Deterministic and probabilistic Lyapunov exponents for linear switched systems

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based on joint works with Yacine Chitour, Fritz Colonius, and Mario Sigalotti

PDMPs, Theory and applications
Seillac — June 2nd, 2017

LMO, Université Paris-Sud
Université Paris-Saclay
This talk is divided in three parts:

1. The deterministic theory of persistently excited control systems.
2. A probabilistic point of view to the same problem (with better results!).
3. Is the probabilistic point of view always better for switched systems?
Persistently excited systems

Control systems

We first recall some classical results for linear time-invariant control systems in continuous time.

\[ \dot{x}(t) = Ax(t) + Bu(t). \]

- \( x \in \mathbb{R}^d \): state; \( u \in \mathbb{R}^m \): control; \( A, B \) matrices.
- \( u \) is chosen to achieve a certain objective (target state, optimal control, stabilization, etc).
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**Definition (Controllability)**

We say that \((A, B)\) is **controllable** in time \( T > 0 \) if \( \forall x_0, x_1 \in \mathbb{R}^d, \exists u : [0, T] \to \mathbb{R}^m \) such that the solution \( x \) of \( \Sigma^{LTI}(A, B) \) with initial condition \( x_0 \) and control \( u \) satisfies \( x(T) = x_1 \).
Persistently excited systems

Control systems

\[ \dot{x}(t) = Ax(t) + Bu(t). \]

\[ \Sigma^{\text{LTI}}(A, B) \]

**Theorem (Kalman criterion)**

\[(A, B) \text{ is controllable} \iff \text{the matrix} \]
\[ C(A, B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{d-1}B \end{pmatrix} \]

*has full rank.*

In particular, controllability does not depend on \( T > 0. \)
Theorem (Kalman criterion)

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has full rank.

In particular, controllability does not depend on \(T > 0\).

Definition (Stabilizability)

We say that \((A, B)\) is stabilizable if there exists \(K \in \mathcal{M}_{m,d}(\mathbb{R})\) such that the system \(\Sigma^{\text{LTI}}(A, B)\) with \(u(t) = Kx(t)\) is asymptotically stable.

That is: \(\dot{x}(t) = (A + BK)x(t)\) and \(A + BK\) is Hurwitz.
Persistently excited systems
Control systems

\[ \dot{x}(t) = Ax(t) + Bu(t). \]

Kalman decomposition: after a linear change of variables,

\[
A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \text{ with } (A_1, B_1) \text{ controllable.}
\]

Theorem (Pole placement theorem)

Let \( r = \text{rank } C(A, B) \). For every unitary polynomial \( q \) of degree \( r \), there exists \( K \in \mathcal{M}_{m,d}(\mathbb{R}) \) such that the characteristic polynomial \( p_{A+BK} \) of \( A + BK \) is \( p_{A+BK}(x) = q(x)p_{A_2}(x) \).
Persistently excited systems

Control systems

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- \((A, B)\) stabilizable \( \iff \) \( A_2 \) Hurwitz.
- \((A, B)\) controllable \( \iff \) any unitary polynomial can be chosen as the characteristic polynomial of \( A + BK \).
Persistently excited systems

Control systems

\[ \dot{x}(t) = Ax(t) \quad \Sigma^{\text{LTI}}(A) \]

**Definition (Lyapunov exponent)**

Lyapunov exponent for \( \Sigma^{\text{LTI}}(A) \) with initial condition \( x_0 \in \mathbb{R}^d \setminus \{0\} \):

\[ \lambda_A^{\text{LTI}}(x_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \| x(t; x_0) \| . \]

- Lyapunov exponents are useful for studying the exponential behavior of a system.
- Clearly, \( \{ \lambda_A^{\text{LTI}}(x_0) \mid x_0 \in \mathbb{R}^d \setminus \{0\} \} = \{ \text{Re } \lambda \mid \lambda \in \sigma(A) \} \).
Persistently excited systems

Control systems

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- Clearly, \( \{ \lambda_A^{\text{LTI}}(x_0) \mid x_0 \in \mathbb{R}^d \setminus \{0\} \} = \{ \Re \lambda \mid \lambda \in \sigma(A) \} \).
- Pole placement theorem \( \implies \) if \((A, B)\) controllable, one can choose the Lyapunov exponents \( \lambda_A^{\text{LTI}} + BK \).
- In particular: arbitrary rate of convergence.
Persistently excited systems

Persistently excited systems

\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t). \]

- \( \alpha \in \mathcal{G} \subset L^\infty(\mathbb{R}_+, [0, 1]) \) (or \{0, 1\}).

Persistently exciting (PE) signals

For \( T > 0 \), we say that \( \mathcal{G} \) if \( L^1(\mathbb{R}_+, [0, 1]) \) and

\[ 8 \int_{t}^{t+T} w(s) ds. \]

Origin of PE signals: identification and adaptive control

[Anderson; 1977]
Persistently excited systems

Persistently excited systems

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\dot{x}(t) = Ax(t) + \alpha(t)Bu(t).
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- \(\alpha \in \mathcal{G} \subset L^\infty(\mathbb{R}_+, [0, 1])\) (or \(\{0, 1\}\)).
- \(\alpha(t)\): activity of the control \(u(t)\) at time \(t\).
- If \(\alpha(t) \equiv 0\), there is no action of the control on the system.
Persistently excited systems

Systems with randomly switching controls

Switched systems in discrete time

Persistently excited systems

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- If \( \alpha(t) \equiv 0 \), there is no action of the control on the system.
- Persistently exciting (PE) signals: for \( T \geq \mu > 0 \), we say that \( \alpha \in \mathcal{G}(T, \mu) \) if \( \alpha \in L^\infty(\mathbb{R}_+, [0, 1]) \) and
  \[ \forall t \in \mathbb{R}_+, \; \int_t^{t+T} \alpha(s) \, ds \geq \mu. \]
Persistently excited systems

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- **Persistently exciting (PE) signals**: for \( T \geq \mu > 0 \), we say that \( \alpha \in \mathcal{G}(T, \mu) \) if \( \alpha \in L^\infty(\mathbb{R}_+, [0, 1]) \) and
  \[ \forall t \in \mathbb{R}_+, \quad \int_t^{t+T} \alpha(s) \, ds \geq \mu. \]
- Origin of PE signals: identification and adaptive control [Anderson; 1977].
- **Persistently excited (PE) system**: system with \( \alpha \in \mathcal{G}(T, \mu) \).
Persistently excited systems

\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha \in \mathcal{G}(T, \mu) \]

**Problem:** given \( T \geq \mu > 0 \), find \( K \in \mathcal{M}_{m,d}(\mathbb{R}) \) s.t. the feedback \( u(t) = Kx(t) \) stabilizes asymptotically the system for every \( \alpha \in \mathcal{G}(T, \mu) \).
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- **Feedback** \( u(t) = Kx(t) \):
  \[ \dot{x}(t) = (A + \alpha(t)BK)x(t). \]

- **Solution with initial condition** \( x_0 \) at \( t = 0 \): \( x(t; x_0, \alpha, K) \).

- **Lyapunov exponents:**
  \[ \lambda_{\mathcal{PE}}^{\mathcal{G}(A,B,K,T,\mu)}(x_0, \alpha) = \limsup_{t \to +\infty} \frac{1}{t} \log \| x(t; x_0, \alpha, K) \|. \]
Persistently excited systems

\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha \in \mathcal{G}(T, \mu) \]

\[ \Sigma^{PE}(A, B, T, \mu) \]

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**Feedback** \( u(t) = Kx(t) \):

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**Lyapunov exponents:**

\[ \lambda^{PE}_{(A,B,K,T,\mu)}(x_0, \alpha) = \limsup_{t \to +\infty} \frac{1}{t} \log \| x(t; x_0, \alpha, K) \|. \]

**Problem:** given \( T \geq \mu > 0 \), find \( K \in \mathcal{M}_{m,d}(\mathbb{R}) \) s.t.

\[ \lambda^{PE}_{\max}(A,B,K,T,\mu) := \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \sup_{\alpha \in \mathcal{G}(T,\mu)} \lambda^{PE}_{(A,B,K,T,\mu)}(x_0, \alpha) < 0. \]
Persistently excited systems

Stabilization

Theorem (Chitour, Sigalotti; 2010)

Suppose that \((A, B)\) is stabilizable and that \(\Re \lambda \leq 0\) for every eigenvalue \(\lambda\) of \(A\). Then, for every \(T \geq \mu > 0\), there exists \(K \in \mathcal{M}_{m,d}(\mathbb{R})\) such that \(\lambda_{\text{PE max}}^{(A,B,K,T,\mu)} < 0\).

- Also holds for delayed feedbacks: \(u(t) = Kx(t - \tau(t))\) [M.; 2014].
- In the LTI setting, holds without the hypothesis \(\Re \lambda \leq 0\). But we cannot drop it here!
Persistently excited systems
Stabilization

\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha \in \mathcal{S}(T, \mu) \quad \Sigma^{PE}(A, B, T, \mu) \]

Can we stabilize it as fast as we want? That is, given \( \gamma > 0 \), can we always choose \( K \) such that \( \lambda_{(A,B,K,T,\mu)}^{PE \max} \leq -\gamma \)?
Persistently excited systems

Stabilization

\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha \in \mathcal{G}(T, \mu) \quad \Sigma^{\text{PE}}(A, B, T, \mu) \]

- Can we stabilize it as fast as we want? That is, given \( \gamma > 0 \), can we always choose \( K \) such that \( \lambda_{\max}^{\text{PE}}(A, B, K, T, \mu) \leq -\gamma \)?
- Recall that, for LTI systems \( \dot{x}(t) = Ax(t) + Bu(t) \), possible \( \iff (A, B) \) is controllable.
Persistently excited systems

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\[ \dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha \in \mathcal{G}(T, \mu) \]

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Single-input case \((m = 1, B \) is a vector \( b)\).

**Theorem (Chitour, Sigalotti; 2010)**

There exists \( \rho^* \in (0, 1) \) (depending only on \( d \)) such that, for every controllable pair \((A, b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d, T > 0, \mu \in (\rho^* T, T] \), and \( \gamma > 0 \), there exists \( K \in \mathcal{M}_{1,d}(\mathbb{R}) \) such that \( \lambda^{PE \max}_{(A,B,K,T,\mu)} \leq -\gamma \).
Persistently excited systems
Stabilization

But...

**Theorem (Chitour, Sigalotti; 2010)**

*There exists* \( \rho_\star \in (0, 1) \) *such that, for every controllable pair* 

\((A, b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2, T > 0, \text{ and } \mu \in (0, \rho_\star T), \text{ there exists } \gamma \in \mathbb{R} \text{ such that } \lambda^{\text{PE max}}_{(A, B, K, T, \mu)} > \gamma \text{ for every } K \in \mathcal{M}_{1,2}(\mathbb{R}) .*
Persistently excited systems
Stabilization

But...

Theorem (Chitour, Sigalotti; 2010)

There exists $\rho_\star \in (0, 1)$ such that, for every controllable pair $(A, b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$, $T > 0$, and $\mu \in (0, \rho_\star T)$, there exists $\gamma \in \mathbb{R}$ such that $\lambda_{PE \max}^{(A, B, K, T, \mu)} > \gamma$ for every $K \in \mathcal{M}_{1, 2}(\mathbb{R})$.

Ideas:

- First result: A Lyapunov function for $\dot{x} = (A + bK)x$ is still a Lyapunov function for $\dot{x} = (A + \alpha(t)bK)x$ if $\frac{\mu}{T}$ is large enough.

- Second result: For every $K$, a signal $\alpha : \mathbb{R}_+ \rightarrow \{0, 1\}$ such that $\lambda_{PE \max}^{(A, B, K, T, \mu)} > \gamma$ is constructed explicitly. As $\|K\|$ increases, $\alpha$ oscillates faster.
The previous negative result is based on the construction of a signal \( \alpha \) with fast switches at very specific moments. In practice, such signals “almost never” happen.
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Recall that

$$\lambda^{\text{PE max}}_{(A,B,K,T,\mu)} = \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \sup_{\alpha \in g(T,\mu)} \lambda^{\text{PE}}_{(A,B,K,T,\mu)}(x_0, \alpha).$$

This gives the worst possible behavior with respect to $\alpha$. Are we asking for too much?
We now consider

\[
    \dot{x}(t) = Ax(t) + \alpha(t)Bu(t)
\]

where \( \alpha \) switches randomly between 0 and 1.
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- Stabilizability by a linear feedback \( u(t) = Kx(t) \): stability of

\[ \dot{x}(t) = (A + \alpha(t)BK)x(t) \]

for \textit{almost every} random \( \alpha \).
Systems with randomly switching controls

Framework

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- Stabilizability by a linear feedback $u(t) = Kx(t)$: stability of
  $$\dot{x}(t) = (A + \alpha(t)BK)x(t)$$
  for almost every random $\alpha$.

- We first consider the almost sure stability analysis of the switched system
  $$\dot{x}(t) = A_{\alpha(t)}x(t),$$
  $A_1, \ldots, A_N \in \mathcal{M}_d(\mathbb{R}); \alpha : \mathbb{R}_+ \to \{1, \ldots, N\}$: switching signal
  (piecewise constant and right continuous).

- [Liberzon; 2003], [Costa, Fragoso, Marques; 2005], [Sun, Ge; 2005], [Shorten, Wirth, Mason, Wulff, King; 2007], [Jungers; 2009],
  [Lin, Antsaklis; 2009], [Costa, Fragoso, Todorov; 2013]...
The stability analysis of switched systems is not trivial.

- Switching between stable systems may lead to instability.
- Switching between unstable systems may lead to stability.

In dimension 2, complete characterization of stability of switched systems:

- deterministic setting: [Balde, Boscain, Mason; 2009]
- Markov switching: [Benaïm, Le Borgne, Malrieu, Zitt; 2014]
Systems with randomly switching controls

Framework

\[ \dot{x}(t) = A_{\alpha(t)}x(t). \]

- \( A_1, \ldots, A_N \in \mathcal{M}_d(\mathbb{R}) \);
- \( \alpha : \mathbb{R}_+ \to \{1, \ldots, N\} \) is piecewise constant and right continuous.
Systems with randomly switching controls

Framework

\[ \dot{x}(t) = A_{\alpha(t)} x(t). \]

- \( A_1, \ldots, A_N \in \mathcal{M}_d(\mathbb{R}) \); \( \alpha : \mathbb{R}_+ \to \{1, \ldots, N\} \) is piecewise constant and right continuous.
- **Markov process for** \( \alpha \): Let \( M \) stochastic irreducible \( N \times N \) matrix, \( p \in [0, 1]^N \) unique invariant probability vector, \( \mu_1, \ldots, \mu_N \) Borel probability measures on \((0, +\infty)\) with finite expectation.
  1. Choose \( \alpha(0) = i \) in \( \{1, \ldots, N\} \) according to the probability \( p \).
  2. Stay in the state \( i \) for a time \( t \) chosen randomly according to the law \( \mu_i \).
  3. After this time, switch to a new state chosen randomly according to the law \( (M_{ij})_{j=1}^N \) (\( i \)-th row of \( M \)).
  4. Repeat from 2.
Systems with randomly switching controls

Framework

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  4. Repeat from 2.

- \((\Omega, \mathcal{F}, \mathbb{P})\): probability space for this process;
  \( \Omega = \{ \omega = (i_n, t_n)_{n=1}^\infty \} \).
- Markov chain in \( \{1, \ldots, N\} \times (0, +\infty) \); \( x \) is a PDMP.
Persistently excited systems

Systems with randomly switching controls

Switched systems in discrete time

Systems with randomly switching controls

Lyapunov exponents and Oseledets’ Multiplicative Ergodic Theorem

\[
\dot{x}(t) = A_{\alpha(t)}x(t).
\]

- Solution with initial condition \(x_0\) and with an \(\alpha\) constructed from an \(\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega: \varphi_c(t; \omega, x_0) = \varphi_c(t; \omega)x_0\).
- \(\varphi_c\) is a linear cocycle: linear in \(x_0\), \(\varphi_c(0; \omega, x_0) = x_0\), and
  \[
  \varphi_c(t + s; \omega, x_0) = \varphi_c(t; \theta_s(\omega), \varphi_c(s; \omega, x_0)),
  \]
where \(\theta_s(\omega) \in \Omega\) corresponds to the shifted signal \(\alpha(s + \cdot)\).
Systems with randomly switching controls

Lyapunov exponents and Oseledets’ Multiplicative Ergodic Theorem

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  where \( \theta_s(\omega) \in \Omega \) corresponds to the shifted signal \( \alpha(s + \cdot) \).
- Lyapunov exponents: for \( x_0 \in \mathbb{R}^d \setminus \{0\}, \omega \in \Omega \),
  \[ \lambda_c(x_0, \omega) = \limsup_{t \to +\infty} \frac{1}{t} \log \| \varphi_c(t; \omega, x_0) \|. \]
- For LTI systems \( \dot{x}(t) = Ax(t) \):
  \( \{ \lambda_{LT}^A(x_0) \mid x_0 \in \mathbb{R}^d \setminus \{0\} \} = \{ \text{Re} \lambda \mid \lambda \in \sigma(A) \} \).
- For random switching: Oseledets’ Multiplicative Ergodic Theorem.
Let $\varphi$ be a linear cocycle and assume that $\theta$ preserves the measure $\mathbb{P}$ and defines an ergodic dynamical system. Under some integrability assumptions, the following hold in a $\theta$-invariant set of full measure:

1. **for every** $x \in \mathbb{R}^d \setminus \{0\}$,
   $$\lambda_c(x, \omega) = \lim_{t \to +\infty} \frac{1}{t} \log \| \varphi(t, \omega, x) \|;$$
Theorem (Oseledets’ Multiplicative Ergodic Theorem)

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- for every $x \in \mathbb{R}^d \setminus \{0\}$, $\lambda_c(x, \omega) = \lim_{t \to +\infty} \frac{1}{t} \log \| \varphi(t, \omega, x) \|$;

- there exist $q \in \{1, \ldots, d\}$ and $q$ vector subspaces $V_q(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d$ such that $\varphi(t, \omega)V_i(\omega) = V_i(\theta_t \omega)$;
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- there exist $\lambda_1 > \cdots > \lambda_q$ such that $\lambda_c(x, \omega) = \lambda_i \iff x \in V_i(\omega) \setminus V_{i+1}(\omega)$.
Oseledets’ theorem is a powerful tool.

Generalizes the LTI case: the set of all possible Lyapunov exponents is the same for a.e. $\omega$.

Largest Lyapunov exponent $\lambda_1$: [Furstenberg, Kesten; 1960].
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But, in our case, the usual time shift \( \theta_t(\alpha) = \alpha(t + \cdot) \) is not measure-preserving! (Measures \( \mu_1, \ldots, \mu_N \) are too general, no reason for time-shift to preserve them).
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Solution: consider instead an associated discrete-time system and apply the discrete-time version of Oseledets’ theorem.
Systems with randomly switching controls

Discrete-time system

\[(\Omega, \mathcal{F}, \mathbb{P})\]

- \(\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega\)
- \(\alpha(t) = i_n\) for \(t\) between \(t_1 + \cdots + t_{n-1}\) and \(t_1 + \cdots + t_n\).
Systems with randomly switching controls
Discrete-time system

- \((\Omega, \mathcal{F}, P)\)
- \(\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega\)
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Idea: discretize the system at the times \(s_n = t_1 + \cdots + t_n\).

Discrete-time system: \(\varphi_d(n; \omega, x_0) = \varphi_c(s_n; \omega, x_0)\).

\(\varphi_d\) is a linear cocycle in discrete time with the time-shift \(\theta : \Omega \to \Omega\) given by

\[\theta((i_n, t_n)_{n=1}^{\infty}) = (i_{n+1}, t_{n+1})_{n=1}^{\infty}.\]
Systems with randomly switching controls

Discrete-time system

- \((\Omega, \mathcal{F}, P)\)
  - \(\omega = (i_n, t_n)_{n=1}^\infty \in \Omega\)
  - \(\alpha(t) = i_n\) for \(t\) between \(t_1 + \cdots + t_{n-1}\) and \(t_1 + \cdots + t_n\).

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- **Discrete-time system**: \(\varphi_d(n; \omega, x_0) = \varphi_c(s_n; \omega, x_0)\).

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  \[\theta((i_n, t_n)_{n=1}^\infty) = (i_{n+1}, t_{n+1})_{n=1}^\infty.\]

- **Lyapunov exponents**:
  \[\lambda_d(x_0, \omega) = \limsup_{n \to \infty} \frac{1}{n} \log \| \varphi_d(n; \omega, x_0) \| .\]

**Proposition**

*The discrete time-shift \(\theta\) preserves the measure \(P\) and defines an ergodic dynamical system.*
Lemma

For every $x_0 \in \mathbb{R}^d \setminus \{0\}$ and a.e. $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$, $\lambda_d(x_0, \omega) = m \lambda_c(x_0, \omega)$, where

$$m = \sum_{i=1}^{N} p_i \int_{(0, +\infty)} t \, d\mu_i(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} t_k \in (0, +\infty).$$
The following hold in an invariant set of full measure:

- For every $x \in \mathbb{R}^d \setminus \{0\}$, $\lambda_c(x, \omega) = \lim_{t \to +\infty} \frac{1}{t} \log \| \varphi_c(t; \omega, x) \|$.
- And $\lambda_d(x, \omega) = \lim_{n \to \infty} \frac{1}{n} \log \| \varphi_d(n; \omega, x) \|$.
Theorem

The following hold in an invariant set of full measure:

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and $\lambda_d(x, \omega) = \lim_{n \to \infty} \frac{1}{n} \log \| \varphi_d(n; \omega, x) \|$

- there exist $q \in \{1, \ldots, d\}$ and $q$ vector subspaces $V_q(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d$ such that $\varphi_d(1, \omega) V_i(\omega) = V_i(\theta(\omega))$.
Systems with randomly switching controls

Discrete-time system

**Theorem**

The following hold in an invariant set of full measure:

- for every \( x \in \mathbb{R}^d \setminus \{0\} \),
  \[
  \lambda_c(x, \omega) = \lim_{t \to +\infty} \frac{1}{t} \log \| \varphi_c(t; \omega, x) \|,
  \]
  and
  \[
  \lambda_d(x, \omega) = \lim_{n \to \infty} \frac{1}{n} \log \| \varphi_d(n; \omega, x) \| ;
  \]

- there exist \( q \in \{1, \ldots, d\} \) and \( q \) vector subspaces
  \( V_q(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d \) such that
  \( \varphi_d(1, \omega) V_i(\omega) = V_i(\theta(\omega)) \);

- there exist \( \lambda_1^d > \cdots > \lambda_q^d \) and \( \lambda_1^c > \cdots > \lambda_q^c \) such that
  \( \lambda_{c/d}(x, \omega) = \lambda_i^{c/d} \iff x \in V_i(\omega) \setminus V_{i+1}(\omega) \).
Maximal Lyapunov exponent $\lambda_{1}^{c/d}$ gives exponential stability: continuous / discrete system is a.s. exponentially stable $\iff \lambda_{1}^{c/d} < 0$. 
Maximal Lyapunov exponent $\lambda_{1}^{c/d}$ gives exponential stability: continuous / discrete system is a.s. exponentially stable $\iff \lambda_{1}^{c/d} < 0$.

Since $\lambda_{1}^{d} = m\lambda_{1}^{c}$ and $m > 0$, exponential stability does not depend on continuous / discrete.
Systems with randomly switching controls

The maximal Lyapunov exponent

- Maximal Lyapunov exponent $\lambda_1^{c/d}$ gives exponential stability: continuous / discrete system is a.s. exponentially stable $\iff \lambda_1^{c/d} < 0$.
- Since $\lambda_1^d = m\lambda_1^c$ and $m > 0$, exponential stability does not depend on continuous / discrete.

Theorem

$$\lambda_1^d \leq \inf_{n \in \mathbb{N}^*} \frac{1}{n} \int_{\Omega} \log \| \varphi_d(n, \omega) \| \, d\mathbb{P}(\omega)$$

$$= \lim_{n \to \infty} \frac{1}{n} \int_{\Omega} \log \| \varphi_d(n, \omega) \| \, d\mathbb{P}(\omega).$$

If $\exists r > 1$ s.t. $\int_{(0, +\infty)} t^r \, d\mu_i(t) < \infty$ for every $i$, then we have equality.
Systems with randomly switching controls
Application to stabilization of control systems

\[
\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha(t) \in \{0, 1\}.
\]
Systems with randomly switching controls
Application to stabilization of control systems

\[
\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \quad \alpha(t) \in \{0, 1\}.
\]

More generally:

\[
\dot{x}_i(t) = A_ix_i(t) + \alpha_i(t)B_iu_i(t), \quad \alpha_i(t) \in \{0, 1\}, \quad \sum_{i=1}^{N} \alpha_i(t) \leq 1.
\]

- \( N \) control systems, at most one of them active at each time.
- Goal: stabilize through \( u_i(t) = K_i x_i(t) \).
Systems with randomly switching controls
Application to stabilization of control systems

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- \( N \) control systems, at most one of them active at each time.
- Goal: stabilize through \( u_i(t) = K_i x_i(t) \).
- \( \dot{x}(t) = Ax(t) + \hat{B}_{\alpha(t)}u_{\alpha(t)}(t) \) where

\[
\begin{align*}
    x(t) &= \begin{pmatrix}
        x_1(t) \\
        x_2(t) \\
        \vdots \\
        x_N(t)
    \end{pmatrix}, \\
    A &= \begin{pmatrix}
        A_1 & 0 & \cdots & 0 \\
        0 & A_2 & \cdots & 0 \\
        \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & A_N
    \end{pmatrix}, \\
    \hat{B}_i &= \begin{pmatrix}
        0 \\
        \vdots \\
        B_i \\
        \vdots \\
        0
    \end{pmatrix},
\end{align*}
\]

\( \hat{B}_0 = 0 \), and \( \alpha(t) = i \) if \( \alpha_i(t) = 1 \), \( \alpha(t) = 0 \) if \( \alpha_i(t) = 0 \) for all \( i \).
Systems with randomly switching controls
Application to stabilization of control systems

Feedback: $u_i(t) = K_i x_i(t) = K_i P_i x(t)$, with $P_i \in \mathcal{M}_{d_i,d}(\mathbb{R})$ projection.

$$\dot{x}(t) = \left( A + \hat{B}_\alpha(t) K_\alpha(t) P_\alpha(t) \right) x(t).$$
Systems with randomly switching controls
Application to stabilization of control systems

Feedback: \( u_i(t) = K_i x_i(t) = K_i P_i x(t) \), with \( P_i \in \mathcal{M}_{d_i,d} (\mathbb{R}) \) projection.

\[
\dot{x}(t) = \left( A + \hat{B}_{\alpha(t)} K_{\alpha(t)} P_{\alpha(t)} \right) x(t).
\]

Theorem (Colonius, M.; preprint)

Assume that \((A_i, B_i)\) is controllable for every \( i \in \{1, \ldots, N\} \) and that \( \alpha : \mathbb{R}_+ \rightarrow \{0, 1, \ldots, N\} \) is generated from a stochastic irreducible matrix \( M \), its invariant probability vector \( p \), and Borel probability measures \( \mu_0, \mu_1, \ldots, \mu_N \) with finite expectation as before. Then, for every \( \gamma > 0 \), there exist matrices \( K_1, \ldots, K_N \) such that the maximal Lyapunov exponent \( \lambda_1^c \) of \( \Sigma^c \) satisfies \( \lambda_1^c \leq -\gamma \).
Ideas of the proof:

- It suffices to show that, for a certain $r \in \mathbb{N}^*$,
  \[
  \int_{\Omega} \log \| \varphi_d(r, \omega) \| \, d\mathbb{P}(\omega) \text{ can be made arbitrarily small by suitably choosing } K_1, \ldots, K_N.
  \]

- Key idea: if $(A, B)$ is controllable, there exists $C, L > 0$ such that, for every $\gamma > 0$, one can find $K$ such that
  \[
  \| e^{(A+BK)t} \| \leq C \gamma^L e^{-\gamma t} \text{ [Cheng, Guo, Lin, Wang, 2004].}
  \]
  If we want to increase the decay rate, the constant increases \textit{polynomially}.
Ideas of the proof:

- It suffices to show that, for a certain \( r \in \mathbb{N}^* \),
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- Key idea: if \((A, B)\) is controllable, there exists \( C, L > 0 \) such that, for every \( \gamma > 0 \), one can find \( K \) such that
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  \]
  [Cheng, Guo, Lin, Wang, 2004]. If we want to increase the decay rate, the constant increases polynomially.

- With random switching, one gets arbitrary rate of convergence. The “bad” signals that prevent stabilization form a set of measure zero.
Switched systems in discrete time

Framework

In the previous problem of stabilization of control systems, the probabilistic approach gives better stability results than the deterministic one. How often is that the case?
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Consider the discrete time switched system
\[ x_{n+1} = A_{\alpha_{n+1}} x_n, \]
with \( A_1, A_2 \in \mathcal{M}_d(\mathbb{R}) \) and \((\alpha_n)_{n \in \mathbb{N}^*}\) a sequence in \(\{1, 2\}\).
In the previous problem of stabilization of control systems, the probabilistic approach gives better stability results than the deterministic one. How often is that the case?

Consider the discrete time switched system

$$x_{n+1} = A_{\alpha_{n+1}} x_n,$$

with $A_1, A_2 \in \mathcal{M}_d(\mathbb{R})$ and $(\alpha_n)_{n \in \mathbb{N}^*}$ a sequence in $\{1, 2\}$.

We want to compare two points of view:

**Deterministic:** worst behavior w.r.t. all sequences $(\alpha_n)_{n \in \mathbb{N}^*}$.

**Probabilistic:** Markov chain

$$\begin{align*}
1 - p_1 &\quad \quad 1 - p_2 \\
p_1 &\quad \quad p_2 \\
1 - p_1 &\quad \quad 1 - p_2 \\
p_2 &\quad \quad p_1
\end{align*}$$

worst behavior w.r.t. $p_1, p_2 \in [0, 1]$. 

Guilherme Mazanti
Switched systems in discrete time

Framework

\[ x_{n+1} = A_{\alpha_{n+1}} x_n. \]

- **Deterministic:**

\[
\lambda_d = \sup_{(\alpha_k) \in \{1,2\}^{N^*}} \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \lim_{n \to \infty} \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} x_0 \| \\
= \lim_{n \to \infty} \sup_{(\alpha_k) \in \{1,2\}^{N^*}} \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \|.
\]

Logarithm of the **joint spectral radius** of \( \{ A_1, A_2 \} \) [Jungers; 2009].
Switched systems in discrete time

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Switched systems in discrete time

Framework

\[ x_{n+1} = A_{\alpha_{n+1}} x_n. \]

- **Probabilistic:** Fix \((p_1, p_2) \in [0, 1]^2 \setminus \{(0, 0)\}\). Markov chain in \{1, 2\}:
  - Transition matrix \(M = \begin{pmatrix} 1 - p_1 & p_1 \\ p_2 & 1 - p_2 \end{pmatrix} \).
  - Initial measure: unique invariant measure \(\left(\frac{p_2}{p_1 + p_2}, \frac{p_1}{p_1 + p_2}\right)\).
  - \(\mathbb{P}_{(p_1, p_2)}\): probability measure in \{1, 2\}^\mathbb{N}^*.

\[ \lambda_p(p_1, p_2) = \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \lim_{n \to \infty} \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} x_0 \| \quad \mathbb{P}_{(p_1, p_2)}\text{-a.s.} \]

\[ = \lim_{n \to \infty} \mathbb{E}_{(p_1, p_2)} \left[ \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \| \right]. \]
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\[
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\]

\[
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\]

Convention: \(\lambda_p(0, 0) = \max\{\log \rho(A_1), \log \rho(A_2)\}\).
Persistently excited systems
Systems with randomly switching controls
Switched systems in discrete time

Switched systems in discrete time

Framework

$$x_{n+1} = A_{\alpha_{n+1}} x_n.$$ 

- **Probabilistic:** Fix $\left( p_1, p_2 \right) \in [0, 1]^2 \setminus \{(0, 0)\}$. Markov chain in $\{1, 2\}$:
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$$\lambda_p(p_1, p_2) = \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \lim_{n \to \infty} \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} x_0 \| \quad \mathbb{P}(p_1, p_2)\text{-a.s.}$$

$$= \inf_{n \in \mathbb{N}^*} \mathbb{E}(p_1, p_2) \left[ \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \| \right].$$

Convention: $\lambda_p(0, 0) = \max\{\log \rho(A_1), \log \rho(A_2)\} \implies \lambda_p$ is upper semi-continuous on $[0, 1]^2 \implies \lambda_p$ attains its maximum.

Guilherme Mazanti
Example \textit{(in continuous time)}: Consider the switched system
\[ \dot{x}(t) = A_{\alpha(t)} x(t) \]
with
\[ A_1 = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a & -\frac{1}{2} \\ 1 & -\frac{a}{2} \end{pmatrix}. \]
There exists a unique \( a \approx 0.1083918 \) for which \( \lambda_d = 0 \).
Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

\[ x_{n+1} = A_{\alpha_{n+1}} x_n. \]

\[ \lambda_d = \inf_{n \in \mathbb{N}^*} \sup_{(\alpha_k) \in \{1,2\}^\mathbb{N}^*} \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \|, \]

\[ \lambda_p(p_1, p_2) = \inf_{n \in \mathbb{N}^*} \mathbb{E}(p_1, p_2) \left[ \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \| \right]. \]

Clearly, \( \lambda_p(p_1, p_2) \leq \lambda_d \) for all \( (p_1, p_2) \in [0, 1]^2. \)
Persistently excited systems
Systems with randomly switching controls
Switched systems in discrete time

Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

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Clearly, \( \lambda_p(p_1, p_2) \leq \lambda_d \) for all \((p_1, p_2) \in [0, 1]^2\).

**Problem**

When does one have

\[ \sup_{(p_1, p_2) \in [0, 1]^2} \lambda_p(p_1, p_2) < \lambda_d? \]

Or, equivalently, when does one have

\[ \sup_{(p_1, p_2) \in [0, 1]^2} \lambda_p(p_1, p_2) = \lambda_d? \] (E)
Persistently excited systems
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Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

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Clearly, \( \lambda_p(p_1, p_2) \leq \lambda_d \) for all \( (p_1, p_2) \in [0, 1]^2 \).

Problem

When does one have

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Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

\[ x_{n+1} = A_{\alpha_{n+1}} x_n \]

The easy ("deterministic") cases:
- \( p_2 \neq 0 \): \( \lambda_p(0, p_2) = \log \rho(A_1) \).
- \( p_1 \neq 0 \): \( \lambda_p(p_1, 0) = \log \rho(A_2) \).
Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

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- \( \lambda_p(1, 1) = \frac{1}{2} \log \rho(A_1 A_2) = \frac{1}{2} \log \rho(A_2 A_1). \)
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Persistently excited systems
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Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

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1 - \( p_1 \)
1 - \( p_2 \)

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- \( \lambda_p(0, 0) = \max\{\log \rho(A_1), \log \rho(A_2)\} \).

If \( \max_{(q_1, q_2) \in [0,1]^2} \lambda_p(q_1, q_2) = \cdots \) then (E) \( \iff \cdots \)

| \( \lambda_p(0, p_2), \ p_2 \neq 0 \) | \( \lambda_d = \log \rho(A_1) \) |
| \( \lambda_p(p_1, 0), \ p_1 \neq 0 \) | \( \lambda_d = \log \rho(A_2) \) |
| \( \lambda_p(1, 1) \) | \( \lambda_d = \frac{1}{2} \log \rho(A_1 A_2) \) |
| \( \lambda_p(0, 0) \) | \( \lambda_d = \log \rho(A_1) \) or \( \lambda_d = \log \rho(A_2) \) |
For the remaining cases, we assume \( \{A_1, A_2\} \) to be irreducible: if \( V \subset \mathbb{R}^d \) is an invariant subspace for both \( A_1 \) and \( A_2 \), then \( V = \{0\} \) or \( V = \mathbb{R}^d \).

**Theorem (Chitour, M., Sigalotti; in preparation)**

Assume that \( \{A_1, A_2\} \) is irreducible. Then (E) holds if and only if
\[
\lambda_d = \log \rho(A_1) \quad \text{or} \quad \lambda_d = \log \rho(A_2) \quad \text{or} \quad \lambda_d = \frac{1}{2} \log \rho(A_1 A_2).
\]
Switched systems in discrete time
Comparison between deterministic and probabilistic behavior

For the remaining cases, we assume \( \{A_1, A_2\} \) to be irreducible: if \( V \subset \mathbb{R}^d \) is an invariant subspace for both \( A_1 \) and \( A_2 \), then \( V = \{0\} \) or \( V = \mathbb{R}^d \).

**Theorem (Chitour, M., Sigalotti; in preparation)**

Assume that \( \{A_1, A_2\} \) is irreducible. Then (E) holds if and only if

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\lambda_d = \log \rho(A_1) \quad \text{or} \quad \lambda_d = \log \rho(A_2) \quad \text{or} \quad \lambda_d = \frac{1}{2} \log \rho(A_1 A_2).
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- The “worst probabilistic behavior” is as bad as the “worst deterministic behavior” only when the “worst probabilistic behavior” is actually deterministic.
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- The “worst probabilistic behavior” is as bad as the “worst deterministic behavior” only when the “worst probabilistic behavior” is actually deterministic.
- Proof of \( \Longleftrightarrow \): take \((\rho_1, \rho_2) = (0, 1)\) or \((1, 0)\) or \((1, 1)\).
- Proof of \( \Longrightarrow \): ok if the maximum of \( \lambda_p(\rho_1, \rho_2) \) is attained at \( \{0\} \times [0, 1] \) or \([0, 1] \times \{0\} \) or \{(1, 1)\}. Remaining cases: \((0, 1)^2\), \( \{1\} \times (0, 1) \), \((0, 1) \times \{1\} \).
Switched systems in discrete time

Barabanov norm

Definition

A Barabanov norm for \( \{A_1, A_2\} \) is a norm \( \| \cdot \|_B \) in \( \mathbb{R}^d \) such that:

- **(Extremality)** \( \forall x_0 \in \mathbb{R}^d, \forall i \in \{1, 2\}, \) one has \( \|A_i x_0\|_B \leq e^{\lambda_d} \|x_0\|_B \);

- \( \forall x_0 \in \mathbb{R}^d, \exists i \in \{1, 2\} \) such that \( \|A_i x_0\|_B = e^{\lambda_d} \|x_0\|_B \).
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**Theorem (Barabanov; 1988)**

*If \( \{A_1, A_2\} \) is irreducible, then there exists a Barabanov norm for \( \{A_1, A_2\}. \)*
Lemma

Assume that \( \{A_1, A_2\} \) is irreducible and let \( \| \cdot \|_B \) be a Barabanov norm for \( \{A_1, A_2\} \). Let \( (p_1, p_2) \in (0, 1)^2 \). Then the following are equivalent:

1. \( \lambda_p(p_1, p_2) = \lambda_d \)

2. For every \( (\alpha_k) \in \{1, 2\}^{\mathbb{N}^*} \) and \( n \in \mathbb{N}^* \), one has
   \[
   \| A_{\alpha_n} \cdots A_{\alpha_1} \|_B = e^{n\lambda_d}
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\[
\lambda_p(p_1, p_2) = \inf_{n \in \mathbb{N}^*} \mathbb{E}_{(p_1, p_2)} \left[ \frac{1}{n} \log \|A_{\alpha_n} \cdots A_{\alpha_1}\|_B \right].
\]

2 \( \implies \) 1: immediate
Switched systems in discrete time

Proof of the main result

\[ \lambda_p(p_1, p_2) = \inf_{n \in \mathbb{N}^*} \mathbb{E}(p_1, p_2) \left[ \frac{1}{n} \log \| A_{\alpha_n} \cdots A_{\alpha_1} \|_B \right] . \]

\( \circ \circ \Rightarrow \circ \circ \circ \): we prove \( \circ \circ \circ \Rightarrow \circ \circ \).

- \( \| \cdot \|_B \) is a Barabanov norm, then
  \[ \| A_{\alpha_n} \cdots A_{\alpha_1} \|_B \leq e^{n\lambda_d}, \quad \forall (\alpha_k) \in \{1, 2\}^{\mathbb{N}^*}, \forall n \in \mathbb{N}^*. \]
Switched systems in discrete time

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1 \implies 2: we prove \( N(2) \implies N(1) \).

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  \]

- If \( (\alpha_k^*) \in \{1, 2\}^{\mathbb{N}^*}, n^* \in \mathbb{N}^* \) are s.t.
  \[
  \| A_{\alpha_n^*} \cdots A_{\alpha_1^*} \|_B < e^{n^*\lambda_d},
  \]
  then
  \[
  \mathbb{E}(p_1, p_2) \left[ \frac{1}{n^*} \log \| A_{\alpha_n^*} \cdots A_{\alpha_1} \|_B \right] < \lambda_d
  \]
  \[
  \implies \lambda_p(p_1, p_2) < \lambda_d.
  \]
Stoppedly excited systems

Systems with randomly switching controls

Switched systems in discrete time

Switched systems in discrete time

Proof of the main result

Theorem (Chitour, M., Sigalotti; in preparation)

Assume that \( \{A_1, A_2\} \) is irreducible. Then (E) holds if and only if

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Proof of \( \iff \), case where

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\max_{(q_1, q_2) \in [0,1]^2} \lambda_p(q_1, q_2) = \max_{(q_1, q_2) \in (0,1)^2} \lambda_p(q_1, q_2):
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- Lemma, \( \square \) \( \implies \) \( \|A_i^n\|_B = e^{n\lambda_d}, \forall n \in \mathbb{N}^*, \forall i \in \{1, 2\}. \)
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- \( \implies \rho(A_i) = e^{\lambda_d}, i \in \{1, 2\} \).

If the maximum is attained at \( \{1\} \times (0, 1) \): similar lemma, but

\[
\|A_{\alpha_n} \cdots A_{\alpha_1}\|_B = e^{n\lambda_d} \text{ only for the } (\alpha_k) \text{ for which two 1's never appear together.}
\]
How large can the difference $\lambda_d - \max_{(p_1,p_2)\in[0,1]^2} \lambda_p(p_1,p_2)$ be?

**Proposition (Chitour, M., Sigalotti; in preparation)**

Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Then $\lambda_d = 0$ and $\lambda_p(p_1, p_2) = -\infty$ for every $(p_1, p_2) \in [0,1]^2$. 
Switched systems in discrete time

Gap between deterministic and probabilistic exponents

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$$A_1^2 A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 A_2 A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 A_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_1^3 = A_1 A_2^2 = A_2 A_1 A_2 = A_2^2 A_1 = A_2^3 = 0.$$
Switched systems in discrete time
Gap between deterministic and probabilistic exponents

\[ \|A_{\alpha_n} \cdots A_{\alpha_1}\|_1 = \begin{cases} 
1, & \text{if } (\alpha_k) = (2, 1, 1, 2, 1, 1, \ldots) \\
& \text{or } (1, 2, 1, 1, 2, 1, \ldots) \\
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Switched systems in discrete time

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0, & \text{otherwise.}
\end{cases} \]

\[
\mathbb{P}_{(p_1,p_2)}(\{(2, 1, 1, 2, 1, 1, \ldots), (1, 2, 1, 1, 2, 1, \ldots), (1, 1, 2, 1, 1, 2, \ldots)\}) = 0
\]

for every \((p_1, p_2) \in [0, 1]^2\), hence

\[
\lim_{n \to \infty} \left[ \frac{1}{n} \log \|A_{\alpha_n} \cdots A_{\alpha_1}\|_B \right] = -\infty \quad \mathbb{P}_{(p_1,p_2)}-\text{a.s.}
\]
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Deterministic and probabilistic Lyapunov exponents for linear switched systems

Guilherme Mazanti