a submanifold of $\mathcal{M}_n(\mathbb{R})$, and it can be identified with the set of volume 1 ellipsoids of $\mathbb{R}^n$ centred at 0 (by identifying $x \in \mathcal{E}_n$ with $\left\{ v \in \mathbb{R}^n \,\middle|\, {}^t x v x \leq 1 \right\}$).

The tangent spaces are easily described, as

$$T_{1_n} \mathcal{E}_n = \left\{ X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X, \, \mathrm{Tr}\, X = 0 \right\},$$

and more generally

$$T_x \mathcal{E}_n = \left\{ X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X, \, \mathrm{Tr}(x^{-1} X) = 0 \right\}.$$

Consider the Riemannian metric on $\mathcal{E}_n$ defined as

$$\langle X | Y \rangle_x = \mathrm{Tr}(x^{-1} X x^{-1} Y)$$

To prove that it is a Riemannian metric (the hard part is the positive definiteness), it is convenient to use the existence of a square root of x, i.e. an element $\sqrt{x} \in \mathcal{E}_n$ such that $\sqrt{x}^2 = x$.

For $g \in \mathrm{SL}(n, \mathbb{R})$ the map $x \mapsto {}^t g x g$ is an isometry of $\mathcal{E}_n$. Since -1_n acts as the identity, this gives an embedding of $\mathrm{PSL}(n, \mathbb{R})$ into $\mathrm{Isom}(\mathcal{E}_n)$. We will see later that this map is surjective.

8.5 Raising and lowering indices

8.5.1 The musical isomorphisms

A non degenerate symmetric bilinear form $\langle\cdot|\cdot\rangle$ on a finite dimensional real vector space V induces isomorphisms:

$$\flat : \left\{ \begin{array}{ccc} V & \to & V^* \\ v & \mapsto & v^\flat = \langle v|\cdot\rangle \end{array} \right. \text{ and } \sharp = \flat^{-1} : \left\{ \begin{array}{ccc} V^* & \to & V \\ \lambda & \mapsto & \lambda_\sharp \end{array} \right.$$

A pseudo-Riemannian metric g on a manifold M therefore defines isomorphisms between TM and T^*M. We can apply these isomorphisms to tensor powers, and find isomorphisms between $\mathcal{T}^{r,s}(M)$ and $\mathcal{T}^{r',s'}(M)$ whenever $r+s = r'+s'$.

In particular we get an isomorphism $\flat : \mathcal{X}(M) = \mathcal{T}^{0,1}(M) \to \Omega^1(M) = \mathcal{T}^{1,0}(M)$ from vector fields to 1-forms. For $X \in \mathcal{X}(M)$, if we write $X = \sum_{i=1}^d X^i \partial_i$ in coordinates, then

$$X^\flat = \sum_{i=1}^d \left(\sum_{j=1}^d g_{i,j} X^j \right) dx^i$$

Similarly, the inverse $\sharp : \Omega^1(M) \to \mathcal{X}(M)$ writes for $\omega = \sum_{i=1}^d \omega_i dx^i$ as

$$\omega_\sharp = \sum_{i=1}^d \left(\sum_{j=1}^d g^{i,j} \omega_j \right) \partial_j$$

Einstein's convention makes these formulae very concise:

$$X^\flat_i = g_{i,j} X^j \text{ and } \omega^i_\sharp = g^{i,j}\omega_j$$

When writing tensors in coordinates, the lower indices correspond to the covariant part and the upper indices to the contravariant part. Because of this convention for notations, applying an isomorphism $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r+1,s-1}(M)$ is called **lowering an index**, and applying an isomorphism $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s+1}(M)$ is called **raising an index** (note that this terminology is consistent with the musical notations $\flat$ and $\sharp$).

Note that there are several isomorphisms $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r+1,s-1}(M)$, depending on which index we lower (i.e. on which factor of $TM^{\otimes q}$ we apply $\flat$). If $s = 1$, there is no possible confusion. For example, if we have $T \in \mathcal{T}^{3,1}(M)$, then there is only one index to lower, and we find $T^\flat \in \mathcal{T}^{4,0}(M)$ defined locally by

$$T^\flat_{i,j,k,l} = \sum_{a=1}^d g_{l,a} T^a_{i,j,k}$$

8.5.2 Contractions of tensors

Given a finite dimension real vector space V, there is a unique linear map $V^* \otimes V \to \mathbb{R}$ sending $\lambda \otimes v$ to $\lambda(v)$. The identification of $V^* \otimes V$ with $\mathrm{End}(V)$ identifies this map with the trace in $\mathrm{End}(V)$.

This map also yields maps $(V^*)^{\otimes r} \otimes V^{\otimes s} \to (V^*)^{\otimes r-1} \otimes V^{\otimes s-1}$ (where we have to choose the factors of $(V^*)^{\otimes r}$ and $V^{\otimes s}$ to which we apply the map $V^* \otimes V \to \mathbb{R}$).

On a manifold M, this defines maps $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ called the **contraction** of a tensor. For $T \in \mathcal{T}^{3,1}(M)$, its contraction $U \in \mathcal{T}^{2,0}(M)$ is given by:

$$U_{i,j} = \sum_{a=1}^{d} T^a_{a,i,j}$$

On a pseudo-Riemannian manifold (M,g), by combining this with the musical isomorphisms, we are able to contract covariant (and contravariant) tensors. For $R \in \mathcal{T}^{4,0}(M)$, its contraction $S \in \mathcal{T}^{2,0}(M)$ is defined by

$$S_{i,j} = \sum_{1 \le a,b \le d} g^{a,b} R_{a,i,j,b}$$

For $V \in \mathcal{T}^{2,0}(M)$, its contraction is a function $W \in \mathcal{C}^\infty(M) = \mathcal{T}^{0,0}(M)$ defined by

$$W = \sum_{1 \le i,j \le d} g^{i,j} R_{i,j}$$

Chapter 9

The Levi-Civita connection

9.1 The fundamental theorem of pseudo-Riemannian geometry

Theorem 9.1.1. *Let (M,g) be a pseudo-Riemannian manifold.*
There is a unique connection ∇ on TM with the following two properties:

1. *∇ is torsion-free:* $\forall X,Y \in \mathcal{X}(M)\ \nabla_X Y - \nabla_Y X = [X,Y]$

2. *g is parallel for ∇:*

$$\forall X,Y,Z \in \mathcal{X}(M) \quad X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Remarks.

- This connection $\nabla = \nabla^g$ is called the **Levi-Civita connection** of g.
- Condition 2. is equivalent to $\nabla g = 0$ (where ∇ also denotes the induced connection on $T^*M \otimes T^*M$).

Proof. Let us start with uniqueness. If ∇ satisfies *1.* and *2.*, we find, for $X,Y,Z \in \mathcal{X}(M)$:

$$\begin{aligned} X \cdot g(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y \cdot g(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z \cdot g(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

By adding the first two lines and subtracting the third, then simplifying because ∇ is torsion-free, we find:

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X \cdot g(Y,Z) + Y \cdot g(Z,X) + Z \cdot g(X,Y) \\ & + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned} \tag{9.1}$$

Since g is non-degenerate, this formula, called the **Koszul formula** defines $\nabla_X Y$, hence the uniqueness.

For the existence, we use the Koszul formula to define ∇. For this we first have to check that the formula is tensorial with respect to Z, so that it defines $\nabla_X Y \in \mathcal{X}(M)$, then that it is tensorial in X, so that it defines $\nabla Y \in \mathcal{T}^{1,1}(M)$. Finally we can check that it satisfies the Leibniz rule. □

Example 9.1.2. For the pseudo-Euclidean space $\mathbb{R}^{p,q}$, the Levi-Civita connection is the trivial connection on $T\mathbb{R}^{p+q} = \mathbb{R}^{p+q} \times \mathbb{R}^{p+q}$.

9.2 Pseudo-Riemannian parallel transport

Proposition 9.2.1. *Let (M,g) be a pseudo-Riemannian manifold, and $c : I \to M$ be a smooth curve. If $X, Y : I \to TM$ are vector fields along c, then:*

$$\frac{d}{dt} g(X,Y) = g\left(\frac{D}{dt}X, Y\right) + g\left(X, \frac{D}{dt}Y\right)$$

Proof. The definition of the intrinsic derivative $\frac{D}{dt}g$ induced on the tensor bundle $T^*M \otimes T^*M$ leads directly to:

$$\frac{d}{dt} g(X,Y) = \left(\frac{D}{dt}g\right)(X,Y) + g\left(\frac{D}{dt}X, Y\right) + g\left(X, \frac{D}{dt}Y\right)$$

Since g is a tensor defined on all of M, we have that $\frac{D}{dt}g = \nabla g(\dot{c}) = 0$. The result follows. □

Proposition 9.2.2. *Let (M,g) be a pseudo-Riemannian manifold and $c : I \to M$ a piecewise smooth curve. For every $t_0, t_1 \in I$, the parallel transport for the Levi-Civita connection*

$$\|_{t_0}^{t_1} : (T_{c(t_0)}M, g_{c(t_0)}) \to (T_{c(t_1)}M, g_{c(t_1)})$$

is isometric.

Remark. Consequently, the holonomy group Hol_x is a subgroup of $\mathrm{O}(g_x) \approx \mathrm{O}(p,q)$.

Proof. If $X, Y : I \to TM$ are vector fields along M, we get:

$$\frac{d}{dt} g_{c(t)}(X(t), Y(t)) = g_{c(t)}\left(\frac{D}{dt}X(t), Y(t)\right) + g_{c(t)}\left(X(t), \frac{D}{dt}Y(t)\right)$$

If X and Y are parallel, it follows that $g(X,Y)$ is constant. □

9.3 The Christoffel symbols

Let (M,g) be a pseudo-Riemannian manifold, and consider local coordinates $(x^1,\dots,x^d)$. The connection form of the Levi-Civita connection is usually denoted by Γ. Its components $\Gamma^k_{i,j}$ are called the **Christoffel symbols.**

$$g_{ij} = g(\partial_i, \partial_j)\ ;\ \ g = \sum g_{i,j} dx^i dx^j\ ;\ \ \nabla_{\partial i}\partial_j = \sum \Gamma^k_{i,j}\partial_k$$

Since the Levi-Civita connection is torsion-free, we have:

$$\Gamma^k_{j,i} = \Gamma^k_{i,j}$$

The Koszul formula (9.1) yields:

$$2g\left(\nabla_{\partial i}\partial_j, \partial_k\right) = \partial_i g_{j,k} + \partial_j g_{k,i} - \partial_k g_{i,j}$$

To obtain the Christoffel symbols, we must consider the inverse matrix $(g^{i,j})_{1\le i,j\le d}$ of $(g_{i,j})_{1\le i,j\le d}$.

$$\Gamma^k_{i,j} = \frac{1}{2}\sum_{l=1}^{d} g^{l,k}(\partial_i g_{j,l} + \partial_j g_{i,l} - \partial_l g_{i,j})$$

9.4 Differential operators on Riemannian manifolds

Definition 9.4.1. Let (M,g) be a Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$. The **gradient** of f is the vector field $\vec{\nabla} f \in \mathcal{X}(M)$ defined by:

$$\forall x \in M\ \forall v \in T_xM \quad d_xf(v) = g_x(\vec{\nabla} f(x), v)$$

Remark. With the notation of the musical isomorphisms, we have $\vec{\nabla} f = df_\sharp$.

Note that the notation $\vec{\nabla}$ has nothing to do with the Levi-Civita connection. In coordinates, we find:

$$\vec{\nabla} f = \sum_{1\le i,j\le d} g^{i,j}\partial_i f \partial_j$$

Definition 9.4.2. Let (M,g) be a Riemannian manifold, and let $X \in \mathcal{X}(M)$. The **divergence** of X is the function $\operatorname{div} X \in \mathcal{C}^\infty(M)$ such that for all $x \in M$, $\operatorname{div} X(x)$ is the trace of the map $v \mapsto \nabla_x X(v)$, where ∇ is the Levi-Civita connection.

In local coordinates, for $X = X^i\partial_i$ (using Einstein's convention), we find $\operatorname{div} X = \partial_i X^i + \Gamma^j_{i,j} X^i$. Note that $\operatorname{div} X$ is the contraction of the tensor $\nabla X \in \mathcal{T}^{1,1}(M)$.

Definition 9.4.3. Let (M,g) be a Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$. The **Laplacian** of f is the function $\Delta f \in \mathcal{C}^\infty(M)$ defined by $\Delta f = \operatorname{div} \vec{\nabla} f$.

We have already seen that a connection ∇ on TM allows us to define a Hessian $\operatorname{Hess} f$. If ∇ is the Levi-Civita connection of a Riemannian manifold, one can check that $\Delta f = \operatorname{Tr}_g(\operatorname{Hess} f)$.

9.5 Examples of Levi-Civita connections

9.5.1 The Levi-Civita connection of a submanifold

Let (M,g) is a pseudo-Riemannian manifold, and let $N \subset M$ be a pseudo-Riemannian submanifold. Recall that we can consider the restriction of the Levi-Civita connection ∇ to N, which is a connection on $TM|_N$. This allows us to define $\nabla_x X(v)$ for $x \in N$, $v \in T_x N$ and $X \in \Gamma(TM|_N)$. In other terms, the vector field X is only defined on N but is not necessarily tangent to N. However, requiring X to be tangent to N does not ensure that $\nabla_x X(v)$ is.

The tangent bundle TN is a vector subbundle of $TM|_N$. Since N is a pseudo-Riemannian submanifold, according to Proposition 8.1.10 the vector bundle $TM|_N$ splits as a direct sum $TM|_N = TN \oplus TN^\perp$. The vector bundle $\nu N = TN^\perp$ is called the **normal bundle** of N. For $x \in N$ and $v \in T_x M$, we write $v = v^\top + v^\perp$ its decomposition according to this direct sum.

Lemma 9.5.1. *Let M be a manifold and $N \subset M$ an immersed submanifold. For all $x \in N$ and $v \in T_x N$, there is a vector field $X \in \mathcal{X}(M)$ and a neighbourhood $U \subset N$ of x such that $X(x) = v$ and $X(y) \in T_y N$ for all $y \in U$.*

Proof. Use the linearisation of immersions and a plateau function on M. □

Proposition 9.5.2. *Let (M,g) be a pseudo-Riemannian manifold, and $N \subset M$ a pseudo-Riemannian submanifold, with induced metric $\overline{g}$. For all $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_x N$, we have $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ where $\overline{\nabla}$ is the Levi-Civita connection of $(N,\overline{g})$ and ∇ is the Levi-Civita connection of (M,g).*

Proof. First check, let us check that the formula $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ defines a connection $\overline{\nabla}$ on TN. It is a map from $\mathcal{X}(N)$ to $\Omega^1(N)$ (because the map $v \mapsto v^\top$ is a vector bundle morphism).

Let $f \in \mathcal{C}^\infty(N)$, $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_x N$. Since ∇ is a connection we have:

$$\nabla_x(fX)(v) = d_x f(v) X(x) + f(x) \nabla_x X(v)$$

Since $X(x) \in T_x N$, we have $X(x)^\top = X(x)$ and projecting on TN yields:

$$\overline{\nabla}_x(fX)(v) = d_x f(v) X(x) + f(x) \overline{\nabla}_x X(v)$$

This shows that $\overline{\nabla}$ is a connection on TN. We now wish to compute its torsion. For this, we first consider vector fields $X, Y \in \mathcal{X}(M)$ whose restrictions $\overline{X}, \overline{Y}$ to N are tangent to N. Since ∇ is torsion free, we find:

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Now evaluating this at some point $x \in N$ and projecting on $T_x N$, the left hand side is $\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}$, and the right hand side is $[\overline{X}, \overline{Y}]$, so we find:

$$\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} = [\overline{X}, \overline{Y}]$$

Hence $\overline{T}(\overline{X}, \overline{Y}) = 0$ where $\overline{T}$ is the torsion of $\overline{\nabla}$. Using the tensoriality of $\overline{T}$ and Lemma 9.5.1, we find that $\overline{\nabla}$ is torsion free.

Now let $X, Y, Z \in \mathcal{X}(M)$ be such that their restrictions $\overline{X}, \overline{Y}, \overline{Z}$ to N are tangent to N. Since g is parallel for ∇, we have:

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

When restricting to N, the left hand side becomes $\overline{X} \cdot \overline{g}(\overline{Y}, \overline{Z})$. Let us compute the first term of the right hand side:

$$\begin{aligned} g\left(\nabla_X Y, \overline{Z}\right) &= \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right) + \underbrace{g\left((\nabla_X Y)^{\perp}, \overline{Z}\right)}_{=0} \\ &= \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right) \end{aligned}$$

In the end, we find:

$$\overline{X} \cdot \overline{g}(\overline{Y}, \overline{Z}) = \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right) + \overline{g}\left(\overline{Y}, \overline{\nabla}_{\overline{X}}\overline{Z}\right)$$

This shows that $\left(\overline{\nabla}_{\overline{X}}\overline{g}\right)(\overline{Y}, \overline{Z}) = 0$. Once again by using the tensoriality of $\overline{\nabla}\overline{g}$ and Lemma 9.5.1, we find that $\overline{\nabla}\overline{g} = 0$, so $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$. □

9.5.2 Conformal metrics

Consider a pseudo-Riemannian manifold (M, g) and a conformal metric $g' = e^{2\sigma} g$ with $\sigma \in \mathcal{C}^{\infty}(M)$. Then the Levi-Civita connections ∇, ∇' of g, g' respectively are related by:

$$\nabla'_x X(v) = \nabla_x X(v) + d_x\sigma(v)X(x) + d_x\sigma(X(x))v - g_x(X(x), v)\vec{\nabla}\sigma(x)$$

Where the gradient $\vec{\nabla}\sigma$ is considered for the metric g.

9.5.3 The space of ellipsoids

For $n \geq 2$, we consider the set

$$\mathcal{E}_n = \left\{x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x,\, x > 0,\, \det x = 1\right\}$$

Recall that the tangent spaces are given by

$$T_x\mathcal{E}_n = \left\{X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \mathrm{Tr}(x^{-1}X) = 0\right\},$$

and the Riemannian metric is defined as:

$$\langle X|Y\rangle_x = \mathrm{Tr}(x^{-1}Xx^{-1}Y).$$

The Levi-Civita connection ∇ of $\mathcal{E}_n$ is given by:

$$\nabla_x X(v) = d_x X(v) - \frac{vx^{-1}X(x) + X(x)x^{-1}v}{2}$$

To prove this, one should first check that it is well defined (i.e. that $\nabla_x X(v) \in T_x\mathcal{E}_n$), that it is a connection, that the torsion vanishes, and that the metric is parallel.

9.6 Pseudo-Riemannian geodesics

9.6.1 The geodesic flow

Definition 9.6.1. Let (M,g) be a pseudo-Riemannian manifold. A **geodesic** of (M,g) is a geodesic of its Levi-Civita connection ∇, i.e. a smooth curve $c : I \to M$ such that $\frac{D}{dt}\dot{c} = 0$.

Example 9.6.2. Since the Levi-Civita connection of $\mathbb{R}^{p,q}$ is the trivial connection, geodesics of $\mathbb{R}^{p,q}$ are affinely parametered straight lines.

Even though it is not possible to solve explicitly the geodesic equation, it does have a first integral.

Proposition 9.6.3. *Let (M,g) be a pseudo-Riemannian manifold, and $c : I \to M$ a geodesic. Then $g(\dot{c},\dot{c})$ is constant.*

Proof. It is a straightforward consequence of Proposition 9.2.2 and the fact that $\frac{D}{dt}\dot{c} = 0$. □

Let us fix a notation: for $v \in TM$, we let $I_v \subset \mathbb{R}$ be the maximal interval on which the geodesic c_v is defined.

Definition 9.6.4. Let (M,g) be a pseudo-Riemannian manifold, and set

$$U = \bigcup_{v \in TM} I_v \times \{v\} \subset \mathbb{R} \times TM$$

The **geodesic flow** of (M,g) is the map

$$\Phi : \begin{cases} U & \to & TM \\ (t,v) & \mapsto & \dot{c}_v(t) \end{cases}$$

Write $\Phi(t,v) = \Phi^t(v)$. It is a local flow: if $t, t+s \in I_v$ then $s \in I_{\Phi^t(v)}$ and $\Phi^s(\Phi^t(v)) = \Phi^{t+s}(v)$.

The corresponding vector field $\mathcal{Z}^g = \frac{d}{dt}\Big|_{t=0} \Phi^t \in \mathcal{X}(TM)$ is called the **geodesic spray**.

If the Levi-Civita connection ∇ is complete, then $U = \mathbb{R} \times TM$, and $(\Phi^t)_{t \in \mathbb{R}}$ is a one parameter subgroup of $\mathrm{Diff}(TM)$.

If g is Riemannian, we can consider the **unit tangent bundle**

$$T^1M = \{v \in TM \,|\, \|v\| = 1\}$$

The geodesic flow Φ_t preserves T^1M (Proposition 9.6.3), so we get a dynamical system $(T^1M, (\Phi_t))$. Note that if M is compact, then so is T^1M.

9.6.2 Normal coordinates and the injectivity radius

Definition 9.6.5. Let (M,g) be a Riemannian manifold. Consider $x \in M$, open sets $U \subset T_xM$ and $V \subset M$ such that $\exp_x|_U : U \to V$ is a diffeomorphism, and an orthonormal basis $(e_1, \ldots, e_d)$ of T_xM. The coordinates $(x^1, \ldots, x^d)$ on V defined by $x^i = g_x\left((\exp_x|_U)^{-1}(\cdot), e_i\right)$ are called **normal coordinates**.

Normal coordinates $(x^1, \ldots, x^d)$ correspond to the chart $(x^1, \ldots, x^d) \mapsto \exp_x(x^1e_1 + \cdots + x^d e_d)$.

Proposition 9.6.6. *Let (M,g) be a Riemannian manifold. Consider $x \in M$ and normal coordinates $(x^1, \ldots, x^d)$ around x. Then*

$$\forall i,j,k \in \{1, \ldots, d\} \quad g_{i,j}(x) = \delta_{i,j} \;\; ; \;\; \Gamma^k_{i,j}(x) = \partial_k g_{i,j}(x) = 0$$

Remark. This means that a Riemannian metric is "Euclidean up to order one". This makes it hopeless to find invariants of Riemannian metrics that only involve first order derivatives.

Proof. By definition of normal coordinates, $(\partial_1(x), \ldots, \partial_d(x))$ is an orthonormal basis of T_xM, so $g_{i,j}(x) = g_x(\partial_i(x), \partial_j(x)) = \delta_{i,j}$.

Now fix some $v = (v^1, \ldots, v^d) \in \mathbb{R}^d$. The curve c defined by

$$c(t) = \exp_x(tv^1\partial_1(x) + \cdots + tv^d\partial_d(x))$$

is a geodesic. Recall the geodesic equation in coordinates:

$$\forall k \in \{1,\ldots,d\} \quad \ddot{c}^k(t) + \sum_{1\leq i,j\leq d} \Gamma_{i,j}^k(c(t))\dot{c}^i(t)\dot{c}^j(t) = 0$$

Here $c^i(t) = tv^i$, so the equation simplifies:

$$\forall k \in \{1,\ldots,d\} \qquad \sum_{1\leq i,j\leq d} \Gamma_{i,j}^k(c(t))v^i v^j = 0$$

In particular we have $\sum_{1\leq i,j\leq d} \Gamma_{i,j}^k(x)v^i v^j = 0$. This being true for all $v \in \mathbb{R}^d$, we find $\Gamma_{i,j}^k(x) = 0$.

Now we also have

$$\forall i,j,k \in \{1,\ldots,d\} \quad \partial_i g_{j,k}(x) + \partial_j g_{i,k}(x) - \partial_k g_{i,j}(x) = 2\sum_{l=1}^{d} g_{k,l}(x)\Gamma_{i,j}^l(x) = 0$$

By permuting the indices we also get

$$\forall i,j,k \in \{1,\ldots,d\} \quad \partial_k g_{i,j}(x) + \partial_i g_{k,j}(x) - \partial_j g_{k,i}(x) = 0$$

By adding these last two equalities and using the symmetry of g, we find $\partial_i g_{j,k}(x) = 0$. □

Definition 9.6.7. Let (M,g) be a Riemannian manifold, and let $x \in M$. The **injectivity radius** of (M,g) at x is $\mathrm{inj}_x = \sup E \in]0,+\infty]$ where $E \subset]0,+\infty[$ is the set of positive real numbers r such that $\exp_x$ is diffeomorphic from $B_{T_xM}(0,r)$ onto its image.

The **injectivity radius** of (M,g) is $\mathrm{inj}M = \inf_{x\in M} \mathrm{inj}_x \in [0,+\infty]$.

Remark. The theory of ODEs shows that the map $x \mapsto \mathrm{inj}_x$ is lower semi-continuous. Therefore if M is compact then $\mathrm{inj}M > 0$.

9.6.3 Isometries and geodesics

Lemma 9.6.8. *Let (M,g) be a pseudo-Riemannian manifold, and $N \subset M$ a pseudo-Riemannian submanifold. Let $c : I \to N$ be a smooth curve. If c is a geodesic of M, then it is a geodesic of N.*

Proof. If c is a geodesic of M, then $\frac{D}{dt}\dot{c} = 0$, and its projection on TN also vanishes, so c is a geodesic of N according to Proposition 9.5.2. □

This is far from being an equivalence. It can be seen in the proof: the geodesic equation on M has more constraints that we did not use (the projection on $TN^\perp$ of $\frac{D}{dt}\dot{c}$).

To find some explicit examples, consider submanifolds in $\mathbb{E}^d$. Most submanifolds do not contain any straight line, yet geodesics always exist.

Proposition 9.6.9. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds. If $\varphi : N \to M$ is an isometric immersion, and $c : I \to N$ is a smooth curve such that $\varphi \circ c$ is a geodesic, then c is a geodesic.*

Furthermore, if φ is a local isometry, then $\varphi \circ c$ is a geodesic if and only if c is a geodesic.

Proof. Let us start with the second statement. Introducing local coordinates on M and pulling them back on N by φ, the geodesic equations for c and $\varphi \circ c$ are the same, hence the result.

If φ is only an isometric immersion, we can work locally and assume that φ is an embedding (since being a geodesic is a local condition). Then $\varphi \circ c$ is a geodesic of the submanifold $\varphi(N)$ of M (Lemma 9.6.8), and since φ is an isometry between N and $\varphi(N)$, we recover the result from the previous case.

□

Corollary 9.6.10. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds. Let $\varphi : N \to M$ be an isometry. If $x \in M$ and $v \in T_xM$ are such that $\exp_x(v)$ is well defined, then:*

$$\varphi(\exp_x(v)) = \exp_{\varphi(x)}(d_x\varphi(v))$$

Proposition 9.6.11. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds, with N connected. Let $\varphi, \psi : N \to M$ be isometries. If there is some $x \in N$ such that $\varphi(x) = \psi(x)$ and $d_x\varphi = d_x\psi$, then $\varphi = \psi$.*

Proof. Consider the set $X = \{x \in N \,\big|\, \varphi(x) = \psi(x) \;\&\; d_x\varphi = d_x\psi\}$. It is open because of Corollary 9.6.10 and closed because φ and ψ are smooth. □

Examples 9.6.12. We can use Proposition 9.6.11 to determine the isometry groups of the sphere and the hyperbolic space. Indeed, in both cases, we have found a subgroup G of $\mathrm{Isom}(M,g)$ such that for all $x, y \in M$ and any linear isometry $L : T_xM \to T_xM$, there is $\varphi \in G$ such that $\varphi(x) = y$ and $d_x\varphi = L$ ($G = \mathrm{O}(n+1)$ for $\mathbb{S}^n$ and $G = \mathrm{O}_+(n,1)$ for $\mathbb{H}^n$). It follows that $G = \mathrm{Isom}(M,g)$.

Proposition 9.6.11 says that an isometry φ can be recovered from the image $\varphi(x_0)$ of a given point x_0 and the differential $d_{x_0}\varphi$. It is actually possible to use this to define charts on the isometry group, and prove that it is a Lie group.

The topology that we consider on $\mathrm{Isom}(M,g)$ is the compact-open topology: a basis is given by $\{\varphi \in \mathrm{Isom}(M,g) \,|\, \varphi(K) \subset U\}$ where $K \subset M$ is compact and $U \subset M$ is open.

Theorem 9.6.13 (Myers-Steenrod)**.** *The isometry group $\mathrm{Isom}(M,g)$ of a connected pseudo-Riemannian manifold (M,g) has a unique Lie group structure for the compact-open topology such that the action on M is smooth.*

9.6.4 Examples of geodesics

We already mentioned that geodesics of $\mathbb{R}^{p,q}$ are affinely parameterised straight lines.

is that For a pseudo-Riemannian submanifold $N \subset \mathbb{R}^{p,q}$ and a smooth curve $c : I \to N$, Proposition 9.5.2 yields:

$$\frac{D}{dt}\dot{c} = \ddot{c}^{\top}$$

It follows that c is a geodesic if and only if:

$$\forall t \in I \ \ddot{c}(t) \in T_{c(t)}N^{\perp}$$

For $x \in \mathbb{S}^d \subset \mathbb{E}^d$ and $v \in T_x\mathbb{S}^d = x^{\perp}$ with $\|v\| = 1$, the geodesic is:

$$c_v(t) = \cos(t)x + \sin(t)v$$

For $x \in \mathcal{H}^d \subset \mathbb{M}^{d+1}$ and $v \in T_x\mathcal{H}^d = x^{\perp}$ with $\|v\| = 1$, the geodesic is:

$$c_v(t) = \cosh(t)x + \sinh(t)v$$

In both cases, we find that images of geodesics are intersections with linear planes in the ambiant vector space.

We can use the isometries between the various models of the hyperbolic space to see that geodesics in the ball and hyperboloid models of $\mathbb{H}^d$ are circle arcs perpendicular to the boundary.

Finally, if $p : \widetilde{M} \to M$ is a Riemannian covering, then geodesics of M are compositions of geodesics of $\widetilde{M}$ with p.

In the space of ellipsoids $\mathcal{E}_n$, geodesics through 1_n are exactly the curves $t \mapsto \exp(tX)$ where X is a traceless symmetric matrix.

Chapter 10

Riemannian manifolds as metric spaces

10.1 The Riemannian distance

10.1.1 Lengths of curves

We now consider a **connected Riemannian manifold** (M,g). Recall that we use the convention that piece-wise smooth paths are continuous.

Definition 10.1.1. Let (M,g) be a Riemannian manifold, and $c:[a,b]\to M$ be a piece-wise smooth curve. The **length** of c is

$$L(c)=\int_a^b \|\dot{c}(t)\|\,dt$$

Definition 10.1.2. Let (M,g) be a Riemannian manifold, and $c:[a,b]\to M$ be a piece-wise smooth curve. We say that c has **constant speed** if $\|\dot{c}\|$ is constant.

Remark. Geodesics have constant speed.

We can use the lengths of curves to define a distance.

Definition 10.1.3. Let (M,g) be a connected Riemannian manifold. The **Riemannian distance** of (M,g) is $d:M\times M\to[0,+\infty[$ given by

$$d(x,y)=\inf\{L(c)\,\big|\,c:[a,b]\to M \text{ piece-wise smooth, } c(a)=x,\ c(b)=y\}$$

Remark. Since M is connected, it is also path connected, so $d(x,y)\in[0,+\infty[$ is well defined (understand: it is finite).

Staring at this definition won't get you very far. The space of curves joining two given points is infinite dimensional in its nature, so minimising a functional on this space is by no means easy.

Before we attempt general methods and subtle definitions, let us work out the case of the Euclidean space.

Theorem 10.1.4. *The Riemannian distance of the Euclidean space $\mathbb{E}^n$ is equal to the Euclidean distance.*

Remark. This can be summarized by the fact that the shortest path joining two points is the straight line.

Proof. Let d be the Riemannian distance on $\mathbb{E}^n$, and $x, y \in \mathbb{E}^n$ such that $x \neq y$. Consider the straight line

$$\gamma : \begin{cases} [0,1] & \to & \mathbb{E}^n \\ t & \mapsto & x + t(y-x) \end{cases}$$

joining x and y. Since $L(\gamma) = \|x-y\|$, we get $d(x,y) \leq \|x-y\|$.

Now let $c : [0,1] \to \mathbb{E}^n$ be a piecewise smooth curve such that $c(0) = x$ and $c(1) = y$. Decompose $c(t) = a(t) + b(t)$ where $a(t)$ is on the line joining x and y, and $b(t)$ is orthogonal to it (i.e. $a(t) = x + \left\langle c(t) - x \middle| \frac{y-x}{\|y-x\|} \right\rangle \frac{y-x}{\|y-x\|}$ and $b(t) = c(t) - a(t)$).

Since $\dot{a}(t)$ and $\dot{b}(t)$ are orthogonal, we have that $\|\dot{c}(t)\| \geq \|\dot{a}(t)\|$, hence $L(c) \geq L(a)$.

Now write $a(t) = x + \lambda(t)(y-x)$, so that

$$L(a) = \int_0^1 |\dot{\lambda}(t)| \left\|y - x\right\| dt \geq \left| \int_0^1 \dot{\lambda}(t) dt \right| \left\|x - y\right\| = \left\|y - x\right\|$$

It follows that $L(c) \geq L(a) = \left\|y - x\right\|$, hence $d(x,y) \geq \left\|y - x\right\|$. $\square$

We will keep this result in mind when we prove that the Riemannian distance is always a distance. The idea is that locally, one can compare the Riemannian metric to a Euclidean metric in charts.

Lemma 10.1.5. *Let (M,g) be a Riemannian manifold, and d^g the Riemannian distance.*

1. *If h is another Riemannian metric on M and $g \geq h$, then $d^g \geq d^h$ where d^h is the Riemannian distance of (M,h).*
2. *If $U \subset M$ is open, then $d^U(x,y) \geq d^g(x,y)$ for all $x, y \in U$, where d^U is the Riemannian distance of $(U, g|_U)$.*

Remark. By $g \geq h$ for Riemannian metrics, we mean that $g_x(v,v) \geq h_x(v,v)$ for all $x \in M$ and $v \in T_xM$.

Proof.

1. If $c : I \to M$ is a piece-wise smooth curve, then we let $L^g(c)$ (resp. $L^h(c)$) be its length with respect to g (resp. h). Since $g_{c(t)}(\dot{c}(t), \dot{c}(t)) \geq h_{c(t)}(\dot{c}(t), \dot{c}(t))$ for all $t \in I$, we find that $L^g(c) \leq L^h(c)$, therefore $L^g(c) \geq d^h(x,y)$ if c joins x and y, and finally $d^g(x,y) \geq d^h(x,y)$.

2. Given $x, y \in U$, a curve in U joining x and y is also a curve in M joining x and y, i.e. $d^U(x,y)$ is the infimum of a subset contained in the one defining $d^g(x,y)$, therefore $d^U(x,y) \geq d^g(x,y)$.

□

We can use this principle to prove that the Riemannian distance is a distance.

Theorem 10.1.6. *Let (M,g) be a connected Riemannian manifold. The Riemannian distance d is a distance on M that defines the manifold topology.*

Proof. One easily checks that d is well defined, non negative, symmetric, and that it satisfies the triangle inequality (this is why we work with piecewise smooth curves: they are stable under concatenation).

Let $x, y \in M$ be such that $x \neq y$. Consider local coordinates $(x^1, \ldots, x^d)$ on an open domain $U \subset M$ such that $g_{i,j}(x) = \delta_{i,j}$ (e.g. normal coordinates) and $y \notin U$. To simplify notations, assume that $U \subset \mathbb{R}^d$.

Since the functions $g_{i,j}$ are continuous, we can shrink U and assume the existence of $a > 0$ such that:

$$\forall z \in U \ \forall v \in T_z U \ \ \frac{1}{a^2} \|v\|^2_{eucl} \leq g_z(v,v) \leq a^2 \|v\|^2_{eucl}$$

Let $\varepsilon > 0$ be such that $B_{eucl}(x,\varepsilon) \subset U$. For $z \in B_{eucl}(x,\varepsilon)$, consider a piecewise smooth path $c : [0,1] \to M$ joining x and z. If $c([0,1]) \subset U$, then the first point of Lemma 10.1.5 implies that $L(c) \geq \frac{\|z-x\|_{eucl}}{a}$. If the path c leaves U, consider t_∂ the smallest parameter such that $c(t_\partial) \in \partial B_{eucl}(x,\varepsilon)$, then the restriction of c to $[0, t_\partial]$ is a path contained in U joining x to $c(t_\partial)$, and it is shorter than c, hence $L(c) \geq \frac{\varepsilon}{a} \geq \frac{\|z-x\|_{eucl}}{a}$.

This shows all for $z \in B_{eucl}(x,\varepsilon)$ satisfy $d(x,z) \geq \frac{\|z-x\|_{eucl}}{a}$.

Since any continuous curve from x to y must cross $\partial B_{eucl}(x,\varepsilon)$, we also find that $d(x,y) \geq \frac{\varepsilon}{a} > 0$, therefore $d(x,y) \neq 0$, and d is a distance.

The fact that $d(x,z) \geq \frac{\|z-x\|_{eucl}}{a}$ for all $z \in B_{eucl}(x,\varepsilon)$ shows that $B_d(x,r) \subset B_{eucl}(x, ar)$ for all $r \leq a\varepsilon$.

The first point of Lemma 10.1.5 and the fact that $g \leq a g_{eucl}$ imply that $B_{eucl}(x,r) \subset B_d(x, ar)$ for all $r \leq \varepsilon$. It follows that d defines the manifold topology. □

10.1.2 Minimising curves

The computation of the Riemannian distance for the Euclidean space is based on the fact that we can find an explicit formula for the shortest path between two points. In other terms, the infimum defining the Riemannian distance is a minimum.

This will not always be the case. Consider $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric. Considering paths that take arbitrarily small detours around the origin, we see that the Riemannian distance is still equal to the Euclidean distance, however given two opposite points, there is no shortest curve in $\mathbb{R}^2 \setminus \{0\}$. This problem will be avoided by working locally.

Definition 10.1.7. Let (M, g) be a connected Riemannian manifold, and d the Riemannian distance.

A piece-wise smooth curve $c : [a, b] \to M$ is called **minimising** if $L(c) = d(c(a), c(b))$.

It will be practical to consider minimising curves defined on infinite intervals, so we need a definition that does not involve the endpoints. Notice that the notion of minimising curve is stable under restrictions.

Lemma 10.1.8. *Let (M, g) be a connected Riemannian manifold, d the Riemannian distance, and $c : [a, b] \to M$ a minimising curve. For all $[a', b'] \subset [a, b]$, the restriction $c|_{[a',b']}$ is minimising.*

Proof. We already have $d(c(a'), c(b')) \leq L\left(c|_{[a',b']}\right)$.

Consider a piece-wise smooth curve $\gamma : [a', b'] \to M$ such that $\gamma(a') = c(a')$ and $\gamma(b') = c(b')$. Let $\widetilde{\gamma} : [a, b] \to M$ be the piece-wise smooth curve defined by $\widetilde{\gamma}(t) = c(t)$ if $t \in [a, a']$ or $t \in [b', b]$ and $\widetilde{\gamma}(t) = \gamma(t)$ for $t \in [a', b']$. Since $\widetilde{\gamma}$ is a piece-wise smooth curve joining $c(a)$ and $c(b)$, we have that $L(\widetilde{\gamma}) \geq d(c(a), c(b)) = L(c)$.

By writing out the integral that defines the length, we find that $L(\gamma) \geq L\left(c|_{[a',b']}\right)$. Therefore $c|_{[a',b']}$ is minimising. $\square$

If $I \subset \mathbb{R}$ is any interval and $c : I \to M$ is a curve, where M is a manifold, we say that c is **piece-wise smooth** if its restriction to any compact interval is piece-wise smooth.

Definition 10.1.9. Let (M, g) be a Riemannian manifold, and d the Riemannian distance.

If $I \subset \mathbb{R}$ is any interval, then a piece-wise smooth curve $c : I \to M$ is called **minimising** if

$$\forall s, t \in I \quad L(c|_{[s,t]}) = d(c(s), c(t))$$

It is **locally minimising** if

$$\forall t \in I \quad \exists \varepsilon > 0 \quad c|_{[t-\varepsilon, t+\varepsilon]} \text{ is minimising}$$

The two definitions of minimising curves coincide for a compact interval. Minimising curves are locally minimising, but we will see that the converse is not true.

10.1.3 Riemannian spherical coordinates

We now wish to find out some more about the relationship between Riemannian geodesics and the Riemannian distance. In the Euclidean space, we used an orthogonal projection to find the Riemannian distance, which is somehow related to Cartesian coordinates. However, these coordinates are not well defined in Riemannian geometry. We can however define some spherical coordinates, and they will be very useful.

Theorem 10.1.10 (Gauß Lemma)**.**
Let (M,g) be a Riemannian manifold, and $x \in M$. If $\exp_x$ is defined at $v \in T_xM$, then:

$$\forall w \in T_xM \quad \left\langle d_v \exp_x(v) \middle| d_x \exp_x(w) \right\rangle_{\exp_x(v)} = \langle v|w\rangle_x$$

Proof. Given $\varepsilon > 0$ small enough, consider:

$$f : \left\{ \begin{array}{ccc}]-\varepsilon, 1+\varepsilon[\times]-\varepsilon, \varepsilon[& \to & M \\ (t,s) & \mapsto & \exp_x(tv + stw) \end{array} \right.$$

We find that $\frac{\partial f}{\partial t}(1,0) = d_v \exp_x(v)$ and $\frac{\partial f}{\partial s}(1,0) = d_v \exp_x(w)$, so we wish to compute $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle$ at $(t,s) = (1,0)$.

Note that for s fixed, $f(\cdot, s)$ is a geodesic, so $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$.

$$\begin{aligned} \frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle &= \underbrace{\left\langle \frac{D}{\partial t}\frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle}_{=0} + \left\langle \frac{\partial f}{\partial t} \middle| \frac{D}{\partial t}\frac{\partial f}{\partial s} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial t} \middle| \frac{D}{\partial s}\frac{\partial f}{\partial t} \right\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle \end{aligned}$$

Since $f(\cdot, s)$ is a geodesic, it has constant speed, i.e. $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle$ does not depend on t. Since $\frac{\partial f}{\partial t}(0,s) = v + sw$, we find $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle = \|v + sw\|^2$, and:

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle = \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle = \langle v|w\rangle + s\|w\|^2$$

Integrating yields:

$$\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (1,s) = \left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (0,s) + \langle v|w\rangle + s\|w\|^2$$

Since $\frac{\partial f}{\partial s}(0,s) = 0$, we finally get:

$$\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (1,0) = \langle v|w\rangle$$

□

If w is a multiple of v then this is a just a consequence of the fact that the geodesic $t \mapsto \exp_x(tv)$ has constant speed. So the relevant information in Theorem 10.1.10 is that the differential of the exponential map $\exp_x$ preserves orthogonality with geodesics going through x.

This has a nice interpretation in terms of spherical coordinates.

Proposition 10.1.11. *Let (M,g) be a Riemannian manifold, and $x \in M$. Define*

$$\Phi : \begin{cases}]0,\mathrm{inj}_x[\times T_x^1 M & \to & M \\ (r,v) & \mapsto & \exp_x(rv) \end{cases}$$

There is a smooth family of Riemannian metrics $(h(r))_{r\in]0,\mathrm{inj}_x[}$ on the sphere $T_x^1 M$ such that $(\Phi^* g)_{r,v} = dr^2 + h(r)_v$.

Remark. By smooth family of Riemannian metrics, we mean that each $h(r)$ is a Riemannian metric on $T_x^1 M$, and the map $(r,v) \mapsto h(r)_v$ is a smooth from $]0,\mathrm{inj}_x[\times T_x^1 M$ to the total space of the vector bundle $S^2 T^*(T_x^1 M)$. In human language, this means that the expressions in coordinates are smooth functions in (r,v).

Proof. Since $T_{(r,v)}\left(]0,\mathrm{inj}_x[\times T_x^1 M\right) = \mathbb{R} \times T_v T_x^1 M$, any Riemannian metric on $]0,\mathrm{inj}_x[\times T_x^1 M$ can be written as

$$\alpha(r,v)dr^2 + dr \otimes \omega(r)_v + h(r)_v$$

Where $\omega(r)$ is a smooth family of 1-forms on $T_x^1 M$ and $h(r)$ is a smooth family of Riemannian metrics.

First, we have that $\alpha(r,v) = \left\| d_{(r,v)}\Phi(1,0) \right\|^2 = \|\dot{c}_v(r)\|^2$ where c_v is the geodesic satisfying $\dot{c}_v(0) = v$. It follows that $\alpha(r,v) = \|v\|^2 = 1$.

Since any $w \in T_v T_x^1 M$ satisfies $\langle v | w \rangle_x = 0$, the Gauss Lemma yields:

$$\omega(r)_v(w) = \left\langle d_{(r,v)}\Phi(0,w) \middle| d_{(r,v)}\Phi(1,0) \right\rangle = \left\langle d_{rv}\exp_x(rw) \middle| d_{rv}\exp_x(rv) \right\rangle = 0$$

□

Examples 10.1.12. In the following examples, the metric $h(r)$ can be computed easily.

$\mathbb{E}^2$	$ds^2 = dr^2 + r^2 d\theta^2$	$(r > 0)$
$\mathbb{S}^2$	$ds^2_{sph} = dr^2 + \sin^2 r d\theta^2$	$(0 < r < \pi)$
$\mathbb{H}^2$	$ds^2_{hyp} = dr^2 + \sinh^2 r d\theta^2$	$(r > 0)$

10.1.4 Shortest paths and geodesics

Theorem 10.1.13. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a piece-wise smooth curve. Then c is a locally minimising if and only if c is a reparametrization of a geodesic.*

It is a consequence of the following more precise statement.

Proposition 10.1.14. *Let (M,g) be a Riemannian manifold. Let $x \in M$, $v \in T_x^1 M$, and let c be the geodesic with initial velocity v. If $t \in]0, \mathrm{inj}_x[$, then:*

1. *The curve $c|_{[0,t]}$ is minimising.*
2. *Any minimising curve joining x and $\exp_x(tv)$ is a reparametrization of $c|_{[0,t]}$.*
3. *$B_M(x,t) = \exp_x\left(B_{T_xM}(0,t)\right)$.*

Proof. Let $\gamma : [a,b] \to M$ be a piece-wise smooth curve such that $\gamma(a) = x$ et $\gamma(b) = \exp_x(tv)$. Let b' be the first time at which γ exits $\exp_x\left(B_{T_xM}(0,t)\right)$, i.e. $b' = \inf\left\{s \in [a,b] \,\middle|\, c(s) \notin \exp_x\left(B_{T_xM}(0,t)\right)\right\} \in]a,b]$.

For $s \in [a,b']$, we set $r(s) = \left\|\exp_x^{-1}(c(s))\right\|$. Using the Riemannian spherical coordinates of Proposition 10.1.11, we can write $c(s) = \exp_x(r(s)u(s))$ for $s \in [a,b']$ where $u(s) \in T_x^1 M$. We find:

$$\left\|\dot\gamma\right\|^2 = \dot r^2 + h(r)_u(\dot u, \dot u)$$

It follows that $\left\|\dot\gamma(s)\right\| \geq |\dot r(s)|$ for all $s \in [a,b']$. We can use this estimate the length of γ:

$$\begin{aligned}
L(\gamma) &= \int_a^b \left\|\dot\gamma(s)\right\| ds \\
&\geq \int_a^{b'} \left\|\dot\gamma(s)\right\| ds \\
&\geq \int_a^{b'} |\dot r(s)| ds \\
&\geq \left|\int_a^{b'} \dot r(s) ds\right| = |r(b') - r(a)| = t
\end{aligned}$$

This shows that $L(\gamma) \geq L\left(c|_{[0,t]}\right)$, i.e. the geodesic $c|_{[0,t]}$ is minimising. If γ is also minimising, then every inequality that we used is an equality. In particular, we have that $\int_{b'}^b \left\|\dot\gamma(s)\right\| ds = 0$, which shows that $b' = b$, and $h(r)_u(\dot u, \dot u) = 0$, i.e. u is constant, however $\gamma(b) = \exp_x(v)$ implies that $u(s) =$

v for all s, hence $\gamma(s) = \exp_x(r(s)v) = c(r(s))$, and γ is a reparametrization of $c|_{[0,t]}$.

Since $c|_{[0,t]}$ is minimising, we find that $d(x, \exp_x(tv)) = t$. This being true for all $t \in]0, \mathrm{inj}_x[$ and $v \in T_x^1 M$, we find $\exp_x\left(B_{T_xM}(0,t)\right) \subset B_M(x,t)$. Since we have shown that every piece-wise smooth curve starting from x and leaving $\exp_x\left(B_{T_xM}(0,t)\right)$ has length at least t, we also find that $B_M(x,t) \subset \exp_x\left(B_{T_xM}(0,t)\right)$.

□

Proposition 10.1.15. *Let (M,g) be a Riemannian manifold. Every $x_0 \in M$ has a neighbourhood U such that:*

1. *Any $x, y \in U$ are joined by a unique unit-speed minimising geodesic $c_{x,y}$.*
2. *There is $\varepsilon > 0$ such that $c_{x,y}$ is defined on $]-\varepsilon, \varepsilon[$ for all $x, y \in U$, and the map $\left\{ \begin{array}{ccc} U \times U \times]-\varepsilon, \varepsilon[& \to & M \\ (x,y,t) & \mapsto & c_{x,y}(t) \end{array} \right.$ is smooth.*
3. *If $x, y, z \in U$ and $d(x,y) + d(y,z) = d(x,z)$, then $y = c_{x,z}(d(x,y))$.*
4. *If $c : I \to U$ satisfies $d(c(t), c(s)) = |t-s|$ for all $t, s \in I$, then c is a unit speed geodesic.*

Proof. Using the lower semi-continuity of the injectivity radius, we can find an open set $V \subset M$ containing x_0 such that for all $x \in V$, we have $V \subset B(x, \mathrm{inj}_x)$.

As seen for connections, we can find an open set $U \subset V$ containing x_0, and a smooth map $\varphi : U \times U \to TM$ such that $\varphi(x,y) \in T_xM$, $\varphi(x,x) = 0$ and $\exp_x(\varphi(x,y)) = y$ for all $x, y \in U$. Now $c_{x,y}(t) = \exp_x(t\varphi(x,y))$ is a smooth function of (x,y,t).

Proposition 10.1.14 implies the fist point because $U \subset V$, and the remark above implies the second point.

Now let $x, y, z \in U$ and $d(x,y) + d(y,z) = d(x,z)$. The concatenation of $c_{x,y}$ and $c_{y,z}$ is a minimising curve from x to z, so it must be a geodesic, hence $y = c_{x,z}(d(x,y))$.

Finally, if a curve $c : I \to U$ satifies $d(c(t), c(s)) = |t-s|$ for all $t, s \in I$, the previous point shows that $c(t) = c_{c(a),c(b)}(t-a)$ for $t \in [a,b] \subset I$. □

Lemma 10.1.16. *Let (M,g) be a Riemannian manifold, and let $x \in M$. For $v, w \in T_xM$, we have:*

$$d(\exp_x(tv), \exp_x(tw)) = t\|v-w\|_x + o(t)$$

Proof. Consider a neighbourhood $U \subset M$ given by Proposition 10.1.15, and the smooth map $\varphi : U \times U \to TM$ used in its proof.

Let $W = \exp_x^{-1}(U)$, and consider the function $F : W \times W \to \mathbb{R}$ defined by $F(u,v) = d(\exp_x(u), \exp_x(v))^2$. Then $F(u,v) = \left\|\varphi(\exp_x(u), \exp_x(v))\right\|^2_{\exp_x(u)}$, which shows that F is smooth.

Using the fact that $F(u,v) = F(v,u)$, $F(0,v) = \|v\|_x^2$ and $F(u,u) = 0$, we can recover the first and second order differentials of F at $(0,0)$. We find $d_{(0,0)}F = 0$ and:

$$d^2_{(0,0)}F((u,v),(z,w)) = 2g_x(u-v, z-w)$$

This leads to:

$$d(\exp_x(tv), \exp_x(tw))^2 = t^2\|v-w\|_x^2 + o(t^2)$$

□

10.1.5 Recovering a Riemannian metric from the distance

Proposition 10.1.17. *Let M be a connected manifold, and g, g' Riemannian metrics on M. If the Riemannian distances d_g and $d_{g'}$ are equal, then $g = g'$.*

Proof. At first we only consider the metric g. Let $x \in M$, and define $f : M \to \mathbb{R}$ by $f(y) = \frac{1}{2}d_g(x,y)^2$. By Proposition 10.1.14, we find that $f(\exp_x(v)) = \frac{1}{2}\|v\|^2$ for $v \in T_xM$ small enough. It follows that f is smooth in a neighbourhood of x, that $d_xf = 0$ and that $d_x^2 f = g_x$ (where $d_x^2 f$ is the Hessian of a function at a critical point).

The same being true for g', we find that $g_x = g'_x$. □

Proposition 10.1.18. *Let (M,g) be a connected Riemannian manifold, and d the Riemannian distance. If $f : M \to M$ is an isometry of the metric space (M,d), then it is an isometry of the Riemannian manifold (M,g).*

Remark. A Riemannian isometry is quite clearly an isometry of the Riemannian distance, but this is not true for local isometries. There is however an inequality for the general case of isometric immersions: if $f : (N,h) \to (M,g)$ is an isometric immersion between connected Riemannian manifolds, then it is 1-Lipschitz:

$$\forall x,y \in N \quad d_g(f(x), f(y)) \leq d_h(x,y).$$

Proof. If f is smooth, then the Riemannian distance of f^*g is equal to the Riemannian distance of g, and Proposition 10.1.17 implies that $f \in \mathrm{Isom}(M,g)$. So it remains to show that f is smooth.

Let $x \in M$. For all $v \in T_xM$, the image of the geodesic $t \mapsto \exp_x(tv)$ under f is a geodesic because of the last point in Proposition 10.1.15, so there is

$L(v) \in T_{f(x)}M$ such that $f(\exp_x(tv)) = \exp_{f(x)}(tL(v))$ for t small enough. The map $L : T_xM \to T_{f(x)}M$ fixes 0, and it satisfies:

$$\begin{aligned} \|L(u) - L(v)\|_{f(x)} &= \lim_{t\to 0} \frac{d(f(\exp_x(tu)), f(\exp_x(tv)))}{t} \\ &= \lim_{t\to 0} \frac{d(\exp_x(tu), \exp_x(tv))}{t} \\ &= \|u - v\|_x \end{aligned}$$

It follows that L is a linear isometry. Since $f(\exp_x(v)) = \exp_{f(x)}(L(v))$ for v small enough and $\exp_x$ is a local diffeomorphism at x, we find that f is smooth around x. □

This implies that $\mathrm{Isom}(M,g)$ is closed in $\mathrm{Homeo}(M)$. This is also true for pseudo-Riemannian manifolds, but considerably more difficult.

If M is compact, then so is $\mathrm{Isom}(M,g)$ (by Ascoli's Theorem). Any compact Lie group acting on a manifold preserves a Riemannian metric.

Proposition 10.1.19. *Let $G \curvearrowright M$ be a smooth action of a compact Lie group on a manifold. There is a Riemannian metric on M for which G acts isometrically.*

Proof. We use a left-invariant volume form ω on G and any Riemannian metric h on M to define:

$$H_x(u,v) = \int_G h_{gx}(d_xg(u), d_xg(v))\, d\omega(g)$$

Then H is a Riemannian metric on M, invariant under the action of G. □

10.1.6 Closed geodesics

Corollary 10.1.20. *Let (M,g) be a compact Riemannian manifold. Any non trivial free homotopy class contains a closed geodesic.*

Remark. The statement is false for non compact manifolds.

Let us sketch the proof. Since M is compact, we know that $r = \mathrm{inj}M > 0$.

Let $\mathcal{C} \subset \mathcal{C}^0([0,1], M)$ be a non trivial free homotopy class. It is a closed subset of $\mathcal{C}^0([0,1], M)$. Let $\mathcal{C}' \subset \mathcal{C}$ be the subset of piecewise $\mathcal{C}^1$ paths.

For $c \in \mathcal{C}'$, we can find a piecewise geodesic $\gamma \in \mathcal{C}'$, with at most $\frac{L(c)}{r}$ pieces, such that $L(\gamma) \le L(c)$. Set:

$$L = \inf\{L(c) \,\big|\, c \in \mathcal{C}'\}$$

Let us prove that $L > 0$. If not, we could find a sequence of paths (c_k) in $\mathcal{C}'$ such that $L(c_k) \to 0$. Since M is compact, up to a considering a subsequence this means that c_k converges to a constant path, which must be in $\mathcal{C}$ because it is closed. This contradicts the non triviality of $\mathcal{C}$.

Consider a sequence $c_k \in \mathcal{C}'$ such that $L(c_k) \to L$. Using the above remark, we can assume that c_k is piecewise geodesic with at most $\frac{L+1}{r}$ pieces.

Using Ascoli's Theorem, we can assume that c_k converges in the space $\mathcal{C}^0([0,1],M)$ to some path $c \in \mathcal{C}$. We find that c is a geodesic.

10.2 Geodesics and calculus of variations

10.2.1 Energy and the variational approach to geodesics

The fact that minimising the length is invariant under a change of parameter is a major technical issue. It means that if we find an equation describing minimising curves, then this equation must have an infinite dimensional space of solutions (a physicist might say that the group of diffeomorphisms of the interval act as gauge transformations, and infinite dimensional gauge is something that one should stay away from).

A first clue towards finding a way around this problem is to consider only curves with constant speed. Indeed, any regular curve (i.e. with non vanishing velocity) can be reparametrized so that it has constant speed, and this reparametrization is unique.

The appropriate solution consists in finding a functional on curves that is minimised by length-minimising curves with constant speed. This functional is the energy.

Definition 10.2.1. Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ be a piece-wise smooth curve. The **energy** of c is

$$E(c) = \frac{1}{2}\int_a^b \|\dot{c}(t)\|^2 \, dt$$

Remark. The energy still makes sense in pseudo-Riemannian manifolds, but the length does not (artificially defining it with an absolute value before the square root can be useful but only if we restrict the study to subspaces of curves).

There is a simple inequality between the length and the energy of a curve, and the equality case is only achieved by curves with constant speed.

Lemma 10.2.2. *Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ be a piece-wise smooth curve. Then $L(c)^2 \leq 2(b-a)E(c)$, and equality holds if and only if c has constant speed.*

Proof. It is a consequence of the Cauchy-Bunyakovsky-Schwarz inequality, and its equality case. □

We can define energy-minimising curves in a similar fashion as for the length.

Definition 10.2.3. Let (M,g) be a connected Riemannian manifold, and d the Riemannian distance.

A piece-wise smooth curve $c : [a,b] \to M$ is called **energy-minimising** if any other piece-wise smooth curve $\gamma : [a,b] \to M$ such that $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$ satisfies $E(\gamma) \geq E(c)$.

Note that considering curves defined on the same interval is vital in this definition, since the energy changes when rescaling to a different interval.

We still have the same properties for restrictions.

Lemma 10.2.4. *Let (M,g) be a connected Riemannian manifold, d the Riemannian distance, and $c : [a,b] \to M$ an energy-minimising curve. For all $[a',b'] \subset [a,b]$, the restriction $c|_{[a',b']}$ is energy-minimising.*

Proof. The proof of Lemma 10.1.8 can be carried out mutatis mutandis. $\square$

This also allows for a notion of energy-minimising curves defined on an arbitrary interval.

Definition 10.2.5. Let (M,g) be a Riemannian manifold, and d the Riemannian distance.

If $I \subset \mathbb{R}$ is any interval, then a piece-wise smooth curve $c : I \to M$ is called **energy-minimising** if for all $s,t \in I$, the restriction $c|_{[s,t]}$ is energy-minimising.

It is **locally energy-minimising** if

$$\forall t \in I \;\; \exists \varepsilon > 0 \;\; c|_{[t-\varepsilon,t+\varepsilon]} \text{ is energy-minimising}$$

Proposition 10.2.6. *Let (M,g) be a Riemannian manifold, and let $c : I \to M$ be a piece-wise smooth curve. Then c is energy-minimising if and only if it is minimising and has constant speed.*

Similarly, it is locally energy-minimising if and only if it is locally minimising and has constant speed.

Proof. Note that the global statement implies the local one, since constant speed is a local property. Without loss of generality we can assume that I is a compact interval $[a,b]$.

First assume that c is minimising and has constant speed. If $\gamma : [a,b] \to M$ is a piece-wise smooth curve such that $\gamma(a) = c(a)$ and $\gamma(b)$, then Lemma 10.2.2 gives $E(\gamma) \geq \frac{L(\gamma)^2}{2(b-a)}$ Since c is minimising, it follows that $E(\gamma) \geq \frac{L(c)^2}{2(b-a)}$. However c has constant speed, so Lemma 10.2.2 implies that $\frac{L(c)^2}{2(b-a)} = E(c)$, hence $E(\gamma) \geq E(c)$, and c is energy-minimising.

Now assume that c is energy-minimising. Let $\widetilde{c} : [a,b] \to M$ be the constant speed reparametrization of c. By Lemma 10.2.2, we get $E(c) \geq \frac{L(c)^2}{2(b-a)} = \frac{L(\widetilde{c})^2}{2(b-a)} = E(\widetilde{c})$. Since c is energy-minimising, these are equalities, i.e. $E(c) = \frac{L(c)^2}{2(b-a)}$, and according to Lemma 10.2.2 c has constant speed.

Let $\gamma : [a,b] \to M$ be a piece-wise smooth curve such that $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$, and let $\widetilde{\gamma}$ be the constant speed reparametrisation of γ. Lemma 10.2.2 and the invariance of the length under reparametrisations each applied twice give:

$$L(\gamma) = L(\widetilde{\gamma}) = \sqrt{2(b-a)E(\widetilde{g})} \geq \sqrt{2(b-a)E(c)} = L(c)$$

It follows that c is minimising. □

10.2.2 The first variation formula

Since we are looking for minimisers of the energy functional, a possible approach is to try to find its critical points, then compute the second order derivative at these points and try to evaluate its sign. For now we will just focus on the first derivative.

It is possible to formalize this in terms of functions on infinite dimensional manifolds of paths, but we will not go down this path.

Instead, we will work with variations of curves.

Definition 10.2.7. Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ a smooth curve.

A **variation** of c is a smooth map $f : [a,b] \times]-\varepsilon,\varepsilon[\to M$ for some $\varepsilon > 0$ such that, if $c_s : I \to M$ is the curve defined by $c_s(t) = f(t,s)$, then $c_0 = c$.

It has **fixed endpoints** if $c_s(a) = c(a)$ and $c_s(b) = c(b)$ for all $s \in]-\varepsilon,\varepsilon[$.

The **variation field** of f the the vector field J along c defined by $J(t) = \frac{\partial}{\partial s}\Big|_{s=0} c_s(t) \in T_{c(t)}M$.

Remarks.

- We consider smooth curves in order to avoid complicated definitions of piece-wise smooth functions of two variables.
- If a variation f has fixed endpoints, then the variation field J satisfies $J(a) = 0$ and $J(b) = 0$.
- One can show that any vector field along c is the variation field of a variation of c, and that the variation can be chosen with fixed endpoints if the vector field vanishes at the endpoints of c. However, it will not be necessary for our applications.

Theorem 10.2.8 (First variation formula for the energy)**.**
Let (M,g) be a Riemannian manifold, $c : [a,b] \to M$ a smooth curve, $f : [a,b] \times]-\varepsilon,\varepsilon[\to M$ a variation of c, and J its variation field. We have:

$$\frac{d}{ds}\Big|_{s=0} E(c_s) = -\int_a^b g_{c(t)}\Big(J(t), \frac{D}{dt}\dot{c}(t)\Big)dt + g_{c(b)}(\dot{c}(b), J(b)) - g_{c(a)}(\dot{c}(a), J(a))$$

Remark. If f has fixed endpoints, then the formula simplifies to:

$$\left.\frac{d}{ds}\right|_{s=0} E(c_s) = -\int_a^b g_{c(t)}\left(J(t), \frac{D}{dt}\dot{c}(t)\right) dt$$

Proof. Since we are considering smooth functions on a compact interval, we can differentiate before integrating:

$$\frac{d}{ds}E(c_s) = \frac{1}{2}\int_a^b \frac{\partial}{\partial s} g_{c_s(t)}(\dot{c}_s(t), \dot{c}_s(t))\, dt$$

We can compute the integrand:

$$\begin{aligned}
\frac{\partial}{\partial s} g_{c_s(t)}(\dot{c}_s(t), \dot{c}_s(t)) &= \frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) \\
&= 2g\left(\frac{\partial f}{\partial t}, \frac{D}{\partial s}\frac{\partial f}{\partial t}\right) \\
&= 2g\left(\frac{\partial f}{\partial t}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\right) \\
&= 2\frac{\partial}{\partial t} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) - 2g\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)
\end{aligned}$$

At $s = 0$, this simplifies as:

$$\left.\frac{\partial}{\partial s}\right|_{s=0} g_{c_s(t)}(\dot{c}_s(t), \dot{c}_s(t)) = 2\frac{d}{dt} g(\dot{c}, J) - 2g\left(\frac{D}{dt}\dot{c}, J\right)$$

Integration yields the desired formula. □

We little effort, one can show that critical points of the energy must satisfy $\frac{D}{dt}\dot{c} = 0$, i.e. be geodesics.

A similar formula can be obtained for the variation of the length, but it is only practical if we assume the curve c to have constant speed.

Theorem 10.2.9 (First variation formula for the length)**.**
Let (M, g) be a Riemannian manifold, $c : [a, b] \to M$ a smooth curve with constant speed, $f : [a, b] \times]-\varepsilon, \varepsilon[\to M$ a variation of c with fixed endpoints, and J its variation field. We have:

$$\left.\frac{d}{ds}\right|_{s=0} L(c_s) = -\sqrt{\frac{b-a}{L(c)}} \int_a^b g_{c(t)}\left(J(t), \frac{D}{dt}\dot{c}(t)\right) dt$$

Proof. We proceed in the same way as we did for Theorem 10.2.8, and find that:

$$\frac{d}{ds}L(c_s) = \int_a^b \frac{\partial}{\partial s}\left[g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)^{\frac{1}{2}}\right] dt$$

Since c has constant speed, we get $g(\dot{c},\dot{c}) = \frac{L(c)}{b-a}$. Using the computations made in the proof of Theorem 10.2.8, we find:

$$\frac{\partial}{\partial s}\bigg|_{s=0}\left[g\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial t}\right)^{\frac{1}{2}}\right] = \sqrt{\frac{b-a}{L(c)}}\left(\frac{d}{dt}g(\dot{c},J) - g\left(\frac{D}{dt}\dot{c},J\right)\right)$$

Integration once again yields the desired formula. □

10.3 Completeness of Riemannian manifolds

10.3.1 The Hopf-Rinow Theorem

Definition 10.3.1. A Riemannian manifold (M,g) is called **geodesically complete** if any geodesic $c: I \to M$ extends to $\mathbb{R}$.

It is called **geodesically connected** if any pair of points in M is linked by a minimising geodesic.

Examples: $\mathbb{E}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ are geodesically complete and connected, but $\mathbb{R}^n \setminus \{0\}$ is neither.

Geodesic completeness and connectedness are related.

Lemma 10.3.2. *Let (M,g) be a Riemannian manifold, and consider two distinct points $x,y \in M$. If $r > 0$ satisfies $r < \mathrm{inj}_x$ and $r < d(x,y)$, then there is $z \in S(x,r)$ such that*

$$d(x,z) + d(z,y) = d(x,y)$$

Proof. Since $r < \mathrm{inj}_x$, Proposition 10.1.14, shows that:

$$S(x,r) = \exp_x\left(S_{T_xM}(0,r)\right)$$

It follows that $S(x,r)$ is compact, and we can find $z \in S(x,r)$ such that:

$$\forall p \in S(x,r) \quad d(y,p) \geq d(y,z)$$

Let $\gamma: [0,1] \to M$ be a piece-wise smooth curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Since $d(\gamma(0),x) = 0$ and $d(\gamma(1),x) > r$, we can consider $t \in [0,1]$ such that $\gamma(t) \in S(x,r)$. We find:

$$\begin{aligned}
L(\gamma) &\geq L\left(\gamma|_{[0,t]}\right) + L\left(\gamma|_{[t,1]}\right) \\
&\geq d(x,\gamma(t)) + d(y,\gamma(t)) \\
&\geq r + d(z,y) \\
&\geq d(x,z) + d(z,y)
\end{aligned}$$

Since the right term does not depend on γ, we find:

$$d(x,y) \geq d(x,z) + d(z,y)$$

The triangle inequality yields $d(x,z)+d(z,y)\geq d(x,y)$, hence $d(x,y)=d(x,z)+d(z,y)$.

□

Lemma 10.3.3. *Let (M,g) be a connected Riemannian manifold. Let $x\in M$ be such that $\exp_x$ is defined on all T_xM. For all $y\in M$, there is a minimising geodesic from x to y.*

Remark. This has the important consequence that for all $R>0$:

$$B_{d_g}(x,R)=\exp_x\Big(B_{T_xM}(0,R)\Big) \text{ and } \overline{B}_{d_g}(x,R)=\exp_x\Big(\overline{B}_{T_xM}(0,R)\Big).$$

Proof. Let $y\in M$ be distinct from x, and consider $r\in\,]0,\min(\mathrm{inj}_x,d(x,y))[$. Let $z_0\in S(x,r)$ be given by Lemma 10.3.2, and consider $v\in T^1_xM$ be such that $\exp_x(rv)=z_0$. Set:

$$I=\{t\in[0,d(x,y)]\,\Big|\,d(y,c_v(t))+t=d(x,y)\}$$

Since $0\in I$, this set is not empty. Let $s=\sup I$. Note that S is closed, so $s\in I$. Assume, by contradiction, that $s<d(x,y)$.

Write $x_1=c_v(s)$. Note that $d(x,x_1)\leq L\Big(c_v|_{[0,s]}\Big)=s$, and $d(x,x_1)\geq d(x,y)-d(x_1,y)=s$, so $d(x,x_1)=s$. In particular $x_1\neq y$.

Let $r_1\in\,]0,\max(\mathrm{inj}_{x_1},d(x_1,y)[$, so that we can apply Lemma 10.3.2 and find $z_1\in S(x_1,r_1)$ such that $d(x_1,z_1)+d(z_1,y)=d(x_1,y)$.

$$\begin{aligned} d(x,z_1)&\geq d(x,y)-d(z_1,y)\\ &\geq d(x,x_1)+d(x_1,y)-(d(x_1,y)-d(x_1,z_1))\\ &\geq d(x,x_1)+d(x_1,z_1)\end{aligned}$$

It follows that the concatenation of $c_v|_{[0,s]}$ and the unit speed minimising geodesic joining x_1 and z_1 is a minimising curve, hence a geodesic. Therefore $z_1=c_v(s+r_1)$, and $s+r_1\in I$. This is a contradiction with $s=\sup I$. □

Theorem 10.3.4 (Hopf-Rinow)**.** *Let (M,g) be a connected Riemannian manifold, ∇ its Levi-Civita connection and d the Riemannian distance. The following assertions are equivalent.*

1. *(M,g) is geodesically complete.*
2. *$\forall x\in M$ $\exp_x$ is defined on T_xM.*
3. *$\exists x\in M$ $\exp_x$ is defined on T_xM.*
4. *$\forall R>0$ $\forall x\in M$ $\overline{B}(x,R)$ is compact.*
5. *$\exists R>0$ $\forall x\in M$ $\overline{B}(x,R)$ is compact.*

6. (M,d) is complete.

Furthermore, if there conditions are satisfied then (M,g) is geodesically connected.

Remark. We will say that (M,g) is **complete** if it satisfies these conditions.

Proof. The last statement is a consequence of the second condition and Lemma 10.3.3.

1.⇒2. is a matter of definitions and *2.⇒3.* is just specification.

3.⇒4. Lemma 10.3.3 implies that $\overline{B}(x,R) = \exp_x\left(\overline{B}_{T_xM}(0,R)\right)$, hence the compactness.

4.⇒5. and *5.⇒6.* are general fact for metric spaces.

6.⇒1. Let $c :]a,b[\to M$ be a geodesic. Without loss of generality, we can assume that $\|\dot{c}\| = 1$ and $0 \in]a,b[$.

If $b < +\infty$, consider a sequence (t_k) such that $t_k \to b$.

Since c is 1-Lispchitz, the sequence $(c(t_k))$ is Cauchy, therefore converges to some $y \in M$. Let $r = \frac{\mathrm{inj}_y}{2}$.

For k large enough, we find both $\mathrm{inj}_{c(t_k)} > r$ (which implies that c is defined on $]a, t_k + r[$) and $b - t_k < r$, so c can be extended. □

Corollary 10.3.5. *Let (M,g) be a complete Riemannian manifold, and d the Riemannian distance. If the metric space (M,d) is bounded, then M is compact.*

Proof. Let $x \in M$ and $R > 0$ be such that $M = \overline{B}(x,R)$. The Hopf-Rinow Theorem implies that $\overline{B}(x,R)$ is compact, and so is M. □

Corollary 10.3.6. *Let (M,g) be a connected Riemannian manifold. If M is compact, then (M,g) is geodesically complete and geodesically connected.*

This is not true for general pseudo-Riemannian manifolds: the Clifton-Pohl torus is a compact Lorentzian manifold yet is not geodesically complete.

Proposition 10.3.7. *Let (M,g) be a Riemannian manifold. There exists $\sigma \in C^\infty(M)$ such that the conformal metric $e^\sigma g$ is complete.*

10.3.2 Riemannian coverings

Proposition 10.3.8. *Let (M,g) and (N,h) be connected Riemannian manifolds, and $f : M \to N$ a local isometry.*

Then (M,g) is complete if and only if f is a Riemannian covering and (N,h) is complete.

Proof. First assume that (N,h) is complete and that f is a Riemannian covering. Let $x \in M$ and $v \in T_xM$. The geodesic $c_{d_xf(v)}$ of N is defined on $\mathbb{R}$. Since f is a covering map, consider the lift $c : \mathbb{R} \to M$ of $c_{d_xf(v)}$ such that

$c(0) = x$. It is a geodesic because of Proposition 9.6.9, and $\dot{c}(0) = v$, i.e. $c = c_v$ is defined on $\mathbb{R}$.

Now assume that (M,g) is complete. Let us start by showing that f is onto, using the connectedness of N. Since f is a local diffeomorphism, $f(M)$ is open in N. The surjectivity of f will follow from the fact that $f(M)$ is also closed.

For this, let us prove that there is no geodesic of N joining a point in $f(M)$ and a point in its complement. Assume by contradiction that there is a geodesic $c : I \to N$ such that $c(0) \in f(M)$ and $c(1) \notin f(M)$. Let $x \in M$ be such that $f(x) = c(0)$ and set $v = d_x f^{-1}(\dot{c}(0))$. The geodesic c_v is defined on $\mathbb{R}$ because (M,g) is complete, and $f \circ c_v$ is a geodesic of (N,h) with the same initial data as c, hence $f \circ c_v = c$ and $c(1) = f(c_v(1)) \in f(M)$, which is a contradiction.

This shows that $f(M)$ is closed: if $y \in N \setminus f(M)$, then $B_{d_h}(y, \mathrm{inj}_y) \cap f(M) = \emptyset$. It also shows that geodesics of N going through $f(M)$ are complete, so (N,h) is complete.

It remains to show that f is a covering map. Let $x \in N$, then set $r = \mathrm{inj}_x$ and $U = B_{d_h}(x,r)$. For $\widehat{x} \in f^{-1}(\{x\})$, consider the set $U_{\widehat{x}} = B_{d_g}(\widehat{x}, r)$. Let us prove that:

$$f^{-1}(U) = \bigcup_{\widehat{x} \in f^{-1}(\{x\})} U_{\widehat{x}}.$$

If $y \in f^{-1}(U)$, we can consider $v \in T_x N$ such that $f(y) = c_v(1)$ and $\|v\| < r$. Let $w = -\dot{c}_v(1) \in T_{f(y)}N$ (so that $x = c_w(1)$) and $\widehat{w} = d_y f^{-1}(w) \in T_y N$. Then $\widehat{x} = c_{\widehat{w}}(1) \in f^{-1}(\{x\})$ (because $f \circ c_{\widehat{w}} = c_w$) and the path $c_{\widehat{w}}|_{[0,1]}$ has length $\|v\| < r$, therefore $y \in U_{\widehat{x}}$.

If $y \in U_{\widehat{x}}$ for some $\widehat{x} \in f^{-1}(\{x\})$, recall that the local isometry f is 1-Lipschitz, so $d_h(f(y), x) \leq d_g(y, \widehat{x}) < r$, and $y \in f^{-1}(U)$.

We now fix $\widehat{x} \in f^{-1}(\{x\})$ and wish to show that f induces a diffeomorphism from $U_{\widehat{x}}$ to U. Since f is a local isometry, hence a local diffeomorphism, we only have to check that its restriction to $U_{\widehat{x}}$ is injective, and that $f(U_{\widehat{x}}) = U$.

The injectivity of $f|_{U_{\widehat{x}}}$ follows from the fact that $U_{\widehat{x}} = \exp_{\widehat{x}}\Big(B_{T_{\widehat{x}}M}(0,r)\Big)$ (because (M,g) is complete), the identity $f \circ \exp_{\widehat{x}} = \exp_x \circ d_x f$ and the injectivity of $\exp_x$ on $B_{T_x M}(0,r)$ (recall that $r = \mathrm{inj}_x$).

If $x \neq x'$, let us prove by contradiction that $U_x \cap U_{x'} = \emptyset$. Consider $z \in U_x \cap U_{x'}$, and let $v, v' \in B_{T_x M}(0,\varepsilon)$ be such that $z = \exp_x(v) = \exp_x(v')$. The geodesics $f \circ c_v$ and $f \circ c_{v'}$ join $f(z)$ to y and $d(f(z), y) < \mathrm{inj}_y$, so they must be equal. Consequently $\dot{c}_v(1) = \dot{c}_{v'}(1)$, and $c_v = c_{v'}$, hence $x = c_v(0) = c_{v'}(0) = x'$.

We have already seen that $U_{\widehat{x}} \subset f^{-1}(U)$, hence $f(U_{\widehat{x}}) \subset U$. Now let $y \in U$ and write $y = c_v(1)$ for $v \in T_x M$ such that $\|v\| < r$. Then $y = f(c_{\widehat{v}}(1))$ where $\widehat{v} = d_{\widehat{x}} f^{-1}(v)$, hence $y \in f(U_{\widehat{x}})$.

Finally, consider two distinct points $\widehat{x_1}, \widehat{x_2} \in f^{-1}(\{x\})$. We wish to show that $U_{\widehat{x_1}} \cap U_{\widehat{x_2}} = \emptyset$.

Let $z \in U_{\widehat{x_1}} \cap U_{\widehat{x_2}}$, and write $z = c_{v_1}(1) = c_{v_2}(1)$ where $v_i \in T_{\widehat{x_i}}M$ and $\|v_i\| < r$ for $i = 1,2$. Then $f \circ c_{v_1}$ and $f \circ c_{v_2}$ are both geodesics in N joining x and $f(z)$ that are included in $B(x,r)$. Since $r = \mathrm{inj}_x$, it follows that these geodesics are equal, hence $\widehat{x_1} = \widehat{x_2}$.

□

Proposition 10.3.9. *Let (M,g) and (N,h) be connected Riemannian manifolds of the same dimension. If (M,g) is complete and $f : M \to N$ is a smooth map such that $f^*h \geq g$, then f is a covering map.*

Proof. Since $f^*h \geq g$, we see that f^*h is a Riemannian metric on M. Now the inequality $f^*h \geq g$ integrates to an inequality on the Riemannian distances:

$$\forall x,y \in M \quad d_{f^*h}(x,y) \geq d_g(x,y)$$

This means that for $x \in M$ and $r > 0$, we have an inclusion of closed balls:

$$\overline{B}_{f^*h}(x,r) \subset \overline{B}_g(x,r)$$

The Hopf-Rinow Theorem assures that $\overline{B}_g(x,r)$ is compact, and so is $\overline{B}_{f^*h}(x,r)$. According to the Hopf-Rinow Theorem, the Riemannian manifold (M, f^*h) is complete. Since f is a local isometry from (M, f^*h) to (N,h) (because they have the same dimension), we can apply Prposition 10.3.8 and find that f is a covering map. □

10.3.3 Completeness and vector fields

The classical property of finite time explosion for ODEs generalizes to complete Riemannian manifolds.

Proposition 10.3.10. *Let (M,g) be a complete Riemannian manifold, and let $X \in \mathcal{X}(M)$. For $x \in M$, let $I \subset \mathbb{R}$ be maximal the interval on which the flow line $\left(\varphi_X^t(x)\right)_{t\in I}$ is defined. If $t_0 = \sup I$ is finite, then*

$$\limsup_{t\to t_0} \left\| X\left(\varphi_X^t(x)\right)\right\|_{\varphi_X^t(x)} = +\infty$$

Proof. Assume the contrary. Then for any sequence $t_k \in I$ with $t_k \to t_0$, we find that $\varphi_X^{t_k}(x)$ is a Cauchy sequence for the Riemannian distance, so it must converge to some $y \in M$ because (M,g) is complete. Then the flow line starting at y extends the flow line of x on a larger interval, hence the contradiction. □

Corollary 10.3.11. *Let (M,g) be a complete Riemannian manifold. If $X \in \mathcal{X}(M)$ is bounded for the metric g, then it is complete.*

Chapter 11

Pseudo-Riemannian curvature

11.1 The various notions of curvature

11.1.1 Symmetries and contractions of the curvature tensor

Definition 11.1.1. Let (M,g) be a pseudo-Riemannian manifold, and ∇ the Levi-Civita connection. The curvature field of ∇ is called the **Riemann tensor** of type (3,1) of (M,g). It is denoted by $R \in \Omega^2(\mathrm{End}(TM))$.

Let us list its symmetries.

Proposition 11.1.2. *Let (M,g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection, and R its type (3,1) Riemann tensor.*

For all $x \in M$ and $u,v,w,z \in T_xM$, we have the following symmetries:

1. $R_x(u,v)w = -R_x(v,u)w$.

2. $R_x(u,v)w + R_x(v,w)u + R_x(w,u)v = 0$.

3. $g_x(R_x(u,v)w,z) = -g_x(R_x(u,v)z,w)$.

4. $g_x(R_x(u,v)w,z) = g_x(R_x(w,z)u,v)$.

Remark. Property *3.* can be written as $R_x(u,v) \in \mathfrak{so}(g_x)$.

Proof. Property *1.* is a consequence of the skew-symmetry of the curvature, i.e. $R \in \Omega^2(\mathrm{End}(TM))$.

Property *2.* is the first Bianchi identity.

In order to prove property *3.* we consider a smooth function $f : \mathbb{R}^2 \to M$ such that $f(0) = x$, $\frac{\partial f}{\partial t}(0) = u$ and $\frac{\partial f}{\partial s}(0) = v$. Also consider sections $\sigma, \tau \in$

$\Gamma(f^*TM)$ such that $\sigma(0) = w$ and $\tau(0) = z$. First we compute:

$$
\begin{aligned}
g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma, \tau\right) &= g\left(\left[\frac{D}{\partial t}, \frac{D}{\partial s}\right]\sigma, \tau\right) \\
&= g\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma, \tau\right) - g\left(\frac{D}{\partial s}\frac{D}{\partial t}\sigma, \tau\right) \\
&= -g\left(\frac{D}{\partial s}\sigma, \frac{D}{\partial t}\tau\right) - \frac{\partial}{\partial t} g\left(\frac{D}{\partial s}\sigma, \tau\right) \\
&\quad + g\left(\frac{D}{\partial t}\sigma, \frac{D}{\partial s}\tau\right) + \frac{\partial}{\partial s} g\left(\frac{D}{\partial t}\sigma, \tau\right)
\end{aligned}
$$

When symmetrizing in σ and τ, half of the terms disappear and the rest yields:

$$
\begin{aligned}
g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma, \tau\right) + g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\tau, \sigma\right) &= \frac{\partial}{\partial s}\left[g\left(\frac{D}{\partial t}\sigma, \tau\right) + g\left(\sigma, \frac{D}{\partial t}\tau\right)\right] \\
&\quad - \frac{\partial}{\partial t}\left[g\left(\frac{D}{\partial s}\sigma, \tau\right) + g\left(\sigma, \frac{D}{\partial s}\tau\right)\right] \\
&= \frac{\partial^2}{\partial s \partial t} g(\sigma, \tau) - \frac{\partial^2}{\partial t \partial s} g(\sigma, \tau) \\
&= 0
\end{aligned}
$$

Evaluating at 0 yields property *3*.

In order to prove property *4.*, let us use normal coordinates $(x^1, \ldots, x^d)$ around x. We then have that $g_{i,j}(x) = g^{i,j}(x) = \delta_{i,j}$ and $\partial_k g_{i,j}(x) = 0$ (also $\Gamma^k_{i,j}(x) = 0$), hence $R^l_{i,j,k}(x) = \partial_i \Gamma^l_{j,k}(x) - \partial_j \Gamma^l_{i,k}(x)$. Derivatives at x simplify a lot:

$$
\begin{aligned}
\partial_i \Gamma^l_{j,k}(x) &= \partial_i \left(\frac{1}{2}\sum_{m=1}^{d} g^{l,m}(\partial_j g_{k,m} + \partial_k g_{j,m} - \partial_m g_{j,j})\right)(x) \\
&= \frac{1}{2}(\partial^2_{i,j} g_{k,l}(x) + \partial^2_{i,k} g_{j,l}(x) - \partial^2_{i,l} g_{j,k}(x))
\end{aligned}
$$

We now get:

$$
\begin{aligned}
g_x(R_x(\partial_i, \partial_j)\partial_k, \partial_l) &= R^l_{i,j,k}(x) \\
&= \frac{1}{2}\left(\partial^2_{i,k} g_{j,l}(x) - \partial^2_{i,l} g_{j,k}(x) - \partial^2_{j,k} g_{i,l}(x) + \partial^2_{j,l} g_{i,k}(x)\right)
\end{aligned}
$$

This formula remains invariant when switching (i, j) and (k, l), which is the desired symmetry (and all other symmetries can be retrieved through this formula). $\square$

These symmetries are all encoded in the type (4,0) tensor obtained by lowering an index of the type (3,1) curvature tensor.

Definition 11.1.3. Let (M,g) be a pseudo-Riemannian manifold, ∇ its Levi-Civita connection, and R its type (3,1) Riemann tensor.

The **type (4,0) Riemann tensor** is the tensor $R \in \mathcal{T}^{4,0}(M)$ defined by:

$$\forall x \in M\ \forall u,v,w,z \in T_xM \quad R_x(u,v,w,z) = g_x(R_x(u,v)w,z)$$

We use the same letter for the type (3,1) and the type (4,0) tensor as there is very little chance of it inducing a confusion.

Proposition 11.1.4. *Let (M,g) be a pseudo-Riemannian manifold. The type (4,0) Riemann tensor R satisfies the following symmetries:*

$$\begin{aligned}\forall x \in M\ \forall u,v,w,z \in T_xM \quad R_x(u,v,w,z) &= -R_x(v,u,w,z)\\ &= -R_x(u,v,z,w)\\ &= R_x(w,z,u,v)\end{aligned}$$

Remark. These symmetries are summarized by $R \in \Gamma\left(S^2(\Lambda^2 T^*M)\right)$.

Proof. These are just concise versions of Proposition 11.1.2. □

In local coordinates, we write $R_{i,j,k,l} = R(\partial_i, \partial_j, \partial_k, \partial_l)$, and find:

$$R_{i,j,k,l} = \sum_{m=1}^{n} g_{m,l} R^m_{i,j,k}$$

Definition 11.1.5. Let (M,g) be a pseudo-Riemannian manifold. For $x \in M$ and a vector plane $P \subset T_xM$ non degenerate for g_x, if (v,w) is a vector basis of P, the **sectional curvature** $K(P)$ is defined by:

$$K(P) = \frac{R_x(v,w,w,v)}{g_x(v,v)g_x(w,w) - g_x(v,w)^2}$$

Remarks.

- It does not depend on the choice of a vector basis (v,w) of P.
- The fact that P is non degenerate for g_x is equivalent to the non vanishing of the denominator.
- If g is Riemannian, and (v,w) is an orthonormal basis of P, then $K(P) = R_x(v,w,w,v)$.
- If g is Riemannian, then planes are always non degenerate, so the sectionnal curvature can be defined as a function $K : \mathcal{G}_2(TM) \to \mathbb{R}$ where $\mathcal{G}_2(\mathbb{R})$ is the (total space of the) fibre bundle above M whose fibre over $x \in M$ is the Grassmannian $\mathcal{G}_2(T_xM)$. The function K is smooth. It follows that for a compact M, the sectional curvature is bounded.

One can show that the sectional curvature determines the Riemann tensor R (it is a consequence of the symmetries of the Riemann tensor).

Definition 11.1.6. Let (M,g) be a pseudo-Riemannian manifold, and R its type (3,1) Riemann tensor. The **Ricci curvature** of (M,g) is the type (2,0) tensor $\mathrm{Ric} \in \Gamma\left((T^*M)^{\otimes 2}\right)$ defined for $x \in M$ and $v,w \in T_xM$ by:

$$\mathrm{Ric}_x(v,w) = \mathrm{Tr}\,(z \mapsto R_x(z,v)w)$$

Proposition 11.1.7. *Let (M,g) be a pseudo-Riemannian manifold. The Ricci curvature* Ric *is symmetric, i.e.* $\forall x \in M\ \forall v,w \in T_xM \quad \mathrm{Ric}_x(v,w) = \mathrm{Ric}_x(w,v)$.

Remark. This is summarized by $\mathrm{Ric} \in \Gamma(S^2T^*M)$.

Proof. Let $(e_1,\ldots,e_d)$ be an orthonormal basis of T_xM. Then we have:

$$\mathrm{Ric}_x(v,w) = \sum_{i=1}^{d} R_x(e_i,v,w,e_i)$$

The symmetries of the type (4,0) tensor imply that Ric_x is symmetric. □

In local coordinates, we write $R_{i,j} = \mathrm{Ric}(\partial_i,\partial_j)$, and we find:

$$R_{i,j} = \sum_{k=1}^{d} R^k_{k,i,j} = \sum_{1\le k,l\le d} g^{k,l} R_{k,i,j,l}$$

Note that the Ricci curvature is of the same type as the metric, so it makes sense to compare them.

Definition 11.1.8. An **Einstein manifold** is a pseudo-Riemannian manifold (M,g) for which there is $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$.

Remarks.

- If (M,g) has constant sectional curvature equal to κ, then $\mathrm{Ric} = (n-1)\kappa g$.
- This implies that $\nabla\mathrm{Ric} = 0$, and it is almost an equivalence (in other terms, an Einstein manifold should be interpreted as having constant Ricci curvature).

Definition 11.1.9. Let (M,g) be a pseudo-Riemannian manifold, and $\mathrm{Ric} \in \Gamma(S^2T^*M)$ its Ricci curvature. The **scalar curvature** of (M,g) is the function $R = \mathrm{Scal} \in \mathcal{C}^\infty(M)$ defined by:

$$\forall x \in M \quad R(x) = \mathrm{Scal}(x) = \mathrm{Tr}_{g_x}(\mathrm{Ric}_x)$$

where Tr_{g_x} is the trace of a quadratic form with reference g_x.

Remarks.

- If $(e_i)_{1\leq i\leq d}$ is a g_x-orthonormal frame of T_xM, then
$$R(x) = \sum_{i=1}^{d} \mathrm{Ric}_x(e_i, e_i).$$
- It can also be defined as $R(x) = \mathrm{Tr}(f_x)$ where $f_x \in \mathrm{End}(T_xM)$ is the g_x-self adjoint operator such that $\mathrm{Ric}_x(v, w) = g_x(v, f_x(w))$ for all $v, w \in T_xM$.
- In local coordinates, $R = \sum_{1\leq i,j\leq d} g^{i,j} R_{i,j}$.

Einstein's equation: in the theory of General Relativity, a spacetime is represented by a 4-dimensional Lorentzian manifold (M, g) (Special Relativity corresponds to the Minkowski space $\mathbb{M}^4$). The physics of a spacetime are encoded in a type (2,0) tensor $T \in \Gamma(S^2(T^*M))$, called the stress-energy tensor, and Einstein's equation is an equation on the Lorentzian metric g:
$$\mathrm{Ric} - \frac{1}{2}R + \Lambda g = T$$
where Λ is called the cosmological constant.

Proposition 11.1.10. *Let (M, g) be a pseudo-Riemannian manifold. For $\lambda > 0$, the following tensors and functions associated to the metrics g and $\lambda^2 g$ are related in the following way:*

1. $\nabla^{\lambda^2 g} = \nabla^g$
2. $\mathrm{dvol}^{\lambda^2 g} = \lambda^n \,\mathrm{dvol}^g$
3. *If g is Riemannian, then $d_{\lambda^2 g} = \lambda d_g$.*
4. $R^{\lambda^2 g} = R^g$ *(where R is the type (3,1) Riemann tensor).*
5. $K^{\lambda^2 g} = \frac{1}{\lambda^2} K^g$
6. $\mathrm{Ric}^{\lambda^2 g} = \mathrm{Ric}^g$
7. $\mathrm{Scal}^{\lambda^2 g} = \frac{1}{\lambda^2}\mathrm{Scal}^g$.

Moreover, isometries preserve R, K, Ric and Scal.

Definition 11.1.11. We say that a Riemannian manifold (M, g) has **pinched sectional curvature** if there are $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq K(P) \leq \beta$ for every plane $P \in \mathcal{G}_2(TM)$.

We say that it has **negative pinched curvature** if there are $b < a < 0$ such that $-b^2 \leq K(P) \leq -a^2$ for every plane $P \in \mathcal{G}_2(TM)$.

The formulae for different types of curvature look quite intimidating when it comes to computing them in examples. For this, we can try to cheat and use the abundance of isometries in the three main examples.

If (M,g) is Riemannian, and $\mathrm{Isom}(M,g) \curvearrowright M$ is transitive, then (M,g) has pinched sectional curvature, and Scal is constant.

If moreover $\mathrm{Isom}(M,g) \curvearrowright \mathcal{G}_2(TM)$ is transitive, then (M,g) has constant sectional curvature.

Consequence: the Euclidean space $\mathbb{E}^n$, the sphere $(\mathbb{S}^n, g_{sph})$ and the hyperbolic space $(\mathbb{H}^n, g_{hyp})$ have constant sectional curvature (we will see that their values are respectively $0, 1$ and -1).

11.1.2 Jacobi fields as variation fields

Consider a Riemannian manifold (M,g) and a geodesic $c : I \to M$.

Reminder: A Jacobi field along c is $J : I \to TM$ such that $\forall t \in I$ $J(t) \in T_{c(t)}M$ and $\frac{D^2}{dt^2}J + R(J,\dot{c})\dot{c} = 0$.

A Jacobi field J is determined by $J(t_0)$ and $\frac{D}{dt}J(t_0)$ for a given $t_0 \in I$.

Definition 11.1.12. Let (M,g) be a Riemannian manifold, and $c : I \to M$ a geodesic. A Jacobi field J along c is called **orthogonal** if

$$\forall t \in I \quad J(t) \perp \dot{c}(t)$$

It is called **tangent** if $J(t)$ is proportional to $\dot{c}(t)$ for all $t \in I$.

Proposition 11.1.13. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a geodesic. Let J be a Jacobi field along c.*

1. *If J is orthogonal, then $\frac{D}{dt}J(t) \perp \dot{c}(t)$ for all $t \in I$.*
2. *If there is $t_0 \in I$ such that $J(t_0) \perp \dot{c}(t_0)$ and $\frac{D}{dt}J(t_0) \perp \dot{c}(t_0)$, then J is orthogonal.*

Proof. Since c is a geodesic, we have that:

$$\frac{d}{dt}g(J,\dot{c}) = g\left(\frac{D}{dt}J,\dot{c}\right) \tag{11.1}$$

If J is orthogonal, this formula shows that $\frac{D}{dt}J$ is also orthogonal to $\dot{c}$.

We also have:

$$\frac{d}{dt}g\left(\frac{D}{dt}J,\dot{c}\right) = g\left(\frac{D^2}{dt^2}J,\dot{c}\right)$$

Using the fact that J is a Jacobi field, it follows that:

$$\frac{d}{dt}g\left(\frac{D}{dt}J,\dot{c}\right) = -R(J,\dot{c},\dot{c},\dot{c}) = 0$$

In other words, $g(\frac{D}{dt}J,\dot{c})$ is constant. If it vanishes at t_0, then it vanishes everywhere. In this case, (11.1) shows that $g(J,\dot{c})$ is also constant, and if it also vanishes at t_0 then J is orthogonal. $\square$

Lemma 11.1.14. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a geodesic. Consider a function $f \in \mathcal{C}^\infty(I)$, and set $J = f\dot{c}$. Then J is a Jacobi field if and only if $\ddot{f} = 0$.*

Proof. Since $\frac{D}{dt}\dot{c} = 0$, we find $\frac{D^2}{dt^2}J = \frac{D}{dt}(\dot{f}\dot{c}) = \ddot{f}\dot{c}$. Now $R(J,\dot{c})\dot{c} = fR(\dot{c},\dot{c})\dot{c} = 0$ because of skew-symmetry. □

So tangent Jacobi field are all of the form $t \mapsto (at+b)\dot{c}(t)$ for some $a,b \in \mathbb{R}$.

Proposition 11.1.15. *Let (M,g) be a Riemannian manifold, $c : I \to M$ a geodesic and J a Jacobi field along c. Then g is tangent if and only if there is $t_0 \in I$ such that $J(t_0)$ and $\frac{D}{dt}J(t_0)$ are proportional to $\dot{c}(t_0)$.*

Proof. If J is tangent, then the computation made in Lemma 11.1.14 shows that J and $\frac{D}{dt}J$ are proportional to $\dot{J}$ everywhere.

If $\frac{D}{dt}J(t_0) = a\dot{c}(t_0)$ and $J(t_0) = (at_0+b)\dot{c}(t_0)$, then the $t \mapsto (at+b)\dot{c}(t)$ is a Jacobi field according to Lemma 11.1.14, and has the same initial condition at t_0 as J, so it must be equal to J, therefore J is tangent. □

Proposition 11.1.16. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a geodesic. If J is a Jacobi field along c, then J decomposes uniquely as $J = J^T + J^\perp$ where J^T is a tangent Jacobi field along c and $J^\perp$ is an orthogonal Jacobi field along c.*

Proof. The uniqueness comes from the fact that tangent and orthogonal Jacobi fields form vector spaces whose intersection is null.

For the existence, fix $t_0 \in I$, and decompose $\frac{D}{dt}J(t_0) = a\dot{c}(t_0) + u$ and $J(t_0) = (at_0+b)\dot{c}(t_0)+v$ where $a,b \in \mathbb{R}$ and $u,v \in \dot{c}(t_0)^\perp$. Let $J^T(t) = (at+b)\dot{c}(t)$, and let $J^\perp$ be the Jacobi field along c such that $J^\perp(t_0) = u$ and $\frac{D}{dt}J^\perp(t_0) = v$. Then $J^T + J^\perp$ is a Jacobi field along c with the same initial data at t_0 as J, therefore $J = J^T + J^\perp$.

By construction J^T is tangent, and $J^\perp$ is orthogonal thanks to Proposition 11.1.13. □

Definition 11.1.17. Let (M,g) be a Riemannian manifold, and $c : I \to M$ a smooth curve.

A **variation** of c is a smooth map $f : I \times]-\varepsilon,\varepsilon[\to M$ for some $\varepsilon > 0$ such that, if $c_s : I \to M$ is the curve defined by $c_s(t) = f(t,s)$, then $c_0 = c$

It is a **geodesic variation** if all the curves c_s are geodesics.

The **variation field** of f the the vector field J along c defined by $J(t) = \frac{\partial}{\partial s}\Big|_{s=0} c_s(t) \in T_{c(t)}M$.

Proposition 11.1.18. *Let (M,g) be a Riemannian manifold, $c : I \to M$ a geodesic, and J a vector field along c.*

Then J is a Jacobi field if and only if for every $t_0 \in I$, there is an open interval $I_0 \subset I$ with $t_0 \in I$ such that $J|_{I_0}$ is the variation field of a geodesic variation of $c|_{I_0}$.

Remark. If (M,g) is geodesically complete, then the local asumption can be removed: J is a Jacobi field if and only if it is the variation field of a geodesic variation of c.

Proof. The fact that the variation field of a geodesic variation is a Jacobi field was proved in section 7.4.

If J is a Jacobi field and $t_0 \in I$, first consider a geodesic $\gamma : [-\varepsilon, \varepsilon] \to M$ such that $\gamma(0) = c(t_0)$ and $\dot\gamma(0) = J(t_0)$.

Let $X, Y :]-\varepsilon, \varepsilon[\to TM$ be the parallel vector fields along γ such that $X(0) = \dot c(t_0)$ and $Y(0) = \frac{D}{dt}J(t_0)$.

Let $I_0 \subset I$ be an open interval such that $t_0 \in I$ and the geodesic c_s with initial condition $c_s(t_0) = \gamma(s)$ and $\dot c_s(t_0) = X(s) + sY(s)$ is defined on I_0 (note that $c_0 = c$).

Let $c_s(t) = \exp_{\gamma(s)}((t-t_0)X(s) + s(t-t_0)Y(s))$. Then $f : I_0 \times]-\varepsilon, \varepsilon[\to M$ defined by $f(t,s) = c_s(t)$ is a geodesic variation of c. It follows that $J_1 = \frac{\partial}{\partial s}\Big|_{s=0} c_s$ is a Jacobi field.

Since $c_s(t_0) = \gamma(s)$, we have that $J_1(t_0) = \dot\gamma(0) = J(0)$. We also have that $\frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial t}(t_0, s) = \dot c_s(t_0) = X(s) + sY(s)$, so $\frac{D}{\partial s}\frac{\partial f}{\partial t}(t_0, 0) = Y(0)$ (because X and Y are parallel along γ). This shows that $\frac{D}{dt}J_1(t_0) = \frac{D}{dt}J(t_0)$, so $J_1 = J$, and J is the variation field of a geodesic variation on I_0. □

11.1.3 The second variation formula

Jacobi fields also arise naturally from the variational study of geodesics.

Definition 11.1.19. Let (M,g) be a Riemannian manifold, R its type $(3,1)$ curvature tensor and $c : [a,b] \to M$ a geodesic. The bilinear form

$$I : \begin{cases} \Gamma(c^*TM) \times \Gamma(c^*TM) & \to & \mathbb{R} \\ (X,Y) & \mapsto & -\int_a^b g\left(X, \frac{D^2}{dt^2}Y + R(Y,\dot c)\dot c\right) dt \end{cases}$$

is called the **index form** of c.

Theorem 11.1.20. *Let (M,g) be a Riemannian manifold, $c : [a,b] \to M$ a geodesic, $f : [a,b] \times]-\varepsilon, \varepsilon[\to M$ a variation of c, and J its variation field. We have:*

$$\frac{d^2}{ds^2}\bigg|_{s=0} E(c_s) = I(J,J)$$

11.1.4 Jacobi fields and sectional curvature

Let (M,g) be a Riemannian manifold, $x \in M$, $P \subset T_xM$ a plane, and (v,w) vector basis of T_xM.

Let $c = c_v$ be the geodesic with initial velocity v, and J a Jacobi field along c such that $J(0) = w$.

$$\begin{aligned} K(P) &= \frac{g_x(R_x(v,w)w,v)}{g_x(v,v)g_x(w,w) - g_x(v,w)^2} \\ &= -\frac{g_x\left(\frac{D^2}{dt^2}J(0), J(0)\right)}{g_x(\dot{c}(0),\dot{c}(0))g_x(J(0),J(0)) - g_x(\dot{c}(0),J(0))^2} \end{aligned}$$

In particular, if $g_x(v,v) = 1$, $g_x(v,w) = 0$, and J is orthogonal along c, we find:

$$K\left(\mathbb{R}\dot{c}(0) \oplus \mathbb{R}J(0)\right) = -\frac{\left\langle \frac{D^2}{dt^2}J(0) \middle| J(0) \right\rangle}{\|J(0)\|^2}$$

If we know the explicit expressions of geodesics of a given Riemannian manifold, then we can compute Jacobi fields using geodesic variations, and this formula gives the sectional curvature.

11.2 Curvature and topology

11.2.1 Riemannian manifolds with constant sectional curvature

Theorem 11.2.1. *For $n \geq 2$, the sectional curvature of $(\mathbb{S}^n, g_{sph})$ is $+1$, that of $(\mathbb{H}^n, g_{hyp})$ is -1.*

Proof. For $x \in \mathbb{S}^n \subset \mathbb{E}^{n+1}$ and $v \in T_x\mathbb{S}^n = x^\perp$ such that $\|v\| = 1$, the geodesic c_v is given by:

$$c(t) = \cos t x + \sin t v$$

If $w \in T_x\mathbb{S}^n$ is such that $\|w\| = 1$ and $\langle v|w \rangle = 0$, then:

$$f(s,t) = \cos t(\cos sx + \sin sw) + \sin tv$$

is a geodesic variation of c. Therefore $J(t) = \cos tw$ is a Jacobi field along c.

Now $\frac{D^2}{dt^2}J = \ddot{J} = -J$, so $K(\mathbb{R}v \oplus \mathbb{R}w) = -\frac{\langle \ddot{J}(0)|J(0)\rangle}{\langle J(0)|J(0)\rangle} = 1$.

For the hyperbolic space $\mathbb{H}^n$, consider the hyperboloïd model $\mathcal{H}^n \subset \mathbb{M}^{n+1}$. The geodesic c_v is now given by:

$$c(t) = \cosh tx + \sinh tv$$

The geodesic variation is:

$$f(s,t) = \cosh t(\cosh sx + \sinh sw) + \sinh tv$$

The associated Jacobi field is $J(t) = \cosh tw$, it satisfies $\frac{D^2}{dt^2}J = \ddot{J} = J$, and the sectional curvature is $K = -1$. □

Theorem 11.2.2. *Let (M,g) be a Riemannian manifold of dimension n with constant sectional curvature equal to $\kappa \in \mathbb{R}$. Then every $x \in M$ has a neighbourhood U isometric to an open set of:*

$$\begin{cases} \mathbb{E}^n & \text{if } \kappa = 0 \\ (\mathbb{S}^n, \frac{1}{\kappa} g_{sph}) & \text{if } \kappa > 0 \\ (\mathbb{H}^n, -\frac{1}{\kappa} g_{hyp}) & \text{if } \kappa < 0 \end{cases}$$

Moreover, if (M,g) is complete and simply connected, then it is globally isometric to this model space.

Lemma 11.2.3. *Let (M,g) be a Riemannian manifold with constant sectional curvature equal to $\kappa \in \mathbb{R}$. For all $x \in M$, if (u,v) is an orthonormal basis of a plane $P \subset T_xM$, then $R_x(u,v)v = \kappa u$.*

Proof. Let us start with showing that $R_x(u,v)v \in \mathbb{R}.v \oplus \mathbb{R}.w$. If $\dim M = 2$, it is automatic. If $\dim M \geq 3$, we can consider $w \in T_xM$ such that $\|w\|_x = 1$ and $g_x(u,w) = g_x(v,w) = 0$.

Notice that since (u,v) and (v,w) are both orthonormal, we have:

$$R_x(u,v,v,u) = R_x(w,v,v,w) = \kappa$$

Expressing the sectional curvature of the plane generated by v and $u+w$ yields $R(u+w,v,v,u+w) = 2\kappa$.

The symmetries of the (4,0) Riemann tensor show that $R_x(w,v,v,u) = R_x(u,v,v,w)$, and multi-linearity leads to:

$$\underbrace{R_x(u+w,v,v,u+w)}_{=2\kappa} = \underbrace{R_x(u,v,v,u)}_{=\kappa} + 2R_x(u,v,v,w) + \underbrace{R_x(w,v,v,w)}_{=\kappa}$$

It follows that $R_x(u,v,v,w) = 0 = g_x(R(u,v)v,w)$. This being true for any unitary vector w orthogonal to u and w, we find that $R_x(u,v)v = \lambda u + \mu v$ for some $\lambda, \mu \in \mathbb{R}$.

Now $R_x(u,v,v,v) = 0$ yields $\mu = 0$, and $R_x(u,v,v,u) = \kappa$ yields $\lambda = \kappa$, hence the result. □

Lemma 11.2.4. *Let (M,g) be a Riemannian manifold with constant sectional curvature equal to $\kappa \in \mathbb{R}$. For $x \in M$, and $u,v \in T_xM$ such that $g_x(u,v) = 0$ and $g_x(v,v) = 1$, we let U be the parallel vector field along c_v such that $U(0) = u$, and J the Jacobi field along c_v such that $J(0) = 0$ and $\frac{D}{dt}J(0) = u$.*

For $t \in I_v$, we have $J(t) = f(t)U(t)$, where

$$f(t) = \begin{cases} t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa} t\right) & \text{if } \kappa > 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa} t\right) & \text{if } \kappa < 0 \end{cases}$$

Proof. Notice that the function f satisfies (and is determined by) $f(0) = 0$, $\dot f(0) = 1$, and $\ddot f = -\kappa f$.

Set $J_1(t) = f(t)U(t)$. Since U is parallel along c_v, we find $\frac{D}{dt}J_1 = \dot f U$, and $\frac{D^2}{dt^2}J_1 = \ddot f U = -\kappa J_1$.

According to Lemma 11.2.3, we have $R(J_1, \dot c_v)\dot c_v = \kappa J_1$. So J_1 is a Jacobi field along c_v. But $J_1(0) = 0$ and $\frac{D}{dt}J_1(0) = u$, therefore $J_1 = J$. □

Proof of Theorem 11.2.2. Multiplying g by some well chosen $\lambda > 0$ if necessary, we can assume that $\kappa = 0$, 1 or -1.
Fix some $x \in M$.

Flat case: Let us show that $\exp_x : (T_xM, g_x) \to (M, g)$ is a local isometry.

Let $v \in T_xM$ be such that $\exp_x(v)$ is well defined. Set $c(t) = \exp_x(tv)$. For $w \in T_xM$, we have $d_v \exp_x(w) = J(1)$, where J is the Jacobi field along c satisfying $J(0) = 0$ and $\frac{D}{dt}J(0) = w$. Write $w = \lambda v + u$ where $g_x(v, u) = 0$.

Lemma 11.2.4 shows that $J(1) = U(1)$ where U is the parallel vector field along c such that $U(0) = w$.

It follows that $d_v \exp_x(w) = J(1) + \lambda \dot c(1)$ is the value at 1 of a parallel vector field along c. Since the parallel transport is isometric, it follows that $d_v \exp_x$ is isometric, i.e. $\exp_x$ is a local isometry.

In the complete case, the map $\exp_x$ is local isometry from $\mathbb{E}^n$ to M, hence a Riemannian covering by Proposition 10.3.8, and an isometry if M is simply connected.

Negative curvature case:

Now Lemma 11.2.4 gives $\left\|d_{tv} \exp_x(tw)\right\| = \sinh t$ when $g_x(v, w) = 0$.

Consider some $\widetilde{x} \in \mathbb{H}^n$. We know that $\exp_{\widetilde{x}} = T_{\widetilde{x}}\mathbb{H}^n \to \mathbb{H}^n$ is a diffeomorphism.

Fix a linear isometry $f : T_{\widetilde{x}}\mathbb{H}^n \to T_xM$. Just as in the flat case, Lemma 11.2.4 shows that $\exp_{\widetilde{x}}^* g_{hyp} = (\exp_x \circ f)^* g$.

The map $F = \exp_x \circ f \circ \exp_{\widetilde{x}}^{-1}$ is an isometry (local or global depending on the hypothesis) with $\mathbb{H}^n$.

Positive curvature case:

Now Lemma 11.2.4 gives $\left\|d_{tv} \exp_x(tw)\right\| = \sin t$ when $g_x(v, w) = 0$.

The construction of a local isometry works exactly as in the negative curvature case.

In the complete and simply connected case, we have to be more careful. Using the inverse of the exponential map, we build a local isometry $F : \mathbb{S}^n \setminus \{-\widetilde{x}\} \to M$.

Fix some $\widetilde{y} \in \mathbb{S}^n \setminus \{\widetilde{x}, -\widetilde{x}\}$. Consider $y = F(\widetilde{y})$, and the same construction as for F gives a local isometry $G : \mathbb{S}^n \setminus \{-\widetilde{y}\}$ such that $G(\widetilde{y}) = y$ and $d_{\widetilde{y}}G = d_{\widetilde{y}}F$.

Since $\mathbb{S}^n \setminus \{\pm\widetilde{x}, \pm\widetilde{y}\}$ is connected, It follows from Proposition 9.6.11 that $F = G$ on this subset, and we have built a local isometry $\mathbb{S}^n \to M$, which is an isometry because M is simply connected. □

Consequence: If a Riemannian manifold (M,g) has constant sectional curvature equal to κ, then (M,g) is isometric to a quotient $\mathbb{X}^n_\kappa/\Gamma$ where $\Gamma \subset \mathrm{Isom}(\mathbb{X}^n_\kappa)$ is isomorphic to $\pi_1(M)$.

All surfaces posses a Riemannian metric with constant sectional curvature. This is not true for higher dimensional manifolds (e.g. $\mathbb{S}^2 \times \mathbb{S}^1$).

Theorem 11.2.5 (Poincaré-Koebe Uniformisation Theorem)**.**
Let (M,g) be a two-dimensional Riemannian manifolds. There is a constant curvature metric g' conformal to g. It is unique up to homothety.

Case $\kappa = 0$: A Riemannian manifold (M,g) with constant sectional curvature equal to 0 is called **flat**.

Theorem 11.2.6 (Bieberbach Theorem)**.**
If $\mathbb{E}^n/\Gamma$ is compact and orientable, then $\Gamma \cap \mathbb{R}^n$ is the group generated by n linearly independent translations, and has finite index in G.

Flat compact surfaces are the torus $\mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}^2$ and the Klein bottle $\mathbb{E}^2/\Gamma$ where Γ is the group generated by $(x,y) \mapsto (x+1,y)$ and $(x,y) \mapsto (1-x, y+1)$.

Case $\kappa = 1$: A Riemannian manifold with constant sectional curvature equal to $+1$ is called **spherical** (note that it is not the negation of aspherical...).

A complete spherical manifold is compact. Compact spherical surfaces are the sphere $\mathbb{S}^2$ and the projective plane $\mathbb{RP}^2$.

Case $\kappa = -1$: A Riemannian manifold with constant sectional curvature equal to -1 is called **hyperbolic**.

Hyperbolic manifolds are plentiful (e.g. all the remaining compact surfaces), their study is still an active area of research.

11.2.2 The topology of non positively curved Riemannian manifolds

Given $\kappa \in \mathbb{R}$, we define the function $f_\kappa : \mathbb{R} \to \mathbb{R}$ by:

$$f_\kappa(t) = \begin{cases} t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa} t\right) & \text{if } \kappa > 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa} t\right) & \text{if } \kappa < 0 \end{cases}$$

It is the solution of the ODE $\ddot{y}+\kappa y=0$ with initial conditions $f_\kappa(0)=0$ and $\dot{f}_\kappa(0)=1$.

Recall that if J is a Jacobi field along a unit speed geodesic in a Riemannian manifold with constant sectional curvature equal to κ, and $J(0)=0$, then $\|J(t)\| = \left\|\frac{D}{dt}J(0)\right\| f_\kappa(t)$ for $t \geq 0$ (and $t < \frac{\pi}{\sqrt{\kappa}}$ when $\kappa > 0$).

Theorem 11.2.7 (Rauch Comparison Theorem)**.** *Let (M,g) be a Riemannian manifold with sectional curvature bounded from above by $\kappa_0 \in \mathbb{R}$.*

If $c : I \to M$ is a unit speed geodesic, and $J : I \to TM$ is an orthogonal Jacobi field along c such that $J(0)=0$, then $\|J(t)\| \geq \left\|\frac{D}{dt}J(0)\right\| f_{\kappa_0}(t)$ for $t \in I$ (and $t < \frac{\pi}{\sqrt{\kappa_0}}$ when $\kappa_0 > 0$).

Proof. Assume that $\left\|\frac{D}{dt}J(0)\right\| = 1$ (which is always possible unless $\frac{D}{dt}J(0)=0$, in which case $J=0$ and the result is straightforward).

The strategy consists in showing that the function $\frac{\|J\|}{f_{\kappa_0}}$ is non decreasing. For this to be useful, we need to know its value at $t=0$, which is a limit since $f_{\kappa_0}(0)=0$. For this, set $v(t) = g_{c(t)}(J(t),J(t))$. Then $\dot{v} = 2\left\langle J \middle| \frac{D}{dt}J\right\rangle$, hence $\ddot{v} = 2\left\|\frac{D}{dt}J\right\|^2 + 2\left\langle J \middle| \frac{D^2}{dt^2}J\right\rangle$. It follows that $\dot{v}(0)=0$ and $\ddot{v}(0)=2$, so $v(t) \sim t^2$ as $t \to 0$, and $\|J(t)\| \sim |t|$. This leads to:

$$\lim_{t\to 0, t>0} \frac{\|J(t)\|}{f_{\kappa_0}(t)} = 1$$

Set $A = \{t > 0 \,|\, J(t) \neq 0\}$. We find that A contains some interval $]0,\varepsilon[$. For $t \in A$, we can differentiate $\|J\|$:

$$\frac{d}{dt}\|J\| = \frac{\left\langle J \middle| \frac{D}{dt}J\right\rangle}{\|J\|}$$

Let us differentiate once more:

$$\begin{aligned}
\frac{d^2}{dt^2}\|J\| &= \frac{\left\langle \frac{D}{dt}J \middle| \frac{D}{dt}J\right\rangle + \left\langle J \middle| \frac{D^2}{dt^2}J\right\rangle}{\|J\|} - \frac{\left\langle J \middle| \frac{D}{dt}J\right\rangle^2}{\|J\|^3} \\
&= \frac{\left\|\frac{D}{dt}J\right\|^2 \|J\|^2 - \left\langle J \middle| \frac{D}{dt}J\right\rangle^2}{\|J\|^3} - \frac{\langle R(J,\dot{c})\dot{c}|J\rangle}{\|J\|} \\
&\geq -\frac{\langle R(J,\dot{c})\dot{c}|J\rangle}{\|J\|} \\
&\geq -\kappa_0 \|J\|
\end{aligned}$$

Hence $\frac{d^2}{dt^2}\|J\| \geq -\kappa_0\|J\|$ on A. We wish to compare $\|J\|$ and f_{κ_0}. We have:

$$\frac{d}{dt}\left(\frac{\|J\|}{f_{\kappa_0}}\right) = \frac{f_{\kappa_0}\frac{d}{dt}\|J\| - \|J\|\frac{d}{dt}f_{\kappa_0}}{f_{\kappa_0}^2}$$

Since $\|J(0)\| = f_{\kappa_0}(0)$ and $\frac{d}{dt}\big|_{t=0}\|J\| = \frac{d}{dt}\big|_{t=0} f_{\kappa_0}$, we find $f_{\kappa_0}(0)\,\frac{d}{dt}\big|_{t=0}\|J\| - \|J(0)\|\,\frac{d}{dt}\big|_{t=0} f_{\kappa_0} = 0$.

$$\frac{d}{dt}\left(f_{\kappa_0}\frac{d}{dt}\|J\| - \|J\|\frac{d}{dt}f_{\kappa_0}\right) = f_{\kappa_0}\left(\frac{d^2}{dt^2}\|J\| + \kappa_0\|J\|\right) \geq 0$$

It follows that $\frac{\|J\|}{f_{\kappa_0}}$ is non-decreasing on A, hence $A = I \cap \mathbb{R}_{>0}$ and $\|J\| \geq k_{\kappa_0}$, which is the desired result. □

Definition 11.2.8. Let (M, g) be a Riemannian manifold. Two points $x, y \in M$ are called **conjugate** if there is a geodesic $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$, and a non trivial Jacobi field J along c such that $J(0) = 0$ and $J(1) = 0$.

Proposition 11.2.9. *Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Then M has no pairs of conjugate points.*

Proof. If c is a geodesic and J a Jacobi field along c such that $J(0) = 0$ and $J(1) = 0$, the Rauch Comparison Theorem yields $\frac{D}{dt}J(0) = 0$, hence $J = 0$. □

Theorem 11.2.10 (Cartan-Hadamard Theorem)**.**
Let (M, g) be a connected complete Riemannian manifold. If the sectional curvature is non-positive, then for all $x \in M$, the map $\exp_x : T_xM \to M$ is a covering map.

In particular, its universal cover is diffeomorphic to $\mathbb{R}^d$ (where $d = \dim M$).

Proof. By completeness, $\exp_x$ is defined on all T_xM. Recall (Proposition 7.4.8) that for $v, w \in T_xM$, the differential $d_v \exp_x(w)$ is equal to $J(1)$ where J is the Jacobi field along the geodesic c_v satisfying $J(0) = 0$ and $\frac{D}{dt}J(0) = w$.

The Rauch Comparison Theorem show that $\left\|d_v \exp_x(w)\right\|_{\exp_x(v)} \geq \|w\|_x$. This means that $\exp_x^* g \geq g_x$. It follows from Proposition 10.3.9 that $\exp_x$ is a covering map. □

Consequence: if M is simply connected and $\kappa \leq 0$, then two points are linked by a unique geodesic.

11.2.3 The topology of positively curved Riemannian manifolds

Theorem 11.2.11 (Myers)**.**
Let (M, g) be a complete Riemannian manifold of dimension d. If there is $r > 0$ such that:

$$\mathrm{Ric} \geq \frac{d-1}{r^2} g$$

Then $\operatorname{diam} M \leq \pi r$. *In particular, M is compact and $\pi_1(M)$ is finite.*

Remarks.

- Under these conditions, $\operatorname{diam} M = \pi r$ if and only if (M,g) is isometric to $\mathbb{S}^n$ (Cheng).

- If $\kappa \geq \frac{1}{r^2}$, then $\mathrm{Ric} \geq \frac{d-1}{r^2} g$.

This result can appear weaker than the Cartan-Hadamard Theorem, since it does not determine the topology of M. The reason for this is that positively curved simply connected Riemannian manifolds can have different topologies (e.g. $\mathbb{S}^n$ and $\mathbb{CP}^n$). There is no topological characterisation of simply connected manifolds admitting a Riemannian metric with positive curvature. In particular, it is still unknown whether $\mathbb{S}^2 \times \mathbb{S}^2$ admits such a metric (this problem is often referred to as Hopf's conjecture).

In even dimension, the Synge Theorem asserts that an oriented complete Riemannian manifold with positive sectional curvature is simply connected (this is not true in odd dimensions, as show the Lens spaces, quotients of $\mathbb{S}^3$ by finite cyclic groups).

To obtain that M is covered by $\mathbb{S}^n$, we need to add a condition on the curvature. The story starts in 1926 with a conjecture of Hopf stating that a simply connected Riemannian manifold with sectional curvature close enough to 1 should be homeomorphic to a sphere. This was first proved in 1951 by Rauch: if the sectional curvature κ of a complete simply connected Riemannian manifold (M,g) satifies $\frac{3}{4} \leq \kappa \leq 1$, then M is homeomorphic to a sphere. The optimal constant was found in 1961, a result of Berger (heavily relying on the work of Klingenberg) states that if the sectional curvature κ satisfies $\frac{1}{4} < \kappa \leq 1$, then M is homeomorphic to a sphere. Berger also showed that if it satisfies $\frac{1}{4} \leq \kappa \leq 1$ but M is not homeomorphic to a sphere, then (M,g) is isometric to a standard Riemannian metric on a projective space $\mathbb{CP}^n$, $\mathbb{HP}^n$ or $\mathbb{OP}^2$.

The question of differentiability in the Sphere Theorem stayed open for many years after the work of Berger. A first version with non optimal pinching constants was obtained by Gromoll and Calabi in 1966. The final version was proved in Brendle and Schoen in 2007: if (M,g) is a complete simply connected Riemannian manifold with sectional curvature κ satisfying $\frac{1}{4} < \kappa \leq 1$, then M is diffeomorphic to a sphere. Note that there are examples of manifolds that are homeomorphic to a sphere $\mathbb{S}^n$ but not diffeomorphic to $\mathbb{S}^n$ (e.g. for $n = 7$), known as exotic spheres (for $n = 4$, we still don't know whether there are exotic spheres). Gromoll and Meyer exhibited in 1974 an exotic sphere with a positively curved Riemannian metric.

The main tool used by Brendle and Schoen is the Ricci flow, which is famous for being used by Perelman in his proof of the Poincaré conjecture.

11.3 The geometry of non positively curved Riemannian manifolds

Definition 11.3.1. A **Cartan-Hadamard manifold** is a simply connected complete Riemannian manifold of non positive sectional curvature.

We have seen that if (M,g) is a Cartan-Hadamard manifold and $x \in M$, then $\exp_x$ is a diffeomorphism. For $x,p,q \in M$, we can define the angle $\sphericalangle_x(p,q)$ to be $\sphericalangle(u,v)$ where $p = \exp_x(u)$ and $q = \exp_x(v)$.

Lemma 11.3.2. *Let (M,g) be a Cartan-Hadamard manifold, $x \in M$ and $u,v \in T_xM$. We have:*

$$d(\exp_x(u), \exp_x(v)) \geq \|u-v\|_x$$

Moreover, the map $t \mapsto d(c_u(t), c_v(t))$ is non decreasing.

Proof. Consider the geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = \exp_x(u)$ and $\gamma(1) = \exp_x(v)$. Let $c(s) = \exp_x^{-1}(\gamma(s)) \in T_xM$.

Since c is a curve in T_xM joining u and v, its length is at least $\|u-v\|_x$. The Rauch Comparison Theorem implies that $\left\|d_w \exp_x(z)\right\|_{\exp_x(w)} \geq \|z\|_x$ for all $w,z \in T_xM$. So we get:

$$\begin{aligned} d(\exp_x(u), \exp_x(v)) &= \int_0^1 \left\|\dot{\gamma}(s)\right\|_{\gamma(s)} ds \\ &\geq \int_0^1 \|\dot{c}(s)\|_x \, ds \\ &\geq \|u-v\|_x \end{aligned}$$

Now consider the geodesic variation $f(t,s) = \exp_x(tc(s))$. The Jacobi field $\frac{\partial f}{\partial s}$ vanishes at $t = 0$, so it follows from the last point of the Rauch Comparison Theorem that for every $s \in [0,1]$, the map $t \mapsto \left\|\frac{\partial f}{\partial s}(t,s)\right\|$ is non decreasing.

For $t \in [0,1]$, the curve $s \mapsto f(s,t)$ joins $\exp_x(tu)$ and $\exp_x(tv)$, so we find:

$$\begin{aligned} d(\exp_x(tu), \exp_x(tv)) &\leq \int_0^1 \left\|\frac{\partial f}{\partial s}(t,s)\right\| ds \\ &\leq \int_0^1 \left\|\frac{\partial f}{\partial s}(1,s)\right\| ds \\ &\leq L(\gamma) \\ &\leq d(\exp_x(u), \exp_x(v)) \end{aligned}$$

This shows that the map $t \mapsto d(\exp_x(tu), \exp_x(tv))$ is non decreasing on $[0,1]$. A simple re-scaling shows that it is non decreasing on $\mathbb{R}_{\geq 0}$. □

Lemma 11.3.3. *Let (M,g) be a Cartan-Hadamard manifold, and let $x,y,z \in M$. Set $a = d(x,z)$, $b = d(y,z)$, $c = d(x,y)$ and $\gamma = \sphericalangle_z(x,y)$. We have the following relation:*

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma$$

Proof. Consider $u,v \in T_zM$ such that $x = \exp_z(u)$ and $y = \exp_z(v)$.

$$\begin{aligned} c^2 &= d(\exp_z(u), \exp_z(v))^2 \\ &\geq \|u-v\|_x^2 = \|u\|_x^2 + \|v\|_x^2 - 2\langle u|v\rangle_x \\ &\geq a^2 + b^2 - 2ab\cos(\gamma) \end{aligned}$$

□

Proposition 11.3.4. *Let (M,g) be a Cartan-Hadamard manifold, and let $S \subset M$ be a non empty bounded subset. There is a unique closed ball of minimal radius containing S.*

Proof. Set $E = \left\{(x,r) \in M \times \mathbb{R}^+ \,\middle|\, S \subset \overline{B}(x,r)\right\}$ and $R = \inf\{r \,|\, \exists x \in M \ (x,r) \in E\}$.

Consider a sequence $(x_k, r_k) \in E$ such that $r_k \to R$. Let us show that (x_k) is a Cauchy sequence.

Let $\varepsilon > 0$, and let $k_0 > 0$ be such that: $\forall k \geq k_0 \ r_k^2 \leq R^2 + \varepsilon$. For $k,l \geq k_0$ and $p \in S$, we let q be the middle point of the geodesic segment joining x_k and x_l, hence

$$\sphericalangle_q(p,x_k) + \sphericalangle_q(p,x_l) = \pi$$

Up to exchanging k and l, we can assume that

$$\cos\sphericalangle_q(p,x_k) \leq 0$$

We now get

$$\begin{aligned} R^2 + \varepsilon &\geq r_k^2 \\ &\geq d(x_k,p)^2 \\ &\geq d(x_k,q)^2 + d(p,q)^2 = \frac{d(x_k,x_l)^2}{4} + d(p,q)^2 \end{aligned}$$

Since S is not included in $B(q, \sqrt{R^2 - \varepsilon})$ (by definition of R), we can choose $p \in S$ that satisfies $d(p,q)^2 \geq R^2 - \varepsilon$. Therefore

$$R^2 + \varepsilon \geq \frac{d(x_k,x_l)^2}{4} + R^2 - \varepsilon$$

It follows that (x_k) is a Cauchy sequence.

Existence: Consider any sequence (x_k, r_k) such as above, and let $x = \lim x_k \in M$. Then $S \subset \cap_{k\geq 0}\overline{B}(x_k,r_k) \subset \overline{B}(x,R)$.

Uniqueness: If $S \subset B(x,R)$ and $S \subset B(y,R)$, then consider the sequence (x_k,r_k) in E such that $r_k = R$, $x_{2k} = x$ and $x_{2k+1} = y$. We find that (x_k) converges in M, so $x = y$.

□

Corollary 11.3.5 (Cartan's Fixed Point Theorem)**.** *Let (M,g) be a Cartan-Hadamard manifold, and let K be a compact group that acts continously on M by isometries. Then K fixes a point in M.*

Remark. By a continuous action, we mean that the map $K \times M \to M$ is continuous.

Proof. Let $x \in M$. The orbit $K.x$ is compact, hence bounded and Proposition 11.3.4 says there is a unique closed ball of minimal radius $\overline{B}(x_0,R)$ containing $K.x$. For $\varphi \in K$, we find that $B(\varphi(x_0),R)$ contains $\varphi(K.x) = K.x$, so $\varphi(x_0) = x_0$ by uniqueness. □

11.3.1 The boundary at infinity of a Cartan-Hadamard manifold

Definition 11.3.6. Let (M,g) be a complete Riemannian manifold.

Two unit speed geodesics $c_1, c_2 : \mathbb{R} \to M$ are called **positively asymptotic** if there is M such that $d(c_1(t),c_2(t)) \leq M$ for all $t \geq 0$.

Lemma 11.3.7. *Let (M,g) be a Cartan-Hadamard manifold. For any unit speed geodesic $c : \mathbb{R} \to M$ and any $x \in M$, there is a unique $v \in T^1_x M$ such that c_v is positively asymptotic to c.*

Proof. For $t \geq 0$, we let $v_t \in T^1_x M$ be such that the geodesic c_{v_t} goes through $c(t)$. Let us show that $t \mapsto v_t$ is Cauchy.

Set $d_0 = d(x,c(0))$. Lemma 11.3.3 yields

$$d(c(t),c(s))^2 \geq d(c(t),x)^2 + d(c(s),x)^2 - 2d(c(t),x)d(c(s),x)\cos\sphericalangle(v_t,v_s)$$

The triangle inequality gives

$$t - d_0 \leq d(c(t),x) \leq t + d_0$$

$$s - d_0 \leq d(c(s),x) \leq s + d_0$$

For t,s large enough, we find

$$\begin{aligned}\cos\sphericalangle(v_t,v_s) &\geq \frac{(t-d_0)^2 + (s-d_0)^2 - (t-s)^2}{2(t+d_0)(s+d_0)} \\ &\geq \frac{(t-d_0)(s-d_0)}{(t+d_0)(s+d_0)} \end{aligned} \tag{11.2}$$

We can set $v = \lim_{t\to+\infty} v_t \in T^1_x M$. For $t,s \geq 0$, we have

$$d(c_v(t), c(t)) \leq d(c_v(t), c_{v_s}(t)) + d(c_{v_s}(t), c(t))$$

For $s > t$, Lemma 11.3.2 yields $d(c_{v_s}(t'), c(t)) \leq d(c_{v_s}(z), c(0))$ where $t' = t + s' - s$ and $z = s' - s$, with $c_{v_s}(s') = c(s)$.

The triangle inequality gives $s - d_0 \leq s' \leq s + d_0$, and we find

$$\begin{aligned} d(c_{v_s}(t), c(t)) &\leq d(c_{v_s}(t), c_{v_s}(t')) + d(c_{v_s}(t'), c(t)) \\ &\leq d_0 + d(c_{v_s}(z), c(0)) \\ &\leq d_0 + d(c_{v_s}(z), x) + d_0 \\ &\leq 3d_0 \end{aligned}$$

It follows that $d(c_v(t), c(t)) \leq d(c_v(t), c_{v_s}(t)) + 3d_0$, and $s \to +\infty$ leads to $d(c_v(t), c(t)) \leq 3d_0$ for all $t \geq 0$.

The uniqueness is also a consequence of Lemma 11.3.3, as for all $v' \in T_x^1 M$, we have

$$d(c_v(t), c_{v'}(t))^2 \geq 2t^2 (1 - \cos \sphericalangle(v, v'))$$

□

We now let $\mathcal{G}(M)$ be the set of unit speed geodesics in M, and $\partial_\infty M = \mathcal{G}(M)/\sim$ where $c_1 \sim c_2$ if c_1 and c_2 are positively asymptotic.

According to Lemma 11.3.7, for all $x \in M$ the map

$$\varphi_x : \begin{cases} T_x^1 M & \to & \partial_\infty M \\ v & \mapsto & [c_v] \end{cases}$$

is a bijection.

Lemma 11.3.8. *Let (M, g) be a Cartan-Hadamard manifold. For $x, y \in M$, the map $\varphi_y^{-1} \circ \varphi_x : T_x^1 M \to T_y^1 M$ is a homeomorphism.*

Proof. We only have to prove that $\varphi_y^{-1} \circ \varphi_x : T_x^1 M \to T_y^1 M$ is continuous.

For this purpose we consider $u_k \to u \in T_x^1 M$. For $t \geq 0$, we let $v_k(t) \in T_y^1 M$ be such that $c_{v_k(t)}$ goes through $c_{u_k}(t)$ and $v(t) \in T_y^1 M$ be such that $c_{v(t)}$ goes through $c_u(t)$.

Now set $v_k = \varphi_y^{-1} \circ \varphi_x(u_k)$ and $v = \varphi_y^{-1} \circ \varphi_x(u)$. Applying Lemma 11.3.3 to the triangle with vertices y, $c_{u_k}(t)$ and $c_u(t)$, we find

$$\cos \sphericalangle(v_k(t), v(t)) \geq \left(\frac{t - d(x, y)}{t + d(x, y)} \right)^2 - \frac{1}{2} \left(\frac{d(c_{u_k}(t), c_u(t))}{t - d(x, y)} \right)^2$$

The uniformity in (11.2) in the proof of Lemma 11.3.7 shows that $v_k(t)$ (resp. $v(t)$) converges to v_k (resp. v) as t goes to $+\infty$.

So for all $\varepsilon > 0$, we can find $t > 0$ such that

$$\cos \sphericalangle(v_k, v) \geq \cos \sphericalangle(v_k(t), v(t)) - \varepsilon$$

and

$$\left(\frac{t - d(x,y)}{t + d(x,y)}\right)^2 \geq 1 - \varepsilon$$

it follows that

$$\cos \sphericalangle(v_k, v) \geq 1 - 2\varepsilon - \frac{1}{2}\left(\frac{d(c_{u_k}(t), c_u(t))}{t - d(x,y)}\right)^2$$

But $c_{u_k}(t) \to c_u(t)$ because $\exp_x$ is continuous, which leads to $v_k \to v$. □

Theorem 11.3.9. *Let (M, g) be a Cartan-Hadamard manifold.*
There is a unique topology on $\overline{M} = M \cup \partial_\infty M$ such that:

- *For all $x \in M$, the map $\varphi_x : T_x^1 M \to \partial_\infty M$ is a homeomorphism.*
- *M is open and dense in $\overline{M}$, and the induced topology is the manifold topology.*
- *For any unit speed geodesic $c : \mathbb{R} \to M$, we have $\lim_{t\to+\infty} c(t) = [c]$.*
- *$\overline{M}$ is compact.*

Note that by uniqueness, the group $\mathrm{Isom}(M)$ acts by homeomorphisms on $\overline{M}$, hence on $\partial_\infty M$.

The compactification $\overline{M}$ is homeomorphic to the closed ball $\overline{B}(0,1)$ in $\mathbb{R}^d$, so we can apply Brower's Fixed Point Theorem to see that any isometry of (M, g) must fix a point in $\overline{M}$.

Chapter 12

Riemannian submanifolds

Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold (immersed or embedded). Recall that the **first fundamental form** of N, also called the **induced metric** is the Riemannian metric $\overline{g}$ on N defined as the restriction of g to TN.

We will use a bar to denote everything that relates to N: the Levi-Civita connection of N is $\overline{\nabla}$, the curvature tensor is $\overline{R}$, the Riemannian distance is $\overline{d}$, etc...

For $x \in N$ and $v \in T_xM$, we will write $v = v^\top + v^\perp$ where $v^\top \in T_xN$ and $v^\perp \in (T_xN)^\perp$.

Recall that the Levi-Civita connection ∇ of (M,g) restricts to N (i.e. $\nabla_x X(v)$ is well defined for $X \in \Gamma(TM|_N)$ and $v \in T_xN$), and that the Levi-Civita connection $\overline{\nabla}$ of $(N,\overline{g})$ satisfies $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ for all $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_xN$.

12.1 The second fundamental form

Definition 12.1.1. Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The **normal bundle** of N is the vector sub-bundle νN of $TM|_N$ defined by $\nu_x N = (T_xN)^\perp \subset T_xM$.

The orthogonal decomposition induces an isomorphism of vector bundles $TN \oplus \nu N = TM|_N$.

Example 12.1.2. the normal bundle of $\mathbb{S}^n \subset \mathbb{E}^{n+1}$ is a trivialisable line bundle ($\nu_x \mathbb{S}^n = \mathbb{R}.x$).

Lemma 12.1.3. *Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The map $A : \mathcal{X}(N \times \mathcal{X}(N) \to \Gamma(\nu N)$ defined by:*

$$\forall X, Y \in \mathcal{X}(N) \quad A(X,Y) = (\nabla_X Y)^\perp$$

is tensorial and symmetric.

Proof. Tensoriality in X comes from the definition of a connection. For $f \in \mathcal{C}^\infty(N)$, we have $A(X, fY) = fA(X,Y) + (X \cdot f)Y^\perp = fA(X,Y)$ since $Y^\perp = 0$. Now that we know that it is tensorial, in order to prove the symmetry we can consider the case where X, Y are vector fields on M whose restriction to N is tangent, thanks to Lemma 9.5.1.

$$\begin{aligned} A(X,Y) - A(Y,X) &= (\nabla_X Y)^\perp - (\nabla_Y X)^\perp \\ &= [X,Y]^\perp \\ &= 0 \end{aligned}$$

□

Definition 12.1.4. Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The **second fundamental form** of N is $\vec{\mathrm{II}} \in \Gamma\left(S^2(T^*N) \otimes \nu N\right)$ defined by:

$$\forall X, Y \in \mathcal{X}(N) \quad \vec{\mathrm{II}}(X,Y) = (\nabla_X Y)^\perp$$

Note that for $X, Y \in \mathcal{X}(N)$, we find $\nabla_X Y = \overline{\nabla}_X Y + \vec{\mathrm{II}}(X,Y)$ (this is known as the Gauß formula).

The second fundamental form has values in the normal bundle. Given a normal vector $n \in \nu_x N$, the map $(u,v) \mapsto g_x\left(n, \vec{\mathrm{II}}_x(u,v)\right)$ is a symmetric bilinear form $T_x N$, so it can be represented by a self adjoint operator of $T_x N$.

Definition 12.1.5. Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The **shape operator** of N is $S \in \Gamma\left((\nu N)^* \otimes \mathrm{End}(TN)\right)$ defined by:

$$\forall x \in N \ \forall n \in \nu_x N \ \forall u, v \in T_x N \quad \overline{g}_x(S_x(n)u, v) = -g_x\left(n, \vec{\mathrm{II}}_x(u,v)\right)$$

Remark. The shape operator is also called the **Weingarten operator**.

Proposition 12.1.6 (Weingarten formula)**.** *Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. For all $n \in \Gamma(\nu N)$, we have*

$$S(n) = (\nabla n)^\top$$

Proof. Consider $X, Y \in \mathcal{X}(N)$. Since the formula is local, we can assume that X, Y, n extend to vector fields on M. Note that we only assume that $g(n,Y) = 0$ on N, but this is enough to find that $X \cdot g(n,Y) = 0$ on N.

$$\begin{aligned} \overline{g}\left((\nabla n(X))^\top, Y\right) &= g(\nabla n(X), Y) \\ &= X \cdot g(n,Y) - g(n, \nabla Y(X)) \\ &= 0 - g\left(n, \vec{\mathrm{II}}(X,Y) + \overline{\nabla} Y(X)\right) \\ &= -\overline{g}(S(n)X, Y) \end{aligned}$$

□

Theorem 12.1.7 (Gauß equation). *Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R, and let $N \subset M$ be a submanifold. Let $\overline{g}$ be the restricted metric on N, and $\overline{R}$ the curvature tensor of $\overline{g}$. All $x \in N$ and $u, v \in T_xN$ satisfy*

$$\overline{R}_x(u,v,v,u) = R_x(u,v,v,u) - g_x\left(\vec{\mathrm{II}}_x(u,v), \vec{\mathrm{II}}_x(u,v)\right) + g_x\left(\vec{\mathrm{II}}_x(u,u), \vec{\mathrm{II}}_x(v,v)\right)$$

Proof. Recall that according to Lemma 9.5.1, we can consider that $u = X(x)$ and $v = Y(x)$ where $X, Y \in \mathcal{X}(M)$ are vector fields that restrict to vector fields of N.

First, we consider two other tangent vectors $w, z \in T_xN$, and use the Weingarten formula to obtain:

$$\begin{aligned} g_x\left(\nabla_x\left(\vec{\mathrm{II}}(X,Y)\right)(w), z\right) &= -\overline{g}_x\left(S_x\left(\vec{\mathrm{II}}(X,Y)\right)(w), z\right) \\ &= g_x\left(\vec{\mathrm{II}}_x(u,v), \vec{\mathrm{II}}_x(w,z)\right) \end{aligned} \tag{12.1}$$

By using the decomposition $\nabla = \overline{\nabla} + \vec{\mathrm{II}}$ for vector fields on N, we find:

$$R_x(u,v,v,u) - \overline{R}_x(u,v,v,u) = g_x\left(\nabla_x\left(\vec{\mathrm{II}}(Y,Y)\right)(u), u\right) - g_x\left(\nabla_x\left(\vec{\mathrm{II}}(Y,X)\right)(v), u\right)$$

Using (12.1), we find the desired formula. □

12.2 Hypersurfaces

Consider a hypersurface $N \subset M$, and $n \in \Gamma(\nu N)$ unitary (it is always possible to find such a field locally, and there are exactly two choices).

We can consider the **scalar second fundamental form** defined by $\mathrm{II} = \left\langle n \middle| \vec{\mathrm{II}} \right\rangle$.

The Eigenvalues of the shape operator $S(n)$ are called the **principal curvatures** of N.

The **Gauß curvature** is $K = \det S(n)$. It is the product of the principal curvatures.

Theorem 12.1.7 in the case $M = \mathbb{R}^3$ says that the Gauß curvature of N is equal to its sectional curvature for the induced metric. This result implies the *Theorema Egregium* of Gauß: the Gauß curvature is invariant under isometries.

Proposition 12.2.1. *Let N be an immersed hypersurface of a Riemannian manifold (M,g), with unitary normal field n. Let $x \in N$, $v \in T_xN$, and consider a smooth curve $c :]-\varepsilon, \varepsilon[\to N$ such that $c(0) = x \in N$ and $\dot{c}(0) = v$. The scalar second fundamental form satisfies:*

$$\mathrm{II}_x(v,v) = g_x\left(\frac{D}{dt}\dot{c}(0), n(x)\right)$$

Proof. Differentiating the fact that $\dot{c}(t)$ and $n(c(t))$ are orthogonal for all t, we find that:

$$\begin{aligned} g\left(\frac{D}{dt}\dot{c}, n\right) &= -g\left(\dot{c}, \frac{D}{dt}n\right) \\ &= -g\left(\dot{c}, \nabla n(\dot{c})\right) \\ &= -g\left(\dot{c}, S(n)\dot{c}\right) \\ &= g\left(n, \vec{\mathrm{II}}(\dot{c},\dot{c})\right) \\ &= \mathrm{II}(\dot{c},\dot{c}) \end{aligned}$$

□

12.3 Euclidean submanifolds

If (M,g) is the Euclidean space $\mathbb{E}^d$, and $N \subset \mathbb{R}^d$ is a hypersurface with normal field n, then the shape operator is simply dn, i.e. it is already tangent to N. Indeed, the normal field n can be seen as a map $n : N \to \mathbb{S}^{d-1}$, and its differential $d_x n$ at $x \in N$ is a map from $T_x N$ to $T_{n(x)}\mathbb{S}^{d-1} = n(x)^\perp = T_x N$.

To compute the second fundamental form, we start with a smooth curve $c :]-\varepsilon, \varepsilon[\to N$ such that $c(0) = x \in N$ and $\dot{c}(0) = v$. Proposition 12.2.1 yields

$$\mathrm{II}_x(v,v) = \langle \ddot{c}(0) | n(x) \rangle$$

i.e. $\mathrm{II}_x(v,v)$ is the curvature of the curve obtained by intersecting N with a plane spanned by the normal direction to N and v.

If $N \subset \mathbb{R}^d$ is a submanifold of arbitrary codimension, then there is also a simpler way of defining the second fundamental form (i.e. not involving covariant derivation). For any vector subspace $V \subset \mathbb{R}^d$, let $p_V : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection on V. Consider the function:

$$p : \left\{ \begin{array}{ccc} N \times \mathbb{R}^d & \to & \mathbb{R}^d \\ (x,v) & \mapsto & p_{T x N^\perp}(v) \end{array} \right.$$

Then $\vec{\mathrm{II}}_x(u,v)$ can be obtained by differentiating p:

$$\vec{\mathrm{II}}_x(u,v) = -d_{(x,u)} p(v,0)$$

Indeed, by definition, if $u = X(x)$ and $v = Y(x)$ for some vector fields $X, Y \in \mathcal{X}(N)$, then $\vec{\mathrm{II}}_x(u,v) = p(x, d_x X(Y(x)))$. By differentiating the fact that $p(x, X(x)) = 0$ and using the linearity of p in its second variable, we get the equality.

This shows that curvature measures the variations of the tangent space. Another way of seeing this is by describing the tangent space of $TN \subset \mathbb{R}^{2d}$ at $(x,v) \in TN$ (i.e. $x \in N$ and $v \in T_x N$):

$$T_{(x,v)} TN = \left\{ (\dot{x}, \dot{v}) \,\middle|\, \dot{x} \in T_x N \;\&\; p_{T_x N^\perp}(\dot{v}) = \vec{\mathrm{II}}_x(v, \dot{x}) \right\}$$

Indeed, $(x,v) \in TN$ is characterized by the equations $x \in N$ and $p(x,v) = 0$, so the tangent space of TN is obtained by differentiating these equations: $\dot{x} \in T_xN$ and $d_{(x,v)}p(\dot{x},\dot{v}) = 0$, i.e. $-\vec{\mathrm{II}}_x(v,\dot{x}) + p_{T_xN^\perp}(\dot{v}) = 0$.

12.4 Moving frames for surfaces in $\mathbb{R}^3$

Consider a surface $S \subset \mathbb{R}^3$. Locally, we can always consider a unitary normal field $n \in \Gamma(\nu S)$, and an orthonormal frame field t_1, t_2 of TN. For each $x \in S$, the vectors $(n(x), t_1(x), t_2(x))$ form an orthonormal basis of $\mathbb{R}^3$. So we can decompose the derivatives of n, t_1, t_2 in this basis. Differentiating $n.n = 1$, we find that $d_x n(v).n(x) = 0$ for all $x \in S$ and $v \in T_xS$, i.e. $d_x n(v) \in T_xS = \mathrm{Vect}(t_1(x), t_2(x))$. So we can decompose:

$$d_x n(v) = \alpha_x(v) t_1(x) + \beta_x(v) t_2(x)$$

This defines (locally) $\alpha, \beta \in \Omega^1(S)$. We can now do the same for t_1 and t_2. Because they are unit vector field, the derivative of each of them must be orthogonal to itself.

$$\begin{aligned} d_x t_1(v) &= \gamma_x(v) t_2(x) + \delta_x(v) n(x) \\ d_x t_2(v) &= \varepsilon_x(v) t_1(x) + \zeta_x(v) n(x) \end{aligned}$$

Since $n.t_1 = 0$, we have that $d_x n(v).t_1(x) + n(x).d_x t_1(v) = 0$, i.e. $\alpha + \delta = 0$. So we can simplify:

$$d_x t_1(v) = \gamma_x(v) t_2(x) - \alpha_x(v) n(x)$$

Similarly, the fact that $n.t_2 = 0$ differentiates to $\beta + \zeta = 0$, and $t_1.t_2 = 0$ yields $\gamma = \varepsilon = 0$. So the differentiation of the frame (n, t_1, t_2) is given by $\alpha, \beta, \gamma \in \Omega^1(S)$ such that:

$$\begin{aligned} d_x n(v) &= \alpha_x(v) t_1(x) + \beta_x(v) t_2(x) \\ d_x t_1(v) &= \gamma_x(v) t_2(x) - \alpha_x(v) n(x) \\ d_x t_2(v) &= -\gamma_x t_1(x) - \beta_x(v) n(x) \end{aligned}$$

One way of understanding this is by considering the matrix $C(x) = (t_1(x)|t_2(x)|n(x)) \in \mathrm{O}(3,\mathbb{R})$. The derivative must lie in the tangent space to $\mathrm{SO}(3,\mathbb{R})$, i.e. $C(x)^{-1} d_x C(v) \in \mathfrak{so}(3,\mathbb{R})$. Now this matrix is:

$$C(x)^{-1} d_x C(v) = \begin{pmatrix} 0 & \gamma_x(v) & -\alpha_x(v) \\ -\gamma_x(v) & 0 & -\beta_x(v) \\ \alpha_x(v) & \beta_x(v) & 0 \end{pmatrix}$$

The left upper 2×2 bloc, given by the form γ, corresponds to the Levi-Civita connection of S. The vector-valued form (α, β) corresponds to the second fundamental form of S. The two are related by the Gauß equation: $d\gamma = -\alpha \wedge \beta$. To prove this formula, we only need to show that $d\gamma(t_1, t_2) = \alpha \wedge \beta(t_1, t_2) = \alpha(t_1)\beta(t_2) - \alpha(t_2)\beta(t_1)$. Now using the Cartan magic formula, we find:

$$d\gamma(t_1, t_2) = d(\gamma(t_2))(t_1) - d(\gamma(t_1))(t_2) - \gamma([t_1, t_2])$$

In order to compute a term like $d(\gamma(t_2))(t_1)$, we can use the fact that functions defined on S (such as vector fields which are $\mathbb{R}^3$ valued functions) can be locally extended to $\mathbb{R}^3$, so it makes sense to use second order derivatives. Since $\gamma = dt_1.t_2$, we find:

$$\begin{aligned} d(\gamma(t_2))(t_1) &= d^2t_1(t_2, t_1).t_2 + dt_1(dt_2(t_1)).t_2 + dt_1(t_2).dt_2(t_1) \\ d(\gamma(t_1))(t_2) &= d^2t_1(t_1, t_2).t_2 + dt_1(dt_1(t_2)).t_2 + dt_1(t_1).dt_2(t_2) \end{aligned}$$

The second order derivatives of t_1 cancel out thanks to Schwartz symmetry, and $d_xt_1(u).d_xt_2(v) = \alpha_x(u)\beta_x(v)$ for any $u, v \in T_xS$, so we get:

$$\begin{aligned} d(\gamma(t_2))(t_1) - d(\gamma(t_1))(t_2) &= \gamma(dt_2(t_1) - dt_1(t_2)) + \alpha(t_2)\beta(t_1) - \alpha(t_1)\beta(t_2) \\ &= \gamma([t_1, t_2]) - \alpha \wedge \beta(t_1, t_2) \end{aligned}$$

This shows that $d\gamma(t_1, t_2) = -\alpha \wedge \beta(t_1, t_2)$, therefore $d\gamma = -\alpha \wedge \beta$.

Now $d\gamma$ is the intrinsic (sectional) curvature of S times the area form, and $\alpha \wedge \beta$ is the Gauß curvature times the area form, so this is a way of proving the Theorema Egregium.

12.5 Mean curvature

Definition 12.5.1. Let $N \subset M$ be an immersed submanifold of a Riemannian manifold (M, g). The **mean curvature** of N is $H \in \Gamma(\nu N)$ given by $H(x) = \operatorname{Tr} \mathrm{II}_x$.

A submanifold is called **minimal** if $H = 0$.

Remark. It is the trace of a quadratic form, i.e. $H(x) = \sum_{i=1}^n \mathrm{II}_x(v_i, v_i)$ where (v_i) is an orthonormal basis of T_xN.

If N is a hypersurface, we locally choose a unit normal field $\vec{n} \in \Gamma(\nu N)$, and consider the scalar mean curvature $H(x) = \operatorname{Tr} \mathrm{II}_x$.

Theorem 12.5.2. *If N is compact, and $X \in \mathcal{X}(M)$ is a complete vector field with flow (φ^t), then:*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Vol}\left(\varphi^t(N)\right) = -\int_N g(H, X)\,\mathrm{dvol}_{\overline{g}}$$

12.6 Totally geodesic submanifolds

Proposition 12.6.1. *Let (M,g) be a Riemannian manifold and $N \subset M$ an immersed submanifold. The following are equivalent:*

1. $\forall x \in M \quad \vec{\mathrm{II}}_x = 0.$
2. *Any geodesic of N is a geodesic of M.*
3. *For all $x \in N$, there are neighbourhoods $V \subset N$ of x and $U \subset T_xN$ of 0 such that $\exp_x(U) = V$ (where $\exp_x$ is the exponential in (M,g)).*
4. *TN is stable under ∇, i.e.*

$$\forall X \in \mathcal{X}(N)\ \forall x \in N\ \forall v \in T_xN \quad \nabla_x X(v) \in T_xN$$

A submanifold satisfying these properties is called **totally geodesic.**

Proof. First note that 1. $\Longleftrightarrow$ 4. comes from the definition of $\vec{\mathrm{II}}$.

2. $\Rightarrow$ 3. is a consequence of the local surjectivity of the exponential map.

3. $\Rightarrow$ 2. is a consequence of the uniqueness of geodesics.

2. $\Longleftrightarrow$ 4. comes from the geodesic equation on a submanifold (see the discussion following Proposition 9.6.9). $\square$

Exercise. Show that the totally geodesic submanifolds of $\mathbb{E}^n$ are open subsets of affine subspaces.

Lemma 12.6.2. *Let (M,g) be a Riemannian manifold, and let $N \subset M$ be a totally geodesic submanifold. For all $x \in N$ and $u, v, w \in T_xN$, we have $R_x(u,v)w \in T_xN$ where R is the Riemann tensor of (M,g).*

Proof. We let $\overline{R}$ be the Riemann tensor of N, so that $\overline{R}_x(u,v)w \in T_xN$. The fact that $\vec{\mathrm{II}} = 0$ assures that $\nabla = \overline{\nabla}$ on N, hence $R_x(u,v)w = \overline{R}_x(u,v)w$, and $R_x(u,v)w \in T_xN$. $\square$

Part IV

Symmetric spaces

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Chapter 13

Globally and locally symmetric spaces

13.1 Globally symmetric spaces

Definition 13.1.1. A **Riemannian symmetric space** is a connected Riemannian manifold $\mathbb{X}$ such that for all $x \in \mathbb{X}$, there is an isometry $s_x \in \mathrm{Isom}(\mathbb{X})$ such that $s_x(x) = x$ and $d_x s_x = -\mathrm{Id}_{T_x\mathbb{X}}$.

Note that such an isometry s_x is unique because of Proposition 9.6.11.

Lemma 13.1.2. *Let $\mathbb{X}$ be a Riemannian symmetric space. Then $\mathbb{X}$ is complete, and if $c : \mathbb{R} \to \mathbb{X}$ is a geodesic, we have $s_{c(t)}(c(s)) = c(2t - s)$ for all $t \in \mathbb{R}$.*

Proof. First notice that if $c : I \to \mathbb{X}$ is a geodesic and $x = c(0)$, then $s_x \circ c$ is also a geodesic, with velocity vector $-\dot{c}(0)$, hence $-I = I$ and $s_x \circ c(t) = c(-t)$. Up to a translation of the parameter, this proves the second point under the assumption of completeness.

Let $c : [a, b] \to \mathbb{X}$ be a geodesic with $a, b \in \mathbb{R}$. Set $z = c(b)$. Now $\gamma : [b, 2b-a] \to \mathbb{X}$ defined by $\gamma(t) = s_z(c(2b - t))$ is a geodesic such that $\gamma(b) = c(b)$ and $\dot{\gamma}(b) = \dot{c}(b)$, so it extends c to $[a, 2b-a]$. Repeating this argument shows that c is extendable to $\mathbb{R}$, i.e. $\mathbb{X}$ is complete. □

Proposition 13.1.3. *If $\mathbb{X}$ is a Riemannian symmetric space, then $\mathbb{X}$ is homogeneous, i.e. the isometry group* $\mathrm{Isom}(\mathbb{X})$ *acts transitively on $\mathbb{X}$.*

Proof. Let $x, y \in \mathbb{X}$. Since $\mathbb{X}$ is complete by Lemma 13.1.2, the Hopf-Rinow Theorem 10.3.4 provides a geodesic $c : \mathbb{R} \to \mathbb{X}$ such that $c(0) = x$ and $c(1) = y$. By Lemma 13.1.2, we find that $y = s_z(x)$ where $z = c\left(\frac{1}{2}\right)$. □

Notation: If $\mathbb{X}$ is a symmetric space, we consider $G = \mathrm{Isom}(\mathbb{X})_\circ$ the identity component of the isometry group. Recall that it is a Lie group and that the action on X is smooth (Myers-Steenrod Theorem). We fix some $o \in \mathbb{X}$, and set $K = \mathrm{Stab}_G(o)$. It is a compact Lie subgroup of G.

For $g \in G$, we have $s_{g(x)} = g \circ s_x \circ g^{-1}$.

Lemma 13.1.4. *Let $\mathbb{X}$ be a Riemannian symmetric space. The map*

$$\begin{cases} \mathbb{X} & \to & \operatorname{Isom}(\mathbb{X}) \\ x & \mapsto & s_x \end{cases}$$

is smooth.

Proof. Set $\overline{G} = \operatorname{Isom}(\mathbb{X})$ and let $o \in \mathbb{X}$. Since $\overline{G} \curvearrowright \mathbb{X}$ is transitive, the orbit map $\varphi_o : \overline{G} \to \mathbb{X}$ is a submersion.

Since $s_{g(o)} = g \circ s_o \circ g^{-1}$, the map $x \mapsto s_x$ lifts through the submersion φ_o to the map $g \mapsto g \circ s_o \circ g^{-1}$ which is smooth, so $x \mapsto s_x$ is smooth. □

Lemma 13.1.5. *Let $\mathbb{X}$ a Riemannian symmetric space, and $G = \operatorname{Isom}(\mathbb{X})_\circ$. For all $x, y \in \mathbb{X}$, we have that $s_x \circ s_y \in G$. The action of G on $\mathbb{X}$ is transitive.*

Proof. Let $c : \mathbb{R} \to X$ be a geodesic such that $c(0) = x$ and $c(1) = y$. Then $s_x \circ s_{c(t)}$ is a continuous path in $\operatorname{Isom}(\mathbb{X})$ that links Id and $s_x \circ s_y$.

By letting $z = c\left(\frac{3}{2}\right)$, we find $y = s_z \circ s_y(x)$, hence the transitivity. □

Examples 13.1.6. $\mathbb{E}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ are symmetric spaces.

13.2 Locally symmetric spaces

Motivated by Proposition 13.1.3, we can try to define symmetries in arbitrary Riemannian manifolds.

Definition 13.2.1. Let (M, g) be a Riemannian manifold, and let $x \in M$. The **geodesic symmetry** through x is the map $s_x = \exp_x \circ \left(-\exp_x^{-1}\right)$ defined on $B(x, \operatorname{inj}_x)$.

In general, it is not a local isometry, but it is in some cases.

Definition 13.2.2. A **Riemannian locally symmetric space** is a Riemannian manifold (M, g) such that for all $x \in M$, the geodesic symmetry s_x is isometric on a neighbourhood of x.

Theorem 13.2.3 (Cartan-Ambrose-Hicks)**.** *Let (M, g) be a Riemannian manifold. Then (M, g) is a locally symmetric space if and only if $\nabla R = 0$.*

If (M, g) is a complete and simply connected Riemannian locally symmetric space, then it is a symmetric space.

The condition $\nabla R = 0$ will be used in the following way:

Lemma 13.2.4. *Let (M, g) be a Riemannian manifold, and $T \in \mathcal{T}^{p,0}(M)$ be a covariant tensor. The following are equivalent:*

1. $\nabla T = 0$

2. *For any smooth curve $c : I \to M$ and parallel vector fields $X_1, \ldots, X_p \in \Gamma(c^*TM)$ along c, $T(X_1, \ldots, X_p)$ is constant.*

Lemma 13.2.5. *Let (M, g) be a Riemannian manifold with curvature tensor R. If $\nabla R = 0$, then for all $x \in M$ and $v \in T_xM$ with $\|v\|_x < \mathrm{inj}_x$, the differential $d_{\exp_x(v)} s_x : T_{\exp_x(v)}M \to T_{\exp_x(-v)}M$, is equal to the opposite of the parallel transport along the geodesic $t \mapsto \exp_x(tv)$.*

Proof. Write $y = \exp_x(v)$, and let $u \in T_yM$. From the definition of s_x and the chain-rule we find that $d_y s_x(u) = -d_{-v} \exp_x(w)$ where $w = (d_v \exp_x)^{-1}(u) \in T_xM$.

Let $c : I \to M$ be the geodesic defined by $\dot{c}(0) = v$, and J the Jacobi field along c such that $J(0) = 0$ and $\frac{D}{dt}J(0) = w$. The formula for the differential of the exponential map (Proposition 7.4.8) yields $w = J(1)$ and $d_y s_x(w) = J(-1)$.

Consider a parallel orthonormal frame $(\varepsilon_1, \ldots, \varepsilon_d)$ along c, and decompose $J(t) = \sum_{i=1}^d J^i(t)\varepsilon_i(t)$. Since we have chosen a parallel frame, we have that $\frac{D}{dt}\frac{D}{dt}J(t) = \sum_{i=1}^d \ddot{J}^i(t)\varepsilon_i(t)$. Using the fact that it is an orthonormal frame, we find:

$$\begin{aligned} R_{c(t)}(J(t), \dot{c}(t))\dot{c}(t) &= \sum_{j=1}^d J^j(t) R_{c(t)}(\varepsilon_j(t), \dot{c}(t))\dot{c}(t) \\ &= \sum_{1 \le i,j \le d} J^j(t) R_{c(t)}(\varepsilon_j(t), \dot{c}(t), \dot{c}(t), \varepsilon_i(t))\varepsilon_i(t) \end{aligned}$$

Let $\alpha_{i,j} = R_x(\varepsilon_j(0), v, v, \varepsilon_i(0)) \in \mathbb{R}$. According to Lemma 13.2.4, we have that:

$$\forall t \in I \quad R_{c(t)}(J(t), \dot{c}(t))\dot{c}(t) = \sum_{1 \le i,j \le d} \alpha_{i,j} J^j(t)\varepsilon_i(t)$$

The Jacobi field equation writes as:

$$\forall i \ \ddot{J}^i + \sum_{j=1}^d \alpha_{i,j} J^j = 0$$

It is a linear differential equation with constant coefficients. It follows that the vector field $\widetilde{J}$ along c defined by $\widetilde{J}(t) = -\sum_{i=1}^d J^i(-t)\varepsilon_i(t)$ is also a Jacobi field.

Since $\widetilde{J}(0) = 0 = J(0)$ and $\frac{D}{dt}\widetilde{J}(0) = \sum_{i=1}^d \dot{J}^i(0)\varepsilon_i(0) = \frac{D}{dt}J(0)$, we find that $\widetilde{J} = J$.

Finally we get:

$$\begin{aligned} d_y s_x(u) &= J(-1) \\ &= \widetilde{J}(-1) \\ &= -\sum_{i=1}^{d} J^i(1)\varepsilon_i(-1) \\ &= -\|_1^{-1} J(1) \\ &= -\|_1^{-1} u \end{aligned}$$

□

Since the parallel transport is isometric (proposition 9.2.2), it follows that s_x is a local isometry.

Lemma 13.2.5 admits the following generalisation:

Lemma 13.2.6. *Let (M,g) be a complete Riemannian manifold, and assume that $\nabla R = 0$. For all $x,y \in M$ and all linear isometry $\varphi : T_xM \to T_yM$ which preserves the Riemann tensor (i.e. $R_y(\varphi(u),\varphi(v))\varphi(w) = \varphi(R_x(u,v)w)$ for all $u,v,w \in T_xM$), there is a local isometry $f : B(x, \mathrm{inj}_x) \to M$ such that $f(x) = y$ and $d_xf = \varphi$.*

If we drop the completeness hypothesis, we can still build f on $B(x,r)$ where $r \leq \mathrm{inj}_x$ and $\exp_y$ is defined on $B_{T_yM}(0,r)$.

Lemma 13.2.7. *Let (M,g) be a complete Riemannian manifold, and assume that $\nabla R = 0$. For all $x,y \in M$ and smooth curve $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$, there is a neighbourhood $U \subset M$ of $c([0,1])$ and a local isometry $f : U \to M$ such that $f(x) = x$ and $d_xf = -\mathrm{Id}$.*

Proof. Considering the open cover $c([0,1]) \subset \bigcup_{t\in[0,1]} B(c(t), \mathrm{inj}_{c(t)})$, we can consider a finite sequence $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $c([t_i,t_{i+1}]) \subset U_i$ where $x_i = c(t_i)$ and $U_i = B(x_i, \mathrm{inj}_{x_i})$.

Our goal is to inductively construct a connected open set $V_i \subset M$ containing $c([0,t_{i+1}])$ and a local isometry $f_i : V_i \to M$ such that $f_i(x) = x$ et $d_xf_i = -\mathrm{Id}$.

The open set $V_0 = U_0$ and the local isometry f_0 are given by Lemma 13.2.5.

Assume that we have V_i and $f_i : V_i \to M$ as described above. Since $x_{i+1} \in V_i$, Lemma 13.2.6 guarantees the existence of a local isometry $\widetilde{f} : U_{i+1} \to M$ such that $\widetilde{f}(x_{i+1}) = y_i$ and $d_{x_{i+1}}\widetilde{f} = d_{x_{i+1}}f_i$.

Let W be the connected component of x_{i+1} in $U_{i+1} \cap V_i$. The restrictions of f_i and $\widetilde{f}$ to W are equal.

We now denote by V_{i+1} the connected component of x_{i+1} in $V_i \cup U_{i+1}$. We can define $f_{i+1} : V_{i+1} \to M$ that extends both f_i and $\widetilde{f}$, since they are equal on the intersection of their domains, and satisfy all the requirements.

Finally $U = V_{N-1}$ and $f = f_{N-1}$ answer the initial problem. □

Proof of Theorem 13.2.3. Assume that (M,g) is locally symmetric. Near $x \in M$, we have $s_x^*(\nabla R) = \nabla R$. Evaluating this at x, since $s_x(x) = x$ and $d_x s_x = -\mathrm{Id}$, we find $(s_x^*(\nabla R))_x = -(\nabla R)_x$ (because ∇R is a type (4,1) tensor). Hence $(\nabla R)_x = 0$.

If $\nabla R = 0$, then Lemma 13.2.5 shows that (M,g) is locally symmetric (because parallel transport is isometric).

We now assume that (M,g) is locally symmetric, complete and simply connected. Let $x \in M$. We wish to construct s_x.

Lemma 13.2.7 assures that for every $y \in M$, we can find a connected open set $U_y \subset M$ containing x and y, and a local isometry $f_y : U_y \to M$ such that $f_y(x) = x$ et $d_x f_y = -\mathrm{Id}$. Let us show that $f_y(y)$ does not depend on the choice of the curve to which we apply 13.2.7.

Consider two curves $c_0, c_1 : [0,1] \to M$ such that $c_0(0) = c_1(0) = x$ and $c_0(1) = c_1(1) = y$. Since M is simply connected, we can consider a smooth homotopy $H : [0,1]^2 \to M$ such that $H(0,\cdot) = c_0$, $H(1,\cdot) = c_1$, $H(s,0) = x$ and $H(s,1) = y$ for all $s \in [0,1]$.

Let c_s be the curve $c_s = H(s,\cdot)$ for $s \in [0,1]$, also U_s the connected open set and $f_s : U_s \to M$ the local isometry obtained by applying Lemma 13.2.7 to c_s.

Let us show that the map $\begin{cases} [0,1] & \to & M \\ s & \mapsto & f_s(y) \end{cases}$ is locally constant. Note that in order to show that $f_s(y) = f_{s'}(y)$, we only need to check that x and y are in the same connected component of $U_s \cap U_{s'}$.

For $s \in [0,1]$, we set $r = \min\left\{\mathrm{inj}_{c_s(t)} \,\middle|\, t \in [0,1]\right\}$. Let $\eta > 0$ be such that:

$$|s - s'| < \eta \Rightarrow \forall t \in [0,1]\; c_{s'}(t) \in B(c_s(t), r)$$

We find that $c_{s'}(t) \in U_s$ for $s \in]s-\eta, s+\eta[$ and $t \in [0,1]$. Hence $U_s \cap U_{s'} \supset c_{s'}([0,1])$, and y is in the connected component of $U_s \cap U_{s'}$ which contains x. Hence $f_{s'}(y) = f_s(y)$.

Since $[0,1]$ is connected, we get $f_0(y) = f_1(y)$.

We now define s_x in the following way: for $y \in M$, we choose a smooth path c_y from x to y and we set $s_x(y) = f_y(y)$ where f_y is given by Lemma 13.2.7.

The map s_x is isometric: for $z \in B(y, \mathrm{inj}_y)$, if we choose the concatenation of a path from x to y and the minimising geodesic from y to z in order to construct $s_x(z)$, we find that $s_x(z) = f_y(z)$, and f_y is isometric on a neighbourhood of y.

Since (M,g) is complete, the local isometry $s_x : M \to M$ is a Riemannian covering, hence a diffeomorphism because M is simply connected.

□

Corollary 13.2.8. *Let $\mathbb{X}$ be a symmetric space. Then $\nabla R = 0$.*

Proposition 13.2.9. *Let $\mathbb{X}$ be a symmetric space. If $c : \mathbb{R} \to \mathbb{X}$ is a geodesic, then for all $t, s \in \mathbb{R}$ the differential at $c(t)$ of $s_{c(s)}$ is equal to the opposite of the parallel transport along c.*

Proof. The same computations as in Lemma 13.2.5 can be carried out. □

13.3 The symmetric space of ellipsoids

For $n \geq 2$, we let $\mathcal{E}_n = \left\{x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x,\, x > 0,\, \det x = 1\right\}$. Let $o = 1_n \in \mathcal{E}_n$, and $\mathfrak{p} = T_o\mathcal{E}_n$. Note that

$$\mathfrak{p} = \left\{X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \operatorname{Tr} X = 0\right\}$$

The map $\left\{\begin{array}{ccc} \mathfrak{p} & \to & \mathcal{E}_n \\ X & \mapsto & \exp(X) \end{array}\right.$ is a diffeomorphism, and we let Log be its inverse. The map $x \mapsto \sqrt{x} = e^{\frac{1}{2}\operatorname{Log} x}$ is a diffeomorphism of $\mathcal{E}_n$.

Consider the action $\mathrm{SL}(n,\mathbb{R}) \curvearrowright \mathcal{E}_n$ defined by $g.x = gx^t g$. The stabiliser $\mathrm{Stab}_{\mathrm{SL}(n,\mathbb{R})}(o)$ is equal to $\mathrm{SO}(n,\mathbb{R})$. This action is transitive, so we can identify $\mathcal{E}_n$ with $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

Endow $\mathfrak{p}$ with the inner product $\langle X|Y\rangle_o = \operatorname{Tr}(XY) = \sum_{i,j} X_{i,j} Y_{i,j}$. It is invariant under the action of $\mathrm{SO}(n,\mathbb{R})$.

For $X \in T_x\mathcal{E}_n$, we set $\|X\|_x = \left\|\sqrt{x}^{-1} X \sqrt{x}^{-1}\right\|_o$. The polarized form is:

$$\langle X|Y\rangle_x = \operatorname{Tr}(Xx^{-1}Yx^{-1})$$

It is a Riemannian metric on $\mathcal{E}_n$, and $\mathrm{SL}(n,\mathbb{R})$ acts isometrically.

The map $s_o : \left\{\begin{array}{ccc} \mathcal{E}_n & \to & \mathcal{E}_n \\ x & \mapsto & x^{-1} \end{array}\right.$ is an isometry. It fixes o and satisfies $d_o s_o = -\mathrm{Id}$. Therefore $\mathcal{E}_n$ is a symmetric space.

Exercise. Show that $\mathcal{E}_2$ is isometric to $\mathbb{H}^2$.

13.4 The algebraic structure of symmetric spaces

13.4.1 Symmetric spaces and involutions of Lie groups

Proposition 13.4.1. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}(\mathbb{X})_\circ$, $o \in \mathbb{X}$ and $K = \mathrm{Stab}_G(o)$. Let $H = \{g \in G \,|\, s_o g = g s_o\}$ and $H_\circ$ its identity component. Then:*

$$H_\circ \subset K \subset H$$

Conversely, if G is a connected Lie group, $\sigma : G \to G$ an involutive Lie group automorphism, and K a compact subgroup of G such that $H_\circ \subset K \subset H$, where $H = \{g \in G \,|\, \sigma(g) = g\}$, then any G-invariant Riemannian metric on G/K is symmetric.

Remarks.

- If K is compact, then G-invariant Riemannian metrics on G/K exist.
- The double inclusion $H_\circ \subset K \subset H$ should be interpreted as the fact that K as the same Lie algebra as H.

Proof. Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $o \in \mathbb{X}$ and $K = \mathrm{Stab}_G(o)$. Let $H = \{g \in G \,|\, s_o g = g s_o\}$ and $H_\circ$ its identity component.

If (γ_t) is a one-parameter subgroup of H, then $s_o(\gamma_t(o)) = \gamma_t(o)$ for all t. Since o is an isolated fixed point of s_o, it follows that $\gamma_t \in K$, hence $H_\circ \subset K$.

If $g \in K$, then $h = s_o g s_o$ is an isometry of $\mathbb{X}$ satisfying $h(o) = o = g(o)$ and $d_o h = d_o g$, hence $h = g$. Therefore $K \subset H$.

We now consider a connected Lie group G, an involutive Lie group automorphism $\sigma : G \to G$, and K a compact subgroup of G such that $H_\circ \subset K \subset H$, where $H = \{g \in G \,|\, \sigma(g) = g\}$. Let $\mathfrak{k}$ be the Lie algebra of K.

Note that the Lie algebra of G^σ is $\{X \in \mathfrak{g} \,|\, \theta(X) = X\}$ where $\theta = d_e\sigma$. Since $G^\sigma_\circ \subset K \subset G^\sigma$, the groups G^σ and K have the same Lie algebra, hence:

$$\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$$

Let $\pi : G \to G/K$ be the projection, and let $o = \pi(e)$. The map $s : \left\{\begin{array}{ccc} G/K & \to & G/K \\ \pi(g) & \mapsto & \pi(\sigma(g)) \end{array}\right.$ is well defined because $K \subset H$. We have $s(o) = o$.

Let us show that $d_o s = -\mathrm{Id}$. Since $s \circ \pi = \pi \circ \sigma$, we have that $d_o s \circ d_e \pi = d_e \pi \circ d_e \sigma$.

Since $(d_e\sigma)^2 = \mathrm{Id}$, and the Lie algebra of H, which is equal to $\mathfrak{k}$, is the eigenspace of $d_e\sigma$ for the eigenvalue 1, the eigenspace of $d_e\sigma$ for the eigenvalue -1 is supplementary to $\ker d_e\pi = \mathfrak{k}$, hence $d_o s = -\mathrm{Id}$.

For $g \in G$, we let $m_g : G/K \to G/K$ be the multiplication by g. We find:

$$\begin{aligned} m_g \circ s \circ \pi &= m_g \circ \pi \circ \sigma \\ &= \pi \circ L_g \circ \sigma \\ &= \pi \circ \sigma \circ L_{\sigma(g)} \\ &= s \circ \pi \circ L_{\sigma(g)} \\ &= s \circ m_{\sigma(g)} \circ \pi \end{aligned}$$

It follows that $m_g \circ s = s \circ m_{\sigma(g)}$. We now consider a G-invariant Riemannian metric Ω on G/K. Then $s^*\Omega$ is also G-invariant:

$$\begin{aligned} m_g^*(s^*\Omega) &= (s \circ m_g)^*\Omega \\ &= (m_{\sigma(g)} \circ s)^*\Omega \\ &= s^*(m_{\sigma(g)}^*\Omega) \\ &= s^*\Omega \end{aligned}$$

Moreover, $(s^*\Omega)_o = \Omega_o$, and a G-invariant metric is characterized by its value at o, hence $s^*\Omega = \Omega$, and $(G/K, \Omega)$ is a Riemannian symmetric space. □

13.4.2 The Cartan involution

Definition 13.4.2. Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $K = \mathrm{Stab}_G(o)$. The **Cartan involution** relatively to o is the map $\theta = d_{\mathrm{Id}}\sigma : \mathfrak{g} \to \mathfrak{g}$ where $\sigma : G \to G$ is defined by $\sigma(g) = s_o \circ g \circ s_o$.

Example 13.4.3. For $\mathcal{E}_n$ and $o = 1_n$, we find $s_o(x) = x^{-1}$. For $[g] \in \mathrm{PSL}(n, \mathbb{R})$, write $\alpha([g]) \in G$ the associated isometry, i.e. $\alpha([g])(x) = gx^t g$. We find:

$$s_o \circ \alpha([g]) \circ s_o(x) = (gx^{-1}\,{}^t g)^{-1} = {}^t g^{-1} x^t ({}^t g^{-1})$$

Therefore $\sigma([g]) = [{}^t g^{-1}]$, and $\theta(X) = -{}^t X$ for $X \in \mathfrak{sl}(n, \mathbb{R})$.

Proposition 13.4.4. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $K = G_o$. Let θ be the Cartan involution relatively to o, and B the Killing form of $\mathfrak{g}$. Set $\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}$.*

1. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, *and this decomposition is* $\mathrm{Ad}(K)$*-invariant.*
2. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ *et* $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.
3. $\mathfrak{k}$ *is the Lie algebra of K and $d_e\varphi_o : \mathfrak{p} \to T_o\mathbb{X}$ is an isomorphism.*
4. $B|_{\mathfrak{k}\times\mathfrak{p}} = 0$.
5. $B|_{\mathfrak{k}\times\mathfrak{k}}$ *is negative definite.*

Remark. The decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is called the **Cartan decomposition** of $\mathfrak{g}$.

Proof.

1. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a consequence of $\theta^2 = \mathrm{Id}$.

   If $\sigma : G \to G$ denotes the conjugacy by s_o, then $i_g \circ \sigma = \sigma \circ i_{\sigma(g)}$ for all $g \in G$. For $g \in K$, we have $\sigma(g) = g$, hence $i_g \circ \sigma = \sigma \circ i_g$. Differentiating at Id yields $\mathrm{Ad}(g) \circ \theta = \theta \circ \mathrm{Ad}(g)$, hence the $\mathrm{Ad}(g)$-invariance of the decomposition $\mathfrak{k} \oplus \mathfrak{p}$.

2. All three inclusions are a consequence of the fact that θ is a Lie algebra morphism.

3. We have seen in the proof of Proposition 13.4.1 that $\mathfrak{k}$ is the Lie algebra of K.

   The map $d_e\varphi_o$ is surjective because the action of G is transitive. Its kernel is the Lie algebra of K, hence supplementary to $\mathfrak{p}$, therefore $d_e\varphi_o : \mathfrak{p} \to T_oX$ is an isomorphism.

4. If $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, then the matrix of $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ in a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ has vanishing diagonal blocs, hence $\mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 0$, i.e. $B(X,Y) = 0$.

5. Let $\langle\langle \cdot | \cdot \rangle\rangle$ be an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathfrak{g}$.

   For $X \in \mathfrak{k}$, we find:

$$\begin{aligned} B(X,X) &= \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X)) \\ &= \sum_{i=1}^{n} \langle\langle \mathrm{ad}(X) \circ \mathrm{ad}(X) e_i \,|\, e_i \rangle\rangle \\ &= -\sum_{i=1}^{n} \langle\langle \mathrm{ad}(X) e_i \,|\, \mathrm{ad}(X) e_i \rangle\rangle \\ &= -\sum_{i=1}^{n} \|[X, e_i]\|^2 \end{aligned}$$

   We get $B(X,X) \leq 0$. Moreover, if $B(X,X) = 0$, then $X \in \mathfrak{z}(\mathfrak{g})$. It follows that for all $t \in \mathbb{R}$ and $Y \in \mathfrak{g}$, $\exp(tX)$ commutes with $\exp(Y)$.

   Since G is connected, we find that $\exp(tX) \in Z(G)$. The G-action on M is transitive, and for all $g \in G$ we have:

$$\exp(tX)(g(o)) = g(\exp(tX)(o)) = g(o)$$

   It follows that $\exp(tX) = \mathrm{Id}$, hence $X = 0$.

□

Example 13.4.5. For $\mathcal{E}_n$ and $o = 1_n$, we find

$$\mathfrak{k} = \mathfrak{so}(n,\mathbb{R}) \text{ and } \mathfrak{p} = \left\{ X \in \mathfrak{sl}(n,\mathbb{R}) \,\middle|\, {}^tX = X \right\}.$$

The Killing form of $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ is:

$$B(X,Y) = 4n \,\mathrm{Tr}(XY)$$

Chapter 14

The geometry of symmetric spaces

14.1 Geodesics in symmetric spaces

Definition 14.1.1. Let (M,g) be a Riemannian manifold. An isometry $\varphi \in \mathrm{Isom}(M,g)$ is a **transvection** if there are a non constant geodesic $c : \mathbb{R} \to M$ and $t_0 \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, we have $\varphi(c(t)) = c(t+t_0)$ and $d_{c(t)}\varphi = \|_t^{t+t_0}$.

Lemma 14.1.2. *Let $\mathbb{X}$ be a symmetric space, and let $c : \mathbb{R} \to M$ be a non constant geodesic. For $t \in \mathbb{R}$, consider $g_t = s_{c\left(\frac{t}{2}\right)} \circ s_{c(0)}$. Then g_t is a transvection, and $t \mapsto g_t$ is a one-parameter subgroup of G.*

Proof. Lemma 13.1.2 shows that $g_t(c(s)) = c(t+s)$, and Proposition 13.2.9 shows that $d_{c(s)}g_t = \|_s^{s+t}$, so g_t is a transvection.

The isometries g_{t+s} and $g_t \circ g_s$ both send $o = c(0)$ to $c(t+s)$, and their differential at o is the parallel transport along c (Proposition 13.2.9). It follows that they are equal, i.e. $t \mapsto g_t$ is a one parameter subgroup of G (it is smooth thanks to Lemma 13.1.4). □

Proposition 14.1.3. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. For all $X \in \mathfrak{p}$ and $g \in G$, the isometry $g\exp_G(X)g^{-1}$ is a transvection of $\mathbb{X}$. Moreover, any transvection of $\mathbb{X}$ has this form.*

Proof. Since the conjugate of a transvection by an isometry is a transvection, so we only have to show that $\exp_G(X)$ is a transvection for $X \in \mathfrak{p} \setminus \{0\}$. Set $v = d_e\varphi_o(X)$, and $c = c_v$ (it is a non constant geodesic because of Proposition 13.4.4). For $t \in \mathbb{R}$, we let $g_t = s_{c\left(\frac{t}{2}\right)} \circ s_o$.

According to Lemma 14.1.2, $t \mapsto g_t$ is a one-parameter subgroup of G, so we can consider $X' \in \mathfrak{g}$ such that $g_t = \exp_G(tX')$. First, we wish to show that X' is in $\mathfrak{p}$. For this we compute $\sigma\left(\exp_G(tX')\right)$

$$\begin{aligned}
\sigma\left(\exp_G(tX')\right) &= s_o \circ s_{c\left(\frac{t}{2}\right)} \circ s_o \circ s_o \\
&= s_o \circ s_{c\left(\frac{t}{2}\right)} \\
&= \left(s_{c\left(\frac{t}{2}\right)} \circ s_o\right)^{-1} \\
&= g_t^{-1} \\
&= g_{-t} \\
&= \exp_G(-tX')
\end{aligned}$$

The derivative at $t = 0$ yields $\theta(X') = -X'$, i.e. $X' \in \mathfrak{p}$. Since $g_t(o) = c(t)$, we find:

$$\begin{aligned}
v &= \left.\frac{d}{dt}\right|_{t=0} g_t(o) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi_o\left(\exp_G(tX')\right) \\
&= d_e\varphi_o(X')
\end{aligned}$$

It follows from Proposition 13.4.4 that $X' = X$. Hence $\exp_G(X) = g_1$ is a transvection.

Reciprocally, let $g \in \mathrm{Isom}(\mathbb{X})$ be a transvection along a geodesic c. Let $t_0 \in \mathbb{R}$ be given by the definition of a transvection. Since G acts transitively on $\mathbb{X}$, we can assume that $c(0) = o$.

Consider $g_t = s_{c\left(\frac{t}{2}\right)} \circ s_o$ as in Lemma 14.1.2. Then g and g_{t_0} have the same one-jet at o, so they are equal.

Let $X \in \mathfrak{g}$ be such that $g_t = \exp_G(tX)$ for all $t \in \mathbb{R}$. The computation above shows that $X \in \mathfrak{p}$. We find that $g = \exp_G(t_0X)$. □

Corollary 14.1.4. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The geodesics going through o are exactly the curves $t \mapsto \exp_G(tX)o$ for $X \in \mathfrak{p}$.*

Proof. Let $v \in T_o\mathbb{X}$, and set $c = c_v$. According to Proposition 13.4.4, there is $X \in \mathfrak{p}$ such that $v = d_e\varphi_o(X)$. Following the proof of Proposition 14.1.3, we find:

$$\begin{aligned}
\exp_G(tX).o &= s_{c\left(\frac{t}{2}\right)} \circ s_o(o) \\
&= s_{c\left(\frac{t}{2}\right)}(o) \\
&= c(t)
\end{aligned}$$

□

14.2 The Levi-Civita connection of a symmetric space

In order to relate the Levi-Civita connection of a symmetric space with computations in the Lie algebra of its isometry group, we can can start by seeing that a smooth action $G \curvearrowright M$ induces a Lie algebra anti-morphism (i.e. reversing the bracket) from the Lie algebra of G to the Lie algebra $\mathcal{X}(M)$ of vector fields on M.

Definition 14.2.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold. Consider a smooth action $G \curvearrowright M$. For $X \in \mathfrak{g}$, the **fundamental vector field** associated to X is $\overline{X} \in \mathcal{X}(M)$ defined by:

$$\overline{X}(x) = d_e\varphi_x(X)$$

Lemma 14.2.2. *Let $G \curvearrowright M$ be a smooth action of a Lie group. For $X, Y \in \mathfrak{g}$, we have $[\overline{X}, \overline{Y}] = \overline{[Y, X]}$.*

Remark. This result can be interpreted by seeing Diff(M) as an infinite dimensional Lie group, whose Lie algebra is $\mathcal{X}(M)$, but the bracket is the opposite of the usual Lie bracket of vector fields.

Proof. First notice that the flow φ^t of $\overline{X}$ is given by:

$$\varphi^t(x) = \exp_G(tX).x$$

Indeed, $(t, x) \mapsto \exp_G(tX).x$ is a flow and we have:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \exp_G(tX).x &= \left.\frac{d}{dt}\right|_{t=0} \varphi_x\Big(\exp_G(tX)\Big) \\ &= d_e\varphi_x(X) \\ &= \overline{X}(x) \end{aligned}$$

So the flow ψ^t of $\overline{Y}$ is also given by:

$$\psi^t(x) = \exp_G(tY).x$$

We can compute $[\overline{X}, \overline{Y}]$ by looking at the commutators of the flows:

$$\begin{aligned} [\overline{X}, \overline{Y}](x) &= \left.\frac{d}{dt}\right|_{t=0^+} \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}}(x) \\ &= \left.\frac{d}{dt}\right|_{t=0^+} (\exp(\sqrt{t}Y)\exp(\sqrt{t}X)\exp(-\sqrt{t}Y)\exp(-\sqrt{t}X)).x \end{aligned}$$

But we also have:

$$\left.\frac{d}{dt}\right|_{t=0^+} \exp(\sqrt{t}Y)\exp(\sqrt{t}X)\exp(-\sqrt{t}Y)\exp(-\sqrt{t}X) = [Y, X]$$

It follows that $[\overline{X}, \overline{Y}] = \overline{[Y, X]}$. $\square$

Definition 14.2.3. Let (M,g) be a Riemannian manifold. A **Killing field** is $X \in \mathcal{X}(M)$ whose flow preserves g, i.e.

$$g_{\varphi_X^t(x)}\left(d_x\varphi_X^t(u), d_x\varphi_X^t(v)\right) = g_x(u,v)$$

whenever $\varphi_X^t(x)$ is defined.

Remark. This condition is equivalent to $\mathcal{L}_X g = 0$, where $\mathcal{L}_X$ is the Lie derivative.

If a Lie group G acts smoothly on a Riemannian manifold (M,g) by isometries, i.e. the maps $x \mapsto g.x$ are isometries, then the fundamental vector fields are Killing fields.

Lemma 14.2.4. *Let (M,g) a Riemmanian manifold with Levi-Civita connection ∇. Let $X \in \mathcal{X}(M)$ be a Killing field. For all $V, W \in \mathcal{X}(M)$, we have:*

$$[X, \nabla_V W] = \nabla_{[X,V]} W + \nabla_V [X, W].$$

Proof. For $\varphi \in \mathrm{Isom}(M,g)$, we have $\varphi^*\nabla = \nabla$ (because $\varphi^*\nabla$ is the Levi-Civita connection of $\varphi^* g = g$). This means $\varphi^*(\nabla W(V)) = \nabla\varphi^* W(\varphi^* V)$.

We can apply this to the flow φ^t of X, and find:

$$(\varphi^t)^*(\nabla W(V)) = \nabla(\varphi^t)^* W\left((\varphi^t)^* V\right)$$

The derivative of the left hand side at $t=0$ is $[X, \nabla W(V)]$ by definition of the Lie bracket. The right hand side derivates to $\nabla W([X,V]) + \nabla[X,W](V)$. □

Theorem 14.2.5. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Consider $X \in \mathfrak{p}$ and $v = d_e\varphi_o(X)$. For all $V \in \mathcal{X}(M)$, we have $\nabla_o V(v) = [\overline{X}, V](o)$.*

Proof. Let φ^t be the flow of $\overline{X}$, i.e. $\varphi^t(x) = \exp_G(tX).x$. The relationship between a connection and its parallel transport yields

$$\nabla_o V(v) = \left.\frac{d}{dt}\right|_{t=0} \Big\|_{\varphi^t(o)}^{o} V\left(\varphi^t(o)\right)$$

Here $\|_{\varphi^t(o)}^{o}$ is the parallel transport along the flow line $t \mapsto \varphi^t(o) = \exp_G(tX)$. According to Proposition 14.1.3, it is equal to the differential of the transvection φ^t, hence:

$$\nabla_o V(v) = \left.\frac{d}{dt}\right|_{t=0} \left(d_o\varphi^t\right)^{-1}\left(V\left(\varphi^t(o)\right)\right)$$

We recognize $\left(\varphi^t\right)^* V(o)$, and find $\nabla_o V(v) = [\overline{X}, V](o)$. □

14.3 The curvature of a symmetric space

Theorem 14.3.1. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. For $X, Y, Z \in \mathfrak{p}$, we set $u = d_e\varphi_o(X), v = d_e\varphi_o(Y)$ and $w = d_e\varphi_o(Z) \in T_o\mathbb{X}$. We have:*

$$R_o(u,v)w = -d_e\varphi_o([[X,Y],Z])$$

Remark. Note that $[X,Y] \in \mathfrak{k}$, hence $[[X,Y],Z] \in \mathfrak{p}$ (Proposition 13.4.4). In particular, we find $R_o(u,v)w = 0 \iff [[X,Y],Z] = 0$.

Proof. We use the fundamental vector fields $\overline{X}, \overline{Y}, \overline{Z}$ to compute the curvature.

$$\begin{aligned} R_o(u,v)w &= R(\overline{X},\overline{Y})\overline{Z}(x_0) \\ &= \nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) - \nabla_o\left(\nabla\overline{Z}(\overline{X})\right)(v) - \nabla_o\overline{Z}\left([\overline{X},\overline{Y}](o)\right) \end{aligned} \tag{14.1}$$

According to Lemma 14.2.2, we find:

$$\begin{aligned} [\overline{X},\overline{Y}](o) &= -\overline{[X,Y]}(o) \\ &= -d_e\varphi_o([X,Y]) \end{aligned}$$

But Proposition 13.4.4 shows that $[X,Y] \in \mathfrak{k} = \ker d_e\varphi_o$, hence $[\overline{X},\overline{Y}](o) = 0$, and (14.1) simplifies:

$$R_o(u,v)w = \nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) - \nabla_o\left(\nabla\overline{Z}(\overline{X})\right)(v) \tag{14.2}$$

Theorem 14.2.5 yields

$$\nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) = \left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o)$$

Since $\overline{X}$ is a Killing field, we can apply Lemma 14.2.4.

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right] = \nabla\overline{Z}\left([\overline{X},\overline{Y}]\right) + \nabla[\overline{X},\overline{Z}](\overline{Y})$$

Evaluating at o, we have seen that $[\overline{X},\overline{Y}](o) = 0$, so all that remains is:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = \nabla_o[\overline{X},\overline{Z}](v)$$

Applying Theorem 14.2.5 once again, we find:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = [\overline{Y},[\overline{X},\overline{Z}]](o)$$

Two applications of Lemma 14.2.2 yield:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = \overline{[Y,[X,Z]]}(o)$$

We have shown:

$$\nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) = \overline{[Y,[X,Z]]}(o) \tag{14.3}$$

Injecting (14.3) back into (14.2), we find:

$$R_o(u,v)w = \overline{[Y,[X,Z]]}(o) - \overline{[X,[Y,Z]]}(o)$$

The Jacobi identity yields the desired formula. □

Example 14.3.2. For $\mathcal{E}_n$, we can compute the sectional curvature of the plane generated by $X, Y \in \mathfrak{p}$:

$$R(X,Y,Y,X) = -\mathrm{Tr}([[X,Y],Y]X) = \mathrm{Tr}([X,Y]^2)$$

We find that the sectional curvature is 0 if $[X,Y] = 0$, and negative otherwise.

14.4 Lie triple systems and totally geodesic submanifolds of symmetric spaces

Recall that a submanifold N of a Riemannian manifold $(M,g)M$ is called **totally geodesic** if any geodesic of N is a geodesic of M.

Now we start with a point $x \in M$ and a vector subspace $V \subset T_xM$, and wonder whether there is a totally geodesic submanifold $N \subset M$ such that $x \in N$ and $T_xN = V$. Actually there is not much choice for N, as it should be an open subset of $\exp_x(V)$. But in general, $\exp_x(V)$ is not totally geodesic.

We have already seen that a necessary condition is for V to be stable under the Riemann tensor: if $u,v,w \in V$, then $R_x(u,v)w \in V$. When (M,g) is a symmetric space, we will show that it is also a sufficient condition. In the general case, it cannot be sufficient as one should at least impose some stability by the covariant derivatives of the Riemann tensor.

This stability under the Riemann tensor has a nice interpretation in Lie algebraic terms for a symmetric space.

Definition 14.4.1. Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{v} \subset \mathfrak{g}$ is called a **Lie triple system** if it satisfies:

$$\forall X,Y,Z \in \mathfrak{v} \quad [[X,Y],Z] \in \mathfrak{v}$$

Remark. This can be summarized as $[\mathfrak{v},[\mathfrak{v},\mathfrak{v}]] \subset \mathfrak{v}$.

Lemma 14.4.2. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{v} \subset \mathfrak{g}$ is a Lie triple system, then $[\mathfrak{v},\mathfrak{v}]$ and $\mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$ are Lie subalgebras of $\mathfrak{g}$.*

Proof. For $X,Y,Z,W \in \mathfrak{v}$, the Jacobi identity yields:

$$[[X,Y],[Z,W]] = -[Z,\underbrace{[W,[X,Y]]}_{\in\mathfrak{v}} - [X,\underbrace{[[X,Y],Z]}_{\in\mathfrak{v}} \in [\mathfrak{v},\mathfrak{v}]$$

For $X,Y \in \mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$, there are three cases to deal with:

- If $X \in \mathfrak{v}$ and $Y \in \mathfrak{v}$ then $[X,Y] \in [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$.
- If $X \in \mathfrak{v}$ and $Y \in [\mathfrak{v},\mathfrak{v}]$ then $[X,Y] \in \mathfrak{v} \subset \mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$ by definition of a Lie triple system.
- If $X \in [\mathfrak{v},\mathfrak{v}]$ and $Y \in [\mathfrak{v},\mathfrak{v}]$ then $[X,Y] \in [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$ because $[\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra.

These three cases show that $\mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra of $\mathfrak{g}$. □

Theorem 14.4.3. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $V \subset T_o\mathbb{X}$ be a vector subspace. The following are equivalent:*

1. *V is stable under the Riemann tensor R_o.*
2. *$(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ is a Lie triple system in $\mathfrak{g}$.*
3. *There is a totally geodesic submanifold N of $\mathbb{X}$ such that $x \in N$ and $T_xN = V$.*

Moreover, if these conditions are met, then $\exp_o(V)$ is an immersed totally geodesic submanifold of $\mathbb{X}$, and it is a symmetric space.

Proof. Note that *1.* $\Longleftrightarrow$ *2.* is a straightforward consequence of Theorem 14.3.1.

3.$\Rightarrow$*1.* is a general fact for Riemannian manifolds (Lemma 12.6.2). We will now prove *2.* implies *3.* and the last statement.

Let $\mathfrak{v} = (d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$, and assume that it is a Lie triple system. According to Lemma 14.4.2, $\mathfrak{h} = \mathfrak{v}+[\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra of $\mathfrak{g}$, so we can consider the connected Lie subgroup $H \subset G$ such that $T_eH = \mathfrak{h}$.

Note that $H.o$ is an immersed submanifold on M (it is an orbit of a smooth action), and that $T_oH.o = d_e\varphi_o\mathfrak{h} = V$ (because $[\mathfrak{v},\mathfrak{v}] \subset \mathfrak{k} = \ker d_e\varphi_o$).

Let us show that $\exp_o(V) \subset H.o$. Let $v \in V$, and consider $X \in \mathfrak{v}$ such that $d_e\varphi_o(X) = v$. Corollary 14.1.4 yields $\exp_o(v) = \exp_G(X).o$. Since $X \in \mathfrak{v} \subset \mathfrak{h}$, we find $\exp_G(X) \in H$ and $\exp_o(v) \in H.o$. So we have proved $\exp_o(V) \subset H.o$.

This means that any geodesic starting from o and tangent to V lies in $H.o$. Given $x \in H.o$, we write $x = g.o$ for some $g \in H$, and we have $T_xH.o =$

$d_o g(V)$. For $v \in V$, by setting $w = d_o g(v)$, we find $c_w(t) = g.c_v(t) \in g(H.o) = H.o$. This shows that $H.o$ is totally geodesic, i.e. we have proven *3*.

Note that $H.o$ is complete. Indeed, for $v \in V$ the geodesic c_v of $H.o$ is defined on $\mathbb{R}$. It follows from the Hopf-Rinow Theorem that $H.o \subset \exp_o(V)$, hence $H.o = \exp_o(V)$. So $\exp_o(V)$ is an immersed totally geodesic submanifold of $\mathbb{X}$.

Now note that $\exp_o(V)$ is stable under s_o. This implies that $H.o$ is also stable under $s_{g.o} = g s_o g^{-1}$ for $g \in H$, so $H.o$ is a symmetric space. $\square$

Definition 14.4.4. Let $\mathbb{X}$ be a symmetric space. A **flat** of $\mathbb{X}$ is a complete and connected totally geodesic submanifold $F \subset M$ which is flat.

The **rank** of $\mathbb{X}$ is the maximal dimension of a flat of $\mathbb{X}$.

Remark. The rank is sometimes called the geometric rank.

Proposition 14.4.5. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $V \subset T_o\mathbb{X}$ be a vector subspace. There is a flat $F \subset \mathbb{X}$ such that $T_oF = V$ if and only if $(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ is an abelian subalgebra of $\mathfrak{g}$.*

Remark. One only need to ask of $(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ to be a Lie subalgebra, since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ shows that a vector subspace of $\mathfrak{p}$ must be abelian in order to be a Lie subalgebra of $\mathfrak{g}$.

Proof. Let $\mathfrak{v} = (d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$. If $\mathfrak{v}$ is an abelian subalgebra of $\mathfrak{g}$, then it is a Lie triple system and Theorem 14.4.3 shows that there is a totally geodesic submanifold $F \subset X$ such that $T_oF = V$, and that F is a symmetric space. The curvature of F can be computed thanks to Theorem 14.3.1, and it vanishes.

Now assume that there is a flat $F \subset \mathbb{X}$ such that $T_oF = V$. Theorem 14.3.1 yields $[[X,Y],Z] = 0$ for all $X,Y,Z \in \mathfrak{v}$. Let B be the Killing form of $\mathfrak{g}$. Since B is ad-invariant, we find for $X,Y \in \mathfrak{v}$:

$$\begin{aligned} B([X,Y],[X,Y]) &= B(\mathrm{ad}(X)Y,[X,Y]) \\ &= -B(Y,\mathrm{ad}(X)[X,Y]) \\ &= B(Y,[[X,Y],X]) \end{aligned}$$

Since $[[X,Y],X] = 0$, we find that $B([X,Y],[X,Y]) = 0$. But we showed in Proposition 13.4.4 that B is negative on $\mathfrak{k}$, and $[X,Y] \in [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. It follows that $[X,Y] = 0$, i.e. $\mathfrak{v}$ is an abelian subalgebra of $\mathfrak{g}$. $\square$

Examples 14.4.6. The rank of $\mathbb{E}^n$ is n, the rank of $\mathbb{S}^n$ and $\mathbb{H}^n$ is 1. For $\mathcal{E}_n$, a flat corresponds to matrices that commute, so they are diagonalisable in the same basis. It follows that the rank of $\mathcal{E}_n$ is $n-1$.

14.5 The sectional curvature of symmetric spaces

In order to relate the geometry of a symmetric space $\mathbb{X}$ and the Lie algebra $\mathfrak{g}$ of its isometry group, we first notice that $\mathfrak{p}$ in the Cartan decomposition has two quadratic forms: the Riemannian metric of $\mathbb{X}$ and the Killing form of $\mathfrak{g}$. To understand their relationship, we must first ask if they have the same signature.

Definition 14.5.1. Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, and let $o \in \mathbb{X}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We say that $\mathbb{X}$ is:

- Of **Euclidean type** if $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$.
- Of **compact type** if $B|_{\mathfrak{p}\times\mathfrak{p}}$ is negative definite.
- Of **non compact type** if $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite.

Remark. By using the homogeneity, we can see that this definition does not depend on the choice of $o \in \mathbb{X}$.

The Euclidean type can be interpreted in terms of curvature.

Proposition 14.5.2. *Let $\mathbb{X}$ be a symmetric space. Then $\mathbb{X}$ is of Euclidean type if and only if $\mathbb{X}$ is flat.*

Proof. According to Proposition 14.4.5 applied to $T_o\mathbb{X}$, we see that $\mathbb{X}$ is flat if and only if $\mathfrak{p}$ is abelian. If $\mathfrak{p}$ is abelian, then $[\mathfrak{p},\mathfrak{k}] \subset \mathfrak{p}$ shows that it is an ideal, which completes the proof. □

Consequently, a simply connected symmetric space of Euclidean type is isometric to the Euclidean space $\mathbb{E}^n$, hence the terminology.

Not every symmetric space is of one of these types, but the classification of symmetric spaces reduces to these three types. Note that the product of symmetric spaces is always a symmetric space, so a classification of symmetric spaces requires the understanding of which symmetric spaces can split into a product.

Definition 14.5.3. Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, and let $o \in \mathbb{X}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We say that $\mathbb{X}$ is **irreducible** if the only vector subspaces $\mathfrak{v} \subset \mathfrak{p}$ satisfying $[\mathfrak{k},\mathfrak{v}] \subset \mathfrak{v}$ are $\{0\}$ and $\mathfrak{p}$.

Remarks.

- By using the homogeneity, we can see that this definition does not depend on the choice of $o \in \mathbb{X}$.

- This is equivalent to stating that the action of the identity component of K on $T_o\mathbb{K}$ is irreducible.

Irreducible symmetric spaces must be of one of the three types described above.

Proposition 14.5.4. *Every irreducible symmetric space is either of Euclidean, compact or non compact type.*

Before we can prove Proposition 14.5.4, we need to introduce some notations.

Definition 14.5.5. Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.

The **Riemannian form** is the inner product $\langle \cdot | \cdot \rangle$ on $\mathfrak{p}$ defined by:

$$\forall X, Y \in \mathfrak{p} \quad \langle X|Y \rangle = \langle d_e\varphi_o(X) | d_e\varphi_o(Y) \rangle_o$$

The **Killing operator** is the self-adjoint (for $\langle \cdot | \cdot \rangle$) operator $b \in \mathrm{End}(\mathfrak{p})$ defined by:

$$\forall X, Y \in \mathfrak{p} \quad B(X,Y) = \langle b(X) | Y \rangle$$

Lemma 14.5.6. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The Killing operator b commutes with* $\mathrm{ad}(X)$ *for all $X \in \mathfrak{k}$.*

Proof. Since $\exp_G(tX) \in \mathrm{Isom}(\mathbb{X})$, we have:

$$\forall v, w \in T_o\mathbb{X} \quad \Big\langle d_o \exp_G(tX)(v) \Big| d_o \exp_G(tX)(w) \Big\rangle_{\exp_G(tX).o} = \langle v|w \rangle_o$$

The derivative at $t = 0$ pulled back by φ_o yields:

$$\forall Y, Z \in \mathfrak{p} \quad \langle \mathrm{ad}(X)Y|Z \rangle + \langle Y|\mathrm{ad}(X)Z \rangle = 0 \tag{14.4}$$

The ad-invariance of the Killing form now translates to b as:

$$\forall Y, Z \in \mathfrak{p} \quad \underbrace{\langle b \circ \mathrm{ad}(X)Y|Z \rangle}_{=B(\mathrm{ad}(X)Y,Z)} + \underbrace{\langle b(Y)|\mathrm{ad}(X)Z \rangle}_{=B(Y,\mathrm{ad}(X)Z)} = 0 \tag{14.5}$$

Putting (14.4) and (14.5) together, we find for all $Y, Z \in \mathfrak{p}$:

$$\begin{aligned} \langle b \circ \mathrm{ad}(X)(Y)|Z \rangle &= -\langle b(Y)|\mathrm{ad}(X)(Z) \rangle \\ &= \ \langle \mathrm{ad}(X) \circ b(Y)|Z \rangle \end{aligned}$$

Since $\langle \cdot | \cdot \rangle$ is positive definite, it follows that $b \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ b$. □

Proof of Proposition 14.5.4. Since b is self-adjoint, there is an orthonormal basis $X_1, \dots, X_n$ of $\mathfrak{p}$ such that $b(X_i) = \lambda_i X_i$ for $\lambda_i \in \mathbb{R}$. We set:

$$\begin{aligned} \mathfrak{v}_0 &= \mathrm{Vect}\{X_i \mid \lambda_i = 0\} \\ \mathfrak{v}_- &= \mathrm{Vect}\{X_i \mid \lambda_i < 0\} \\ \mathfrak{v}_+ &= \mathrm{Vect}\{X_i \mid \lambda_i > 0\} \end{aligned}$$

We have a direct sum $\mathfrak{p} = \mathfrak{v}_0 \oplus \mathfrak{v}_- \oplus \mathfrak{v}_+$, this decomposition is orthogonal for both $\langle\cdot|\cdot\rangle$ and $B|_{\mathfrak{p}\times\mathfrak{p}}$. It is also $\mathrm{ad}(\mathfrak{k})$-invariant because of Lemma 14.5.6.

If $\mathbb{X}$ is irreducible, then one of the three K-invariant spaces $\mathfrak{v}_0$, $\mathfrak{v}_-$, $\mathfrak{v}_+$ is equal to $\mathfrak{p}$. If $\varphi = \mathfrak{v}_0$, then $\mathbb{X}$ is of Euclidean type. If $\varphi = \mathfrak{v}_+$, then $\mathbb{X}$ is of compact type. If $\varphi = \mathfrak{v}_-$, then $\mathbb{X}$ is of non compact type.

□

Theorem 14.5.7. *Let $\mathbb{X}$ be an irreducible symmetric space of compact or non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*

There is $\alpha \in \mathbb{R} \setminus \{0\}$ such that $B(X,Y) = \alpha \langle X|Y\rangle$ for all $X, Y \in \mathfrak{p}$.

If (u,v) is an orthonormal basis of a plane $P \subset T_o\mathbb{X}$, and $X, Y \in \mathfrak{p}$ are such that $d_e\varphi_o(X) = u$ and $d_e\varphi_o(Y) = v$, then:

$$K(P) = \frac{1}{\alpha} B([X,Y],[X,Y])$$

If $\mathbb{X}$ is of compact (resp. non compact) type, then it has non negative (resp. non positive) sectional curvature.

Proof. The Killing operator b is diagonalisable, and $\mathrm{ad}(X)$ commutes with b so it must preserve its eigenspaces for each $X \in \mathfrak{k}$, so there is $\alpha \in \mathbb{R}$ such that $b = \alpha\,\mathrm{Id}$. Therefore $B(X,Y) = \alpha \langle X|Y\rangle$ for all $X, Y \in \mathfrak{p}$.

By looking at the sign of B, we see that $\alpha < 0$ (resp. $\alpha > 0$) when $\mathbb{X}$ is of compact (resp. non compact) type.

$$\begin{aligned} K(P) &= R_o(u,v,v,u) \\ &= \langle R_o(u,v)v|u\rangle_o \\ &= \langle -[[X,Y],Y]|X\rangle \\ &= \frac{1}{\alpha} B([Y,[X,Y]],X) \\ &= \frac{1}{\alpha} B([X,Y],[X,Y]) \end{aligned}$$

The sign of the sectional curvature comes from the fact that B is negative definite on $\mathfrak{k}$. □

Chapter 15

Classification of symmetric spaces

15.1 Decomposition into irreducible factors

Recall that the universal cover of a symmetric space is still a symmetric space, and that the product of symmetric spaces is also a symmetric space. Up to these manipulations, the classification of symmetric spaces reduces to the irreducible ones.

Theorem 15.1.1. *Every simply connected symmetric space is isometric to a product $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$ where each $\mathbb{X}_i$ is an irreducible symmetric space.*

Lemma 15.1.2. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*

The vector space $\mathfrak{p}$ admits a decomposition

$$\mathfrak{p} = \mathfrak{v}_0 \oplus \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k_+} \oplus \mathfrak{v}_{-1} \oplus \cdots \oplus \mathfrak{v}_{k_-}$$

such that:

- $[\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$ *and* $\mathfrak{v}_i$ *is irreducible for* $i \neq 0$.
- *If $i \neq j$ then $\mathfrak{v}_i$ et $\mathfrak{v}_j$ are orthogonal for both the Riemannian form $\langle \cdot | \cdot \rangle$ and the Killing form B.*
- $\mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k_+}$ *is equal to the sum $\mathfrak{v}_+$ of eigenspaces attached to positive eigenvalues of the Killing operator b and $\mathfrak{v}_{-1} \oplus \cdots \oplus \mathfrak{v}_{k_-}$ is equal to the sum $\mathfrak{v}_-$ of eigenspaces attached to negative eigenvalues of b.*
- $\mathfrak{v}_0 = \ker b$.

Proof. Since $\mathrm{ad}(X)$ preserves the Riemannian form on $\mathfrak{p}$ for all $X \in \mathfrak{k}$, the orthogonal of an $\mathrm{ad}(\mathfrak{k})$-invariant subspace is also invariant, and we can find a decomposition of $\mathfrak{p}$ into irreducible subspaces. Each of them must be included in an eigenspace of b. □

Definition 15.1.3. Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.

The decomposition $\mathfrak{p} = \bigoplus_{k_- \le i \le k_+} \mathfrak{v}_i$ given by Lemma 15.1.2 is called the **decomposition into irreducible factors of** $\mathfrak{p}$.

Remark. The space $\mathfrak{v}_0$ may not be irreducible.

Lemma 15.1.4. *Let $\mathbb{X}$ be an irreducible symmetric space of compact or non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*

The kernel of the Killing operator $\mathfrak{v}_0 = \ker b \subset \mathfrak{p}$ is equal to $\ker B$. It is an abelian ideal of $\mathfrak{g}$, and $[\mathfrak{v}_0, \mathfrak{p}] = \{0\}$.

Proof. The definition of $\mathfrak{v}_0$ yields $B(X,Y) = 0$ for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{p}$. Since $\mathfrak{k}$ is B-orthogonal to $\mathfrak{p}$, we also have $B(X,Y) = 0$ for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{k}$, hence $\mathfrak{v}_0 \subset \ker B$.

Since B is negative definite on $\mathfrak{k}$, we find that $\ker B$ is included in the orthogonal for B of $\mathfrak{k}$, i.e. in $\mathfrak{p}$. It follows that $\ker B \subset \ker b = \mathfrak{v}_0$.

This implies that $\mathfrak{v}_0$ is an ideal of $\mathfrak{g}$. Moreover, for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{p}$, we have $[X,Y] \in \mathfrak{k} \cap \mathfrak{v}_0 = \{0\}$, hence $[\mathfrak{v}_0, \mathfrak{p}] = \{0\}$, and $\mathfrak{v}_0$ is abelian. □

Lemma 15.1.5. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p} = \bigoplus_{k_- \le i \le k_+} \mathfrak{v}_i$ the decomposition into irreducible factors.*

For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$.

1. *$\mathfrak{v}_i$ is a Lie triple system of $\mathfrak{g}$.*
2. *$\mathfrak{l}_i$ is an ideal of $\mathfrak{k}$.*
3. *$\mathfrak{h}_i$ is an ideal of $\mathfrak{g}$.*
4. *If $i \neq j$ are both different from 0, then $[\mathfrak{h}_i, \mathfrak{h}_j] = \{0\}$.*
5. *If $i \neq j$ are both different from 0, the ideals $\mathfrak{l}_i$ and $\mathfrak{l}_j$ of $\mathfrak{k}$ are orthogonal for B.*

Proof. 1. Since $[\mathfrak{v}_i, \mathfrak{v}_i] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have $[[\mathfrak{v}_i, \mathfrak{v}_i], \mathfrak{v}_i] \subset [\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$, so $\mathfrak{v}_i$ is a Lie triple system.

2. For $X \in \mathfrak{k}$ and $Y, Z \in \mathfrak{v}_i$, we find:

$$[X,[Y,Z]] = -[\underbrace{Y}_{\in \mathfrak{v}_i}, \underbrace{[Z,X]}_{\in [\mathfrak{v}_i, \mathfrak{k}] \subset \mathfrak{v}_i}] - [\underbrace{Z}_{\in \mathfrak{v}_i}, \underbrace{[X,Y]}_{\in [\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i}] \in [\mathfrak{v}_i, \mathfrak{v}_i] = \mathfrak{l}_i$$

3. Let us start by proving that $[\mathfrak{v}_i, \mathfrak{v}_j] = \{0\}$. Let $X \in \mathfrak{v}_j$ and $Y \in \mathfrak{v}_i$. Since $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are B-orthogonal, we find:

$$B([X,Y],[X,Y]) = B(X, \underbrace{[Y,[X,Y]]}_{\in[\mathfrak{v}_i,\mathfrak{k}]\subset\mathfrak{v}_i}) = 0$$

Since B is negative definite on $\mathfrak{k}$, we find $[X,Y] = 0$, and $[\mathfrak{v}_j, \mathfrak{v}_i] = \{0\}$.

We already know that $[\mathfrak{k}, \mathfrak{l}_i] \subset \mathfrak{l}_i$ and $[\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$, so we find that $[\mathfrak{k}, \mathfrak{h}_i] \subset \mathfrak{h}_i$.

The fact that $[\mathfrak{v}_i, \mathfrak{v}_j] = \{0\}$ for $j \neq i$ shows that $[\mathfrak{p}, \mathfrak{v}_i] \subset [\mathfrak{v}_i, \mathfrak{v}_i] = \mathfrak{l}_i$. Finally, for $X \in \mathfrak{p}$ and $Y, Z \in \mathfrak{v}_i$, we have:

$$[X,[Y,Z]] = -[\underbrace{Y}_{\in\mathfrak{v}_i}, \underbrace{[Z,X]}_{\in[\mathfrak{v}_i,\mathfrak{p}]\subset\mathfrak{l}_i}] - [\underbrace{Z}_{\in\mathfrak{v}_i}, \underbrace{[X,Y]}_{\in[\mathfrak{p},\mathfrak{v}_i]\subset\mathfrak{l}_i}] \in [\mathfrak{v}_i, \mathfrak{l}_i] \subset \mathfrak{v}_i$$

4. Since $\mathfrak{h}_i$ and $\mathfrak{h}_j$ are both ideals, we have $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$.

5. For $X_i, Y_i \in \mathfrak{v}_i$, and $X_j, Y_j \in \mathfrak{v}_j$, we find:

$$B([X_i,Y_i],[X_j,Y_j]) = B(X_i, \underbrace{[Y_i,[X_j,Y_j]]}_{=0} = 0$$

□

Thanks to Theorem 14.4.3 and Lemma 15.1.5, we know that the $\mathbb{X}_i = \exp_o(\mathfrak{v}_i)$ are totally geodesic subspaces of $\mathbb{X}$, and symmetric spaces.

Lemma 15.1.6. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p} = \bigoplus_{k_- \le i \le k_+} \mathfrak{v}_i$ the decomposition into irreducible factors.*

For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0} \mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0 = \mathfrak{l}_0 \oplus \mathfrak{v}_0$.

1. *$\mathfrak{h}_0$ is an ideal of $\mathfrak{g}$.*

2. *For $i \neq 0$ we have $[\mathfrak{h}_i, \mathfrak{h}_0] = \{0\}$.*

3. *The decomposition $\mathfrak{g} = \bigoplus_{k_- \le i \le k_+} \mathfrak{h}_i$ is B-orthogonal.*

Proof. Let us start by showing that for $i \neq 0$, we have $[\mathfrak{h}_i, \mathfrak{v}_0] = \{0\}$. It follows from $[\mathfrak{v}_i, \mathfrak{v}_0] \subset [\mathfrak{p}, \mathfrak{v}_0] = \{0\}$ and:

$$[\mathfrak{l}_i, \mathfrak{v}_0] = [[\mathfrak{v}_i, \mathfrak{v}_i], \mathfrak{v}_0] \subset [\underbrace{[\mathfrak{v}_i, \mathfrak{v}_0]}_{=\{0\}}, \mathfrak{v}_i] + [\underbrace{[\mathfrak{v}_0, \mathfrak{v}_i]}_{=\{0\}}, \mathfrak{v}_i] = \{0\}$$

1. Thanks to Lemma 15.1.4 we know that $[\mathfrak{g},\mathfrak{v}_0]\subset\mathfrak{v}_0$, so it only remains to show that $[\mathfrak{g},\mathfrak{l}_0]\subset\mathfrak{h}_0$. Let $X\in\mathfrak{k}$, $Y\in\mathfrak{l}_0$ and $Z\in\mathfrak{l}_i$ for some $i\neq 0$.

$$B([X,Y],Z)=B(Y,\underbrace{[Z,X]}_{\in[\mathfrak{l}_i,\mathfrak{k}]\subset\mathfrak{l}_i})=0$$

It follows that $[\mathfrak{k},\mathfrak{l}_0]\subset\mathfrak{l}_0$. Now let $X\in\mathfrak{p}$, $Y\in\mathfrak{l}_0$ and $Z\in\mathfrak{v}_i$ for some $i\neq 0$.

$$B([X,Y],Z)=B(Y,\underbrace{[Z,X]}_{\in[\mathfrak{v}_i,\mathfrak{p}]\subset\mathfrak{l}_i})=0$$

It follows that $[\mathfrak{p},\mathfrak{k}_0]$ is orthogonal to $\mathfrak{v}_i$. Since it is included in $\mathfrak{p}$, we find $[\mathfrak{p},\mathfrak{l}_0]\subset\mathfrak{v}_0$.

2. Since $\mathfrak{h}_i$ and $\mathfrak{h}_0$ are both ideals of $\mathfrak{g}$ we have $[\mathfrak{h}_i,\mathfrak{h}_0]\subset\mathfrak{h}_i\cap\mathfrak{h}_0=\{0\}$.

3. Since the decompositions $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ and $\mathfrak{k}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{l}_i$ are B-orthogonal, and so is $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$, the same goes for $\mathfrak{g}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{h}_i$.

□

Lemma 15.1.7. *Let $\mathbb{X}$ be a symmetric space, $G=\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o\in\mathbb{X}$, $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ the decomposition into irreducible factors.*

For $i\neq 0$, write $\mathfrak{l}_i=[\mathfrak{v}_i,\mathfrak{v}_i]$ and $\mathfrak{h}_i=\mathfrak{v}_i\oplus\mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0}\mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0=\mathfrak{l}_0\oplus\mathfrak{v}_0$.

Let H_0 be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_0$. The symmetric space $\mathbb{X}_0=H_0.o$ is of Euclidean type.

Proof. This is a consequence of Proposition 14.4.5 and Lemma 15.1.4. □

Lemma 15.1.8. *Let $\mathbb{X}$ be a symmetric space, $G=\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o\in\mathbb{X}$, $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ the decomposition into irreducible factors.*

For $i\neq 0$, write $\mathfrak{l}_i=[\mathfrak{v}_i,\mathfrak{v}_i]$ and $\mathfrak{h}_i=\mathfrak{v}_i\oplus\mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0}\mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0=\mathfrak{l}_0\oplus\mathfrak{v}_0$.
Let H_i be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_i$. If $i<0$ (resp. $i>0$), the symmetric space $\mathbb{X}_i=H_i.o$ is of compact (resp. non compact) type and irreducible.

Proof. Since $\mathfrak{h}_i$ is an ideal of $\mathfrak{g}$, its Killing form is the restriction of the Killing form of $\mathfrak{g}$. It is non degenerate, so according to Cartan's criterion $\mathfrak{h}_i$ is semi-simple.

Let $\mathfrak{g}_i$ be the Lie algebra of the isometry group of $\mathbb{X}_i$, $G_i = \mathrm{Isom}_\circ(\mathbb{X}_i)$, and $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i$ the Cartan decomposition associated to o. Since H_i acts isometrically on $\mathbb{X}_i$, we have a Lie group morphism $f_i : H_i \to G_i$. The kernel of f_i acts trivially on $\mathbb{X}_i$ so it fixes o, hence $\ker f_i \subset K$. It follows that $d_e f_i$ is injective on $\mathfrak{v}_i$. Hence $d_e f_i(\mathfrak{v}_i) = \mathfrak{p}_i$ by equality of dimensions. We also have $d_e f_i(\mathfrak{l}_i) \subset \mathfrak{k}_i$.

Let us show that $\mathbb{X}_i$ is irreducible. If $\mathfrak{v} \subset \mathfrak{p}_i$ is $\mathrm{ad}(\mathfrak{k}_i)$-invariant, then $\mathfrak{v}' = (d_e f_i)^{-1}(\mathfrak{v}) \cap \mathfrak{v}_i$ must be $\mathrm{ad}(\mathfrak{l}_i)$-invariant. Indeed, given $X \in \mathfrak{v}'$ and $Y \in \mathfrak{l}_i$, we have $[X, Y] \in [\mathfrak{v}_i, \mathfrak{l}_i] \subset \mathfrak{v}_i$ and since $d_e f_i$ is a Lie algebra morphism we find:

$$d_e f_i([X,Y]) = [\underbrace{d_e f_i(X)}_{\in \mathfrak{v}}, \underbrace{d_e f_i(Y)}_{\in \mathfrak{k}_i}] \in \mathfrak{v}$$

Since $[\mathfrak{l}_j, \mathfrak{v}_i] = \{0\}$ for $j \neq i$, it follows that $\mathfrak{v}'$ is $\mathrm{ad}(\mathfrak{k})$-invariant, so $\mathfrak{v}' = \{0\}$ or $\mathfrak{v}' = \mathfrak{v}_i$, hence $\mathfrak{v} = \{0\}$ or $\mathfrak{v} = \mathfrak{p}_i$, and $\mathbb{X}_i$ is irreducible.

The same computation as in Theorem 14.5.7 shows that $\mathbb{X}_i$ has non negative (resp. non positive) sectional curvature when $i < 0$ (resp. $i > 0$), so $\mathbb{X}_i$ must have compact (resp. non compact) type. □

Lemma 15.1.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $H_1, H_2 \subset G$ be connected Lie subgroups with respective Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$.*
If $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$, then any elements $g_1 \in H_1$ and $g_2 \in H_2$ commute:

$$g_1 g_2 = g_2 g_1$$

Proof. Since H_1 is connected, we find $\mathrm{Ad}(g_1)|_{\mathfrak{h}_2} = \mathrm{Id}_{\mathfrak{h}_2}$. By considering the one-parameter subgroup associated to $X_2 \in \mathfrak{h}_2$, we find that g_1 commutes with $\exp_G(X_2)$. Since H_2 is connected, g_1 commutes with any element of H_2. □

Proof of Theorem 15.1.1. Consider the decomposition $\mathfrak{p} = \bigoplus_{k_- \le i \le k_+} \mathfrak{v}_i$ into irreducible factors. For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i \neq 0} \mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0 = \mathfrak{l}_0 \oplus \mathfrak{v}_0$.

Let H_i be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_i$, and recall that the symmetric space $\mathbb{X}_i = H_i.o$ is a totally geodesic submanifold of $\mathbb{X}$. Consider $L_i = H_i \cap K_i$, so that $\mathbb{X}_i$ can be identified with $L_i \backslash H_i$.

We will now show that $\mathbb{X}$ is isometric to the product $\mathbb{X}_{k_-} \times \cdots \times \mathbb{X}_{k_+}$. This implies that $\mathbb{X}_0$ is simply connected, so it is isometric to the Euclidean space $\mathbb{E}^k$, and therefore is irreducible. So the proof of Theorem 15.1.1 will be complete thanks to Lemma 15.1.7 and 15.1.8.

We start by considering the map:

$$\overline{\varphi} : \begin{cases} H_{k_-} \times \cdots \times H_{k_+} & \to & \mathbb{X} \\ (g_{k_-}, \ldots, g_{k_+}) & \mapsto & g_{k_-} \cdots g_{k_+} o \end{cases}$$

It is smooth, and Lemma 15.1.9 shows that for all $(\ell_{k_-},\dots,\ell_{k_+}) \in L_{k_-}\times\cdots\times L_{k_+}$ we have $\varphi(\ell_{k_-}g_{k_-},\dots,\ell_{k_+}g_{k_+}) = \varphi(g_{k_-},\dots,g_{k_+})$. Thus the map:

$$\varphi : \begin{cases} \mathbb{X}_{k_-}\times\cdots\times\mathbb{X}_{k_+} & \to & \mathbb{X} \\ (g_{k_-}o,\dots,g_{k_+}o) & \mapsto & g_{k_-}\cdots g_{k_+}o \end{cases}$$

is well defined and smooth.

Let us prove that it is isometric (then the completeness of $\mathbb{X}_{k_-}\times\cdots\times\mathbb{X}_{k_+}$, the simple connectedness of $\mathbb{X}$ and a count of dimensions will imply that φ is an isometry).

We start by computing the differential of $\overline{\varphi}$. Since the elements in different groups H_i commute, we can easily compute the partial derivatives:

$$\begin{aligned} d_{(g_{k_-},\dots,g_{k_+})}\overline{\varphi}(0,\dots,0,X_i,0,\dots,0) &= d_o\left(g_{k_-}\cdots\widehat{g_i}\cdots g_{k_+}\right)\circ d_{g_i}\varphi_o(X_i) \\ &= d_o\left(g_{k_-}\cdots g_{k_+}\right)\circ (d_o g_i)^{-1}\left[d_{g_i}\varphi_o(X_i)\right] \end{aligned}$$

This leads to:

$$d_{(g_{k_-}o,\dots,g_{k_+}o)}\varphi(0,\dots,0,u_i,0,\dots,0) = d_o\left(g_{k_-}\cdots g_{k_+}\right)\circ (d_o g_i)^{-1}(u_i)$$

Since the spaces $T_o\mathbb{X}_i$ are pairwise orthogonal, we find that φ is isometric. □

15.2 Symmetric spaces without Euclidean factors

Definition 15.2.1. A symmetric space $\mathbb{X}$ has **no Euclidean factor** if none of the factors in the decomposition of its universal cover $\widetilde{\mathbb{X}}$ given by Theorem 15.1.1 is of Euclidean type.

Lemma 15.2.2. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition.*

For $X \in \mathfrak{p}\setminus\{0\}$, the following are equivalent:

1. *$X \in \ker B$.*
2. *Any plane $P \subset T_o\mathbb{X}$ containing $d_e\varphi_o(X)$ has vanishing sectional curvature.*

Proof. First assume that $X \in \ker B$, and consider a plane $P \subset T_o\mathbb{X}$ spanned by $d_e\varphi_o(X)$ and $d_e\varphi_o(Y)$ for some $Y \in \mathfrak{p}$. According to Lemma 15.1.4, we have $[X,Y] = 0$, so Theorem 14.3.1 shows that the curvature of P is 0.

Now assume that any plane $P \subset T_o\mathbb{X}$ containing $d_e\varphi_o(X)$ has vanishing sectional curvature. Then according to Proposition 14.4.5, we find that $[X,Y] = 0$ for all $Y \in \mathfrak{p}$. Let $\mathfrak{c} = \{X' \in \mathfrak{p} | \forall Y \in \mathfrak{p}\ [X',Y] = 0\}$. Let us see that $\mathfrak{c}$

is stable under the Killing operator b. For $X' \in \mathfrak{c}$, $Y \in \mathfrak{p}$ and $Z \in \mathfrak{k}$ we can compute:

$$\begin{aligned} B(Z,[Y,b(X')]) &= B([Z,Y],b(X')) \\ &= \langle b([Z,Y]) \big| b(X')\rangle \\ &= \langle [Z,b(Y)] \big| b(X')\rangle \\ &= B([Z,b(Y)],X') \\ &= B(Z,[b(Y),X']) \\ &= 0 \end{aligned}$$

Since B is negative definite on $\mathfrak{k}$, we find $[Y,b(X')] = 0$, i.e. $b(X') \in \mathfrak{c}$. It follows that $\mathfrak{c}$ is the direct sum of its intersections with eigenspaces of b. Let $X' \in \mathfrak{c}$ be such that $b(X') = \lambda X'$ for some $\lambda \neq 0$. For $Z \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, we find:

$$B(Y,[X',Z]) = B([Y,X'],Z) = 0$$

It follows that $[X',Z] \in \ker b$. However the eigenspaces of b are invariant under $\mathrm{ad}(Z)$, hence $[X',Z] = 0$, and $X' \in \mathfrak{z}(\mathfrak{g}) \subset \ker B$, therefore $X' = 0$. It follows that $\mathfrak{c} \subset \ker B$, and $X \in \ker B$. □

Proposition 15.2.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$ and $\mathfrak{g}$ its Lie algebra. Then $\mathbb{X}$ has no Euclidean factor if and only if $\mathfrak{g}$ is semi-simple.*

Proof. Cartan's criterion states that $\mathfrak{g}$ is semi-simple if and only if $\ker B = \{0\}$. But Lemma 15.2.2 shows that the condition $\ker B = \{0\}$ remains invariant under coverings, so we can assume that $\mathbb{X}$ is simply connected. If $\mathbb{X}$ has a Euclidean factor $\mathbb{X}_0$, then any element $X \in \mathfrak{v}_0$ is in $\ker B$, so $\mathfrak{g}$ is not semi-simple. If $\mathbb{X}$ has no Euclidean factor, then B is non degenerate on $\mathfrak{p}$, so it is non degenerate on $\mathfrak{g}$ and $\mathfrak{g}$ is semi-simple. □

Proposition 15.2.4. *A symmetric space with no Euclidean factor is of compact (resp. non compact) type if and only if it has non negative (resp. non positive) sectional curvature.*

For a symmetric space $\mathbb{X}$ of compact or non compact type, we find that all the irreducible factors must be of the same type.

Proposition 15.2.5. *Let $\mathbb{X}$ be a simply connected symmetric space. If $\mathbb{X}$ is of compact (resp. non compact) type, there are irreducible symmetric spaces $\mathbb{X}_1,\dots,\mathbb{X}_k$ of compact (resp. non compact) type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of compact (resp. non compact) type, then $\mathfrak{p} = \mathfrak{v}_-$ (resp. $\mathfrak{p} = \mathfrak{v}_+$) in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Geometrically, the three types of symmetric spaces are determined by the sign of the curvature.

Proposition 15.2.6. *Let $\mathbb{X}$ be a symmetric space. If $\mathbb{X}$ is of compact (resp. non compact) type, then the sectional curvature of $\mathbb{X}$ is non negative (resp. non positive).*

Proof. This is a consequence of the fact that the universal cover of $\mathbb{X}$ has the same property, so we can apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non negative (resp. non positive) sectional curvature also has non negative (resp. non positive) sectional curvature). □

Lemma 15.2.7. *Let $\mathbb{X}$ be a symmetric space. The universal cover $\widetilde{\mathbb{X}}$ is of Euclidean (resp. compact, non compact) type if and only if $\mathbb{X}$ is of Euclidean (resp. compact, non compact) type.*

Proof. The Euclidean case is a consequence of Proposition 14.5.2. □

Proposition 15.2.8. *If $\mathbb{X}_1, \dots, \mathbb{X}_k$ are symmetric spaces and are all of Euclidean (resp. compact, non compact) type, then $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$ is of Euclidean (resp. compact, non compact) type.*

Proof. □

Theorem 15.2.9. *Any simply connected symmetric space $\mathbb{X}$ is isometric to a product $\mathbb{X}_0 \times \mathbb{X}_- \times \mathbb{X}_+$ where:*

- *$\mathbb{X}_0$ is a symmetric space of Euclidean type.*
- *$\mathbb{X}_-$ is a symmetric space of compact type.*
- *$\mathbb{X}_+$ is a symmetric space of non compact type.*

15.3 Symmetric spaces of compact type

Proposition 15.3.1. *Let $\mathbb{X}$ be a simply connected symmetric space of compact type. There are irreducible symmetric spaces $\mathbb{X}_1, \dots, \mathbb{X}_k$ of compact type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of compact type, then $\mathfrak{p} = \mathfrak{v}_-$ in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Proposition 15.3.2. *A symmetric space of compact type has non negative sectional curvature.*

Proof. Non negative sectional curvature for a symmetric space $\mathbb{X}$ or for its universal cover $\widetilde{\mathbb{X}}$ are equivalent, so we can apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non negative sectional curvature also has non negative sectional curvature. □

Proposition 15.3.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$ and $\mathfrak{g}$ its Lie algebra. If $\mathbb{X}$ is of compact type, then G is compact, and $\mathfrak{g}$ is semi-simple.*

Remark. Since G acts transitively on $\mathbb{X}$, it is also compact.

Proof. The Killing form of $\mathfrak{g}$ is negative definite, so $\mathfrak{g}$ is semi-simple.

The opposite of the Killing form of $\mathfrak{g}$ induces a bi-invariant Riemannian metric on G, and computations show that its sectional curvature must be positive. The Myers Theorem implies that G is compact. □

Definition 15.3.4. A real Lie algebra $\mathfrak{g}$ is called **compact** if its Killing form is negative definite.

If $\mathfrak{g}$ is compact, then it is semi-simple, and so is $\mathfrak{g} \otimes \mathbb{C}$.

Theorem 15.3.5. *The map $\mathfrak{g} \mapsto \mathfrak{g} \otimes \mathbb{C}$ is a bijection from the set of compact Lie algebras (up to isomorphism) to the set of complex semi-simple Lie algebras (up to isomorphism).*

Compact Lie algebras	Complex simple Lie algebras
$\mathfrak{so}(n)$	$\mathfrak{so}(n, \mathbb{C})$
$\mathfrak{su}(n)$	$\mathfrak{sl}(n, \mathbb{C})$
$\mathfrak{sp}(n)$	$\mathfrak{sp}(n, \mathbb{C})$

Note that even if $\mathbb{X}$ is irreducible, the group G may not be simple. The main example is a compact Lie group itself. If $\mathfrak{h}$ is a compact Lie algebra, there is a unique (up to isomorphism) Lie group H with trivial centre and Lie algebra isomorphic to $\mathfrak{h}$. We can endow H with a bi-invariant Riemannian metric whose value at $\mathfrak{h}$ is the opposite of the Killing form. Then H is a symmetric space, and $G = H \times H$ acts on the left and on the right on H.

Classical examples include spheres $\mathbb{S}^n = SO(n+1)/SO(n)$ and projective spaces $\mathbb{CP}^n = SU(n+1)/U(n)$.

15.4 Symmetric spaces of non compact type

We will admit the following result.

Proposition 15.4.1. *A symmetric space of non compact type is simply connected.*

This means that a symmetric space of non compact type is a Cartan-Hadamard manifold, hence diffeomorphic to the Euclidean space.

Proposition 15.4.2. *Let $\mathbb{X}$ be a symmetric space of non compact type. There are irreducible symmetric spaces $\mathbb{X}_1,\dots,\mathbb{X}_k$ of non compact type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1\times\cdots\times\mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of non compact type, then $\mathfrak{p}=\mathfrak{v}_+$ in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Proposition 15.4.3. *A symmetric space of non compact type has non positive sectional curvature.*

Proof. Simply apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non positive sectional curvature also has non positive sectional curvature. □

Proposition 15.4.4. *Let $\mathbb{X}$ be a symmetric space of non compact type, and $G=\mathrm{Isom}_\circ(\mathbb{X})$. Any compact subgroup $K\subset G$ fixes a point of $\mathbb{X}$.*

Proof. This was already shown to be true for any Cartan-Hadamard manifold. □

Proposition 15.4.5. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G=\mathrm{Isom}_\circ(\mathbb{X})$ and $o\in\mathbb{X}$. The stabiliser $K=G_o$ is a maximal compact subgroup of G, i.e. any compact subgroup $L\subset G$ containing K is equal to K.*

Definition 15.4.6. A semi-simple Lie algebra has **no compact factor** if it has no compact ideal.

Theorem 15.4.7. *The map sending $\mathbb{X}$ to the Lie algebra $\mathfrak{g}$ of its isometry group is a bijection from the set of symmetric spaces of non compact type (up to multi-homothety) to the set of non compact semi-simple real Lie algebras (up to isomorphism).*

A multi-homothety is a map that is homothetic on each irreducible factor, but possibly with different constants .

Starting with a semi-simple Lie algebra with no compact factor $\mathfrak{g}$, we can choose a Lie group G whose Lie algebra is $\mathfrak{g}$, and with trivial centre $Z(G)=\{1\}$. We then consider the symmetric space $\mathbb{X}=G/K$ where $K\subset G$ is a maximal compact subgroup.

Chapter 16

Real semi-simple Lie algebras

16.1 The structure of real semi-simple Lie algebras

16.1.1 Restricted roots

Definition 16.1.1. Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra with Killing form B. A **Cartan involution** of $\mathfrak{g}$ is a Lie algebra automorphism θ of $\mathfrak{g}$ such that $\theta^2 = \mathrm{Id}$ and such that the bilinear form $\langle\cdot|\cdot\rangle_\theta$ on $\mathfrak{g}$ defined by

$$\forall X, Y \in \mathfrak{g} \quad \langle X|Y\rangle_\theta = -B(X, \theta(Y))$$

is positive definite.

Given a Cartan involution θ, we write $\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\}$ and $\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\}$. The decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{g}$ is called the **Cartan decomposition**.

Note that it is symmetric because θ is a Lie algebra morphism. The same computations as in Proposition 13.4.4 show that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}]^{\subset}\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Proposition 16.1.2. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, and $\mathfrak{g}$ its Lie algebra. For all $o \in \mathbb{X}$, the Cartan involution of $\mathbb{X}$ associated to o is a Cartan involution of $\mathfrak{g}$.*

Lemma 16.1.3. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
For all $X \in \mathfrak{p}$, the map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is self-adjoint for the inner product $\langle\cdot|\cdot\rangle_\theta$.

Proof. Recall that θ is a Lie algebra automorphism of $\mathfrak{g}$, so $\mathrm{ad}(\theta(X)) \circ \theta = \theta \circ \mathrm{ad}(X)$, hence $\mathrm{ad}(X) \circ \theta = -\theta \circ \mathrm{ad}(X)$ since $X \in \mathfrak{p}$.

For $Y, Z \in \mathfrak{g}$, we find:

$$\begin{aligned}\langle \mathrm{ad}(X)Y|Z\rangle_\theta &= -B(\mathrm{ad}(X)Y, \theta(Z))\\ &= B(Y, \mathrm{ad}(X)\circ\theta(Z))\\ &= -B(Y, \theta(\mathrm{ad}(X)Z))\\ &= B(Y, \mathrm{ad}(X)Z)\end{aligned}$$

□

Consequently, the map $\mathrm{ad}(X)$ is diagonalisable for all $X \in \mathfrak{p}$.

Definition 16.1.4. Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition.

A **Cartan subspace** of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{p}$.

Definition 16.1.5. Let $\mathbb{X}$ be a symmetric space. A **maximal flat** of $\mathbb{X}$ is a flat $F \subset \mathbb{X}$ that is maximal for the inclusion.

Proposition 16.1.6. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a vector subspace. Then $\mathfrak{a}$ is Cartan subspace if and only if $d_e\varphi_o(\mathfrak{a}) \subset T_o\mathbb{X}$ is the tangent space of a maximal flat.

Proof. This is a consequence of Proposition 14.4.5. □

For $\alpha \in \mathfrak{a}^*$, we write:

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \,|\, \forall X \in \mathfrak{a}\ [X,Y] = \alpha(X)Y\}$$

Definition 16.1.7. Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace. A **restricted root** is $\alpha \in \mathfrak{a}^*\setminus\{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$.

We will denote by $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. We have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma}\mathfrak{g}_\alpha$$

Moreover, this decomposition is orthogonal for the Cartan form.

Theorem 16.1.8. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. We equip $\mathfrak{a}^*$ with the inner product obtained by duality from the inner product on $\mathfrak{a}$.

1. *Σ spans $\mathfrak{a}^*$ as a vector space.*

2. $\forall \alpha, \beta \in \mathfrak{a}^*$ $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

3. $\forall \alpha \in \mathfrak{a}^*$ $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$

4. $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$ *and* $\mathfrak{g}_0 = \mathfrak{a} \oplus (\mathfrak{g}_0 \cap \mathfrak{k})$.

5. *If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal for B.*

6. *If $\alpha, \beta \in \Sigma$, then $\frac{\langle \alpha | \beta \rangle}{2\langle \beta | \beta \rangle} \in \mathbb{Z}$ and $\alpha - \frac{\langle \alpha | \beta \rangle}{2\langle \beta | \beta \rangle}\beta \in \Sigma$.*

Remark. The last property means that Σ is a root system in $\mathfrak{a}^*$. Contrary to the complex case, it is not always reduced. Recall that in a Euclidean space, for a fixed vector x, the map $y \mapsto y - \frac{\langle y|x \rangle}{2\langle x|x \rangle}x$ is the reflection with respect to the hyperplane $x^\perp$.

For $\alpha \in \Sigma$, the root space $\mathfrak{g}_\alpha$ does not decompose as the sum of its intersections with $\mathfrak{k}$ and $\mathfrak{p}$. The root space decomposition and the Cartan decomposition are related in a more complicated way.

Lemma 16.1.9. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. For $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}_\alpha$, we write $Y = Y_\mathfrak{k} + Y_\mathfrak{p} \in \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of Y. The Lie bracket $[X, Y_\alpha]$ decomposes as:

$$[X, Y_\mathfrak{k}] = \alpha(X) Y_\mathfrak{p} \text{ and } [X, Y_\mathfrak{p}] = \alpha(X) Y_\mathfrak{k}$$

Proof. We have $[X, Y_\mathfrak{k}] + [X, Y_\mathfrak{p}] = \alpha(X)Y_\mathfrak{p} + \alpha(X)Y_\mathfrak{k}$. Since $[X, Y_\mathfrak{k}] \in [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ and $[X, Y_\mathfrak{p}] \in [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we can identify the factors. □

16.1.2 Some examples

The hyperbolic space $\mathbb{H}^n$

To understand the description of the Lie algebra $\mathfrak{g} = \mathfrak{so}(n,1)$, we will use a decomposition in blocks of size n and 1. For $A \in \mathfrak{gl}(n, \mathbb{R})$, $u, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we find:

$$\begin{pmatrix} A & {}^t v \\ u & \lambda \end{pmatrix} \in \mathfrak{so}(n,1) \iff A \in \mathfrak{so}(n),\ v = u,\ \lambda = 0$$

For the symmetric space $\mathbb{H}^n$, by fixing the point $o = (0, \ldots, 0, 1)$ in the hyperboloid model, we find that the Cartan involution of is simply $\theta(X) = -{}^t X$.

$$\mathfrak{k} = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & 0 \end{array} \right) \middle| A \in \mathfrak{so}(n) \right\} \ ; \ \mathfrak{p} = \left\{ \left(\begin{array}{cc} 0 & {}^t u \\ u & 0 \end{array} \right) \middle| u \in \mathbb{R}^n \right\}$$

The bracket of two elements in $\mathfrak{p}$ can be computed explicitly.

$$\left[\left(\begin{array}{cc} 0 & {}^t u \\ u & 0 \end{array} \right), \left(\begin{array}{cc} 0 & {}^t v \\ v & 0 \end{array} \right) \right] = \left(\begin{array}{cc} {}^t u v - {}^t v u & 0 \\ 0 & 0 \end{array} \right)$$

This bracket can only vanish if $u = 0$ or $v = 0$. It follows that a maximal abelian subalgebra of $\mathfrak{p}$ has dimension 1. Set $\mathfrak{a} = \mathbb{R}.X$ where

$$H = \left(\begin{array}{cc} 0 & {}^t h \\ h & 0 \end{array} \right) \ ; \ h = (0, \ldots, 0, 1)$$

The bracket between H and an arbitrary element of $\mathfrak{g}$ can be computed explicitly.

$$\left[H, \left(\begin{array}{cc} A & {}^t v \\ v & 0 \end{array} \right) \right] = \left(\begin{array}{cc} {}^t h v - {}^t v h & -A {}^t h \\ hA & 0 \end{array} \right)$$

The line hA is simply the last line of A, and for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have:

$${}^t h v - {}^t v h = \left(\begin{array}{cccc} & & & -v_1 \\ & 0 & & \vdots \\ & & & -v_{n-1} \\ v_1 & \cdots & v_{n-1} & 0 \end{array} \right)$$

The centraliser of $\mathfrak{a}$ can be found easily:

$$\mathfrak{g}_0 \cap \mathfrak{k} = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & 0 \end{array} \right) \middle| A \in \mathfrak{so}(n-1) \right\}$$

Here we identify $\mathfrak{so}(n-1)$ with the top left block diagonal embedding in $\mathfrak{so}(n)$. The roots are $\pm\alpha$ where $\alpha(H) = 1$, and the root spaces are:

$$\mathfrak{g}_\alpha = \left\{ \left(\begin{array}{ccc} 0 & -{}^t u & {}^t u \\ u & 0 & 0 \\ u & 0 & 0 \end{array} \right) \middle| u \in \mathbb{R}^{n-1} \right\}$$

$$\mathfrak{g}_{-\alpha} = \left\{ \left(\begin{array}{ccc} 0 & {}^t u & {}^t u \\ -u & 0 & 0 \\ u & 0 & 0 \end{array} \right) \middle| u \in \mathbb{R}^{n-1} \right\}$$

The space of ellipsoids $\mathcal{E}_n$

The decomposition that we find for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ is exactly the same as for $\mathfrak{sl}(n, \mathbb{C})$. Indeed, we can choose $\mathfrak{a}$ to be the space of diagonal traceless matrices. It happens to be a Cartan subalgebra, i.e. $\mathfrak{g}_0 = \mathfrak{a}$. A real semi-simple Lie algebra with this property is called a **real split** Lie algebra. There is a one to one correspondence between real split Lie algebras and complex semi-simple Lie algebras.

Real split Lie algebras	Complex Lie algebras
$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{C})$
$\mathfrak{so}(n, n+1)$	$\mathfrak{so}(2n+1, \mathbb{C})$
$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{C})$
$\mathfrak{so}(n, n)$	$\mathfrak{so}(2n, \mathbb{C})$

16.1.3 Restricted roots and geometry

Lemma 16.1.10. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots.

For $\alpha \in \Sigma$, consider $Y_\alpha \in \mathfrak{g}_\alpha$ such that $\langle Y_\alpha | Y_\alpha \rangle_\theta = 1$. Then the element $X_\alpha = [\theta(Y_\alpha), Y_\alpha] \in \mathfrak{a}$ satisfies:

$$\forall X \in \mathfrak{a} \quad \langle X_\alpha | X \rangle_\theta = \alpha(X)$$

Proof. It follows from Theorem 16.1.8 that $[\theta(Y_\alpha), Y_\alpha] \in \mathfrak{g}_0$. By using the Cartan decomposition of Y_α we also find $[\theta(Z), Z] \in \mathfrak{p}$ for any $Z \in \mathfrak{g}$, so $[\theta(Y_\alpha), Y_\alpha] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

$$\begin{aligned}
\langle X_\alpha | X \rangle_\theta &= -B([\theta(Y_\alpha), Y_\alpha], X) \\
&= -B(\theta(Y_\alpha), [Y_\alpha, X]) \\
&= B(\theta(Y_\alpha), \alpha(X) Y_\alpha \\
&= \alpha(X) \langle Y_\alpha | Y_\alpha \rangle_\theta \\
&= \alpha(X)
\end{aligned}$$

□

Definition 16.1.11. Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition. We say that $\mathbb{X}$ is **normalized** if the Killing form and the Riemannian form are equal on $\mathfrak{p}$.

Proposition 16.1.12. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, $\theta : \mathfrak{g} \to \mathfrak{g}$ the associated Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots.

For $\alpha \in \Sigma$, consider $Y_\alpha \in \mathfrak{g}_\alpha$ such that $\langle Y_\alpha | Y_\alpha \rangle_\theta = 1$, and $X_\alpha = [\theta(Y_\alpha), Y_\alpha]$. There is a totally geodesic surface $S \subset \mathbb{X}$ containing o such that $T_o S$ is spanned by $d_e\varphi_o(X_\alpha)$ and $d_e\varphi_o(Y_\alpha)$.

If $\mathbb{X}$ is normalized, then the sectional curvature of S is $-\|\alpha\|^2$.

Proof. Use $B(Y, \theta(Y)) = 1$ and $B(Y,Y) = 0$. □

16.1.4 Regular elements and Weyl chambers

Proposition 16.1.13. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

For $X \in \mathfrak{p}$, the following assertions are equivalent.

1. *$\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian.*
2. *X belongs to a unique Cartan subspace.*
3. *For any Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ containing X, we have $\forall \alpha \in \Sigma\ \alpha(X) \neq 0$.*
4. *There is a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ containing X, such that $\forall \alpha \in \Sigma\ \alpha(X) \neq 0$.*

Definition 16.1.14. Such an element $X \in \mathfrak{p}$ is called **regular**.

Proof. *1.⇒2.*If $\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian, then it is a Cartan subspace. If $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace containing X, then $\mathfrak{a} \subset \mathfrak{z}(X)$, hence $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$ because of maximality.

(2)⇒(1) Let $\mathfrak{a} \subset \mathfrak{p}$ be the Cartan subspace containing X. For $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$, the abelian subalgebra $\mathbb{R}.X + \mathbb{R}.Y$ is contained in Cartan subalgebra, so $Y \in \mathfrak{a}$, and $\mathfrak{z}(X) \cap \mathfrak{p} \subset \mathfrak{a}$ is abelian.

(3)⇒(4) is just specification.

(1)⇒(3) Assume that $\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian, and consider a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ that contains X.

If $\alpha(X) = 0$ for some $\alpha \in \Sigma$, then Lemma 16.1.9 shows that $Y_\mathfrak{p} \in \mathfrak{z}(X) \cap \mathfrak{p} = \mathfrak{a}$ for all $Y \in \mathfrak{g}_\alpha$. Hence $[X', Y_\mathfrak{p}] = \alpha(X') Y_\mathfrak{k} = 0$ for all $X' \in \mathfrak{a}$, and $\mathfrak{g}_\alpha \subset \mathfrak{p}$, which leads to $\alpha = 0$.

(4)⇒(1) For $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$, we write $Y = Y_0 + \sum_{\alpha \in \Sigma} Y_\alpha$ its decomposition in $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Since $0 = [X, Y] = \sum_{\alpha \in \Sigma} \alpha(X) Y_\alpha$, we get $Y_\alpha = 0$, hence $Y \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. □

Note that a consequence of the fourth point is that every Cartan subspace contains regular elements (because $\bigcup_{\alpha \in \Sigma} \ker \alpha$ has empty interior in $\mathfrak{a}$).

Now consider a symmetric space of non compact type $\mathbb{X}$. The stabiliser K of a point $o \in \mathbb{X}$ acts on the set of Cartan subspaces of $\mathfrak{p}$ (an element $g \in K$ acts on a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ by $g.\mathfrak{a} = \mathrm{Ad}(g)\mathfrak{a}$).

Proposition 16.1.15. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

The action of K on the set of Cartan subspaces of $\mathfrak{p}$ is transitive.

Proof. Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{p}$ be Cartan subspaces. Consider regular elements $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$, and the function

$$f : \left\{ \begin{array}{ccc} K & \to & \mathbb{R} \\ g & \mapsto & B(\mathrm{Ad}(g)X, Y) \end{array} \right.$$

Since K is compact, f reaches its maximum at some $g_0 \in K$. Up to replacing X with $\mathrm{Ad}(g_0)X$ and $\mathfrak{a}$ with $\mathrm{Ad}(g_0)\mathfrak{a}$, we can assume that $g_0 = \mathrm{Id}$.

For all $Z \in \mathfrak{k}$, we have $\frac{d}{dt}\big|_{t=0} f(\exp_G(tZ)) = 0$. This derivative can be computed:

$$\left.\frac{d}{dt}\right|_{t=0} f(\exp_G(tZ)) = B(\mathrm{ad}(Z)X, Y) = B(Z, [X, Y])$$

Since $[X, Y] \in \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$, it follows that $[X, Y] = 0$. The elements X and Y being regular, we find $\mathfrak{a} = \mathfrak{b}$. $\square$

Definition 16.1.16. Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and Σ the set of restricted roots.

A **Weyl chamber** of $\mathfrak{a}$ is a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha$.

Proposition 16.1.17. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

The group K acts transitively on the set of pairs $(\mathfrak{a}^+, \mathfrak{a})$ where $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace and $\mathfrak{a}^+ \subset \mathfrak{a}$ is a Weyl chamber.

Proof. Consider $X \in \mathfrak{a}^+ \subset \mathfrak{a}$, $Y \in \mathfrak{b}^+ \subset \mathfrak{b}$, and the function f introduced in the proof of Proposition 16.1.15.

$$f : \left\{ \begin{array}{ccc} K & \to & \mathbb{R} \\ g & \mapsto & B(\mathrm{Ad}(g)X, Y) \end{array} \right.$$

We can still assume that f reaches its maximum at e (hence $\mathfrak{a} = \mathfrak{b}$). For $Z \in \mathfrak{k}$, we have $\frac{d^2}{dt^2}\big|_{t=0} f(\exp_G(tZ)) \leq 0$. Since $\mathrm{Ad}(\exp_G(tZ)) = \exp_{\mathrm{GL}(\mathfrak{g})}(t\,\mathrm{ad}(Z))$, we find:

$$\begin{aligned} \left.\frac{d^2}{dt^2}\right|_{t=0} f(\exp_G(tZ)) &= B(\mathrm{ad}(Z)^2 X, Y) \\ &= B([Z, [Z, X]], Y) \\ &= -B([Z, X], [Z, Y]) \\ &= -B([X, Z], [Y, Z]) \end{aligned}$$

We are left with:

$$B([X,Z],[Y,Z]) \geq 0$$

For $\alpha \in \Sigma$, choose $Y \in \mathfrak{g}_\alpha \setminus \{0\}$ and decompose $Y = Y_\mathfrak{k} + Y_\mathfrak{p} \in \mathfrak{k} \oplus \mathfrak{p}$. We can apply the previous inequality with $Z = Y_\mathfrak{k}$.

Since $[X, Y_\mathfrak{k}] = \alpha(X) Y \mathfrak{p}$, we find:

$$\alpha(X)\alpha(Y) B(Y_\mathfrak{p}, Y_\mathfrak{p}) \geq 0$$

Note that $Y_\mathfrak{p} \neq 0$. Since B is positive definite on $\mathfrak{p}$, we find that $\alpha(X)\alpha(Y) \geq 0$. It follows that X and Y are in the same Weyl chamber.

□

Proposition 16.1.18 (Polar decomposition)**.** *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

For all $g \in G$, there is a unique pair $(k, X) \in K \times \mathfrak{p}$ such that $g = k \exp_G(X)$.

Proof. Let $X \in \mathfrak{p}$ be such that $g(o) = \exp_G(X).o$ (i.e. $X = (d_e \varphi_o^{-1}(v)$ where $g(o) = \exp_o(v)$). Now $k = g \exp_G(-X) \in K$ satisfies $g = k \exp_G(X)$.

The uniqueness comes from the fact that $\exp_o$ is a diffeomorphism. □

Proposition 16.1.19 (KAK decomosition)**.** *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber.

For all $g \in G$, there are $k, k' \in K$ and $X \in \overline{\mathfrak{a}^+}$ such that :

$$g = k \exp_G(X) k'$$

Proof. Following Proposition 16.1.18, we write $g = l \exp_G(Y)$ with $Y \in \mathfrak{p}$ and $l \in K$.

Note that any element of $\mathfrak{a}$ is in the closure of a Weyl chamber, so according to Proposition 16.1.17 there is $k' \in K$ such that $X = \mathrm{Ad}(k')Y \in \overline{\mathfrak{a}^+}$. Set $k = lk'^{-1} \in K$, so that we find:

$$\begin{aligned} k \exp_G(X) k' &= lk'^{-1} \exp_G(X) k' \\ &= lk' \exp_G(\mathrm{Ad}(k'^{-1})X) \\ &= l \exp_G(Y) \\ &= g \end{aligned}$$

□

16.2 Compactifications of symmetric spaces

16.2.1 The visual boundary of a symmetric space of non compact type

Proposition 16.2.1. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $o \in \mathbb{X}$ and $K = G_o$ its stabiliser.*

The map $\psi_o : T^1_o\mathbb{X} \to \partial_\infty\mathbb{X}$ is K-equivariant.

Proof. For $g \in K$ and $v \in T^1_{x_0}\mathbb{X}$, we find $g(c_v(t)) = c_{g.v}(t)$ where $g.v = d_{x_0}g(v)$. □

Proposition 16.2.2. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $o \in \mathbb{X}$ and $K = G_o$ its stabiliser.*

The following assertions are equivalent.

1. *$G \curvearrowright \partial_\infty\mathbb{X}$ is transitive.*
2. *$K \curvearrowright T^1_o\mathbb{X}$ is transitive.*
3. *$\mathbb{X}$ has rank 1.*
4. *The sectional curvature of $\mathbb{X}$ is negative.*

Proof. *(1)* $\Longleftrightarrow$ *(2)* is a consequence of Proposition 16.2.1.

(3) $\Longleftrightarrow$ *(4)* is a consequence of the formula for sectional curvature (Theorem 14.5.7).

(3)$\Rightarrow$*(2)* is a consequence of Proposition 16.1.17.

(2)$\Rightarrow$*(3)* is a consequence of Proposition 16.1.17. □

Theorem 16.2.3. *Any rank 1 symmetric space of non compact type is homothetic to one of the following:*

- *The real hyperbolic space $\mathbb{H}^n$ (and $\mathfrak{g} = \mathfrak{so}(n,1)$).*
- *The complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ (and $\mathfrak{g} = \mathfrak{su}(n,1)$).*
- *The quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$ (and $\mathfrak{g} = \mathfrak{sp}(n,1)$).*
- *The octonionic hyperbolic plane $\mathbb{H}^2_{\mathbb{O}}$ (and $\mathfrak{g} = \mathfrak{f}_4^{-20}$).*

In particular, a rank 1 symmetric space of non compact type is irreducible (this is the first part of the proof, and a consequence of the more general fact that the rank is additive under products).

Proposition 16.2.4. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $o \in \mathbb{X}$ and $K = G_o$ its stabiliser.*

For all $\xi \in \partial_\infty\mathbb{X}$, we have $G.\xi = K.\xi$, i.e.

$$\forall g \in G\ \exists k \in K \quad g\xi = k\xi$$

More over, the stabiliser $G_\xi \subset G$ acts transitively on $\mathbb{X}$.

Proof. Let γ be the unit speed geodesic such that $\gamma(0) = o$ and $\gamma \in \xi$. Let γ_t be the 1-parameter group of transvections along γ (i.e. $\dot\gamma(0) = d_e\varphi_o(X)$ and $\gamma_t = \exp_G(tX)$ for some $X \in \mathfrak{p}$).

Let $g \in G$. For $t > 0$, we let σ_t be the unit speed geodesic such that $\sigma_t(0) = x_0$ and passing through $g\gamma(t) = g\gamma_t(x_0)$. Set $\xi_t = [\sigma_t]$.

Denote by q_t the transvection along σ_t such that $q_t(x_0) = g\gamma_t(x_0)$. Note that $q_t.\xi_t = \xi_t$.

Set $k_t = q_t^{-1}g\gamma_t$. We have $k_t(x_0) = x_0$, i.e. $k_t \in K$. Hence:

$$\sphericalangle_{x_0}(k_t\xi, \xi_t) = \sphericalangle_{q_t(x_0)}(g\xi, q_t\xi_t) = \sphericalangle_{g\gamma(t)}(x_0, g(x_0)) \to 0$$

We also find:

$$\sphericalangle_{x_0}(\xi_t, g\xi) = \sphericalangle_{x_0}(g\gamma(t), g\xi) \le \pi - \sphericalangle_{g\gamma(t)}(g\xi, x_0) = \sphericalangle_{g\gamma(t)}(x_0, g(x_0)) \to 0$$

It follows that $\sphericalangle_{x_0}(k_t\xi, g\xi) \to 0$, hence $k_t\xi \to g\xi$. Therefore $g\xi \in \overline{K.\xi} = K.\xi$ (because K is compact).

The transitivity of $G_\xi \curvearrowright \mathbb{X} = G/K$ is a rewriting of the transitivity of $K \curvearrowright G.\xi = G_\xi\backslash G$:

$$\begin{aligned}
\forall g \in G\, \exists k \in K \quad gk \in G_\xi \iff & \forall g \in G\, \exists k \in K\, \exists p \in G_\xi \quad gk = p \\
\iff & \forall g \in G\, \exists p \in G_\xi\, \exists k \in K \quad gk = p \\
\iff & \forall g \in G\, \exists p \in G_\xi\, \exists k \in K \quad pg = k
\end{aligned}$$

Therefore $G_\xi \curvearrowright K\backslash G = M$ is transitive. □

16.2.2 The Furstenberg boundary

Definition 16.2.5. Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, and $\mathfrak{g}$ its Lie algebra.

An **asymptotic Weyl chamber** is $\psi_o(d_e\varphi_o(\mathfrak{a}^+)) \subset \partial_\infty\mathbb{X}$ where $\mathfrak{a}^+ \subset \mathfrak{p}$ is a Weyl chamber and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition associated to some $o \in \mathbb{X}$.

The **Furstenberg boundary** $\partial_F\mathbb{X}$ is the set of asymptotic Weyl chambers.

An asymptotic Weyl chamber can be described geometrically: it is the boundary at infinity of a maximal flat.

Proposition 16.2.6. *Let $\mathbb{X}$ be a symmetric space of non compact type. The group $G = \mathrm{Isom}_\circ(\mathbb{X})$ acts transitively on the Furstenberg boundary $\partial_F\mathbb{X}$.*

Definition 16.2.7. Let $\mathbb{X}$ be a symmetric space of non compact type. A point $\xi \in \partial_\infty\mathbb{X}$ is called **regular** if there are $o \in \mathbb{X}$ and a regular vector $v \in T^1_o\mathbb{X}$ such that $\xi = \psi_o(v)$.

Proposition 16.2.8. *Let $\mathbb{X}$ be a symmetric space of non compact type, and $G = \mathrm{Isom}_\circ(\mathbb{X})$. If $\xi \in \partial_\infty \mathbb{X}$ is regular, then ξ belongs to a unique asymptotic Weyl chamber C, and $\{g \in G \,|\, gC = C\} = G_\xi$.*

If $\xi \in \partial_\infty M$ is regular, we fix $o \in \mathbb{X}$ and the Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a} \subset \mathfrak{p}$ such that $\xi = \lim_{t\to+\infty} \exp_G(tX)o$ with $X \in \mathfrak{a}^+$. We set:

$$A_\xi = \exp_G(\mathfrak{a})$$

$$\mathfrak{n}_\xi = \bigoplus_{\alpha(X)>0} \mathfrak{g}_\alpha$$

$$N_\xi = \exp_G(\mathfrak{n}_\xi)$$

Theorem 16.2.9 (Iwasawa decomposition)**.** *N_ξ is a subgroup of G, and the multiplication $K \times A_\xi \times N_\xi \to G$ is a diffeomorphism.*

16.3 Lattices in semi-simple Lie groups

If (M,g) is a complete locally symmetric space, then its universal cover is a symmetric space $\mathbb{X}$. So the study of locally symmetric spaces is related to the study of discrete subgroups of the isometry groups of symmetric spaces. For the Euclidean type, these groups are, up to finite index, abelian (Bieberbach's Theorem). For the compact type, they must be finite. The non compact type leads to a very rich theory.

Definition 16.3.1. Let G be a Lie group. A **lattice** of G is a discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G$ has finite volume. We say that Γ is **uniform** (or **cocompact**) if $\Gamma\backslash G$ est compact.

Theorem 16.3.2 (Borel, Harish-Chandra)**.** *Every semi-simple Lie group possesses a lattice.*

If $G = \mathrm{Isom}_\circ(\mathbb{X})$ where $\mathbb{X}$ is a symmetric space of non compact type, then a lattice $\Gamma \leq G$ is the fundamental group of a complete locally symmetric space $\Gamma\backslash\mathbb{X}$ if Γ is also required to be torsion-free (i.e. non trivial elements have infinite order).

Examples 16.3.3.

1. $\mathrm{SL}(n,\mathbb{Z})$ is a non uniform lattice in $\mathrm{SL}(n,\mathbb{R})$.
2. If $P \subset \mathbb{H}^2$ is a regular right angled polygon with $4g$ sides (it exists for any $g \geq 2$) labelled $A_1, B_1, A_1^{-1}, B_1^{-1}, A_2, \dots B_g^{-1}$, consider the isometry a_i (resp. b_i) sending A_i to A_i^{-1} (resp. B_i to B_i^{-1}) and reversing the orientation of the edges. The subgroup Γ of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}_\circ(\mathbb{H}^2)$ generated by $a_1, b_1, \dots, a_g, b_g$ is a torsion-free uniform lattice, and the quotient $\Gamma\backslash\mathbb{H}^2$ is a compact orientable surface of genus g.

Consider two torsion-free lattices $\Gamma_1, \Gamma_2 \le G$, and the associated locally symmetric spaces $M_i = \Gamma_i \backslash \mathbb{X}$, $i = 1, 2$. If Γ_1 and Γ_2 are conjugate, i.e. if $\Gamma_2 = g\Gamma_1 g^{-1}$ for some $g \in G$, then the isometry g of $\mathbb{X}$ induces an isometry from M_1 to M_2. Reciprocally, an isometry $\varphi : M_1 \to M_2$ induces an isometry $g \in \mathrm{Isom}(\mathbb{X})$ such that $\Gamma_2 = g\Gamma_1 g^{-1}$.

Theorem 16.3.4 (Mostow rigidity)**.** *Let $\mathbb{X}$ be an irreducible symmetric space of non compact type which is not homothetic to $\mathbb{H}^2$. If $\Gamma_1, \Gamma_2 \subset G = \mathrm{Isom}_\circ(\mathbb{X})$ are lattices, and $\theta : \Gamma_1 \to \Gamma_2$ is a group isomorphism, there is $g \in G$ such that:*

$$\forall \gamma \in \Gamma_1 \quad \theta(\gamma) = g\gamma g^{-1}$$

For $\mathbb{H}^2$, the situation is very different. If S is a closed orientable surface of genus $g \ge 2$ and $\Gamma = \pi_1(S)$, then the space of representations $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ such that $\rho(\Gamma)$ is a lattice, up to conjugation in $\mathrm{PGL}(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$, is homeomorphic (for a suitable topology) to $\mathbb{R}^{6g-6}$. This space is called the **Teichmüller space** of S.