Groups and geometry

Mid-term exam

Exercise 1

Let $\mathfrak{g}$ be a real Lie algebra.

1. Prove that if $\dim\mathfrak{g} = 2$ and $\mathfrak{g}$ is not abelian, then there is a basis (X, Y) of $\mathfrak{g}$ satisfying $[X, Y] = Y$.

   Solution: If (A, B) is a basis of $\mathfrak{g}$, then every Lie bracket is a multiple of $[A, B]$, so $\mathfrak{g}$ is 1-dimensional. Let $Y \in [\mathfrak{g}, \mathfrak{g}] \setminus \{0\}$, and consider $Z \in \mathfrak{g}$ not proportional to Y. Then $[Z, Y] = \lambda Y$ for some $\lambda \neq 0$ (otherwise $\mathfrak{g}$ would be abelian), and the basis (X, Y) with $X = \frac{1}{\lambda} Z$ does the job.

2. Prove that if $\dim\mathfrak{g} \leq 3$ and $\mathfrak{g}$ is not solvable, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

   Hint: you can use the solvable radical of $\mathfrak{g}$ to show that $\mathfrak{g}$ is either solvable or semi-simple.

   Solution: Warning: this statement is completely false. The answer should be that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{so}(3, \mathbb{R})$. If $\dim\mathfrak{g} = 1$ then $\mathfrak{g}$ is abelian, therefore solvable. If $\dim\mathfrak{g} = 2$, then either $\mathfrak{g}$ is abelian or there is a basis (X, Y) of $\mathfrak{g}$ such that $[X, Y] = Y$. In both cases, $\mathfrak{g}$ is solvable (in the second case, $[\mathfrak{g}, \mathfrak{g}]$ is abelian).

   If $\dim\mathfrak{g} = 3$, consider its solvable radical $R \subset \mathfrak{g}$. If $\mathfrak{g}$ is not solvable, then $R \neq \mathfrak{g}$. Therefore $\mathfrak{g}/R$ has dimension 1, 2 or 3. If it has dimension 1 or 2, then it solvable by the previous discussion. Since there is a short exact sequence $0 \to R \to \mathfrak{g} \to \mathfrak{g}/R \to 0$, we find that $\mathfrak{g}$ is solvable, hence the contradiction.

   We now see that if $\dim\mathfrak{g} \leq 3$ and $\mathfrak{g}$ is not solvable, then $\dim\mathfrak{g} = 3$ and $\mathfrak{g}$ is semi-simple. We can either use the classification of semi-simple Lie algebras, or use the Killing form of $\mathfrak{g}$ to show that it must be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{so}(3, \mathbb{R})$.

   Since $\mathfrak{g}$ is semi-simple, its Killing form B is non degenerate, so its signature can be $(3, 0), (2, 1), (1, 2)$ or $(0, 3)$. If the signature is $(0, 3)$ or $(3, 0)$, then the subalgebra $\mathfrak{so}(B) \subset \mathfrak{gl}(\mathfrak{g})$ is isomorphic to $\mathfrak{so}(3, \mathbb{R})$, thus the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ can be seen as a Lie algebra morphism into $\mathfrak{so}(3, \mathbb{R})$. This morphism is injective (because $\mathfrak{g}$ is semi-simple), and $\dim\mathfrak{g} = 3 = \dim\mathfrak{so}(3, \mathbb{R})$, so it is an isomorphism from $\mathfrak{g}$ to $\mathfrak{so}(3, \mathbb{R})$.

   If the signature is $(2, 1)$ or $(1, 2)$, then $\mathfrak{so}(B)$ is isomorphic to $\mathfrak{so}(2, 1) \approx \mathfrak{sl}(2, \mathbb{R})$, and we conclude like in the previous case.

3. Prove that if $\mathfrak{g}$ is nilpotent, but not abelian, then $\dim\mathfrak{g} \geq 3$.

   Solution: It suffices to show that $\mathfrak{g}$ such as in question 1. is not nilpotent. This is true because $C_2(\mathfrak{g}) = C_1(\mathfrak{g}) \neq \emptyset$.

4. Prove that for any $t \in \mathbb{R}$, there is a 3-dimensional Lie algebra $\mathfrak{g}_t$ which has a basis (X, Y, Z) satisfying:

   $$[X, Y] = Z \ ; \quad [Y, Z] = tZ \ ; \quad [Z, X] = 0.$$

   Solution: These formulae define an antisymmetric bilinear map on a 3-dimensional vector space. We have to show that it satisfies the Jacobi identity. Since the map $(u, v, w) \mapsto [u, [v, w]] + [v, [w, u]] + [w, [u, v]]$ is trilinear and antisymmetric, and the dimension is 3, it is enough to show the Jacobi identity on a given basis, i.e. to show that $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. This is true because all three terms are 0.

5. Prove that for any $t \neq 0$, the Lie algebra $\mathfrak{g}_t$ is isomorphic to $\mathfrak{g}_1$. Are the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_0$ isomorphic to each other?

   Solution: Changing the basis to $(tX, \frac{1}{t}Y, Z)$ shows that $\mathfrak{g}_t$ is isomorphic to $\mathfrak{g}_1$ for $t \neq 0$. The result of the next question implies that $\mathfrak{g}_1$ is not isomorphic to $\mathfrak{g}_0$.

6. Prove that $\mathfrak{g}_t$ is solvable. For which values of t is it nilpotent?

   Solution: We have that $C_1(\mathfrak{g}_t) = D_1(\mathfrak{g}_t) = \mathbb{R}.Z$. Therefore $D_2(\mathfrak{g}_t) = \{0\}$, and $\mathfrak{g}_t$ is solvable.

   If $t = 0$, then $C_2(\mathfrak{g}_t) = \{0\}$, and $\mathfrak{g}_0$ is nilpotent. If $t \neq 0$, then $C_2(\mathfrak{g}_t) = C_1(\mathfrak{g}_t)$, and $\mathfrak{g}_t$ is not nilpotent.

7. Prove that if $\dim \mathfrak{g} = 3$ and $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is either abelian or isomorphic to $\mathfrak{g}_t$ for some $t \in \mathbb{R}$.

   Hint: you can start by proving that $[\mathfrak{g}, \mathfrak{g}]$ *is abelian.*

   Solution: If $\mathfrak{g}$ is nilpotent but not abelian, then $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$ is nilpotent and $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, so it has dimension 1 or 2. In both cases, since $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, it must be abelian because of question 3.

   If $\dim[\mathfrak{g}, \mathfrak{g}] = 2$, then choose a basis (X, Y, Z) of $\mathfrak{g}$ such that (Y, Z) is a basis of $[\mathfrak{g}, \mathfrak{g}]$. Since $[Y, Z] = 0$, we get that $[\mathfrak{g}, \mathfrak{g}]$ is spanned by $[X, Y]$ and $[X, Z]$. It follows that $C_2(\mathfrak{g}) = C_1(\mathfrak{g})$, and $\mathfrak{g}$ is not nilpotent, which is a contradiction.

   We now know that $\dim[\mathfrak{g}, \mathfrak{g}] = 1$. Since $\mathfrak{g}$ is nilpotent, we have that $C_2(\mathfrak{g}) \neq C_1(\mathfrak{g})$, hence $C_2(\mathfrak{g}) = \{0\}$, i.e. $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$. If $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}.Z$, there are $X, Y \in \mathfrak{g}$ such that $[X, Y] = Z$, and (X, Y, Z) is a basis of $\mathfrak{g}$ satisfying $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. Therefore $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_0$.

Exercise 2

Let G be the Lie group $\mathrm{SL}(2, \mathbb{R})$ and V the vector space $\mathcal{M}_2(\mathbb{R})$. Denote by $\rho : G \times G \to \mathrm{GL}(V)$ the map defined by $\rho(g_1, g_2)v = g_1 v g_2^{-1}$ for all $g_1, g_2 \in G$ and $v \in V$.

1. Prove that ρ is a Lie group representation. What is its kernel?

   Solution: It is a morphism built from algebraic operations, therefore smooth. If $\rho(g_1, g_2) = \mathrm{Id}$, then $g_1 1_2 g_2^{-1} = 1_2$ shows that $g_1 = g_2$, and g_1 must commute with every matrix, therefore $g_1 = \pm 1_2$. It follows that $\ker \rho = \{(1_2, 1_2); (-1_2, -1_2)\}$.

2. Prove that the subgroup $\mathrm{O}(\det) \subset \mathrm{GL}(V)$ consisting of maps $f : V \to V$ such that $\det(f(v)) = \det(v)$ for all $v \in V$ is isomorphic to $\mathrm{O}(2, 2)$.

   Solution: From the expression $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, we see that the map

$$\varphi : \left\{ \begin{array}{ccc} V & \to & \mathbb{R}^4 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \left(\frac{a+d}{2}, \frac{b+c}{2}, \frac{a-d}{2}, \frac{b-c}{2}\right) \end{array} \right.$$

   satisfies $\det\left(\varphi^{-1}(x_1, x_2, x_3, x_4)\right) = x_1^2 + x_2^2 - x_3^2 - x_4^2$. If follows that conjugation by φ is an isomorphism between $\mathrm{O}(\det)$ and $\mathrm{O}(2, 2)$.

3. Prove that the Lie algebras $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{o}(2, 2)$ are isomorphic to each other.

   Solution: Let $\Phi : \mathrm{O}(\det) \to \mathrm{O}(2, 2)$ be the isomorphism considered above (i.e. $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$). Then $\Phi \circ \rho : \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \to \mathrm{O}(2, 2)$ is a Lie group morphism with discrete kernel ($\ker \Phi \circ \rho = \ker \rho$). It follows that $d_{(1_2, 1_2)}(\Phi \circ \rho) : \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, R)$ is an injective Lie algebra morphism. But $\dim \mathfrak{o}(2, 2) = 6 = 2 \dim \mathfrak{sl}(2, \mathbb{R})$, so it is an isomorphism.

4. What is the Lie group isomorphism that we obtained?

   Solution: We get an isomorphism from $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\ker\rho$ to the image $H\subset \mathrm{O}(2,2)$ of $\Phi\circ\rho$. Since $\mathrm{SL}(2,\mathbb{R})$ is connected, we have $H\subset\mathrm{O}(2,2)_\circ$. Now $\Phi\circ\rho$ is a local diffeomorphism, so H is open in $\mathrm{O}(2,2)_\circ$ and closed because $\mathrm{O}(2,2)_\circ$ is connected, i.e. $H=\mathrm{O}(2,2)_\circ$. The Lie group isomorphism we obtained is therefore:

$$(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\{\pm(1_2,1_2)\}\simeq\mathrm{O}(2,2)_\circ.$$

Exercice 3

Endow $\mathcal{M}_n(\mathbb{R})$ with the inner product $\langle X,Y\rangle=\mathrm{Tr}\left({}^tXY\right)$. Consider the subgroup $\mathrm{O}(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$, and the connection ∇ on the tangent bundle $T\mathrm{O}(n,\mathbb{R})$ defined as:

$$\nabla_x\sigma(v)=p_{T_x\mathrm{O}(n,\mathbb{R})}(d_x\sigma(v))$$

where $p_{T_x\mathrm{O}(n,\mathbb{R})}:\mathcal{M}_n(\mathbb{R})\to T_x\mathrm{O}(n,\mathbb{R})$ is the orthogonal projection.

1. Recall why this formula defines a connection on $T\mathrm{O}(n,\mathbb{R})$.

   Solution: The linearity comes from the linearity of differentiation and projection. The smoothness comes from the smoothness of $x\mapsto p_{T_x\mathrm{O}(n,\mathbb{R})}\in\mathrm{End}(\mathcal{M}_n(\mathbb{R}))$. For $f\in\mathcal{C}^\infty(\mathrm{O}(n,\mathbb{R}))$ and $\sigma\in\mathcal{X}(\mathrm{O}(n,\mathbb{R}))$, we find:

$$\begin{aligned}\nabla_x(f\sigma)(v)&=p_{T_x\mathrm{O}(n,\mathbb{R})}(d_x(f\sigma)(v))\\&=p_{T_x\mathrm{O}(n,\mathbb{R})}(d_xf(v)\sigma(x)+f(x)d_x\sigma(v))\\&=d_xf(v)\underbrace{p_{T_x\mathrm{O}(n,\mathbb{R})}(\sigma(x))}_{=\sigma(x)}+d_xf(v)p_{T_x\mathrm{O}(n,\mathbb{R})}(d_x\sigma(v))\\&=d_xf(v)\sigma(x)+f(x)\nabla_x\sigma(v).\end{aligned}$$

2. Given two left-invariant vector fields $\overline{X},\overline{Y}$ on $\mathrm{O}(n,\mathbb{R})$, compute $\nabla_{\overline{X}}\overline{Y}$.

   Solution: Let $X,Y\in\mathfrak{so}(n,\mathbb{R})$ and consider the associated left-invariant vector fields $\overline{X},\overline{Y}\in\mathcal{X}(\mathrm{O}(n,\mathbb{R}))$. The explicit formula at $x\in\mathrm{O}(n,\mathbb{R})$ is

$$\overline{X}(x)=d_{1_n}L_x(X)=xX\in T_x\mathrm{O}(n,\mathbb{R})=x\mathfrak{so}(n,\mathbb{R}).$$

   So $\overline{X}(x)=xX$ and $\overline{Y}(x)=xY$, and $d_x\overline{Y}(v)=vY$.

$$\begin{aligned}\nabla_{\overline{X}}\overline{Y}(x)&=p_{T_x\mathrm{O}(n,\mathbb{R})}\left(d_x\overline{Y}(\overline{X}(x))\right)\\&=p_{x\mathfrak{so}(n,\mathbb{R})}\left(d_x\overline{Y}(xX)\right)\\&=p_{x\mathfrak{so}(n,\mathbb{R})}(xXY)\end{aligned}$$

   Since $x\in\mathrm{O}(n,\mathbb{R})$, the left-multiplication $L_x:\mathcal{M}_n(\mathbb{R})\to\mathcal{M}_n(\mathbb{R})$ is an isometry for the inner product that we chose, so:

$$p_{x\mathfrak{so}(n,\mathbb{R})}(xXY)=xp_{\mathfrak{so}(n,\mathbb{R})}(XY).$$

Now the orthogonal complement of $\mathfrak{so}(n,\mathbb{R})$ is the space of symmetric matrices, i.e. $p_{\mathfrak{so}(n,\mathbb{R})}(Z) = \frac{Z - {}^tZ}{2}$ for any $Z \in \mathcal{M}_n(\mathbb{R})$. But for $X, Y \in \mathfrak{so}(n,\mathbb{R})$, we have $XY - {}^t(XY) = XY - YX$. Finally:

$$\nabla_{\overline{X}}\overline{Y}(x) = x\frac{XY - YX}{2} = \frac{1}{2}\overline{[X,Y]}(x) = \frac{1}{2}[\overline{X},\overline{Y}](x).$$

3. Compute the curvature and the torsion of ∇.

   Solution: Consider left-invariant vector fields $\overline{X}, \overline{Y}, \overline{Z}$.

$$\begin{aligned}
R(\overline{X},\overline{Y})\overline{Z} &= \nabla_{\overline{X}}\nabla_{\overline{Y}}\overline{Z} - \nabla_{\overline{Y}}\nabla_{\overline{X}}\overline{Z} - \nabla_{[\overline{X},\overline{Y}]}\overline{Z} \\
&= \frac{1}{2}\nabla_{\overline{X}}\overline{[Y,Z]} - \frac{1}{2}\nabla_{\overline{Y}}\overline{[X,Z]} - \nabla_{\overline{[X,Y]}}\overline{Z} \\
&= \frac{1}{4}\left(\overline{[X,[Y,Z]]} - \overline{[Y,[X,Z]]}\right) - \frac{1}{2}\overline{[[X,Y],Z]} \\
&= \frac{1}{4}\left(\overline{[X,[Y,Z]]} + \overline{[Y,[Z,X]]}\right) - \frac{1}{2}\overline{[[X,Y],Z]} \\
&= -\frac{1}{4}\overline{[Z,[X,Y]]} - \frac{1}{2}\overline{[[X,Y],Z]} \\
&= \frac{1}{4}\overline{[Z,[X,Y]]}.
\end{aligned}$$

Since the curvature is tensorial, this expression is enough to compute the curvature. Given $x \in O(n,\mathbb{R})$ and $u, v, w \in T_xO(n,\mathbb{R})$, the curvature is

$$R_x(u,v)w = \frac{1}{4}x[x^{-1}w,[x^{-1}u,x^{-1}v]] \in T_xO(n,\mathbb{R}).$$

Now consider left-invariant vector fields $\overline{X}, \overline{Y} \in \mathcal{X}(O(n,\mathbb{R}))$.

$$\begin{aligned}
\nabla_{\overline{X}}\overline{Y} - \nabla_{\overline{Y}}\overline{X} &= \frac{1}{2}\overline{[X,Y]} - \frac{1}{2}\overline{[Y,X]} \\
&= \overline{[X,Y]} \\
&= [\overline{X},\overline{Y}].
\end{aligned}$$

This shows that the torsion T satisfies $T(\overline{X},\overline{Y}) = 0$, hence $T = 0$.