

Products of Geck-Rouquier conjugacy classes and the Hecke algebra of composed permutations

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Abstract. We show the q -analog of a well-known result of Farahat and Higman: in the center of the Iwahori-Hecke algebra $\mathcal{H}_{n,q}$, if $(a_{\lambda\mu}^\nu(n, q))_\nu$ is the set of structure constants involved in the product of two Geck-Rouquier conjugacy classes $\Gamma_{\lambda,n}$ and $\Gamma_{\mu,n}$, then each coefficient $a_{\lambda\mu}^\nu(n, q)$ depend on n and q in a polynomial way. Our proof relies on the construction of a projective limit of the Hecke algebras; this projective limit is inspired by the Ivanov-Kerov algebra of partial permutations.

Résumé. Nous démontrons le q -analogue d'un résultat bien connu de Farahat et Higman : dans le centre de l'algèbre d'Iwahori-Hecke $\mathcal{H}_{n,q}$, si $(a_{\lambda\mu}^\nu(n, q))_\nu$ est l'ensemble des constantes de structure mises en jeu dans le produit de deux classes de conjugaison de Geck-Rouquier $\Gamma_{\lambda,n}$ et $\Gamma_{\mu,n}$, alors chaque coefficient $a_{\lambda\mu}^\nu(n, q)$ dépend de façon polynomiale de n et de q . Notre preuve repose sur la construction d'une limite projective des algèbres d'Hecke ; cette limite projective est inspirée de l'algèbre d'Ivanov-Kerov des permutations partielles.

Keywords: Iwahori-Hecke algebras, Geck-Rouquier conjugacy classes, symmetric functions.

In this paper, we answer a question asked in [FW09] that concerns the products of Geck-Rouquier conjugacy classes in the Hecke algebras $\mathcal{H}_{n,q}$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a partition with $|\lambda| + \ell(\lambda) \leq n$, we consider the completed partition

$$\lambda \rightarrow n = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_r + 1, 1^{n-|\lambda|-\ell(\lambda)}),$$

and we denote by $C_{\lambda,n} = C_{\lambda \rightarrow n}$ the corresponding conjugacy class, that is to say, the sum of all permutations with cycle type $\lambda \rightarrow n$ in the center of the symmetric group algebra $\mathbb{C}\mathfrak{S}_n$. Notice that in particular, $C_{\lambda,n} = 0$ if $|\lambda| + \ell(\lambda) > n$. It is known since [FH59] that the products of completed conjugacy classes write as

$$C_{\lambda,n} * C_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} a_{\lambda\mu}^\nu(n) C_{\nu,n},$$

where the structure constants $a_{\lambda\mu}^\nu(n)$ depend on n in a polynomial way. In [GR97], some deformations Γ_λ of the conjugacy classes C_λ are constructed. These central elements form a basis of the center $\mathcal{L}_{n,q}$ of the Iwahori-Hecke algebra $\mathcal{H}_{n,q}$, and they are characterized by the two following properties, see [Fra99]:

1. The element Γ_λ is central and specializes to C_λ for $q = 1$.
2. The difference $\Gamma_\lambda - C_\lambda$ involves no permutation of minimal length in its conjugacy class.

As before, $\Gamma_{\lambda,n} = \Gamma_{\lambda \rightarrow n}$ if $|\lambda| + \ell(\lambda) \leq n$, and 0 otherwise. Our main result is the following:

Theorem 1 *In the center of the Hecke algebra $\mathcal{H}_{n,q}$, the products of completed Geck-Rouquier conjugacy classes write as*

$$\Gamma_{\lambda,n} * \Gamma_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} a_{\lambda\mu}^\nu(n, q) \Gamma_{\nu,n},$$

and the structure constants $a_{\lambda\mu}^\nu(n, q)$ are in $\mathbb{Q}[n, q, q^{-1}]$.

The first part of Theorem 1 — that is to say, that elements $\Gamma_{\nu,n}$ involved in the product satisfy the inequality $|\nu| \leq |\lambda| + |\mu|$ — was already in [FW09, Theorem 1.1], and the polynomial dependance of the coefficients $a_{\lambda\mu}^\nu(n, q)$ was Conjecture 3.1; our paper is devoted to a proof of this conjecture. We shall combine two main arguments:

- We construct a projective limit $\mathcal{D}_{\infty,q}$ of the Hecke algebras, which is essentially a q -version of the algebra of Ivanov and Kerov, see [IK99]. We perform *generic computations* inside various subalgebras of $\mathcal{D}_{\infty,q}$, and we project then these calculations on the algebras $\mathcal{H}_{n,q}$ and their centers.
- The centers of the Hecke algebras admit numerous bases, and these bases are related one to another in the same way as the bases of the symmetric function algebra Λ . This allows to separate the dependance on q and the dependance on n of the coefficients $a_{\lambda\mu}^\nu(n, q)$.

Before we start, let us fix some notations. If n is a non-negative integer, \mathfrak{P}_n is the set of partitions of n , \mathfrak{C}_n is the set of compositions of n , and \mathfrak{S}_n is the set of permutations of the interval $\llbracket 1, n \rrbracket$. The **type** of a permutation $\sigma \in \mathfrak{S}_n$ is the partition $\lambda = t(\sigma)$ obtained by ordering the sizes of the orbits of σ ; for instance, $t(24513) = (3, 2)$. The **code** of a composition $c \in \mathfrak{C}_n$ is the complementary in $\llbracket 1, n \rrbracket$ of the set of descents of c ; for instance, the code of $(3, 2, 3)$ is $\{1, 2, 4, 6, 7\}$. Finally, we denote by $\mathcal{Z}_n = Z(\mathbb{C}\mathfrak{S}_n)$ the center of the algebra $\mathbb{C}\mathfrak{S}_n$; the conjugacy classes C_λ form a linear basis of \mathcal{Z}_n when λ runs over \mathfrak{P}_n .

1 Partial permutations and the Ivanov-Kerov algebra

Since our argument is essentially inspired by the construction of [IK99], let us recall it briefly. A **partial permutation** of order n is a pair (σ, S) where S is a subset of $\llbracket 1, n \rrbracket$, and σ is a permutation in $\mathfrak{S}(S)$. Alternatively, one may see a partial permutation as a permutation σ in \mathfrak{S}_n together with a subset containing the non-trivial orbits of σ . The product of two partial permutations is

$$(\sigma, S)(\tau, T) = (\sigma\tau, S \cup T),$$

and this operation yield a semigroup whose complex algebra is denoted by \mathcal{B}_n . There is a natural projection $\text{pr}_n : \mathcal{B}_n \rightarrow \mathbb{C}\mathfrak{S}_n$ that consists in forgetting the support of a partial permutation, and also natural compatible maps

$$\phi_{N,n} : (\sigma, S) \in \mathcal{B}_N \mapsto \begin{cases} (\sigma, S) \in \mathcal{B}_n & \text{if } S \subset \llbracket 1, n \rrbracket, \\ 0 & \text{otherwise,} \end{cases}$$

whence a projective limit $\mathcal{B}_\infty = \varprojlim \mathcal{B}_n$ with respect to this system $(\phi_{N,n})_{N \geq n}$ and in the category of filtered algebras. Now, one can lift the conjugacy classes C_λ to the algebras of partial permutations. Indeed, the symmetric group \mathfrak{S}_n acts on \mathcal{B}_n by

$$\sigma \cdot (\tau, S) = (\sigma\tau\sigma^{-1}, \sigma(S)),$$

and a linear basis of the invariant subalgebra $\mathcal{A}_n = (\mathcal{B}_n)^{\mathfrak{S}_n}$ is labelled by the partitions λ of size less than or equal to n :

$$\mathcal{A}_n = \bigoplus_{|\lambda| \leq n} \mathbb{C}A_{\lambda,n}, \quad \text{where } A_{\lambda,n} = \sum_{\substack{|S|=|\lambda| \\ \sigma \in \mathfrak{S}(S), t(\sigma)=\lambda}} (\sigma, S).$$

Since the actions $\mathfrak{S}_n \curvearrowright \mathcal{B}_n$ are compatible with the morphisms $\phi_{N,n}$, the inverse limit $\mathcal{A}_\infty = (\mathcal{B}_\infty)^{\mathfrak{S}_\infty}$ of the invariant subalgebras has a basis $(A_\lambda)_\lambda$ indexed by all partitions $\lambda \in \mathfrak{P} = \bigsqcup_{n \in \mathbb{N}} \mathfrak{P}_n$, and such that $\phi_{\infty,n}(A_\lambda) = A_{\lambda,n}$ (with by convention $A_{\lambda,n} = 0$ if $|\lambda| > n$). As a consequence, if $(a'_{\lambda\mu})_{\lambda,\mu,\nu}$ is the family of structure constants of the **Ivanov-Kerov algebra**⁽ⁱ⁾ \mathcal{A}_∞ in the basis $(A_\lambda)_{\lambda \in \mathfrak{P}}$, then

$$\forall n, A_{\lambda,n} * A_{\mu,n} = \sum_{\nu} a'_{\lambda\mu} A_{\nu,n},$$

with $A_{\lambda,n} = 0$ if $|\lambda| \geq n$. Moreover, it is not difficult to see that $a'_{\lambda\mu} \neq 0$ implies $|\nu| \leq |\lambda| + |\mu|$, and also $|\nu| - \ell(\nu) \leq |\lambda| - \ell(\lambda) + |\mu| - \ell(\mu)$, cf. [IK99, §10], for the study of the filtrations of \mathcal{A}_∞ . Now, $\text{pr}_n(\mathcal{A}_n) = \mathcal{L}_n$, and more precisely,

$$\text{pr}_n(A_{\lambda,n}) = \binom{n - |\lambda| + m_1(\lambda)}{m_1(\lambda)} C_{\lambda-1,n}.$$

where $\lambda - 1 = (\lambda_1 - 1, \dots, \lambda_s - 1)$ if $\lambda = (\lambda_1, \dots, \lambda_s \geq 2, 1, \dots, 1)$. The result of Farahat and Higman follows immediately, and we shall try to mimic this construction in the context of Iwahori-Hecke algebras.

2 Composed permutations and their Hecke algebra

We recall that the **Iwahori-Hecke algebra** of type A and order n is the quantized version of the symmetric group algebra defined over $\mathbb{C}(q)$ by

$$\mathcal{H}_{n,q} = \left\langle S_1, \dots, S_{n-1} \left| \begin{array}{l} \text{braid relations: } \forall i, S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \\ \text{commutation relations: } \forall |j-i| > 1, S_i S_j = S_j S_i \\ \text{quadratic relations: } \forall i, (S_i)^2 = (q-1) S_i + q \end{array} \right. \right\rangle.$$

When $q = 1$, we recover the symmetric group algebra $\mathbb{C}\mathfrak{S}_n$. If $\omega \in \mathfrak{S}_n$, let us denote by T_ω the product $S_{i_1} S_{i_2} \cdots S_{i_r}$, where $\omega = s_{i_1} s_{i_2} \cdots s_{i_r}$ is any reduced expression of ω in elementary transpositions $s_i = (i, i + 1)$. Then, it is well known that the elements T_ω do not depend on the choice of reduced expressions, and that they form a $\mathbb{C}(q)$ -linear basis of $\mathcal{H}_{n,q}$, see [Mat99].

⁽ⁱ⁾ It can be shown that \mathcal{A}_∞ is isomorphic to the algebra of shifted symmetric polynomials, see Theorem 9.1 in [IK99].

In order to construct a *projective* limit of the algebras $\mathcal{H}_{n,q}$, it is very tempting to mimic the construction of Ivanov and Kerov, and therefore to build an Hecke algebra of partial permutations. Unfortunately, this is not possible; let us explain why by considering for instance the transposition $\sigma = 1432$ in \mathfrak{S}_4 . The possible supports for σ are $\{2, 4\}$, $\{1, 2, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$. However,

$$\sigma = s_2 s_3 s_2,$$

and the support of s_2 (respectively, of s_3) contains at least $\{2, 3\}$ (resp., $\{3, 4\}$). So, if we take account of the Coxeter structure of \mathfrak{S}_4 — and it should obviously be the case in the context of Hecke algebras — then the only valid supports for σ are the connected ones, namely, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$. This problem leads to consider *composed permutations* instead of *partial permutations*. If c is a composition of n , let us denote by $\pi(c)$ the corresponding set partition of $\llbracket 1, n \rrbracket$, i.e., the set partition whose parts are the intervals $\llbracket 1, c_1 \rrbracket$, $\llbracket c_1 + 1, c_1 + c_2 \rrbracket$, etc. A **composed permutation** of order n is a pair (σ, c) with $\sigma \in \mathfrak{S}_n$ and c composition in \mathfrak{C}_n such that $\pi(c)$ is coarser than the set partition of orbits of σ . For instance, $(32154867, (5, 3))$ is a composed permutation of order 8; we shall also write this $32154|867$. The product of two composed permutations is then defined by

$$(\sigma, c)(\tau, d) = (\sigma\tau, c \vee d),$$

where $c \vee d$ is the finest composition of n such that $\pi(c \vee d) \geq \pi(c) \vee \pi(d)$ in the lattice of set partitions. For instance,

$$321|54|867 \times 12|435|687 = 42153|768.$$

One obtains so a semigroup of composed permutations; its complex semigroup algebra will be denoted by⁽ⁱⁱ⁾ \mathcal{D}_n , and the dimension of \mathcal{D}_n is the number of composed permutations of order n .

Now, let us describe an Hecke version $\mathcal{D}_{n,q}$ of the algebra \mathcal{D}_n . As for $\mathcal{H}_{n,q}$, one introduces generators $(S_i)_{1 \leq i \leq n-1}$ corresponding to the elementary transpositions s_i , but one has also to introduce generators $(I_i)_{1 \leq i \leq n-1}$ that allow to join the parts of the composition of a composed permutation. Hence, the **Iwahori-Hecke algebra of composed permutations** is defined (over the ground field $\mathbb{C}(q)$) by $\mathcal{D}_{n,q} = \langle S_1, \dots, S_{n-1}, I_1, \dots, I_{n-1} \rangle$ and the following relations:

$$\begin{aligned} \forall i, \quad S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ \forall |j - i| > 1, \quad S_i S_j &= S_j S_i \\ \forall i, \quad (S_i)^2 &= (q - 1) S_i + q I_i \\ \forall i, j, \quad S_i I_j &= I_j S_i \\ \forall i, j, \quad I_i I_j &= I_j I_i \\ \forall i, \quad S_i I_i &= S_i \\ \forall i, \quad (I_i)^2 &= I_i \end{aligned}$$

The generators S_i correspond to the composed permutations $1|2| \dots |i - 1|i + 1, i|i + 2| \dots |n$, and the generators I_i correspond to the composed permutations $1|2| \dots |i - 1|i, i + 1|i + 2| \dots |n$.

⁽ⁱⁱ⁾ If one considers pairs (σ, π) where π is any set partition of $\llbracket 1, n \rrbracket$ coarser than $\text{orb}(\sigma)$ (and not necessarily a set partition associated to a composition), then one obtains an algebra of *split permutations* whose subalgebra of invariants is related to the connected Hurwitz numbers $H_{n,g}(\lambda)$.

Proposition 2 *The algebra $\mathcal{D}_{n,q}$ specializes to the algebra of composed permutations \mathcal{D}_n when $q = 1$; to the Iwahori-Hecke algebra $\mathcal{H}_{n,q}$ when $I_1 = I_2 = \dots = I_{n-1} = 1$; and to the algebra $\mathcal{D}_{m,q}$ of lower order $m < n$ when $I_m = I_{m+1} = \dots = I_{n-1} = 0$ and $S_m = S_{m+1} = \dots = S_{n-1} = 0$.*

In the following, we shall denote by pr_n the specialization $\mathcal{D}_{n,q} \rightarrow \mathcal{H}_{n,q}$; it generalizes the map $\mathcal{D}_n \rightarrow \mathbb{C}\mathfrak{S}_n$ of the first section. The first part of Proposition 2 is actually the only one that is non trivial, and it will be a consequence of Theorem 3. If ω is a permutation with reduced expression $\omega = s_{i_1} s_{i_2} \dots s_{i_r}$, we denote as before by T_ω the product $S_{i_1} S_{i_2} \dots S_{i_r}$ in $\mathcal{D}_{n,q}$. On the other hand, if c is a composition of $\llbracket 1, n \rrbracket$, we denote by I_c the product of the generators I_j with j in the code of c (so for instance, $I_{(3,2,3)} = I_1 I_2 I_4 I_6 I_7$ in $\mathcal{D}_{8,q}$). These elements are central idempotents, and I_c correspond to the composed permutation (id, c) . Finally, if (σ, c) is a composed permutation, $T_{\sigma,c}$ is the product $T_\sigma I_c$.

Theorem 3 *In $\mathcal{D}_{n,q}$, the products T_σ do not depend on the choice of reduced expressions, and the products $T_{\sigma,c}$ form a linear basis of $\mathcal{D}_{n,q}$ when (σ, c) runs over composed permutations of order n . There is an isomorphism of $\mathbb{C}(q)$ -algebras between*

$$\mathcal{D}_{n,q} \quad \text{and} \quad \bigoplus_{c \in \mathfrak{C}_n} \mathcal{H}_{c,q},$$

where $\mathcal{H}_{c,q}$ is the Young subalgebra $\mathcal{H}_{c_1,q} \otimes \mathcal{H}_{c_2,q} \otimes \dots \otimes \mathcal{H}_{c_r,q}$ of $\mathcal{H}_{n,q}$.

Proof: If $\sigma \in \mathfrak{S}_n$, the Matsumoto theorem ensures that it is always possible to go from a reduced expression $s_{i_1} s_{i_2} \dots s_{i_r}$ to another reduced expression $s_{j_1} s_{j_2} \dots s_{j_r}$ by braid moves $s_i s_{i+1} s_i \leftrightarrow s_{i+1} s_i s_{i+1}$ and commutations $s_i s_j \leftrightarrow s_j s_i$ when $|j - i| > 1$. Since the corresponding products of S_i in $\mathcal{D}_{n,q}$ are preserved by these substitutions, a product T_σ in $\mathcal{D}_{n,q}$ does not depend on the choice of a reduced expression. Now, let us consider an arbitrary product Π of generators S_i and I_j (in any order). As the elements I_j are central idempotents, it is always possible to reduce the product to

$$\Pi = S_{i_1} S_{i_2} \dots S_{i_p} I_c$$

with c composition of n — here, $s_{i_1} s_{i_2} \dots s_{i_p}$ is *a priori* not a reduced expression. Moreover, since $S_i I_i = S_i$, we can suppose that the code of c contains $\{i_1, \dots, i_p\}$. Now, suppose that $\sigma = s_{i_1} s_{i_2} \dots s_{i_p}$ is not a reduced expression. Then, by using braid moves and commutations, we can transform the expression in one with two consecutive letters that are identical, that is to say that if $j_k = j_{k+1}$,

$$\sigma = s_{j_1} \dots s_{j_k} s_{j_{k+1}} \dots s_{j_p} = s_{j_1} \dots s_{j_{k-1}} s_{j_{k+2}} \dots s_{j_p}.$$

We apply the same moves to the S_i in $\mathcal{D}_{n,q}$ and we obtain $\Pi = S_{j_1} \dots S_{j_k} S_{j_{k+1}} \dots S_{j_p} I_c$; notice that the code of c still contains $\{j_1, \dots, j_p\} = \{i_1, \dots, i_p\}$. By using the quadratic relation in $\mathcal{D}_{n,q}$, we conclude that if $j_k = j_{k+1}$,

$$\begin{aligned} \Pi &= (q - 1) S_{j_1} \dots S_{j_{k-1}} S_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c + q S_{j_1} \dots S_{j_{k-1}} I_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c \\ &= (q - 1) S_{j_1} \dots S_{j_{k-1}} S_{j_k} S_{j_{k+2}} \dots S_{j_p} I_c + q S_{j_1} \dots S_{j_{k-1}} S_{j_{k+2}} \dots S_{j_p} I_c \end{aligned}$$

because $I_{j_k} I_c = I_c$. Consequently, by induction on p , any product Π is a $\mathbb{Z}[q]$ -linear combination of products $T_{\tau,c}$ (and with the same composition c for all the terms of the linear combination). So, the

reduced products $T_{\sigma,c}$ span linearly $\mathcal{D}_{n,q}$ when (σ, c) runs over composed permutations of order n . If c is in \mathfrak{C}_n , we define a morphism of $\mathbb{C}(q)$ -algebras from $\mathcal{D}_{n,q}$ to $\mathcal{H}_{c,q}$ by

$$\psi_c(S_i) = \begin{cases} S_i & \text{if } i \text{ is in the code of } c, \\ 0 & \text{otherwise,} \end{cases} \quad ; \quad \psi_c(I_i) = \begin{cases} 1 & \text{if } i \text{ is in the code of } c, \\ 0 & \text{otherwise.} \end{cases}$$

The elements $\psi_c(S_i)$ and $\psi_c(I_i)$ satisfy in $\mathcal{H}_{c,q}$ the relations of the generators S_i and I_i in $\mathcal{D}_{n,q}$. So, there is indeed such a morphism of algebras $\psi_c : \mathcal{D}_{n,q} \rightarrow \mathcal{H}_{c,q}$, and one has in fact $\psi_c(T_{\sigma,b}) = T_\sigma$ if $\pi(b) \leq \pi(c)$, and 0 otherwise. Let us consider the direct sum of algebras $\mathcal{H}_{\mathfrak{C}_n,q} = \bigoplus_{c \in \mathfrak{C}_n} \mathcal{H}_{c,q}$, and the direct sum of morphisms $\psi = \bigoplus_{c \in \mathfrak{C}_n} \psi_c$. We denote the basis vectors $[0, 0, \dots, (T_\sigma \in \mathcal{H}_{c,q}), \dots, 0]$ of $\mathcal{H}_{\mathfrak{C}_n,q}$ by $T_{\sigma \in \mathcal{H}_{c,q}}$; in particular,

$$\psi(T_{\sigma,c}) = \sum_{d \geq c} T_{\sigma \in \mathcal{H}_{d,q}}$$

for any composed permutation (σ, c) . As a consequence, the map ψ is surjective, because

$$\psi \left(\sum_{d \geq c} \mu(c, d) T_{\sigma,c} \right) = T_{\sigma \in \mathcal{H}_{c,q}}$$

where $\mu(c, d) = \mu(\pi(c), \pi(d)) = (-1)^{\ell(c) - \ell(d)}$ is the Möbius function of the hypercube lattice of compositions. If σ is a permutation, we denote by $\text{orb}(\sigma)$ the set partition whose parts are the orbits of σ . Since the families $(T_{\sigma,c})_{\text{orb}(\sigma) \leq \pi(c)}$ and $(T_{\sigma \in \mathcal{H}_{c,q}})_{\text{orb}(\sigma) \leq \pi(c)}$ have the same cardinality $\dim \mathcal{D}_n$, we conclude that $(T_{\sigma,c})_{\text{orb}(\sigma) \leq \pi(c)}$ is a $\mathbb{C}(q)$ -linear basis of $\mathcal{D}_{n,q}$ and that ψ is an isomorphism of $\mathbb{C}(q)$ -algebras. \square

Notice that the second part of Theorem 3 is the q -analog of Corollary 3.2 in [IK99]. To conclude this part, we have to build the inverse limit $\mathcal{D}_{\infty,q} = \varprojlim \mathcal{D}_{n,q}$, but this is easy thanks to the specializations evoked in the third part of Proposition 2. Hence, if $\phi_{N,n} : \mathcal{D}_{N,q} \rightarrow \mathcal{D}_{n,q}$ is the map that sends the generators $I_{i \geq n}$ and $S_{i \geq n}$ to zero and that preserves the other generators, then $(\phi_{N,n})_{N \geq n}$ is a system of compatible maps, and these maps behave well with respect to the filtration $\text{deg } T_{\sigma,c} = |\text{code}(c)|$. Consequently, there is a projective limit $\mathcal{D}_{\infty,q}$ whose elements are the infinite linear combinations of $T_{\sigma,c}$, with σ finite permutation in \mathfrak{S}_∞ and c infinite composition compatible with σ and with almost all its parts of size 1.

It is not true that two elements x and y in $\mathcal{D}_{\infty,q}$ are equal if and only if their projections $\text{pr}_n(\phi_{\infty,n}(x))$ and $\text{pr}_n(\phi_{\infty,n}(y))$ are equal for all n : for instance,

$$T[21|34|5|6|\dots] = S_1 I_1 I_3 \quad \text{and} \quad T[2134|5|6|\dots] = S_1 I_1 I_2 I_3$$

have the same projections in all the Hecke algebras (namely, S_1 if $n \geq 4$ and 0 otherwise), but they are not equal. However, the result is true if we consider only the subalgebras $\mathcal{D}'_{n,q} \subset \mathcal{D}_{n,q}$ spanned by the $T_{\sigma,c}$ with $c = (k, 1, \dots, 1)$ — then, σ may be considered as a partial permutation of $[[1, k]]$.

Proposition 4 *For any n , the vector space $\mathcal{D}'_{n,q}$ spanned by the $T_{\sigma,c}$ with $c = (k, 1^{n-k})$ is a subalgebra of $\mathcal{D}_{n,q}$. In the inverse limit $\mathcal{D}'_{\infty,q} \subset \mathcal{D}_{\infty,q}$, the projections $\text{pr}_{\infty,n} = \text{pr}_n \circ \phi_{\infty,n}$ separate the vectors:*

$$\forall x, y \in \mathcal{D}'_{\infty,q}, \quad (\forall n, \text{pr}_{\infty,n}(x) = \text{pr}_{\infty,n}(y)) \iff (x = y).$$

Proof: The supremum of two compositions $(k, 1^{n-k})$ and $(l, 1^{n-l})$ is $(m, 1^{n-m})$ with $m = \max(k, l)$; consequently, $\mathcal{D}'_{n,q}$ is indeed a subalgebra of $\mathcal{D}_{n,q}$. Any element x of the projective limit $\mathcal{D}'_{\infty,q}$ writes uniquely as

$$x = \sum_{k=0}^{\infty} \sum_{\sigma \in \mathfrak{S}_k} a_{\sigma,k}(x) T_{\sigma,(k,1^\infty)}.$$

Suppose that x and y have the same projections, and let us fix a permutation σ . There is a minimal integer k such that $\sigma \in \mathfrak{S}_k$, and $a_{\sigma,k}(x)$ is the coefficient of T_σ in $\text{pr}_{\infty,k}(x)$; consequently, $a_{\sigma,k}(x) = a_{\sigma,k}(y)$. Now, $a_{\sigma,k}(x) + a_{\sigma,k+1}(x)$ is the coefficient of T_σ in $\text{pr}_{\infty,k+1}(x)$, so one has also $a_{\sigma,k}(x) + a_{\sigma,k+1}(x) = a_{\sigma,k}(y) + a_{\sigma,k+1}(y)$, and $a_{\sigma,k+1}(x) = a_{\sigma,k+1}(y)$. By using the same argument and by induction on l , we conclude that $a_{\sigma,k+l}(x) = a_{\sigma,k+l}(y)$ for every l , and therefore $x = y$. We have then proved that the projections separate the vectors in $\mathcal{D}'_{\infty,q}$. \square

3 Bases of the center of the Hecke algebra

In the following, $\mathcal{Z}_{n,q}$ is the center of $\mathcal{H}_{n,q}$. We have already given a characterization of the **Geck-Rouquier central elements** Γ_λ , and they form a linear basis of $\mathcal{Z}_{n,q}$ when λ runs over \mathfrak{P}_n . Let us write down explicitly this basis when $n = 3$:

$$\Gamma_3 = T_{231} + T_{312} + (q-1)q^{-1}T_{321} \quad ; \quad \Gamma_{2,1} = T_{213} + T_{132} + q^{-1}T_{321} \quad ; \quad \Gamma_{1,1,1} = T_{123}$$

The first significative example of Geck-Rouquier element is actually when $n = 4$. Thus, if one considers

$$\begin{aligned} \Gamma_{3,1} = & T_{1342} + T_{1423} + T_{2314} + T_{3124} + q^{-1}(T_{2431} + T_{4132} + T_{3214} + T_{4213}) \\ & + (q-1)q^{-1}(T_{1432} + T_{3214}) + (q-1)q^{-2}(T_{3421} + T_{4312} + 2T_{4231}) + (q-1)^2q^{-3}T_{4321}, \end{aligned}$$

the terms with coefficient 1 are the four minimal 3-cycles in \mathfrak{S}_4 ; the terms whose coefficients specialize to 1 when $q = 1$ are the eight 3-cycles in \mathfrak{S}_4 ; and the other terms are not minimal in their conjugacy classes, and their coefficients vanish when $q = 1$.

It is really unclear how one can lift these elements to the Hecke algebras of composed permutations; fortunately, the center $\mathcal{Z}_{n,q}$ admits other linear bases that are easier to pull back from $\mathcal{H}_{n,q}$ to $\mathcal{D}_{n,q}$. In [Las06], seven different bases for $\mathcal{Z}_{n,q}$ are studied⁽ⁱⁱⁱ⁾, and it is shown that up to diagonal matrices that depend on q in a polynomial way, the transition matrices between these bases are the same as the transition matrices between the usual bases of the algebra of symmetric functions. We shall only need the **norm basis** N_λ , whose properties are recalled in Proposition 5. If c is a composition of n and \mathfrak{S}_c is the corresponding Young subgroup of \mathfrak{S}_n , it is well-known that each coset in $\mathfrak{S}_n/\mathfrak{S}_c$ or $\mathfrak{S}_c \backslash \mathfrak{S}_n$ has a unique representative ω of minimal length which is called the **distinguished representative** — this fact is even true for parabolic double cosets. In what follows, we rather work with right cosets, and the distinguished representatives of $\mathfrak{S}_c \backslash \mathfrak{S}_n$ are precisely the permutation words whose recoils are contained in the set of descents of c . So for instance, if $c = (2, 3)$, then

$$\mathfrak{S}_{(2,3)} \backslash \mathfrak{S}_5 = \{12345, 13245, 13425, 13452, 31245, 31425, 31452, 34125, 34152, 34512\} = 12 \sqcup \sqcup 345.$$

⁽ⁱⁱⁱ⁾ One can also consult [Jon90] and [Fra99].

Proposition 5 [Las06, Theorem 7] *If c is a composition of n , let us denote by N_c the element*

$$\sum_{\omega \in \mathfrak{S}_c \setminus \mathfrak{S}_n} q^{-\ell(\omega)} T_{\omega^{-1}} T_{\omega}$$

in the Hecke algebra $\mathcal{H}_{n,q}$. Then, N_c does not depend on the order of the parts of c , and the N_{λ} form a linear basis of $\mathcal{Z}_{n,q}$ when λ runs over \mathfrak{P}_n — in particular, the norms N_c are central elements. Moreover,

$$(\Gamma_{\lambda})_{\lambda \in \mathfrak{P}_n} = D \cdot M2E \cdot (N_{\mu})_{\mu \in \mathfrak{P}_n},$$

where $M2E$ is the transition matrix between monomial functions m_{λ} and elementary functions e_{μ} , and D is the diagonal matrix with coefficients $(q/(q-1))^{n-\ell(\lambda)}$.

So for instance, $\Gamma_3 = q^2 (q-1)^{-2} (3N_3 - 3N_{2,1} + N_{1,1,1})$, because $m_3 = 3e_3 - 3e_{2,1} + e_{1,1,1}$. Let us write down explicitly the norm basis when $n = 3$:

$$\begin{aligned} N_3 &= T_{123} & ; & & N_{2,1} &= 3T_{123} + (q-1)q^{-1}(T_{213} + T_{132}) + (q-1)q^{-2}T_{321} \\ N_{1,1,1} &= 6T_{123} + 3(q-1)q^{-1}(T_{213} + T_{132}) + (q-1)^2q^{-2}(T_{231} + T_{312}) + (q^3-1)q^{-3}T_{321} \end{aligned}$$

We shall see hereafter that these norms have natural preimages by the projections pr_n and $\text{pr}_{\infty,n}$.

4 Generic norms and the Hecke-Ivanov-Kerov algebra

Let us fix some notations. If c is a composition of size $|c|$ less than n , then $c \uparrow n$ is the composition $(c_1, \dots, c_r, n - |c|)$, $J_c = I_1 I_2 \cdots I_{|c|-1}$, and

$$M_{c,n} = \sum_{\omega \in \mathfrak{S}_{c \uparrow n} \setminus \mathfrak{S}_n} q^{-\ell(\omega)} T_{\omega^{-1}} T_{\omega} J_c,$$

the products T_{ω} being considered as elements of $\mathcal{D}_{n,q}$. So, $M_{c,n}$ is an element of $\mathcal{D}_{n,q}$, and we set $M_{c,n} = 0$ if $|c| > n$.

Proposition 6 *For any N, n and any composition c , $\phi_{N,n}(M_{c,N}) = M_{c,n}$, and $\text{pr}_n(M_{c,n}) = N_{c \uparrow n}$ if $|c| \leq n$, and 0 otherwise. On the other hand, $M_{c,n}$ is always in $\mathcal{D}'_{n,q}$.*

Proof: Because of the description of distinguished representatives of right cosets by positions of recoils, if $|c| \leq n$, then the sum $M_{c,n}$ is over permutation words ω with recoils in the set of descents of c (notice that we include $|c|$ in the set of descents of c). Let us denote by $R_{c,n}$ this set of words, and suppose that $|c| \leq n-1$. If $\omega \in R_{c,n}$ is such that $\omega(n) \neq n$, then T_{ω} involves S_{n-1} , so the image by $\phi_{n,n-1}$ of the corresponding term in $M_{c,n}$ is zero. On the other hand, if $\omega(n) = n$, then any reduced decomposition of T_{ω} does not involve S_{n-1} , so the corresponding term in $M_{c,n}$ is preserved by $\phi_{n,n-1}$. Consequently, $\phi_{n,n-1}(M_{c,n})$ is a sum with the same terms as $M_{c,n}$, but with ω running over $R_{c,n-1}$; so, we have proved that $\phi_{n,n-1}(M_{c,n}) = M_{c,n-1}$ when $|c| \leq n-1$. The other cases are much easier: thus, if $|c| = n$, then

$M_{c,n-1} = 0$, and $\phi_{n,n-1}(M_{c,n})$ is also zero because $\phi_{n,n-1}(J_c) = 0$. And if $|c| > n$, then $M_{c,n}$ and $M_{c,n-1}$ are both equal to zero, and again $\phi_{n,n-1}(M_{c,n}) = M_{c,n-1}$. Since

$$\phi_{N,n} = \phi_{n+1,n} \circ \phi_{n+2,n+1} \circ \cdots \circ \phi_{N,N-1},$$

we have proved the first part of the proposition, and the second part is really obvious.

Now, let us show that $M_{c,n}$ is in $\mathcal{D}'_{n,q}$. Notice that the result is trivial if $|c| > n$, and also if $|c| = n$, because we have then $J_c = I_{(n)}$, and therefore $d = (n)$ for any composed permutation (σ, d) involved in $M_{c,|c|}$. Suppose then that $|c| \leq n - 1$. Because of the description of $\mathfrak{S}_d \backslash \mathfrak{S}_{|d|}$ as a shuffle product, any distinguished representative ω of $\mathfrak{S}_{c \uparrow n} \backslash \mathfrak{S}_n$ is the shuffle of a distinguished representative ω_c of $\mathfrak{S}_c \backslash \mathfrak{S}_{|c|}$ with the word $|c| + 1, |c| + 2, \dots, n$. For instance, 5613724 is the distinguished representative of a right $\mathfrak{S}_{(2,2,3)}$ -coset, and it is a shuffle of 567 with the distinguished representative 1324 of a right $\mathfrak{S}_{(2,2)}$ -coset. Let us denote by $s_{i_1} \cdots s_{i_r}$ a reduced expression of ω_c , and by $j_{|c|+1}, \dots, j_n$ the positions of $|c| + 1, \dots, n$ in ω . Then, it is not difficult to see that

$$s_{i_1} \cdots s_{i_r} \times (s_{|c|} s_{|c|-1} \cdots s_{j_{|c|+1}}) (s_{|c|+1} s_{|c|} \cdots s_{j_{|c|+2}}) \cdots (s_{n-1} s_{n-2} \cdots s_{j_n})$$

is a reduced expression for ω ; for instance, s_2 is the reduced expression of 1324, and

$$s_2 \times (s_4 s_3 s_2 s_1) (s_5 s_4 s_3 s_2) (s_6 s_5)$$

is a reduced expression of 5613724. From this, we deduce that $T_\omega J_c = T_{\omega, (k, 1^{n-k})}$, where k is the highest integer in $[|c| + 1, n]$ such that $j_k < k$ — we take $k = |c|$ if $\omega = \omega_c$. Then, the multiplication by $T_{\omega^{-1}}$ cannot fatten the composition anymore, so $T_{\omega^{-1}} T_\omega J_c$ is a linear combination of $T_{\tau, (k, 1^{n-k})}$, and we have proved that $M_{n,c}$ is indeed in $\mathcal{D}'_{n,q}$. \square

From the previous proof, it is now clear that if we consider the infinite sum $M_c = \sum q^{-\ell(\omega)} T_{\omega^{-1}} T_\omega J_c$ over permutation words $\omega \in \mathfrak{S}_\infty$ with their recoils in the set of descents of c , then M_c is the unique element of $\mathcal{D}_{\infty,q}$ such that $\phi_{\infty,n}(M_c) = M_{c,n}$ for any positive integer n , and also the unique element of $\mathcal{D}'_{\infty,q}$ such that $\text{pr}_{\infty,n}(M_c) = N_{c \uparrow n}$ for any positive integer n (with by convention $N_{c \uparrow n} = 0$ if $|c| > n$). In particular, M_c does not depend on the order of the parts of c , because this is true for the $N_{c \uparrow n}$ and the projections separate the vectors in $\mathcal{D}'_{\infty,q}$. Consequently, we shall consider only elements M_λ labelled by partitions λ of arbitrary size, and call them **generic norms**. For instance:

$$M_{(2),3} = T_{12|3} + 2T_{123} + (1 - q^{-1})(T_{132} + T_{213}) + (q^{-1} - q^{-2})T_{321}$$

In what follows, if $i < n$, we denote by $(S_i)^{-1}$ the element of $\mathcal{D}_{n,q}$ equal to:

$$(S_i)^{-1} = q^{-1} S_i + (q^{-1} - 1) I_i$$

The product $S_i (S_i)^{-1} = (S_i)^{-1} S_i$ equals I_i in $\mathcal{D}_{n,q}$, and by the specialization $\text{pr}_n : \mathcal{D}_{n,q} \rightarrow \mathcal{H}_{n,q}$, one recovers $S_i (S_i)^{-1} = 1$ in the Hecke algebra $\mathcal{H}_{n,q}$.

Theorem 7 *The M_λ span linearly the subalgebra $\mathcal{C}_{\infty,q} \subset \mathcal{D}'_{\infty,q}$ that consists in elements $x \in \mathcal{D}'_{\infty,q}$ such that $I_i x = S_i x (S_i)^{-1}$ for every i . In particular, any product $M_\lambda * M_\mu$ is a linear combination of M_ν , and moreover, the terms M_ν involved in the product satisfy the inequality $|\nu| \leq |\lambda| + |\mu|$.*

Proof: If $I_i x = S_i x (S_i)^{-1}$ and $I_i y = S_i y (S_i)^{-1}$, then

$$I_i xy = I_i x I_i y = S_i x (S_i)^{-1} S_i y (S_i)^{-1} = S_i x I_i y (S_i)^{-1} = S_i xy (S_i)^{-1},$$

so the elements that “commute” with S_i in $\mathcal{D}_{\infty,q}$ form a subalgebra. As an intersection, $\mathcal{C}_{\infty,q}$ is also a subalgebra of $\mathcal{D}_{\infty,q}$; let us see why it is spanned by the generic norms. If $\mathcal{D}'_{\infty,q,i}$ is the subspace of $\mathcal{D}_{\infty,q}$ spanned by the $T_{\sigma,c}$ with $c = (k, 1^\infty) \vee (1^{i-1}, 2, 1^\infty)$, then the projections separate the vectors in this subspace — this is the same proof as in Proposition 4. For $\lambda \in \mathfrak{P}$, $I_i M_\lambda$ and $S_i M_\lambda (S_i)^{-1}$ belong to $\mathcal{D}'_{\infty,q,i}$, and they have the same projections in $\mathcal{H}_{n,q}$, because $\text{pr}_{\infty,n}(M_\lambda)$ is a norm and in particular a central element. Consequently, $I_i M_\lambda = S_i M_\lambda (S_i)^{-1}$, and the M_λ are indeed in $\mathcal{C}_{\infty,q}$. Now, if we consider an element $x \in \mathcal{C}_{\infty,q}$, then for $i < n$, $\text{pr}_n(x) = S_i \text{pr}_n(x) (S_i)^{-1}$, so $\text{pr}_n(x)$ is in $\mathcal{X}_{n,q}$ and is a linear combination of norms:

$$\forall n \in \mathbb{N}, \text{pr}_n(x) = \sum_{\lambda \in \mathfrak{P}_n} a_\lambda(x) N_\lambda$$

Since the same holds for any difference $x - \sum b_\lambda M_\lambda$, we can construct by induction on n an infinite linear combination S_∞ of M_λ that has the same projections as x :

$$\begin{aligned} \text{pr}_1(x) = \sum_{|\lambda|=1} b_\lambda N_\lambda &\Rightarrow \text{pr}_1\left(x - \sum_{|\lambda|=1} b_\lambda M_\lambda\right) = 0, \quad S_1 = \sum_{|\lambda|=1} b_\lambda M_\lambda \\ \text{pr}_2(x - S_1) = \sum_{|\lambda|=2} b_\lambda N_\lambda &\Rightarrow \text{pr}_{1,2}\left(x - \sum_{|\lambda|\leq 2} b_\lambda M_\lambda\right) = 0, \quad S_2 = \sum_{|\lambda|\leq 2} b_\lambda M_\lambda \\ &\vdots \\ \text{pr}_{n+1}(x - S_n) = \sum_{|\lambda|=n+1} b_\lambda N_\lambda &\Rightarrow S_{n+1} = S_n + \sum_{|\lambda|=n+1} b_\lambda M_\lambda = \sum_{|\lambda|\leq n+1} b_\lambda M_\lambda \end{aligned}$$

Then, $S_\infty = \sum_{\lambda \in \mathfrak{P}} b_\lambda M_\lambda$ is in $\mathcal{D}'_{\infty,q}$ and has the same projections as x , so $S_\infty = x$. In particular, since $\mathcal{C}_{\infty,q}$ is a subalgebra, a product $M_\lambda * M_\mu$ is in $\mathcal{C}_{\infty,q}$ and is an *a priori* infinite linear combination of M_ν :

$$\forall \lambda, \mu, M_\lambda * M_\mu = \sum g'_{\lambda\mu} M_\nu$$

Since the norms N_λ are defined over $\mathbb{Z}[q, q^{-1}]$, by projection on the Hecke algebras $\mathcal{H}_{n,q}$, one sees that the $g'_{\lambda\mu}$ are also in $\mathbb{Z}[q, q^{-1}]$ — in fact, they are *symmetric* polynomials in q and q^{-1} . It remains to be shown that the previous sum is in fact over partitions $|\nu|$ with $|\nu| \leq |\lambda| + |\mu|$; we shall see why this is true in the last paragraph^(iv). □

For example, $M_1 * M_1 = M_1 + (q + 1 + q^{-1}) M_{1,1} - (q + 2 + q^{-1}) M_2$, and from this generic identity one deduces the expression of any product $(N_{(1)\uparrow n})^2$, e.g.,

$$N_{1,1}^2 = (q + 2 + q^{-1}) (N_{1,1} - N_2) \quad ; \quad N_{3,1}^2 = N_{3,1} + (q + 1 + q^{-1}) N_{2,1,1} - (q + 2 + q^{-1}) N_{2,2}.$$

Let us denote by $\mathcal{A}_{\infty,q}$ the subspace of $\mathcal{C}_{\infty,q}$ whose elements are *finite* linear combinations of generic norms; this is in fact a subalgebra, which we call the **Hecke-Ivanov-Kerov algebra** since it plays the same role for Iwahori-Hecke algebras as \mathcal{A}_∞ for symmetric group algebras.

^(iv) Unfortunately, we did not succeed in proving this result with adequate filtrations on $\mathcal{D}_{\infty,q}$ or $\mathcal{D}'_{\infty,q}$.

5 Completion of partitions and symmetric functions

The proof of Theorem 1 and of the last part of Theorem 7 relies now on a rather elementary property of the transition matrices $M2E$ and $E2M$. By convention, we set $e_{\lambda \uparrow n} = 0$ if $|\lambda| > n$, and $m_{\lambda \rightarrow n} = 0$ if $|\lambda| + \ell(\lambda) > n$. Then:

Proposition 8 *There exists polynomials $P_{\lambda\mu}(n) \in \mathbb{Q}[n]$ and $Q_{\lambda\mu}(n) \in \mathbb{Q}[n]$ such that*

$$\forall \lambda, n, \quad m_{\lambda \rightarrow n} = \sum_{\mu' \leq_d \lambda} P_{\lambda\mu}(n) e_{\mu \uparrow n} \quad \text{and} \quad e_{\lambda \uparrow n} = \sum_{\mu \leq_d \lambda'} Q_{\lambda\mu}(n) m_{\mu \rightarrow n},$$

where $\mu \leq_d \lambda$ is the domination relation on partitions.

This fact follows from the study of the Kotska matrix elements $K_{\lambda, \mu \rightarrow n}$, see [Mac95, §1.6, in particular the example 4. (c)]. It can also be shown directly by expanding $e_{\lambda \uparrow n}$ on a sufficient number of variables and collecting the monomials; this simpler proof explains the appearance of binomial coefficients $\binom{n}{k}$. For instance,

$$\begin{aligned} m_{2,1 \rightarrow n} &= e_{2,1 \uparrow n} - 3e_{3 \uparrow n} - (n-3)e_{1,1 \uparrow n} + (2n-8)e_{2 \uparrow n} + (2n-5)e_{1 \uparrow n} - n(n-4)e_{\uparrow n}, \\ e_{2,1 \uparrow n} &= \frac{n(n-1)(n-2)}{2} m_{\rightarrow n} + \frac{(n-2)(3n-7)}{2} m_{1 \rightarrow n} + (3n-10)m_{1,1 \rightarrow n} + 3m_{1,1,1 \rightarrow n} \\ &\quad + (n-3)m_{2 \rightarrow n} + m_{2,1 \rightarrow n}. \end{aligned}$$

In the following, $N_{\lambda,n} = N_{\lambda \uparrow n}$ if $|\lambda| \leq n$, and 0 otherwise. Because of the existence of the projective limits M_λ , we know that $N_{\lambda,n} * N_{\mu,n} = \sum_{\nu} g_{\lambda\mu}^\nu N_{\nu,n}$, where the sum is not restricted. But on the other hand, by using Proposition 5 and the second identity in Proposition 8, one sees that

$$N_{\lambda,n} * N_{\mu,n} = \sum_{|\rho| \leq |\lambda|, |\sigma| \leq |\mu|} h_{\lambda\mu}^{\rho\sigma}(n) \Gamma_{\rho,n} * \Gamma_{\sigma,n}, \quad \text{with the } h_{\lambda\mu}^{\rho\sigma}(n) \in \mathbb{Q}[n, q, q^{-1}].$$

Because of the result of Francis and Wang, the latter sum may be written as $\sum_{|\tau| \leq |\lambda| + |\mu|} i_{\lambda\mu}^\tau(n) \Gamma_{\tau,n}$, and by using the first identity of Proposition 8, one has finally

$$N_{\lambda,n} * N_{\mu,n} = \sum_{|\nu| \leq |\lambda| + |\mu|} j_{\lambda\mu}^\nu(n) N_{\nu,n}, \quad \text{with the } j_{\lambda\mu}^\nu(n) \in \mathbb{Q}[n, q, q^{-1}].$$

From this, it can be shown that the first sum $\sum_{\nu} g_{\lambda\mu}^\nu N_{\nu,n}$ is in fact restricted on partitions $|\nu|$ such that $|\nu| \leq |\lambda| + |\mu|$, and because the projections separate the vectors of $\mathcal{D}'_{\infty,q}$, this implies that $M_\lambda * M_\mu = \sum_{|\nu| \leq |\lambda| + |\mu|} g_{\lambda\mu}^\nu M_\nu$, so the last part of Theorem 7 is proved. Finally, by reversing the argument, one sees that the $a_{\lambda\mu}^\nu(n, q)$ are in $\mathbb{Q}[n](q)$:

$$\begin{aligned} \Gamma_{\lambda,n} * \Gamma_{\mu,n} &= (q/(q-1))^{| \lambda | + | \mu |} \sum_{\rho, \sigma} P_{\lambda\rho}(n) P_{\mu\sigma}(n) N_{\rho,n} * N_{\sigma,n} \\ &= (q/(q-1))^{| \lambda | + | \mu |} \sum_{\rho, \sigma, \tau} P_{\lambda\rho}(n) P_{\mu\sigma}(n) g_{\rho\sigma}^\tau N_{\tau,n} \\ &= \sum_{\rho, \sigma, \tau, \nu} (q/(q-1))^{| \lambda | + | \mu | - | \nu |} P_{\lambda\rho}(n) P_{\mu\sigma}(n) g_{\rho\sigma}^\tau(q) Q_{\tau\nu}(n) \Gamma_{\nu,n} = \sum_{\nu} a_{\lambda\mu}^\nu(n, q) \Gamma_{\nu,n} \end{aligned}$$

with $a_{\lambda\mu}^{\nu}(n, q) = (q/(q-1))^{|^{\lambda}|+|^{\mu}|-|^{\nu}|} (P^{\otimes 2}(n) g(q) Q(n))_{\lambda\mu}^{\nu}$ in tensor notation. And since the Γ_{λ} are known to be defined over $\mathbb{Z}[q, q^{-1}]$, the coefficients $a_{\lambda\mu}^{\nu}(n, q) \in \mathbb{Q}[n](q)$ are in fact^(v) in $\mathbb{Q}[n, q, q^{-1}]$. Using this technique, one can for instance show that

$$(\Gamma_{(1),n})^2 = \frac{n(n-1)}{2} q \Gamma_{(0),n} + (n-1)(q-1) \Gamma_{(1),n} + (q+q^{-1}) \Gamma_{(1,1),n} + (q+1+q^{-1}) \Gamma_{(2),n},$$

and this is because $m_{1 \rightarrow n} = e_{1 \uparrow n} - n e_{\uparrow n}$ and $e_{1 \uparrow n} = n m_{\rightarrow n} + m_{1 \rightarrow n}$. Let us conclude by two remarks. First, the reader may have noticed that we did not construct generic conjugacy classes $F_{\lambda} \in \mathcal{A}_{\infty, q}$ such that $\text{pr}_{\infty, n}(F_{\lambda}) = \Gamma_{\lambda, n}$; since the Geck-Rouquier elements themselves are difficult to describe, we had little hope to obtain simple generic versions of these Γ_{λ} . Secondly, the Ivanov-Kerov projective limits of other group algebras — e.g., the algebras of the finite reductive Lie groups $\text{GL}(n, \mathbb{F}_q)$, $\text{U}(n, \mathbb{F}_{q^2})$, etc. — have not yet been studied. It seems to be an interesting open question.

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^(v) They are even in $\mathbb{Q}_{\mathbb{Z}}[n] \otimes \mathbb{Z}[q, q^{-1}]$, where $\mathbb{Q}_{\mathbb{Z}}[n]$ is the \mathbb{Z} -module of polynomials with rational coefficients and integer values on integers; indeed, the matrices $M2E$ and $E2M$ have integer entries. It is well known that $\mathbb{Q}_{\mathbb{Z}}[n]$ is spanned over \mathbb{Z} by the binomials $\binom{n}{k}$.