TD3 : Subharmonic functions

We let $\Delta(0,r)$ denote the open disk of radius $r > 0$. When $r = 1$ we simply denote by $\mathbb{D}$ the unit open disk. Its boundary will be denoted by $C(0,r)$. The support for these exercises can be found in the second chapter of Ransford's book [Ran95] or the notes of Merker [Mer15].

Exercise 1. Let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a upper semicontinuous function in a domain $\Omega \subset \mathbb{C}$. Prove that the following conditions are equivalent.

1. u is subharmonic in Ω.
2. For any $z_0 \in \Omega$ with $u(z_0) > -\infty$ one has
$$\limsup_{r\to 0} \frac{1}{2\pi r^2}\left(\int_0^{2\pi} u(z_0 + re^{it})dt - u(z_0)\right) \geq 0.$$
3. For any subdomains $\bar{D}(0,r) \Subset \omega \Subset \Omega$ and any harmonic function h on ω such that $u \leq h$ on $\partial D(z_0,r)$, we have $u \leq h$ on $D(z_0,r)$.
4. For any $z_0 \in \Omega$ and any $\mathcal{C}^2$ function q defined in an open neighborhood V of z_0 such that $u(z)-q(z) \leq 0$ with equality at z_0, one has $\Delta q(z_0) \geq 0$.

Exercise 2. Let $h : \Omega \to \mathbb{R}$ be a harmonic function. Prove that h^2 is subharmonic in Ω.

Exercise 3. Let u, v be subharmonic functions in a domain Ω such that $u + v$ attains a maximum at some point in Ω. Prove that both u and v are harmonic on Ω.

Exercise 4. Let $-\infty \leq a < b \leq b$, $h : \Omega \to]a,b[$ be a harmonic function on a domain Ω and let $\psi :]a,b[\to \mathbb{R}$ be a convex function (not necessarily increasing). Prove that $\psi \circ h$ is subharmonic in Ω.

Exercise 5. Let $\Omega \subset \mathbb{C}$ be an open set. Prove that the function
$$u(z) := -\log \operatorname{dist}(z, \partial\Omega)$$
is subharmonic in Ω.

Exercise 6. Let $u : \Omega \to [0,+\infty[$ be a subharmonic function. Prove that $\log u$ is subharmonic iff u^α is subharmonic for any $\alpha > 0$.

Exercise 7. Prove that if u and v are subharmonic in a domain $\Omega \subset \mathbb{C}$, then so is $\log(e^u + e^v)$.

Exercise 8. The purpose of this exercise is to prove that any upper semicontinuous function on a complete metric space is continuous on a dense subset.

Let (X,d) and (X',d') be two metric spaces and let $f : X \to X'$ be an application. The oscillation of f at $x \in X$ is defined as
$$\omega_f(x) := \lim_{r\to 0^+}\left(\sup_{y,z\in B_d(x,r)} d'(f(y), f(z))\right).$$

1. Prove that f is continuous at x iff $\omega_f(x) = 0$.
2. Prove that for any $c > 0$ the set $U_c(f) := \{x \in X \mid \omega_f(x) < c\}$ is open in X.
3. Prove that the set of continuous points of f is a G_δ set, i.e. countable intersection of open sets.
4. We assume from now on that (X,d) is complete and f is the pointwise limit of a sequence of continuous functions (f_n). Prove that for any $c > 0$ the set $U_c(f)$ is dense in X. Then deduce that the set of continuous points of f is a G_δ dense subset of X.
5. Conclude that any upper semicontinuous function $f : X \to \mathbb{R}$ is continuous on a G_δ dense subset of X.

Exercise 9. For $\xi \in \mathbb{C}$ and $r \geq 0$ prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{it} - \xi| dt = \max(\log|\xi|, \log r).$$

Then deduce that the function

$$u(z) := \sum_{n=1}^{+\infty} 2^{-n} \log|z - 2^{-n}|$$

is subharmonic in $\mathbb{C}$. Prove that u is not continuous at 0.

Exercise 10. Let u be a subharmonic function in $\mathbb{D}$ such that $u(z) < 0$ for all $z \in \mathbb{D}$. Prove that

$$\limsup_{r\to 1} \frac{u(re^{i\theta})}{1-r} < 0, \ \forall \theta \in [0, 2\pi[.$$

Exercise 11. The goal of this exercise is to prove that any holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z) = z + o(|1-z|^3)$ near 1 is $\equiv z$. Assume that f is such a function and set

$$u(z) := \operatorname{Re}\left(\frac{1+z}{1-z} - \frac{1+f(z)}{1-f(z)}\right).$$

1. Prove that $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial\mathbb{D}$ and $u(z) = o(|1-z|)$ as $z \to 1$.
2. Deduce that $u \leq 0$ in $\mathbb{D}$ and using Exercise 10 to show that $u \equiv 0$.
3. Deduce that $f(z) \equiv z$.
4. Give an example that the conclusion fails if we merely assume that $f(z) = z + O(|1-z|^3)$.

Exercise 12. Let u be a subharmonic function in $\mathbb{D}$ such that $u(z) \leq -\log|\operatorname{Im}(z)|$ for all $z \in \mathbb{D}$. Prove that

$$u(z) \leq -\log\left|\frac{1-z^2}{2}\right|, \ \forall z \in \mathbb{D}.$$

Exercise 13. Let $u : \mathbb{D} \to \mathbb{R}$ be a function such that $x \mapsto u(x+iy)$ and $y \mapsto u(x+iy)$ are convex. Prove that u is subharmonic in $\mathbb{D}$. Give a counter example for the converse.

Exercise 14. Prove that all convex bounded from above functions on $\mathbb{R}$ are constant. Reprove the Liouville theorem for subharmonic functions in $\mathbb{C}$.

Exercise 15. Let u be a subharmonic function in $\mathbb{D}$ and set $B_u(r) := \frac{1}{\pi r^2}\int_{\mathbb{D}(0,r)} u(z)dV(z)$, $C_u(r) := \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})dt$. Prove that

$$B_u(r) \geq C_u(e^{-1/2}r), \ \forall 0 \leq r < 1.$$

Exercise 16. Let $u \geq 0$ be a subharmonic function such that $\log u$ is also subharmonic. Prove that the functions

$$t \mapsto \log \max_{|z|=e^t} u(z) \ ; \ t \mapsto \log B_u(e^t) \ ; \ t \mapsto \log C_u(e^t)$$

are convex.

Exercise 17. Let u be a subharmonic function and v be a upper semicontinuous function in a domain Ω such that $u \leq v$ almost everywhere in Ω. Prove that $u \leq v$ everywhere in Ω.

A subset $E \subset \mathbb{C}$ is called (globally) polar if there exists a subharmonic function $u \in \mathrm{SH}(\mathbb{C})$ which is not identically $-\infty$ such that $E \subset \{u = -\infty\}$. We know (or we read the proof in e.g. Ransford's book [Ran95]) that globally polar is the same as locally polar.

Exercise 18. Prove that an open line segment in $\mathbb{C}$ is not a polar set.

References

[Mer15] Joël Merker, *Fonctions sous-harmoniques*, François de Marçay, 2015, Notes de cours M2.

[Ran95] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR 1334766