Stability of difference equations and applications to transport and wave propagation on networks

Guilherme Mazanti
joint work with Yacine Chitour and Mario Sigalotti

Stability of non-conservative systems
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Outline

1. Introduction
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Stability of difference equations and applications to transport and wave propagation on networks

Guilherme Mazanti
Introduction
Linear difference equations

\[ \Sigma(\Lambda, A) : \quad x(t) = \sum_{j=1}^{N} A_j(t)x(t - \Lambda_j), \quad t \geq 0. \]

- \(\Lambda_1, \ldots, \Lambda_N\): positive delays.
- \(A_1(t), \ldots, A_N(t)\): time-dependent \(d \times d\) matrices.
- \(x(t) \in \mathbb{C}^d\).
- Notation: \(\Lambda_{\text{min}} = \min_i \Lambda_i, \quad \Lambda_{\text{max}} = \max_i \Lambda_i\).
Introduction

Linear difference equations

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Motivation:

- Applications to some hyperbolic PDEs.
- Generalization of previous results: \( N = 1 \), autonomous.
Introduction
Motivation: transport systems

Hyperbolic PDEs $\rightarrow$ difference equations: [Cooke, Krumme; 1968], [Slemrod; 1971], [Greenberg, Li; 1984], [Coron, Bastin, d’Andréa Novel; 2008], [Fridman, Mondié, Saldivar; 2010], [Gugat, Sigalotti; 2010]...
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\[
\begin{aligned}
\partial_t u_i(t, \xi) + \partial_\xi u_i(t, \xi) + \alpha_i(t, \xi) u_i(t, \xi) &= 0, \\
&\quad t \in \mathbb{R}_+, \; \xi \in [0, \Lambda_i], \; i \in [1, N], \\
u_i(t, 0) &= \sum_{j=1}^{N} m_{ij}(t) u_j(t, \Lambda_j), \quad t \in \mathbb{R}_+, \; i \in [1, N].
\end{aligned}
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u_i(t, 0) &= \sum_{j=1}^{N} m_{ij}(t) u_j(t, \Lambda_j), \quad t \in \mathbb{R}_+, \ i \in \llbracket 1, N \rrbracket. \\
\end{align*}
\]

Method of characteristics: for \( t \geq \Lambda_{\text{max}} \),

\[
\begin{align*}
\nu_i(t, 0) &= \sum_{j=1}^{N} m_{ij}(t) u_j(t, \Lambda_j) = \sum_{j=1}^{N} m_{ij}(t) e^{-\int_{0}^{\Lambda_j} \alpha_j(t-s, \Lambda_j-s) ds} u_j(t - \Lambda_j, 0).
\end{align*}
\]

Set \( x(t) = (u_i(t, 0))_{i \in \llbracket 1, N \rrbracket} \). Then \( x \) satisfies a difference equation.
Introduction
Motivation: wave propagation on networks
Introduction
Motivation: wave propagation on networks

\[ \partial_{tt}^2 u_i(t, \xi) = \partial_{\xi\xi}^2 u_i(t, \xi) \]
\[ u_i(t, q) = u_j(t, q), \quad \forall q \in \mathcal{V}, \forall i, j \in \mathcal{E}_q \]
\[ + \text{ conditions on vertices.} \]
Motivation: wave propagation on networks
Introduction
Motivation: wave propagation on networks

D’Alembert decomposition on travelling waves:
Introduction
Motivation: wave propagation on networks

D’Alembert decomposition on travelling waves:

System of $2N$ transport equations.
D’Alembert decomposition on travelling waves:

System of $2N$ transport equations.
Can be reduced to a system of difference equations.
Motivation: case $N = 1$

- When $N = 1$: $x(t) = A(t)x(t - \Lambda)$.
- Can be reduced to $x_n = A_n x_{n-1}$.
Introduction

Motivation: case \( N = 1 \)

- When \( N = 1 \): \( x(t) = A(t)x(t - \Lambda) \).
- Can be reduced to \( x_n = A_n x_{n-1} \).

**Autonomous system**
\[
x_n = A x_{n-1} \\
A \in \mathcal{M}_d(\mathbb{C})
\]

**Exponential stability**
\[ \iff \rho(A) < 1 \]

**Finite-time stability**
\[ \iff \rho(A) = 0 \]

\[
\rho(A) = \lim_{n \to +\infty} |A^n|^\frac{1}{n} \\
= \max_{\lambda \in \sigma(A)} |\lambda|
\]
Introduction
Motivation: case $N = 1$

- When $N = 1$: $x(t) = A(t)x(t - \Lambda)$.
- Can be reduced to $x_n = A_n x_{n-1}$.

### Autonomous system

$x_n = A x_{n-1}$

$A \in \mathcal{M}_d(\mathbb{C})$

### Arbitrary switching

$x_n = A_n x_{n-1}$

$A_n \in \mathcal{B} \subset \mathcal{M}_d(\mathbb{C})$

**Exponential stability**

$\iff \rho(A) < 1$

**Finite-time stability**

$\iff \rho(A) = 0$

**Uniform exponential stability**

$\iff \rho_J(\mathcal{B}) < 1$

**Finite-time stability**

$\iff \rho_J(\mathcal{B}) = 0$

$$\rho(A) = \lim_{n \to +\infty} |A^n|^\frac{1}{n} = \max_{\lambda \in \sigma(A)} |\lambda|$$

$$\rho_J(\mathcal{B}) = \lim_{n \to +\infty} \sup_{A_1, \ldots, A_n \in \mathcal{B}} |A_1 A_2 \cdots A_n|^\frac{1}{n}$$

(cf. [Jungers; 2009])

Stability of difference equations and applications to transport and wave propagation on networks
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Motivation: autonomous case

\[ \Sigma^{\text{aut}}(\Lambda, A) : \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j), \quad t \geq 0 \]

- [Cruz, Hale; 1970], [Henry; 1974], [Michiels et al.; 2009]...
- Studied through spectral methods.
- Stability: real parts of the roots of \[ \det \left( \text{Id} - \sum_{j=1}^{N} A_j e^{-s\Lambda_j} \right) = 0 \]
  \((\text{exponential polynomial}, \text{ see [Avellar, Hale; 1980]})).\]
Introduction
Motivation: autonomous case

Let $\rho_{HS}(A) = \max_{(\theta_1, \ldots, \theta_N) \in [0,2\pi]^N} \rho \left( \sum_{j=1}^{N} A_j e^{i\theta_j} \right)$.

Theorem ([Hale; 1975], [Silkowski; 1976])

The following are equivalent:

- $\rho_{HS}(A) < 1$;
- $\Sigma^{\text{aut}}(\Lambda, A)$ is exponentially stable for some $\Lambda \in (0, +\infty)^N$ with rationally independent components;
- $\Sigma^{\text{aut}}(\Lambda, A)$ is exponentially stable for every $\Lambda \in (0, +\infty)^N$. 
Introduction
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Still true if we replace $\rho_{HS}(A) < 1$ by $\rho_{HS}(A) = 0$ and exponential by finite-time stability.

For rationally dependent delays: [Michiels et al.; 2009].

Can this be generalized to the non-autonomous case?
Main problem: **exponential stability** of the non-autonomous system \( \Sigma(\Lambda, A) \) **uniformly** with respect to a given class \( \mathcal{A} \) of functions 
\[ A : \mathbb{R} \rightarrow \mathcal{M}_d(\mathbb{C})^N. \]

- The techniques from the autonomous case cannot be applied.
- Our approach: **explicit formula** for solutions of \( \Sigma(\Lambda, A) \).
- When \( \mathcal{A} = L^\infty(\mathbb{R}, \mathcal{B}) \), we obtain a generalization of Hale–Silkowski’s Theorem.
Main problem: **exponential stability** of the non-autonomous system $\Sigma(\Lambda, A)$ uniformly with respect to a given class $\mathcal{A}$ of functions $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$.

- The techniques from the autonomous case cannot be applied.
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### Exponential stability criteria:

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<td>$\rho_J(\mathcal{B}) &lt; 1$</td>
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<td>any $N$</td>
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Stability analysis

Explicit solution (I)

\[ \sum(\Lambda, A) : \quad x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \quad t \geq 0 \]

**Solution** with initial condition \( x_0 : [-\Lambda_{\text{max}}, 0) \rightarrow \mathbb{C}^d : x \) satisfying \( \sum(\Lambda, A) \) for \( t \geq 0 \) and \( x(t) = x_0(t) \) for \(-\Lambda_{\text{max}} \leq t < 0.\)
Stability analysis
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**Lemma**

The solution \( x : [-\Lambda_{\text{max}}, +\infty) \rightarrow \mathbb{C}^d \) of \( \Sigma(\Lambda, A) \) with initial condition \( x_0 : [-\Lambda_{\text{max}}, 0) \rightarrow \mathbb{C}^d \) is, for \( t \geq 0 \),

\[ x(t) = \sum_{n \in \mathbb{N}^N} \sum_{j \in \{1, \ldots, N\}} \Xi_{n-e_j, t} \Lambda_n A_j(t - \Lambda \cdot n + \Lambda_j) x_0(t - \Lambda \cdot n), \]

where the matrices \( \Xi_{n,t}^{\Lambda,A} \) are defined recursively by

\[ \Xi_{n,t}^{\Lambda,A} = \sum_{n_k \geq 1} A_k(t) \Xi_{n-e_k, t-\Lambda_k}^{\Lambda,A}, \quad \Xi_{0,t}^{\Lambda,A} = \text{Id}_d. \]
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\[
x(t) = \sum_{n \in \mathbb{N}^N} \sum_{\substack{j \in [1,N] \\t < \Lambda \cdot n \leq t + \Lambda_{\text{max}} \\Lambda \cdot n - \Lambda_j \leq t}} \Xi_{n - e_j, t}^{\Lambda, A} A_j(t - \Lambda \cdot n + \Lambda_j)x_0(t - \Lambda \cdot n),
\]

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\]

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Stability analysis

Explicit solution (I)

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Lemma

The solution $x : [-\Lambda_{\text{max}}, +\infty) \to \mathbb{C}^d$ of $\Sigma(\Lambda, A)$ with initial condition $x_0 : [-\Lambda_{\text{max}}, 0) \to \mathbb{C}^d$ is, for $t \geq 0$,

$$x(t) = \sum_{n \in \mathbb{N}^N} \sum_{j \in [1,N]} \Xi^{\Lambda, A}_{n-e_j, t} A_j(t - \Lambda \cdot n + \Lambda_j) x_0(t - \Lambda \cdot n),$$

where the matrices $\Xi^{\Lambda, A}_{n,t}$ are defined recursively by

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Stability analysis

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\[
    x(t) = \sum_{\substack{n \in \mathbb{N} \times \mathbb{N} \mid t - \Lambda \cdot n \leq t + \Lambda_{\text{max}}}} \Theta_{n,t}^{\Lambda,A} x_0(t - \Lambda \cdot n),
\]

where the matrices \( \Theta_{n,t}^{\Lambda,A} \) are defined recursively by

\[
    \Theta_{n,t}^{\Lambda,A} = \sum_{n_k \geq 1} A_{k}(t) \Theta_{n-e_k,t-\Lambda_k}^{\Lambda,A}, \quad \Theta_{0,t}^{\Lambda,A} = \text{Id}_d.
\]
Stability analysis (I)

\[ \Sigma(\Lambda, A) : \quad x(t) = \sum_{j=1}^{N} A_j(t)x(t - \Lambda_j), \quad t \geq 0 \]

- \( X_p = L^p([-\Lambda_{\text{max}}, 0], \mathbb{C}^d), \quad p \in [1, +\infty] \)
- \( \mathcal{A} \): set of uniformly locally bounded functions taking values in \( N \)-tuples of matrices
- \( \Sigma(\Lambda, \mathcal{A}) \): family of systems \( \Sigma(\Lambda, A) \) for \( A \in \mathcal{A} \).
- For \( x \) solution of \( \Sigma(\Lambda, A) \), \( x_t = x(t + \cdot)|_{[-\Lambda_{\text{max}}, 0]} \in X_p \).
Stability analysis
Stability analysis (I)

Definition

\[ \Sigma(\Lambda, \mathcal{A}) \text{ is of:} \]

- **exponential type** \( \gamma \) in \( X_p \) if \( \forall \varepsilon > 0 \ \exists K > 0 \text{ s.t. } \forall A \in \mathcal{A}, \ 
\forall x_0 \in X_p, \text{ the solution } x \text{ satisfies } \| x_t \|_{X_p} \leq Ke^{(\gamma+\varepsilon)t} \| x_0 \|_{X_p}; \)

- **\( \Theta \)-exponential type** \( \gamma \) if \( \forall \varepsilon > 0 \ \exists K > 0 \text{ s.t. } \forall A \in \mathcal{A}, \ 
\forall n \in \mathbb{N}^N, \text{ a.e. } t \in (\Lambda \cdot n - \Lambda_{\text{max}}, \Lambda \cdot n), \text{ one has } \left| \Theta_{n,t}^{\Lambda,A} \right| \leq Ke^{(\gamma+\varepsilon)t}; \)

- **\( \Xi \)-exponential type** \( \gamma \) if \( \forall \varepsilon > 0 \ \exists K > 0 \text{ s.t. } \forall A \in \mathcal{A}, \ 
\forall n \in \mathbb{N}^N, \text{ a.e. } t \in \mathbb{R}, \text{ one has } \left| \Xi_{n,t}^{\Lambda,A} \right| \leq Ke^{(\gamma+\varepsilon)\Lambda \cdot n}. \)

**Exponential stability:** exponential type \( \gamma < 0 \).
Stability analysis
Stability analysis (I)

\[ x(t) = \sum_{\substack{n \in \mathbb{N}^N \cap \{t < \Lambda \cdot n \leq t + \Lambda_{\text{max}}\}}} \Theta_{n,t}^{\Lambda,A} x_0(t - \Lambda \cdot n), \quad t \geq 0. \]

Theorem (Chitour, M., Sigalotti; 2015)

Let \( \Lambda \in (0, +\infty)^N \) and \( A \) be uniformly locally bounded.

- If \( \Sigma(\Lambda, A) \) is of \( \Theta \)-exponential type \( \gamma \) then \( \forall p \in [1, +\infty] \) it is of exponential type \( \gamma \) in \( X_p \).
Stability analysis
Stability analysis (I)

\[ x(t) = \sum_{n \in \mathbb{N}^N, t < \Lambda \cdot n \leq t + \Lambda_{\text{max}}} \Theta_{n,t}^{\Lambda,A} x_0(t - \Lambda \cdot n), \quad t \geq 0. \]

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- If \( \Sigma(\Lambda, A) \) is of \( \Theta \)-exponential type \( \gamma \) then \( \forall p \in [1, +\infty] \) it is of exponential type \( \gamma \) in \( X_p \).

- Suppose that \( \Lambda_1, \ldots, \Lambda_N \) are rationally independent. If \( \exists p \in [1, +\infty] \) such that \( \Sigma(\Lambda, A) \) is of exponential type \( \gamma \) in \( X_p \), then it is of \( \Theta \)-exponential type \( \gamma \).
Stability analysis

Stability analysis (I)

\[ x(t) = \sum_{\substack{n \in \mathbb{N}^N \ \ t < \Lambda \cdot n \leq t + \Lambda_{\text{max}}}} \Theta_{n,t}^{\Lambda,A} x_0(t - \Lambda \cdot n), \quad t \geq 0. \]

**Theorem (Chitour, M., Sigalotti; 2015)**

Let \( \Lambda \in (0, +\infty)^{\mathbb{N}} \) and \( \mathcal{A} \) be uniformly locally bounded.

- If \( \Sigma(\Lambda, \mathcal{A}) \) is of \( \Theta \)-exponential type \( \gamma \) then \( \forall p \in [1, +\infty] \) it is of exponential type \( \gamma \) in \( X_p \).

- Suppose that \( \Lambda_1, \ldots, \Lambda_N \) are rationally independent. If \( \exists p \in [1, +\infty] \) such that \( \Sigma(\Lambda, \mathcal{A}) \) is of exponential type \( \gamma \) in \( X_p \), then it is of \( \Theta \)-exponential type \( \gamma \).

- Suppose that \( \mathcal{A} \) is shift-invariant. Then \( \Theta \) - and \( \Xi \)-exponential types \( \gamma \) are equivalent.
Let $\Lambda = (\Lambda_1, \ldots, \Lambda_N) \in (0, +\infty)^N$. We define

$$Z(\Lambda) = \{ n \in \mathbb{Z}^N \mid \Lambda \cdot n = 0 \},$$

$$V(\Lambda) = \{ L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L) \}, \quad \text{(more rationally dependent)}$$

$$W(\Lambda) = \{ L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L) \}, \quad \text{(as rationally dependent)}$$

$$V_+(\Lambda) = V(\Lambda) \cap (0, +\infty)^N, \quad W_+(\Lambda) = W(\Lambda) \cap (0, +\infty)^N.$$
Stability analysis
Rational dependence of the delays

Let \( \Lambda = (\Lambda_1, \ldots, \Lambda_N) \in (0, +\infty)^N \). We define

\[
Z(\Lambda) = \{ n \in \mathbb{Z}^N \mid \Lambda \cdot n = 0 \},
\]

\[
V(\Lambda) = \{ L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L) \}, 
\text{ (more rationally dependent)}
\]

\[
W(\Lambda) = \{ L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L) \}, 
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\]

\[
V_+(\Lambda) = V(\Lambda) \cap (0, +\infty)^N, 
W_+(\Lambda) = W(\Lambda) \cap (0, +\infty)^N.
\]

Example: \( \Lambda = (1, \sqrt{2}, 1 + \sqrt{2}) \).

- \( Z(\Lambda) = \{ (n, m, -n - m) \mid n, m \in \mathbb{Z} \} \);
- \( V(\Lambda) = \{ (a, b, a + b) \mid a, b \in \mathbb{R} \} \);
- \( W(\Lambda) = \{ (a, b, a + b) \mid a, b \in \mathbb{R} \text{ rationally independent} \} \).
For $\Lambda \in (0, +\infty)^N$, define the following equivalence relations on $\llbracket 1, N \rrbracket$ and $\mathbb{Z}^N$,

\begin{align*}
    i \sim j & \iff \Lambda_i = \Lambda_j, \\
    \mathcal{J} & = \llbracket 1, N \rrbracket / \sim, \\
    n \approx n' & \iff \Lambda \cdot n = \Lambda \cdot n', \\
    \mathcal{Z} & = \mathbb{Z}^N / \approx.
\end{align*}
For $\Lambda \in (0, +\infty)^N$, define the following equivalence relations on $\mathbb{N}$ and $\mathbb{Z}^N$, 

$$i \sim j \text{ iff } \Lambda_i = \Lambda_j, \quad n \approx n' \text{ iff } \Lambda \cdot n = \Lambda \cdot n',$$

$$J = \mathbb{N} / \sim, \quad \mathbb{Z} = \mathbb{Z}^N / \approx.$$

For $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $L \in V_+(\Lambda)$, $[n] \in \mathbb{Z}$, $[i] \in J$, and $t \in \mathbb{R}$,

$$\hat{\Xi}^{L,\Lambda,A}_{[n],t} = \sum_{n' \in [n]} \Xi^{L,A}_{n',t}, \quad \hat{A}^\Lambda_{[i]}(t) = \sum_{j \in [i]} A_j(t),$$

$$\hat{\Theta}^{L,\Lambda,A}_{[n],t} = \sum_{[j] \in J} \sum_{L \cdot n - L_j \leq t} \Xi^{L,\Lambda,A}_{[n-e_j],t} \hat{A}^\Lambda_{[j]}(t - L \cdot n + L_j).$$
 Lemma (Chitour, M., Sigalotti; 2015)

\[
\text{Let } \Lambda \in (0, +\infty)^N, \ L \in V_+(\Lambda), \ A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N, \text{ and } \ \\
x_0 : [-L_{\text{max}}, 0) \to \mathbb{C}^d. \text{ The corresponding solution } \ \\
x : [-L_{\text{max}}, +\infty) \to \mathbb{C}^d \text{ of } \Sigma(L, A) \text{ is, for } t \geq 0, \ \\
x(t) = \sum_{[n] \in \mathbb{Z}} \sum_{[j] \in J} \delta_{L \cdot n \leq t + L_{\text{max}}, L \cdot n - L_j \leq t} \hat{A}_{[n-e_j], t}^L \hat{A}_{[j]}^\Lambda (t - L \cdot n + L_j) x_0(t - L \cdot n) 
\]
Stability analysis
Explicit solution (II)

Lemma (Chitour, M., Sigalotti; 2015)

Let \( \Lambda \in (0, +\infty)^N \), \( L \in V_+ (\Lambda) \), \( A : \mathbb{R} \to \mathcal{M}_d (\mathbb{C})^N \), and \( x_0 : [-L_{\text{max}}, 0) \to \mathbb{C}^d \). The corresponding solution \( x : [-L_{\text{max}}, +\infty) \to \mathbb{C}^d \) of \( \Sigma (L, A) \) is, for \( t \geq 0 \),

\[
x(t) = \sum_{[n] \in \mathbb{Z}} \sum_{[j] \in \mathcal{J}} \quad \begin{cases} \Xi L, \Lambda, A \\ \hat{A}^\Lambda_{[n-e_j], t} \\ \hat{A}^\Lambda_{[j]}(t - L \cdot n + L_j) x_0 (t - L \cdot n) \end{cases}
\]

subject to \( t < L \cdot n \leq t + L_{\text{max}} \) and \( L \cdot n - L_j \leq t \).
Stability analysis
Explicit solution (II)

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\[
x(t) = \sum_{[n] \in \mathbb{Z}} \sum_{\substack{[j] \in J \backslash \{n-e_j\} \cap \mathbb{Z}_+^d \leq \mathbb{Z}_+^d \leq t \leq L \cdot n \leq t + L_{\text{max}}, L \cdot n - L_j \leq t}} \hat{A}_j^\Lambda(t - L \cdot n + L_j)x_0(t - L \cdot n)
\]
Stability analysis
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$$
x(t) = \sum_{[n] \in \mathbb{Z}, \ t < L \cdot n \leq t + L_{\text{max}}} \tilde{\Theta}^{L, \Lambda, A}_{[n], t} x_0(t - L \cdot n)
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We can define exponential types for $\Theta$ and $\Xi$ similarly. Since they depend on $\Lambda$, we write $(\Theta, \Lambda)$- and $(\Xi, \Lambda)$-exponential types.
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**Theorem (Chitour, M., Sigalotti; 2015)**

Let $\Lambda \in (0, +\infty)^N$ and $\mathcal{A}$ be uniformly locally bounded.

- Let $L \in V_+(\Lambda)$. If $\Sigma(L, \mathcal{A})$ is of $(\Theta, \Lambda)$-exponential type $\gamma$ then $\forall p \in [1, +\infty]$ it is of exponential type $\gamma$ in $X_p$.
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- Suppose that $A$ is shift-invariant. Then $(\Theta, \Lambda)$- and $(\Xi, \Lambda)$-exponential types $\gamma$ are equivalent.
Stability analysis
Maximal Lyapunov exponent

Definition

The maximal Lyapunov exponent of \( \Sigma(L, A) \) in \( X_p \) is

\[
\lambda_p(L, A) = \lim_{t \to +\infty} \sup_{A \in A} \sup_{x_0 \in X_p} \frac{\log \| x_t \|_{X_p}}{t}.
\]

In particular, \( \Sigma(L, A) \) exponentially stable \( \iff \lambda_p(L, A) < 0 \).
Stability analysis
Maximal Lyapunov exponent

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The maximal Lyapunov exponent of $\Sigma(L, A)$ in $X_p$ is

$$\lambda_p(L, A) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{A}} \sup_{x_0 \in X_p} \frac{\log \|x_t\|_{X_p}}{t}.$$  

Proposition
$$\lambda_p(L, A) = \inf \{ \gamma \in \mathbb{R} | \Sigma(L, A) \text{ is of exponential type } \gamma \text{ in } X_p \}.$$  
In particular,

$$\Sigma(L, A) \text{ exponentially stable } \iff \lambda_p(L, A) < 0.$$  

By the previous results, $\lambda_p(L, A)$ does not depend on $p$.  

Stability of difference equations and applications to transport and wave propagation on networks
Guilherme Mazanti
Stability analysis
Maximal Lyapunov exponent

**Theorem (Chitour, M., Sigalotti; 2015)**

Let $\Lambda \in (0, +\infty)^N$ and suppose that $\mathcal{A}$ is shift-invariant. For every $L \in W_+(\Lambda)$ and $p \in [1, +\infty]$, \[
\lambda_p(L, \mathcal{A}) = \limsup_{|n|_1 \to +\infty} \sup_{A \in \mathcal{A}} \text{ess sup}_{t \in \mathbb{R}} \frac{\log \left| \Xi_{L, \Lambda, A}^n, t \right|}{L \cdot n}.
\]
Stability analysis

Arbitrary switching

\[ \Sigma(L, A) : \quad x(t) = \sum_{j=1}^{N} A_j(t)x(t - L_j), \quad t \geq 0. \]

- \( \mathcal{B} \subset M_d(\mathbb{C})^N \): bounded set of \( N \)-tuples of matrices.
- \( \mathcal{A} = L^\infty(\mathbb{R}, \mathcal{B}) \).
- \( (A_1(t), \ldots, A_N(t)) \) is any measurable function taking values on \( \mathcal{B} \): switched system with arbitrary switching signal.
- In this case, one can obtain more precise results.
Stability analysis

Arbitrary switching

Using the recurrence relation for $\Xi_{n,t}$, we obtain:

$$\Xi_{[n],t} = \sum_{n' \in [n] \cap \mathbb{N}^N} \sum_{v \in V_{n'}} |n'|_{1} \prod_{k=1}^{k-1} A_{v_k} \left(t - \sum_{r=1}^{k-1} L_{v_r}\right).$$

$V_n$: set of all permutations of $(1, \ldots, 1, 2, \ldots, 2, \ldots, N, \ldots, N)$.

- $n_1$ times
- $n_2$ times
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Stability analysis

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$|n|_1$ times $n_N$ times

**Definition**

$$
\mu(\Lambda, \mathcal{B}) = \limsup_{\xi \to +\infty} \sup_{\xi \in \mathcal{L}(\Lambda) \in \mathcal{B}} \left\{ \sum_{n \in \mathbb{N}^N} \sum_{v \in V_n} \prod_{k=1}^{\lfloor |n|_1 \rfloor} B_{v_k}^{\Lambda_{v_1} + \cdots + \Lambda_{v_{k-1}}} \right\}^{\frac{1}{\xi}},
$$

where $\mathcal{L}(\Lambda) = \{ \Lambda \cdot n \mid n \in \mathbb{N}^N \}$ and $\mathcal{L}_\xi(\Lambda) = \mathcal{L}(\Lambda) \cap [0, \xi)$. 

Stability analysis
Arbitrary switching

Theorem (Chitour, M., Sigalotti; 2015)

\[ \lambda_p(\Lambda, A) = \log \mu(\Lambda, \mathcal{B}); \]
Stability analysis

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for every \( L \in V_+(\Lambda) \), \( \lambda_p(L, A) \leq m_1 \log \mu(\Lambda, \mathcal{B}); \)

for every \( L \in W_+(\Lambda) \), \( m_2 \lambda_p(\Lambda, A) \leq \lambda_p(L, A) \leq m_1 \lambda_p(\Lambda, A). \)

Here, \( \{ m_1, m_2 \} = \left\{ \min_{j \in [1,N]} \frac{\Lambda_j}{L_j}, \max_{j \in [1,N]} \frac{\Lambda_j}{L_j} \right\}. \)
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**Corollary**

The following statements are equivalent:

- \( \mu(\Lambda, B) < 1; \)
- \( \Sigma(\Lambda, A) \) is exponentially stable in \( X_p \) for some \( p \in [1, +\infty] \);
- \( \Sigma(L, A) \) is exponentially stable in \( X_p \) for every \( p \in [1, +\infty] \) and \( L \in V_+(\Lambda) \).
### Exponential stability criteria:

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#### Interesting questions:

- Both $\rho(A)$ and $\rho_J(B)$ are limits and $\lim_{n \to +\infty}$ can be replaced by $\inf_{n \in \mathbb{N}^*}$. Is the same true for $\mu(\Lambda, B)$?
- $\rho(A) = 0, \rho_J(B) = 0,$ and $\rho_{HS}(A) = 0$ are equivalent to convergence in finite time. Is this also true for $\mu(\Lambda, B)$?
- Can we numerically compute or approximate $\mu(\Lambda, B)$? (For $\rho_J$, this problem is NP-hard, Turing-undecidable, and non-algebraic, but several useful bounds and approximations exist, see [Jungers; 2009]).
- What can we say if $\Lambda_1, \ldots, \Lambda_N$ are time-dependent?
Stability analysis

Conclusion

Exponential stability criteria:

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Application to a transport system

Transport system
Application to a transport system

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Stability of difference equations and applications to transport and wave propagation on networks

Guilherme Mazanti
Application to a transport system

Transport system

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Transport system

\[
\begin{align*}
\partial_t u_i(t, \xi) + \partial_\xi u_i(t, \xi) + \alpha_i(t) \chi_i(\xi) u_i(t, \xi) &= 0, \\
\quad t \in \mathbb{R}_+, \ \xi \in [0, L_i], \ i \in [1, N_d], \\
\partial_t u_i(t, \xi) + \partial_\xi u_i(t, \xi) &= 0, \\
\quad t \in \mathbb{R}_+, \ \xi \in [0, L_i], \ i \in [N_d + 1, N], \\
u_i(t, 0) &= \sum_{j=1}^{N} m_{ij} u_j(t, L_j), \quad t \in \mathbb{R}_+, \ i \in [1, N].
\end{align*}
\]

- \( \chi_i \): characteristic function of an interval \([a_i, b_i] \subset [0, L_i]\).
- \( M = (m_{ij})_{1 \leq i, j \leq N} \): transmission matrix.
- \( \alpha_i \) is persistently exciting for \( i \in [1, N_d] \).
Persistently exciting (PE) signals: for $T \geq \mu > 0$, we say that $\alpha \in \mathcal{G}(T, \mu)$ if $\alpha \in L^\infty(\mathbb{R}; [0, 1])$ and
\[
\forall t \in \mathbb{R}, \quad \int_{t}^{t+T} \alpha(s)ds \geq \mu.
\]
$\mathcal{G}(T, \mu)$ is shift-invariant.
Persistent exciting (PE) signals: for $T \geq \mu > 0$, we say that $\alpha \in \mathcal{G}(T, \mu)$ if $\alpha \in L^\infty(\mathbb{R}; [0, 1])$ and
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Introduced in the context of identification and adaptive control [Anderson; 1977].

Much studied in finite-dimensional control systems [Chitour, Sigalotti; 2010], [Chitour, M., Sigalotti; 2013].
Application to a transport system
Main result

Hypotheses:
- There exist $i, j \in [1, N]$ such that $\frac{L_i}{L_j} \notin \mathbb{Q}$.
- $|M|_1 \leq 1$ and $m_{ij} \neq 0$ for every $i, j \in [1, N]$.

Theorem
\[ \forall T \geq \mu > 0, \exists C, \gamma > 0 \text{ s.t., } \forall p \in [1, +\infty], \forall u_{i,0} \in L^p(0, L_i), i \in [1, N], \text{ and } \forall \alpha_k \in G(T, \mu), k \in [1, N_d], \text{ the corresponding solution satisfies} \]
\[ \sum_{i=1}^{N} \|u_i(t)\|_{L^p(0, L_i)} \leq C e^{-\gamma t} \sum_{i=1}^{N} \|u_{i,0}\|_{L^p(0, L_i)}, \quad \forall t \geq 0. \]
Application to a transport system

Technique of the proof

- For $t \geq L_{\text{max}}$:
  \[
  u_i(t, 0) = \sum_{j=1}^{N} m_{ij} u_j(t, L_j) = \sum_{j=1}^{N} m_{ij} e^{-\int_{t-L_j+a_j}^{t-L_j+b_j} \alpha_j(s) ds} u_j(t - L_j, 0)
  \]

- Set $x(t) = (u_i(t, 0))_{i \in [1,N]}$. Then $x$ satisfies the difference equation
  \[
  x(t) = \sum_{k=1}^{N} A_k(t)x(t - L_k)
  \]
  with
  \[
  A_k(t) = \left( \delta_{jk} m_{ij} e^{-\int_{t-L_j+a_j}^{t-L_j+b_j} \alpha_j(s) ds} \right)_{i,j \in [1,N]}
  \]

- It suffices to show that such difference equation is $(\hat{\Xi}, L)$-exponentially stable. We study the behavior of the coefficients $\Xi_{n,t}^A$ as $|n|_1 \to +\infty$. 
Application to a transport system

Technique of the proof

Decomposition of the set $\mathbb{N}^N$. 
Decomposition of the set $\mathbb{N}^N$. 

$\mathcal{N}_b(\rho) = \{ n \in \mathbb{N}^N \mid \exists k \in [1, N] \text{ s.t. } n_k \leq \rho |n|_1 \}$

$\mathcal{N}_c(\rho) = \{ n \in \mathbb{N}^N \mid n_k > \rho |n|_1, \forall k \in [1, N] \}$
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Application to a transport system

Technique of the proof

\[ \mathcal{H}_c(\rho) : \text{"box lemma"} \]
Application to a transport system

Technique of the proof

In $\mathcal{N}_c(\rho)$: “box lemma”

\[ e^{-\int_{t-L\cdot n+a_k}^{t-L\cdot n+b_k} \alpha_k(s) ds} \]
In $\mathcal{N}_c(\rho)$: “box lemma”

$$e^{-\int_{t-L\cdot n+a_k}^{t-L\cdot n+b_k} \alpha_k(s) ds} \leq \eta$$
In $\mathcal{H}_c(\rho)$: “box lemma”

Find $\eta \in (0, 1)$ such that $e^{-\int_{t-L}^{t-L} (n+b_k) \alpha_k(s) ds} \leq \eta$ “often enough”
Application to a transport system

Technique of the proof

In $\mathcal{N}_c(\rho)$: "box lemma"

\[ \exists \eta \in (0, 1), \]
\[ \exists \text{ size } K \text{ of box s.t } \]
\[ \forall \text{ box of size } K, \]
\[ \exists \text{ point in the box where } \]
\[ e^{-\int_{t-L \cdot n + a_k}^{t-L \cdot n + b_k} \alpha_k(s) ds} \leq \eta \]

Important:

\[ \frac{L_i}{L_j} \notin \mathbb{Q} \]
\[ \alpha \in \mathcal{G}(T, \mu) \]
Application to a transport system

Technique of the proof

In \( \mathcal{M}_c(\rho) \): “box lemma”

\[ \exists \eta \in (0, 1), \exists \text{ size } K \text{ of box s.t } \forall \text{ box of size } K, \exists \text{ point in the box where } e^{-\int_{t-L\cdot n+b_k}^{t-L\cdot n+a_k} \alpha_k(s) ds} \leq \eta \]

\[ \Xi_{n,t}^{L,A} \text{ decreases exponentially with } n \text{ in } \mathcal{M}_c(\rho) \]

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$\Xi^{L,A}_{n,t}$ decreases exponentially with $n$ in $\mathcal{M}_c(\rho)$

$\Rightarrow$ the solutions converge exponentially
Relative controllability

Definition

\[ \Sigma_{\text{contr}} : \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \quad t \geq 0. \]
Relative controllability

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\[ \Sigma_{\text{contr}} : \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + B u(t), \quad t \geq 0. \]

For every initial condition \( x_0 : [-\Lambda_{\text{max}}, 0) \rightarrow \mathbb{C}^d \) and control \( u : [0, T] \rightarrow \mathbb{C}^m \), \( \Sigma_{\text{contr}} \) admits a unique solution \( x : [-\Lambda_{\text{max}}, T] \rightarrow \mathbb{C}^d \) (no regularity assumptions!).
Relative controllability

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Definition

We say that \( \Sigma_{\text{contr}} \) is relatively controllable in time \( T > 0 \) if, for every \( x_0 : [-\Lambda_{\text{max}}, 0) \to \mathbb{C}^d \) and \( x_1 \in \mathbb{C}^d \), there exists \( u : [0, T] \to \mathbb{C}^m \) such that the unique solution \( x \) of \( \Sigma_{\text{contr}} \) with initial condition \( x_0 \) and control \( u \) satisfies \( x(T) = x_1 \).
Similarly to the stability analysis, we use an explicit formula for the solutions in order to characterize relative controllability.
Relative controllability

Explicit formula

Similarly to the stability analysis, we use an explicit formula for the solutions in order to characterize relative controllability.

**Lemma (Explicit solution)**

Let $u : [0, T] \rightarrow \mathbb{C}^m$. The solution $x : [-\Lambda_{\text{max}}, T] \rightarrow \mathbb{C}^d$ of $\Sigma_{\text{contr}}$ with zero initial condition and control $u$ is, for $t \in [0, T]$,

$$x(t) = \sum_{[n] \in \mathbb{Z}, \Lambda \cdot n \leq t} \hat{x}^L,\Lambda,A_{n,t} Bu(t - \Lambda \cdot n).$$
Similarly to the stability analysis, we use an explicit formula for the solutions in order to characterize relative controllability.

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Similarly to the stability analysis, we use an explicit formula for the solutions in order to characterize relative controllability.

**Lemma (Explicit solution)**

Let \( u : [0, T] \rightarrow \mathbb{C}^m \). The solution \( x : [-\Lambda_{\text{max}}, T] \rightarrow \mathbb{C}^d \) of \( \Sigma_{\text{contr}} \) with zero initial condition and control \( u \) is, for \( t \in [0, T] \),

\[
x(t) = \sum_{[n] \in \mathbb{Z}^d \atop \Lambda \cdot n \leq t} \frac{n!}{n_1! \cdots n_d!} \Lambda^{n - n} A^n Bu(t - \Lambda \cdot n).
\]

By linearity, solution with initial condition \( x_0 \) and control \( u \) is the sum of this formula with the previous one.
Relative controllability

The following statements are equivalent:

1. $\Sigma_{\text{contr}}$ is relatively controllable in time $T$;
2. \[ \text{Span} \left\{ \Xi^{\Lambda, A}_{[n]} Bw \mid n \in \mathbb{N}^N, \Lambda \cdot n \leq T, w \in \mathbb{C}^m \right\} = \mathbb{C}^d; \]
Theorem (M.; 2016)

The following statements are equivalent:

- $\Sigma_{\text{contr}}$ is relatively controllable in time $T$;

- $\text{Span} \left\{ \tilde{\Xi}_{[n]}^{\Lambda, A} B w \mid n \in \mathbb{N}^N, \Lambda \cdot n \leq T, w \in \mathbb{C}^m \right\} = \mathbb{C}^d$;

- $\exists \varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, $x_0 : [-\Lambda_{\text{max}}, 0) \rightarrow \mathbb{C}^d$, and $x_1 : [0, \varepsilon] \rightarrow \mathbb{C}^d$, there exists $u : [0, T + \varepsilon] \rightarrow \mathbb{C}^m$ such that the solution $x$ of $\Sigma_{\text{contr}}$ with initial condition $x_0$ and control $u$ satisfies $x(T + \cdot)|_{[0, \varepsilon]} = x_1$.
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The following statements are equivalent:

- $\Sigma_{\text{contr}}$ is relatively controllable in time $T$;

- $\text{Span} \left\{ \Xi^{,\Lambda A_n}Bw \mid n \in \mathbb{N}^N, \Lambda \cdot n \leq T, w \in \mathbb{C}^m \right\} = \mathbb{C}^d$;

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- $\exists \varepsilon_0 > 0$ such that, for every $p \in [1, +\infty]$, $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in L^p((-\Lambda_{\text{max}}, 0), \mathbb{C}^d)$, and $x_1 \in L^p((0, \varepsilon), \mathbb{C}^d)$, there exists $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$ such that the solution $x$ of $\Sigma_{\text{contr}}$ with initial condition $x_0$ and control $u$ satisfies $x \in L^p((-\Lambda_{\text{max}}, T + \varepsilon), \mathbb{C}^d)$ and $x(T + \cdot)|_{[0,\varepsilon]} = x_1$. 
Relative controllability
Relative controllability criterion

- Can also be generalized to other spaces (e.g., $C^k$).
Relative controllability

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- Can also be generalized to other spaces (e.g., $\mathcal{C}^k$).
- Generalizes Kalman criterion: for $x(t) = Ax(t-1) + Bu(t)$, one has
  \[
  \text{Span}\left\{\sum_{n=1}^{N} A^nw \mid n \in \mathbb{N}^N, \Lambda \cdot n \leq T, w \in \mathbb{C}^m\right\} = \text{Ran}\left(B\ A B\ A^2 B \cdots A^{T} B\right).
  \]

Theorem (M.; 2016)

- If $\Sigma_{\text{contr}}$ is relatively controllable in some time $T > 0$, then it is also relatively controllable in time $T = (d - 1)\Lambda_{\text{max}}$. 
Relative controllability

Can also be generalized to other spaces (e.g., $\mathbb{C}^k$).

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$$\text{Span} \left\{ \Xi_{\Lambda, A}^n Bw \mid n \in \mathbb{N}^N, \Lambda \cdot n \leq T, w \in \mathbb{C}^m \right\} = \text{Ran} \left( B \quad AB \quad A^2B \quad \cdots \quad A^{\lfloor T \rfloor}B \right).$$

**Theorem (M.; 2016)**

- If $\Sigma_{\text{contr}}$ is relatively controllable in some time $T > 0$, then it is also relatively controllable in time $T = (d - 1)\Lambda_{\text{max}}$.

- If $\Lambda_1, \ldots, \Lambda_N$ are rationally independent, then $\Sigma_{\text{contr}}$ is relatively controllable in some time $T > 0$ if and only if

$$\text{Span} \left\{ \Xi_{\Lambda}^A Be_j \mid n \in \mathbb{N}^N, \|n\|_1 \leq d - 1, j \in [1, m] \right\} = \mathbb{C}^d.$$
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Stability of difference equations and applications to transport and wave propagation on networks

Guilherme Mazanti