Special surfaces in special cubic fourfolds

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(joint work with Arend Bayer, Aaron Bertram, Alexander Perry)

In this talk, I reported on work in progress on a possible characterization of Hassett divisors on the moduli space of cubic fourfolds by the property of containing special surfaces. I sketched the construction of such special surfaces for infinitely many divisors and the relation with the work of Russo and Staglianò on rationality of such cubics in low discriminant.

1. The Main Theorem

Let $Y \subset \mathbb{P}^5$ denote a complex smooth cubic fourfold and let $h := [\mathcal{O}_Y(1)]$ be the class of a hyperplane section. By following [7], we say that Y is *special* of discriminant d, and use the notation $Y \in \mathcal{C}_d$, if there exists a surface $\Sigma \subset Y$ not homologous to a complete intersection such that the determinant of the intersection matrix

$$\begin{pmatrix} h^2 & h.\Sigma \\ h.\Sigma & \Sigma^2 \end{pmatrix}$$

is equal to d. The locus $\mathcal{C}_d$ is non-empty if and only if $d \equiv 0, 2 \pmod 6$ and $d > 6$; moreover, in such a case, $\mathcal{C}_d$ is an irreducible divisor in the moduli space of cubic fourfolds, which can also be described purely in terms of Hodge theory and periods (by the Global Torelli Theorem [17] and the surjectivity of the period map [10, 13]; we refer to the book in progress [8] for the general theory of cubic fourfolds).

The main result gives an actual surface defining the divisor $\mathcal{C}_d$, for special values of d.

Theorem 1. *Let $a \geq 1$ be an integer and let $d := 6a^2 + 6a + 2$. Let Y be a general cubic fourfold in $\mathcal{C}_d$. Then there exists a surface $\Sigma \subset Y$ such that*

- $\deg(\Sigma) := h.\Sigma = 1 + \frac{3}{2}a(a+1)$ *and* $\Sigma^2 = \frac{d+\deg(\Sigma)^2}{3}$*;*
- $H^*(Y, \mathcal{I}_\Sigma(a-j)) = 0$*, for all* $j = 0, 1, 2$.

For discriminant d as in Theorem 1, by [1, 2] there is a polarized K3 surface S of degree d associated to each cubic fourfold in $\mathcal{C}_d$. To be precise the surface Σ is not unique but it is a family of surfaces in Y, parameterized by the K3 surface S. Moreover, if the cubic fourfold deforms in the divisor $\mathcal{C}_d$, the K3 surface S and the family of surfaces Σ deform along as well.

Example 2. Let Σ be the surface in Theorem 1.

(1) Let $a = 1$, and so $d = 14$. Then the surface Σ is a smooth quartic scroll, whose existence was observed in [5, 6, 16]; explicitly, in the general case, this is $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the linear system $|\mathcal{O}(1, 2)|$.

(2) Let $a = 2$, and so $d = 38$. Then the surface Σ is a smooth "generalized" Coble surface, whose existence was observed in [14]; explicitly, this is the blow-up of $\mathbb{P}^2$ in 10 general points embedded in $\mathbb{P}^5$ by the linear system $|10L - 3(E_1 + \ldots + E_{10})|$.

A geometric description for the surfaces Σ is not known when $a \geq 3$. In particular, we currently do not know whether the surface is smooth or even integral. If it is smooth, all numerical invariants can be computed; in particular, it will not be rational for any $a \geq 3$. On the positive side, the construction does conjecturally generalize to any $d \equiv 2 \pmod 6$. In particular, it works in general for small discriminant (e.g., $d \leq 44$) and recovers well known surfaces (e.g., the ones in [14]). Moreover, it does provide as well many rational morphisms from the cubic fourfold, which can be described and studied by using the associated K3 surface S and [4]. For example, in the case $a = 2$, this recovers completely the picture described in [15].

The key insight in our construction comes from derived categories; in particular, the construction of Σ arises from understanding the *Kuznetsov component* $\mathrm{Ku}(Y)$ of Y ([9]) and moduli spaces therein ([3, 2]). For d as in Theorem 1, the K3 surface S associated to Y has indeed the property that $\mathrm{Ku}(Y) \cong \mathrm{D^b}(S)$ and the surface Σ arises from a Brill–Noether locus in a special moduli space of stable objects in $\mathrm{Ku}(Y)$. The second property in the statement of the theorem can in fact be rephrased by saying that the ideal sheaf $\mathcal{I}_\Sigma(a)$ belongs to $\mathrm{Ku}(Y)$.

In what follows, we will give an outline of the construction of Σ in Section 2 and how to induce rational morphisms from Y in Section 3.

2. K3 categories and Brill–Noether loci

Let Y be a cubic fourfold and let $\mathrm{D^b}(Y)$ denote the bounded derived category of coherent sheaves on Y. The Kuznetsov component $\mathrm{Ku}(Y)$ of Y is defined as the right orthogonal

$$\mathrm{Ku}(Y) := \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle^\perp \subset \mathrm{D^b}(Y).$$

We denote by i_* the inclusion functor $\mathrm{Ku}(Y) \to \mathrm{D^b}(Y)$ and by i^* its left adjoint $\mathrm{D^b}(Y) \to \mathrm{Ku}(Y)$.

The basic properties of the Kuznetsov component are the following.

- $\mathrm{Ku}(Y)$ is a *K3 category*, i.e., it is smooth, proper triangulated category over $\mathbb{C}$, with Serre functor given by [2], the shift by 2 functor ([9]).
- There is a cohomology lattice $(H(\mathrm{Ku}(Y), \mathbb{Z}), (-,-))$ naturally associated to $\mathrm{Ku}(Y)$, given by topological K-theory

$$H(\mathrm{Ku}(Y), \mathbb{Z}) := K_{\mathrm{top}}(\mathrm{Ku}(Y)) := \langle [\mathcal{O}_Y], [\mathcal{O}_Y(1)], [\mathcal{O}_Y(2)]\rangle^\perp \subset K_{\mathrm{top}}(Y),$$

  where $(-,-)$ denotes the Mukai pairing. It has a Hodge structure of weight 2, given by Hochschild homology, and the Mukai vector gives a morphism $K(\mathrm{Ku}(Y)) \to H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ ([1]).
- The classes

$$\lambda_1 := i^*[\mathcal{O}_{\mathrm{line}}(1)] \qquad \lambda_2 := i^*[\mathcal{O}_{\mathrm{line}}(2)]$$

  define a sublattice $A_2 := \langle \lambda_1, \lambda_2\rangle \subset H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ ([7, 1]).

- There is a "canonical" (orbit of) stability condition σ_0 which deforms over all cubics; we denote by $\mathrm{Stab}(\mathrm{Ku}(Y))$ the connected component of the space of Bridgeland stability conditions containing σ_0 ([3]).
- Given a Mukai vector $v \in H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ and $\sigma \in \mathrm{Stab}(\mathrm{Ku}(Y))$, the moduli space $M_\sigma(v)$ behaves "as nice as" a moduli space of semistable sheaves on a K3 surface. In particular, if v is primitive and σ generic with respect to v, then $M_\sigma(v) \neq \emptyset$ if and only if $v^2 + 2 \geq 0$; in such a case, $M_\sigma(v)$ is a projective irreducible holomorphic symplectic manifold of dimension $v^2 + 2$, deformation equivalent to a Hilbert scheme of points on a K3 surface ([2]).
- If Y does not contain a plane, then for all $y \in Y$, the projection of the skyscraper sheaf $i^*k(y)$ is σ_0-stable of Mukai vector $\lambda_2 - \lambda_1$; in particular, we obtain an embedding $Y \hookrightarrow M_{\sigma_0}(\lambda_2 - \lambda_1)$ ([12]; the geometric construction is in [11]).

The last property is the starting point for our construction. Indeed, let us fix $v := \lambda_2 - \lambda_1$. If we could find another Mukai vector $u \in H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$ such that $(u, v) = -1$, $u^2 + 2 \geq 0$, and the slope of u with respect to σ_0 is larger than the slope of v, then for $F \in M_{\sigma_0}(u)$, the Brill–Noether locus

$$\mathrm{BN}_F := \left\{ E \in M_{\sigma_0}(v) \,:\, \min(\hom(E, F), \mathrm{ext}^1(E, F)) \geq 1 \right\} \subset M_{\sigma_0}(v)$$

has expected codimension 2; in particular, the intersection $Y \cap \mathrm{BN}_F$ has expected codimension 2 as well, and so it could define a surface Σ_F (parameterized by $M_{\sigma_0}(u)$).

To make this argument work, we need to prove the existence of such u and study the non-triviality of BN_F and its intersection with Y. The existence of u is a straightforward computation: u exists if and only if $d \equiv 2 \pmod 6$.

This Brill–Noether locus can be studied directly in low discriminant; in general, we have to assume that $d = 6a^2 + 6a + 2$, $a \geq 1$. In such a case, we can choose u such that $u^2 = 0$. Let $S := M_{\sigma_0}(u)$. Then (up to in case slightly deform σ_0) S is a smooth projective K3 surface and the universal family $\mathcal{U}$ (which exists) gives a Fourier–Mukai equivalence

$$\Phi_{\mathcal{U}} \colon \mathrm{D}^{\mathrm{b}}(S) \xrightarrow{\cong} \mathrm{Ku}(Y).$$

Lemma 3. *Let $a \geq 2$. Then, for all $E \in M_{\sigma_0}(v)$, we have $\Phi_{\mathcal{U}}^{-1}(E) \cong \mathcal{I}_\Gamma$, where $\Gamma \subset S$ is a 0-dimensional closed subscheme of length 4.*

In particular, by Lemma 3, we can identify $M_{\sigma_0}(v)$ with the Hilbert scheme $S^{[4]}$ (in the case $a = 1$ this is not true; this case has to be studied separately). We can use this to show the following.

Lemma 4. *Let $F \in M_{\sigma_0}(u)$ general. Then (up to shift and taking derived dual) we have*

$$i_* F \cong \mathcal{I}_{\Sigma_F}(a),$$

where $\Sigma_F \subset Y$ is a surface.

Theorem 1 follows then directly from Lemma 4.

3. Morphisms

We keep the notation as in the previous section, with $v = \lambda_2 - \lambda_1$, u, and $F \in M_{\sigma_0}(u)$. Then in "optimal situations" by taking extensions with F gives a well-defined rational map

$$g = g_F \colon M_{\sigma_0}(v) \dashrightarrow M_{\sigma_0}(v-u)$$

which induces a diagram

$$\begin{tikzcd}
\mathrm{Bl}_{\Sigma_F} Y \arrow[r] \arrow[d, "\sigma"] & \mathrm{Bl}_{\mathrm{BN}_F} M_{\sigma_0}(v) \arrow[d] \arrow[dr] & \\
Y \arrow[r] & M_{\sigma_0}(v) \arrow[r, dashed, "g"] & M_{\sigma_0}(v-u)
\end{tikzcd}$$

and so a closed embedding $f \colon \mathrm{Bl}_{\Sigma_F} Y \hookrightarrow M_{\sigma_0}(v-u)$.

If this is the case, and we let Δ denote the exceptional divisor of σ, we have the following result.

Lemma 5. *For a divisor classe $D \in \mathrm{NS}(M_{\sigma_0}(v-u)) \cong (v-u)^\perp \subset H_{\mathrm{alg}}(\mathrm{Ku}(Y), \mathbb{Z})$, we have*

$$f^* D = -\frac{(D + (D,u)\, u, \lambda_1 + \lambda_2)}{2}\, \sigma^* h + (D,u)\, \Delta.$$

If $d = 6a^2 + 6a + 2$ as in Theorem 1, then g is indeed well-defined: F corresponds to a skyscraper sheaf at a point $p \in S$, and the morphism g is nothing but the map

$$S^{[4]} \dashrightarrow S^{[5]}, \qquad \Gamma \mapsto p + \Gamma.$$

Moreover, by fixing p, the morphism f gives a closed embedding $\mathrm{Bl}_\Sigma Y \hookrightarrow S^{[5]}$.

To get rational maps from Y, we can study morphisms from $S^{[5]}$ and these can be studied by simply looking at the base locus decomposition of the movable cone $\mathrm{Mov}(S^{[5]})$, which has been completely described in [4].

In the example when $a = 2$ (and so, $d = 38$), we have the following diagram:

$$\begin{tikzcd}
S^{[5]} \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dr] & & M_1 \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dl] \arrow[dr] & & M_2 \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dl] \arrow[drr] & & \dots \\
& \overline{M}_1 & & \overline{M}_2 & & & \dots
\end{tikzcd}$$

where the leftmost diagram is a Mukai flop at a $\mathbb{P}^3$-bundle over the Fano variety of lines $F(Y)$ of Y, and the next diagram is a Mukai flop at a $\mathbb{P}^2$-bundle over the product $S \times F(Y)$.

By taking the restriction of the above sequence of morphisms to Y, we find exactly the "trisecant flop" description in [15]

$$\begin{tikzcd}
& S^{[5]} \supset \mathrm{Bl}_\Sigma Y \arrow[rr, dashed, leftrightarrow, "\cong"] \arrow[dl, "\sigma"'] \arrow[dr] & & Y_1 \subset M_1 \arrow[dl] \arrow[dr] & \\
Y & & \overline{Y}_1 \subset \overline{M}_1 & & \overline{Y}_2 \cong \mathbb{P}^4 \subset \overline{M}_2.
\end{tikzcd}$$

The divisors associated to the two birational maps from Y to $\overline{Y}_1$, respectively $\overline{Y}_2$, correspond, by using Lemma 5, to the linear systems $|\mathcal{I}_\Sigma(3)|$, respectively $|\mathcal{I}_\Sigma^2(5)|$ on Y. Concretely, trisecant lines and 5-secant conics to Σ in Y.

In [15] this was used to show the rationality of Y in $\mathcal{C}_{38}$. Conjecturally ([9]) all cubic fourfolds in $\mathcal{C}_d$, where $d = 6a^2+6a+2$, should be rational. The corresponding picture already in the case $a = 3$ ($d = 74$) is not understood: $S^{[5]}$ has only one interesting morphism, which corresponds to the linear system $|\mathcal{I}_\Sigma^3(10)|$ on Y, and the rationality of Y in $\mathcal{C}_{74}$ is not known.

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