TD2 : Harmonic functions

We let $\Delta(0,r)$ denote the open disk of radius $r>0$. When $r=1$ we simply denote by $\mathbb{D}$ the unit open disk. Its boundary will be denoted by $C(0,r)$. The support for these exercises can be found in the first chapter of Ransford's book [Ran95], or in the notes of Merker [Mer15].

Exercise 1. Prove that the function $h(x+iy):=e^x(x\cos y-y\sin y)$ is harmonic in $\mathbb{C}$. Find a holomorphic function f in $\mathbb{C}$ such that $h=\mathrm{Re}(f)$.

Exercise 2. Let $A(r_1,r_2)$, $0<r_1<r_2$ be the annulus $\{z\in\mathbb{C} \ : \ r_1<|z|<r_2\}$. Prove that there exists a unique $b\in\mathbb{R}$ and a unique sequence of complex numbers $(a_n)_{n\in\mathbb{Z}}$ such that

- $a_0\in\mathbb{R}$;
- $\sum_{n=-\infty}^{+\infty} a_n z^n$ is uniformly convergent on each compact subset of $A(r_1,r_2)$;
- $h(z)=\mathrm{Re}\sum_{n=-\infty}^{+\infty} a_n z^n + b\log|z|$.

Exercise 3. Let h,k be two harmonic functions in a domain $\Omega\subset\mathbb{C}$. Prove that hk is harmonic in Ω if and only if there exists a constant $c\in\mathbb{R}$ such that $h+ick$ is holomorphic in Ω.

Exercise 4. Using the fact that harmonic functions are real analytic give an alternative proof of the uniqueness theorem for harmonic functions.

Exercise 5. Let $\mathbb{D}$ be the unit disk and let $\phi:\partial\Delta\to\mathbb{C}$ be the function $\phi(z)=\bar{z}$. Prove that there is no holomorphic function f in $\mathbb{D}$ which is continuous on $\bar{\mathbb{D}}$ and such that $f=\phi$ on ∂D. Compare with the Dirichlet problem for harmonic functions.

Exercise 6. **(a)** Prove the following formula for the Poisson kernel of $\mathbb{D}$:

$$P_{\mathbb{D}}(re^{it},e^{i\theta})=\sum_{n=-\infty}^{+\infty} r^{|n|}e^{in(t-\theta)},\ \forall r\in[0,1),\theta,t\in[0,2\pi].$$

(b) Prove that if $\phi:\partial\mathbb{D}\to\mathbb{R}$ is a Lebesgue-integrable function then

$$P_{\mathbb{D}}(re^{it})=\sum_{n=-\infty}^{+\infty} a_n r^{|n|}e^{int},$$

where the coefficients a_n are given by the Fourier transform of ϕ.
(c) Suppose $\phi\in\mathcal{C}^0(\partial\mathbb{D},\mathbb{R})$. Prove that $P_{\mathbb{D}}(re^{it})$ converge uniformly to ϕ on $\partial\mathbb{D}$ as $r\to r^-$.
(d) Prove that any continuous function on $\partial\mathbb{D}$ can be approximated uniformly on $\partial\mathbb{D}$ by trigonometric polynomials, i.e. for each $\phi\in\mathcal{C}^0(\partial\mathbb{D},\mathbb{R})$ and $\varepsilon>0$ there exists complex numbers $(c_n)_{n=-N}^{N}$ such that $c_{-n}=\overline{c_n}$ and

$$\sup_{t\in[0,2\pi]}\left|\phi(e^{it})-\sum_{k=-N}^{N} c_n e^{int}\right|<\varepsilon.$$

Exercise 7. Let f be a holomorphic function in an open neighborhood of $\overline{\Delta(0,r)}$ and $h:=\mathrm{Re}f$. Prove that, for all $z\in\Delta(0,r)$,

$$f(z)-f(0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{2z}{re^{it}-z}h(re^{it})dt.$$

Exercise 8. Prove that if h is a positive harmonic function in $\Delta(0,r)$ then

$$|\nabla h(z)|:=\sqrt{h_x^2+h_y^2}\leq\frac{2rh(z)}{r^2-|z|^2},\ \forall z\in\Delta(0,r).$$

Exercise 9. Let (h_n) be a sequence of positive harmonic functions in the unit disk $\mathbb{D}$ and let U be a non-empty open subset of $\mathbb{D}$. If (h_n) converges pointwise in U to some harmonic function in U then it converges uniformly on any compact subset of $\mathbb{D}$.

Hint: Harnack's upper inequality.

Exercise 10. Use Harnack's inequality to reprove Liouville's theorem in $\mathbb{C}$.

Exercise 11. Let f be a holomorphic function in $\Delta(0,r)$, $r>0$ such that $0<|f|<1$. Prove that

$$|f(z)| \le |f(0)|^{\frac{r-|z|}{r+|z|}}, \ \forall z \in \Delta(0,r).$$

Exercise 12. Prove that the Harnack's distance (Joël calls it distorsion de Harnack) between two points $z_1, z_2 \in \mathbb{D}$ is given by

$$\tau_{\mathbb{D}}(z_1,z_2) = \frac{1+\left|\frac{z_1-z_2}{1-z_1\bar{z}_2}\right|}{1-\left|\frac{z_1-z_2}{1-z_1\bar{z}_2}\right|}.$$

Exercise 13. Let h be a harmonic function in an open neighborhood of the closed disk $\overline{\Delta(0,R)}$. For $0 \le r \le R$ set

$$M_h(r) := \sup_{|z|=r} h(z).$$

(a) Prove that

$$M_h(r) \le \frac{2r}{R+r} M_h(R) + \frac{R-r}{R+r} h(0), \ \forall 0 \le r \le R.$$

(b) Deduce the following generalization of Liouville's theorem for harmonic functions: if h is harmonic in $\mathbb{C}$ such that

$$\liminf_{R\to+\infty} \frac{M_h(R)}{R} \le 0,$$

then h is constant.

Exercise 14. Let f be a holomorphic function in an open neighborhood of the closed disk $\overline{\Delta(0,R)}$. For $0 \le r \le R$ set

$$M_f(r) := \sup_{|z|=r} |f(z)|, \quad \text{and} \quad A_f(r) := \sup_{|z|=r} \mathrm{Re} f(z).$$

Prove the Borel-Carathéodory inequality :

$$M_f(r) \le \frac{2r}{R-r} A_f(r) + \frac{R+r}{R-r} |f(0)|, \quad \forall 0 \le r < R.$$

Exercise 15. Let h be a harmonic function on a neighborhood of $\bar{\Delta}(0,2R)$ with $h(0)=0$. Prove that there exists $w \in \Delta(0,2R)$ and $r>0$ such that $\Delta(w,r) \subset \Delta(0,2R)$, $h(w)=0$ and

$$M_h(w,r) \ge 3^{-11} M_h(0,R) \ ; \ M_h(w,r/2) \ge 3^{-11} M_h(w,r).$$

Here $M_h(w,r) := \sup_{\partial\Delta(w,r)} h$. Hints are given below:

Proof. For $z \in \Delta(0,2R)$ denote $\delta(z) := \mathrm{dist}(z, \partial\Delta(0,2R))$. Set

$$Z := \{z \in \Delta(0,2R) \ | \ h(z)=0\} \ ; \ U := \bigcup_{z\in Z} \Delta(z, \delta(z)/4) \ ; \ \gamma := \sup_U h = \sup_{z\in Z} M_h(z, \delta(z)/4).$$

Choose $w \in Z$ such that $M_h(w, \delta(w)/4) \ge \gamma/3$ and set $r = \delta(w)/2$. We shall prove that $\Delta(w,r)$ satisfies the required conditions. It suffices to show that

$$M_h(0,R) \le 3^{10}\gamma \tag{1}$$

and

$$M_h(w,r) \le 3^{10}\gamma \tag{2}$$

To prove (1) fix $z \in \Delta(0,R) \setminus \bar{U}$ and let $z' \in \bar{U}$ be a point on the segment $[0,z]$ such that $[z,z') \cap \bar{U} = \emptyset$.

- Prove that $h>0$ on $\Delta(\zeta, R/5)$ for any $\zeta \in [z,z')$.
- Apply Harnack's inequality for h on $\Delta(\zeta, R/5)$ to deduce that

$$\sup_{\Delta(\zeta,R/10)} h \le 3^2 \inf_{\Delta(\zeta,R/10)} h.$$

- Conclude the proof of (1).
- Proceed similarly to prove (2).

□

Exercise 16 (Picard's little theorem)**.** Prove that if $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function which avoids two points, i.e. there exists $a \neq b \in \mathbb{C}$ such that $f(z) \neq a, f(z) \neq b$ for all $z \in \mathbb{C}$, then f is constant.

Proof. Suppose by contradiction that f is non-constant and there exist $a, b \in \mathbb{C}$ such that $a \neq b$ and $f(z) \neq a, f(z) \neq b$ for all $z \in \mathbb{C}$. Set $h(z) := \log|f(z) - a|, k(z) := \log|f(z) - b|$.

1. Prove that h, k are harmonic on $\mathbb{C}$ and
$$|h^+ - k^+| \leq |a - b| \ ; \ \max(h, k) \geq \log(|a - b|/2).$$
2. Prove that there exists $z_0 \in \mathbb{C}$ such that $h(z_0) = 0$. Without loss of generality we can assume that $z_0 = 0$.
3. Apply Exercise 15 on each disk $\Delta(0, 2^{j+1})$ to produce new disks $\Delta(w_j, r_j)$ such that $h(w_j) = 0$ and
$$M_h(w_j, r_j) \geq 3^{-11} M_h(0, 2^j) \ ; \ M_h(w_j, r_j/2) \geq 3^{-11} M_h(w_j, r_j).$$
4. Prove that $M_j := M_h(w_j, r_j) \to +\infty$ as $j \to +\infty$.
5. Define two sequences of harmonic functions (h_j) and (k_j) on $\Delta(0, 1)$ by
$$h_j(z) := \frac{h(w_j + r_j z)}{M_j} \text{ and } k_j(z) := \frac{k(w_j + r_j z)}{M_j}.$$
Prove that h_j and k_j have the following properties :
   (a) $h_j(0) = 0$
   (b) $M_{h_j}(0, 1/2) \geq 3^{-11}$
   (c) $|h_j^+ - k_j^+| \leq \frac{|a-b|}{M_j}$
   (d) $\max(h_j, k_j) \geq \frac{\log(|a-b|/2)}{M_j}$.
6. Prove that a subsequence of (h_j) converges locally uniformly to a harmonic function $\tilde{h}$ on $\Delta(0, 1)$. The same for (k_j).
7. Find a contradiction and conclude.

□

References

[Mer15] Joël Merker, *Fonctions harmoniques*, François de Marçay, 2015, Notes de cours M2.

[Ran95] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR 1334766