TD4 : Riemann surfaces 1

The main source is the book of Forster [For91].

Exercise 1. For $n \geq 1$ let ∞ be a "symbol" not belonging to $\mathbb{R}^n$. Introduce the following topology on the set $X := \mathbb{R}^n \cup \{\infty\}$. A set $U \subset X$ is open if either $\infty \notin U$ and U is open in $\mathbb{R}^n$ or $\infty \in U$ and $X \setminus U$ is compact in $\mathbb{R}^n$. Show that X is a compact Hausdorff topological space.

Exercise 2. Consider the unit n-sphere $\mathbb{S}^n := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^n \mid \sum_{j=1}^{n+1} x_j^2 = 1\}$ and the stereographic projection $\sigma : \mathbb{S}^n \to \mathbb{R}^n \cup \{\infty\}$ given by

$$\sigma(x_1, ..., x_{n+1}) = \begin{cases} \frac{(x_1,...,x_n)}{1-x_{n-1}}, \text{ if } x_{n+1} \neq 1, \\ \infty \text{ if } x_{n+1} = 1. \end{cases}$$

Show that σ is a homeomorphism of $\mathbb{S}^n$ onto $\mathbb{R}^n \cup \{\infty\}$.

Exercise 3. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Show that the linear fractional transformation

$$f(z) := \frac{az + b}{cz + d}$$

defines a meromorphic function $\mathbb{P}^1 \to \mathbb{C}$. Show that f is biholomorphic $\mathbb{P}^1 \to \mathbb{P}^1$. Prove that any biholomorphism of $\mathbb{P}^1$ is of this form.

Exercise 4. Identify $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with the unit sphere in $\mathbb{R}^3$ by the stereographic projection $\sigma : \mathbb{S}^2 \to \mathbb{P}^1$. Let $\mathrm{SO}(3, \mathbb{R})$ be the group of orthonormal (3×3)-matrices having determinant 1. For every $A \in \mathrm{SO}(3, \mathbb{R})$ show that the map $\sigma \circ A \circ \sigma^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic.

Exercise 5.

1. Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega'_1 + \mathbb{Z}\omega'_2$ be two lattices in $\mathbb{C}$. Prove that $\Gamma = \Gamma'$ if and only if there exists a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ with $|\det(A)| = 1$ such that

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

2. Let Γ and Γ' be two lattices. Suppose $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$. Prove that the map $z \mapsto \alpha z$ on $\mathbb{C}$ induces a holomorphic map $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$. Prove that f is biholomorphic iff $\alpha\Gamma = \Gamma'$.

3. Show that any torus is isomorphic to a torus of the form $\mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$.

4. Prove that two tori defined by two lattices $\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$ and $\Gamma_{\tau'}$ (with $\mathrm{Im}(\tau) > 0$, $\mathrm{Im}(\tau') > 0$) are biholomorphic iff

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

Exercise 6. Let $\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

1. Prove that the series of meromorphic functions on $\mathbb{C}$

$$\rho := \rho_\Gamma(z) := \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left[\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]$$

defines a doubly periodic meromorphic function with respect to Γ. Prove that the poles of ρ_Γ are the points of Γ.

2. Let f be a doubly periodic (with respect to Γ) meromorphic function on $\mathbb{C}$ which has its poles at the points of Γ and has the following Laurent expansion around the origin

$$f(z) = \frac{1}{z^2} + \sum_{k=1}^{+\infty} a_k z^k.$$

Prove that $f = \rho_\Gamma$.

3. Prove that ρ has the following Laurent expansion near zero :

$$\rho(z) = \frac{1}{z^2} + \sum_{n=1}^{+\infty} (2n+1) G_{2(n+1)}(\Gamma) z^{2n},$$

where $G_n(\Gamma)$ is the Einstein series

$$G_n(\Gamma) := \sum_{\omega \in \Gamma \setminus \{0\}} \omega^{-n}.$$

4. Prove the following differential equation

$$(\rho'(z))^2 = 4(\rho(z))^3 - 60 G_4(\Gamma)\rho(z) - 140 G_6(\Gamma).$$

Exercise 7. Let $\{p_1, ..., p_n\}$ be points on a compact Riemann surface X, and $f : X' = X \setminus \{p_1, ..., p_n\} \to \mathbb{C}$ be a holomorphic function. Prove that $f(X')$ is dense in $\mathbb{C}$.

Exercise 8. Let X be a compact Riemann surface and X' the complement of a finite set of points $\{p_1, ..., p_n\}$ in X. Prove that every automorphism $f : X' \to X'$ extends to an automorphism of X.

Exercise 9. Determine the automorphisms of $\mathbb{C}$, of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and of $\mathbb{C} \setminus \{0, 1\}$. Show that $\mathrm{Aut}(\mathbb{C} \setminus \{0, 1, z\})$ (for any $z \notin \{0, 1\}$) contains at least one non-trivial element.

Exercise 10. Let Γ be a lattice in $\mathbb{C}$. Prove that any holomorphic function $f : \mathbb{P}^1 \to \mathbb{C}/\Gamma$ is constant.

Exercise 11. Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. A point $y \in Y$ is called a branch point or ramification point of p, if there is no neighborhood V of y such that $p_{|V}$ is injective. The map p is called a unbranched holomorphic map if it has no branch points.

1. Determine the ramification points of the map $f : \mathbb{C} \to \mathbb{P}^1$ given by $f(z) = z + 1/z$.
2. Let Γ, Γ' be lattices in $\mathbb{C}$ and $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ be a holomorphic function such that $f([0]) = [0]$. Prove that there exists $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$. Then deduce that f has no ramification points.
3. Consider ρ as a holomorphic map $\mathbb{C}/\Gamma \to \mathbb{P}^1$. Show that it is two-sheeted, and use the fact that its derivative is an odd function to show that it has exactly four branch points.

References

[For91] Otto Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York, 1991, Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation. MR 1185074