TD7 : Cartan's merging lemma.

The main source is the book of Noguchi [Nog16]. The goal of these exercises is to prove Cartan's merging lemma (Lemma 4.2.19 in [Nog16]). The simplified proof given on the blackboard is due to François de Marçay.

We first introduce a few notations. Let F be a closed cube in $\mathbb{C}^{n-1}$, $n \geq 1$ and $E_n^{'}, E_n^{''}$ be two closed rectangles in $\mathbb{C}$ sharing an edge ℓ, i.e. $\ell = E_n^{'} \cap E_n^{''}$. We set

$$E' := F \times E_n' \text{ and } E'' := F \times E_n''.$$

We recall elementary results on matrix valued holomorphic functions. For a complex matrix A of rank $p \geq 1$ we define its norm as

$$\|A\| := \sup_{\xi \in \mathbb{C}^p, |\xi|=1} |A\xi|.$$

If $z \mapsto A(z)$ is a (p,p)-matrix valued function on E we define its norm on E as

$$\|A\|_E := \sup\{\|A(z)\| \ \mid \ z \in E\}.$$

Exercise 1. Let $A = (a_{jk}), B = (b_{jk})$ be two matrices of rank (p,p). Then

1. $$\frac{1}{p}\|A\| \leq \max_{j,k} |a_{jk}| \leq \|A\|.$$
2. $$\|A + B\| \leq \|A\| + \|B\|.$$
3. $$\|AB\| \leq \|A\|\|B\|.$$
4. There exists a positive constant $\varepsilon = \varepsilon(p) > 0$ such that whenever $\|A\|_E < \varepsilon$ then $(1 - A(z))$ is invertible and the series
$$(\mathbf{1}_p - A)^{-1}(z) = \mathbf{1}_p + A(z) + A(z)^2 + \cdots$$
converges uniformly on E. Moreover,
$$\|(\mathbf{1}_p - A(z))^{-1}\| \leq \frac{1}{1-\varepsilon} \text{ and } \|(\mathbf{1}_p - A(z))^{-1} - \mathbf{1}_p\| \leq \frac{\|A(z)\|}{1-\varepsilon}.$$
5. Suppose that (ε_k) is a sequence of real numbers in $(0,1)$ such that $\sum \varepsilon_k < +\infty$ and (A_k) be a sequence of (p,p)-matrix valued holomorphic functions on E such that $\|A_k\|_E \leq \varepsilon_k$. Then
$$\lim_{k\to+\infty} (\mathbf{1}_p - A_0(z)) \cdots (\mathbf{1}_p - A_k(z)) \text{ and } \lim_{k\to+\infty} (\mathbf{1}_p - A_k(z)) \cdots (\mathbf{1}_p - A_0(z))$$
converge uniformly on E.

Exercise 2. For (p,p)-matrices P, Q such that $(\mathbf{1}_p - P)$ and $(\mathbf{1}_p - Q)$ are invertible we set

$$M(P,Q) = (\mathbf{1}_p - P)^{-1}(1 - P - Q)(\mathbf{1}_p - Q)^{-1} = \mathbf{1}_p - N(P,Q).$$

Prove that if $\max(\|P\|, \|Q\|) \leq 1/2$ then

$$\|N(P,Q)\| \leq 2^2 \left(\max(\|P\|, \|Q\|)\right)^2.$$

Exercise 3 (Cartan's matrix decomposition)**.** Fix U an open neighborhood of $F \times \ell$. There exists positive constants η, C and closed cube neighborhoods U' of E' and U'' of E'' such that $U' \cap U'' \subset U$ and

for any (p,p)-matrix valued holomorphic function A such that $\|\mathbf{1}_p - A\|_U \leq \eta$, there exist (p,p)-matrix valued functions A' on U' and A'' on U'' such that

$$A(z) = A'(z)A''(z), \ \forall z \in U' \cap U''$$

and

$$\max\left(\|\mathbf{1}_p - A'\|_{U'}, \|\mathbf{1}_p - A''\|_{U''}\right) \leq C\|\mathbf{1}_p - A\|_U.$$

Exercise 4 (Cartan's merging lemma)**.** Let $E' \subset U', E'' \subset U''$ as above (open subsets of Ω) and $\mathscr{F}$ be a coherent sheaf over Ω. Assume that finitely many sections $\sigma'_j \in \Gamma(U', \mathscr{F})$, $1 \le j \le p'$, generate $\mathscr{F}$ over U', and similarly $\sigma''_k \in \Gamma(U'', \mathscr{F})$ generate $\mathscr{F}$ over U''. Furthermore, assume the existence of a_{jk} and $b_{kj} \in \mathscr{O}(U' \cap U'')$, $1 \le j \le p'$, $1 \le k \le p''$ such that

$$\sigma'_j = \sum_{k=1}^{p''} a_{jk}\sigma''_k, \ \sigma''_j = \sum_{j=1}^{p'} b_{kj}\sigma^{'}_j.$$

Then there are neighborhood $E' \cup E'' \subset W \subset U' \cup U''$ and finitely many sections σ_l on W, $1 \le l \le p = p' + p''$ which generate $\mathscr{F}$ over W.

Exercise 5 (A generalized version of Runge's theorem)**.** Let $\Omega = \prod_j \Omega_j$ be a cylinder domain such that all $\Omega_j \subset \mathbb{C}$ are simply connected. Then any $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on each compact subset by polynomials.

Hint: Use Riemann mapping theorem and Runge's theorem in dimension 1.

References

[Nog16] Junjiro Noguchi, *Analytic function theory of several variables*, Springer, Singapore, 2016, Elements of Oka's coherence. MR 3526579