TD5 : SHEAVES.

The main source is the book of Noguchi [Nog16] and the notes of Merker [Mer15].

Exercise 1. Prove that, for $n = 1$, $\mathscr{O}_n = \mathscr{O}_{n,0}$ is principal, noetherian, and factorial. Give an example showing that in the case $n > 1$, $\mathscr{O}_n$ is no longer principal.

Exercise 2. Let $P \in \mathscr{O}_{n-1}^{\mathrm{Wei}}(z_n)$ be a Weierstrass polynomial of degree $p > 1$. Prove that if $\mathrm{Res}(P, P_{z_n}) \equiv 0$ in $\mathscr{O}(\mathbb{C}^{n-1})_0$ then P is reducible in $\mathscr{O}(\mathbb{C}^n)_0$.

Exercise 3. Let $\Omega \subset \mathbb{C}^n$ be an open subset. Set, for $U \subset \Omega$ open,

$$\mathscr{O}_\Omega(U) := \{f : U \to \mathbb{C} \text{ holomorphic}\}.$$

1. Prove that $\mathscr{O}_\Omega$ is a sheaf of unitary commutative rings.
2. What is the germ $\mathscr{O}_{\Omega,z_0}$?
3. Prove that $\mathscr{O}_{\Omega,z_0}$ is a local ring, i.e. there is a unique maximal ideal.
4. Is the topology of the sheaf $\mathscr{O}_\Omega$ separable?

Exercise 4. For $U \subset \mathbb{C}$ open let $\mathcal{G}(U)$ the vector space of bounded holomorphic functions in U. Prove that $\mathcal{G}(U)$ (with the usual restriction maps) is a presheaf but not a sheaf.

Exercise 5. Let X be a Hausdorff topological space and $x_0 \in X$. Let E be a subset of X and fix $e \in E$. If $U \subset X$ is open we set $\mathscr{E}(U) := E$ if $x_0 \in U$ and $\mathscr{E}(U) := \{e\}$ if $x_0 \notin U$. If $V \subset U \subset X$ are open sets then $\rho_{V,U}$ is a map $\mathscr{E}(U) \to \mathscr{E}(V)$ defined as $\rho_{V,U} := \mathrm{Id}_E$ if $x_0 \in V$ and $\rho_{V,U} \equiv e$ if $x_0 \notin V$.

1. Prove that $\mathscr{E}$ is a sheaf.
2. What is $\mathscr{E}_x$?

Exercise 6. For open $U \subset \mathbb{C}$ let $\mathscr{O}_\mathbb{C}(U)$ be the ring of holomorphic functions on U and $\mathscr{O}_\mathbb{C}^*(U)$ the subset of holomorphic functions which are nowhere vanishing in U. Set

$$\mathscr{F}(U) := \mathscr{O}_\mathbb{C}^*(U)/\exp(\mathscr{O}_\mathbb{C}(U)).$$

Prove that $\mathcal{F}$ (with the usual restriction maps) is a presheaf but not a sheaf.

Exercise 7. Let $\Omega \subset \mathbb{C}^n$ be open and E be any subset of Ω. For open subset $U \subset \Omega$ we set

$$\mathscr{I}_E(U) := \{f \in \mathscr{O}(U) \mid f(z) = 0, \ \forall z \in U \cap E\}.$$

1. Prove that $\mathscr{I}_E$, with the usual restriction maps, is a sheaf.
2. Prove that each germ $\mathscr{I}_{E,z_0}$, $z_0 \in \Omega$, is an ideal of $\mathscr{O}_\Omega$.
3. Assume now that $E = \{a\}$ is a singleton in Ω. What is $\mathscr{I}_{E,z}$, $z \in \Omega$?

Exercise 8. Prove that $\mathcal{E}_\mathbb{R}^\infty$, the presheaf of smooth functions on $\mathbb{R}$ with usual restriction maps, is a sheaf. Is the topology of $\mathcal{E}_\mathbb{R}^\infty$ separable?

Exercise 9. Let $\mathscr{C}(\mathbb{S}^1)$ be the sheaf of continuous functions on the circle $\mathbb{S}^1$ and $\mathscr{Z}(\mathbb{S}^1)$ be the sheaf of continuous functions with integer values. Prove that the quotient $\mathscr{C}(\mathbb{S}^1)/\mathscr{Z}(\mathbb{S}^1)$ is not a sheaf.

Exercise 10. Let X be a topological space and E is an arbitrary set having at least two elements $0 \neq e$. Define $\mathscr{E}(\emptyset) = \{0\}$ and $\mathscr{E}(U) = E$ if U is open nonempty together with the following restriction maps $\rho_{U,V} = \mathrm{Id}_E$ if $\emptyset \neq U \subset V$ and $\rho_{U,V} = 0$ if $U = \emptyset$. Prove that, if $X = V_1 \cup V_2$ is the union of two open disjoint subsets then $\mathscr{E}$ is not a sheaf.

Exercise 11. Let X be a topological space and E be an arbitrary set having at least two elements. For an open set $U \subset X$ we define $\mathscr{E}(U)$ to be the set of all locally constant functions on U with values in E (i.e. constant on each connected component of U). Prove that with the usual restriction maps $\mathscr{E}$ is a sheaf, called the sheaf of locally constant functions on X with values in E.

Exercise 12. Let $\mathscr{A}$ be a sheaf of unitary commutative rings on X and $\mathscr{Q} \subset \mathscr{P}$ be sheaves of $\mathscr{A}_X$-modules. Introduce the presheaf $\mathscr{F}(U) = \mathscr{P}(U)/\mathscr{Q}(U)$.

1. Prove that the presheaf $\mathscr{F}$ is in general not a sheaf.
2. Let $\mathscr{P}/\mathscr{Q}$ be the sheafification of $\mathscr{F}$. Prove that

$$(\mathscr{P}/\mathscr{Q})_x \simeq \mathscr{P}_x/\mathscr{Q}_x.$$

3. Assume that $\mathscr{F}$ is not a sheaf. Prove that there exists an open subset $U \subset X$ such that

$$(\mathscr{P}/\mathscr{Q})(U) \not\simeq \mathscr{P}(U)/\mathscr{Q}(U).$$

Exercise 13. Let $X = \mathbb{R}$ with its standard topology and $\mathbb{Z}_X$ is the constant sheaf (the sheafification of the constant presheaf in exercise 10). Introduce the following presheaf

$$\mathscr{F}(U) := \{f : U \to \mathbb{R} \text{ locally constant } f(0) = f(1) = 0 \text{ if } 0 \in U\}.$$

Prove that $\mathscr{F}$ is a sheaf. Prove that

$$\Gamma(X, \mathbb{Z}_X)/\Gamma(\mathbb{R}, \mathscr{F}) \simeq \mathbb{Z}\ ;\ \Gamma(\mathbb{R}, \mathbb{Z}_X/\mathscr{F}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Exercise 14. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X.

- Prove that ϕ is injective iff ϕ_x is injective for all x.
- Prove that ϕ is an isomorphism iff ϕ_x is an isomorphism for every x.
- Let $X = \mathbb{C}^*$ and $\mathscr{O}_X$ be the sheaf of holomorphic functions on X and $\mathscr{O}_X^*$ be the sheaf of hiolomorphic functions on X which are nowhere vanishing. Prove that the exponential map is surjective on the germs but it is not surjective as morphism between sheaves.

Exercise 15. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{F}$ be a presheaf on X. For each open subset $V \subset Y$ we introduce $f_*(\mathscr{F})(V) := \mathscr{F}(f^{-1}(V))$. The restriction maps are given by

$$\rho_{U,V}^{f_*(\mathscr{F})} = \rho_{f^{-1}(U), f^{-1}(V)}^{\mathscr{F}},\ U \subset V.$$

Prove that $f_*(\mathscr{F})$ is a presheaf on X and $f_*(\mathscr{F})$ is a sheaf if $\mathscr{F}$ is so.

Exercise 16. Let $f : X \to Y$ be a continuous map between topological spaces and $\mathscr{G}$ be a presheaf on Y. For each open subset $V \subset X$ we introduce

$$f^{-1}(\mathscr{G})(V) := \varinjlim_{f(V) \subset W \text{ open}} \mathscr{G}(W).$$

Prove that $f^{-1}(\mathscr{G})$ is a presheaf on X.

1. Prove that $f^{-1}(\mathscr{G})$ is a presheaf on X.
2. Give an example showing that even when $\mathscr{G}$ is a sheaf, $f^{-1}(\mathscr{G})$ may not be a sheaf.
3. How to define a natural sheaf structure on $f^{-1}(\mathscr{G})$?

References

[Mer15] Joël Merker, *Faisceaux*, François de Marçay, 2015, Notes de cours M2.

[Nog16] Junjiro Noguchi, *Analytic function theory of several variables*, Springer, Singapore, 2016, Elements of Oka's coherence. MR 3526579