# TD1 : Holomorphic functions of one complex variable

The course is based on the notes by Joël Merker [Mer15]. Notations and definitions are also taken from [Mer15].

We let $\mathbb{D}(0,r)$ denote the open disk of radius $r>0$. When $r=1$ we simply denote by $\mathbb{D}$ the unit open disk. Its boundary will be denoted by $C(0,r)$.

**Exercise 1.** Check that the function $z\mapsto 1/z$ can not be approximated uniformly by polynomials on the circle $\partial\mathbb{D}$.

**Exercise 2.** Let $C,D$ be non-empty convex subsets of $\mathbb{R}^2$ with empty intersection $C\cap D=\emptyset$. Construct an affine function $f:\mathbb{R}^2\to\mathbb{R}$ such that $f(c)\le 0\le f(d)$, for all $(c,d)\in C\times D$. If $C$ is closed and $D$ is a point show further that we can choose $f$ such that the inequalities are strict.

**Exercise 3.** Let $K$ be a convex compact subset of $\mathbb{C}\simeq\mathbb{R}^2$. Prove that $\hat{K}_{\mathcal{O}(\Omega)}=K$. Prove that if $E\subset\mathbb{C}$ is bounded then $\hat{E}_{\mathcal{O}(\Omega)}$ is also bounded for any domain $\Omega\supset E$.

**Exercise 4.** Let $\Omega\subset\mathbb{C}$ be a non-empty open connected set and $f$ be a holomorphic function in $\Omega$, which is nowhere vanishing in $\Omega$. Prove that the following conditions are equivalent:

1. there exists $g\in\mathcal{O}(\Omega)$ such that $f(z)=e^{g(z)}$ for all $z\in\Omega$,
2. for any closed curve $\gamma$ of class $\mathcal{C}^1$ in $\Omega$ we have

$$\int_\gamma \frac{f'(z)dz}{f(z)}=0.$$

**Exercise 5.** Let $\gamma:[0,1]\to\mathbb{C}$ be a function of class $\mathcal{C}^1$ such that $\gamma'(t)\neq 0$ for all $t\in[0,1]$ and $\gamma$ is injective. Prove that $\Omega:=\mathbb{C}\setminus\gamma([0,1])$ is connected.

**Exercise 6.** Let $\gamma:[0,1]\to\mathbb{C}$ be a continuous function such that $\Omega:=\mathbb{C}\setminus\gamma([0,1])$ is connected. Prove that there exists a holomorphic function $f$ in $\Omega$ such that

$$e^{f(z)}=\frac{z-\gamma(0)}{z-\gamma(1)},\ \forall z\in\Omega.$$

**Exercise 7.** Let $\Omega$ be a domain in $\mathbb{C}$ and $(z_j)$ be a discrete sequence of points in $\Omega$. For each $j$ let $f_j$ be a holomorphic functions in an open neighborhoods of $z_j$ and let $n_j\in\mathbb{N}$. Prove that there exists $f\in\mathcal{O}(\Omega)$ such that $f(z)-f_j(z)=O(|z-z_j|^{n_j+1}$ as $z\to z_j$, for each $j\in\mathbb{N}$.

**Exercise 8.** Let $\Omega$ be a domain in $\mathbb{C}$ and let $(z_j)$ be a discrete sequence of points in $\Omega$. Prove that there exists a sequence of sets $(K_j)$ such that

1. each $K_j$ is compact in $\Omega$ : $K_j\Subset\Omega,\ \forall j\in\mathbb{N}$;
2. $K_j\Subset\mathrm{Int}(K_{j+1}),\forall j\in\mathbb{N}$;
3. $\hat{K}_j=K_j$ (the holomorphic envelope of $K_j$ with respect to $\mathcal{O}(\Omega)$ is itself);
4. $\cup_{j\in\mathbb{N}}K_j=\Omega$;
5. $z_k\notin K_j$ if $k\ge j\ge 1$.

**Exercise 9.** Let $\Omega$ be a domain in $\mathbb{C}$ and $(\Omega_j)_{j\in\mathbb{N}}$ be a sequence of open subsets of $\Omega$ such that $\cup_{j\in\mathbb{N}}\Omega_j=\Omega$.

**(a)** Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of meromorphic functions $f_j\in\mathcal{O}(\Omega_j)$, $j\in\mathbb{N}$ such that

$$\frac{f_j}{f_k}\in\mathcal{O}(\Omega_j\cap\Omega_k),\ \forall j,k\in\mathbb{N}.$$

Prove that there exists $f\in\mathcal{O}(\Omega)$ such that $f/f_j\in\mathcal{O}(\Omega_j)$ for all $j\in\mathbb{N}$.

**(b)** Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of meromorphic functions $f_j\in\mathcal{O}(\Omega_j)$, $j\in\mathbb{N}$ such that

$$f_j-f_k\in\mathcal{O}(\Omega_j\cap\Omega_k),\ \forall j,k\in\mathbb{N}.$$

Prove that there exists $f\in\mathcal{O}(\Omega)$ such that $f-f_j\in\mathcal{O}(\Omega_j)$ for all $j\in\mathbb{N}$.

Any domain $\Omega \subset \mathbb{C}$ is a domain of holomorphy (see Section 7 in [Mer15]). This means there exists a holomorphic function defined in $\Omega$ which can not be extended holomorphically across the boundary. The aim of the following exercise is to construct such a function when $\Omega = \mathbb{D}$ is the open unit disk.

**Exercise 10.** Prove that the series

$$\sum_{n=1}^{+\infty} d(n) z^n,$$

where $d(n)$ is the number of divisors of $n$, has radius of convergence 1 and hence it defines a holomorphic function $f$ in the open unit disk $\mathbb{D}$. The goal of this exercise is to prove that $f$ can not be extended holomorphically in any open neighborhood of any boundary point $\zeta \in \partial\mathbb{D}$. In other words, $\mathbb{D}$ is the domain of definition of $f$ (or $\mathbb{D}$ is a domain of holomorphy).

1. Prove that $f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{1-z^2}$, for all $z \in \mathbb{D}$.
2. Prove that for $r \in ]0,1[$, we have
$$f(r) \geq \frac{\log(1-r)}{\log(1/r)}.$$
3. Prove that for any $p \in \mathbb{Z}, q \in \mathbb{N}^*$ there exists a constant $a = a(p,q)$ (which depends only on $p$ and $q$) such that
$$f(re^{ip/q}) \geq a \frac{\log(1-r)}{\log(1/r)},$$
for all $r < 1$ sufficiently near 1.
4. Conclude that for any $e^{i\theta} \in \partial\mathbb{D}$ and any open neighborhood $V$ in $\mathbb{C}$ of $e^{i\theta}$, $f$ can not be extended holomorphically on $\mathbb{D} \cup V$.

# References

[Mer15] Joël Merker, *Fonctions holomorphes d'une variable complexe*, François de Marçay, 2015, Notes de cours M2.