TD8 : Hartogs' theorem for subharmonic functions

The main source is the book of Hörmander [Hör73] and the notes [Mer15]. The goal is to prove Hartogs' theorem saying that separately holomorphic functions are holomorphic.

Exercise 1 (Schwarz's lemma)**.** Let $\mathbb{D}_r$ be the open disk of radius r centered at 0. If $f : \mathbb{D}_r \to \mathbb{D}_R$ is holomorphic and $f(z_0) = 0$ for some $z_0 \in \mathbb{D}_r$ then

$$|f(z)| \leq Rr\frac{|z - z_0|}{|r^2 - \bar{z}_0 z|},\ \forall z \in \mathbb{D}_r.$$

Exercise 2 (Hartogs' theorem for subharmonic functions)**.** Let (u_j) be a sequence of subharmonic functions in $\Omega \subset \mathbb{C}$ which are uniformly bounded from above on every compact subset of Ω and assume that $\limsup_j u_j(z) \leq C$, for every $z \in \Omega$. For every $\varepsilon > 0$ and every compact set $K \Subset \Omega$ one can then find N such that

$$u_j(z) \leq C + \varepsilon,\ z \in K,\ j > N.$$

As application, we will prove that any function of complex variables $f : \Omega \to \mathbb{C}$ $(\Omega \subset \mathbb{C}^n)$ which is holomorphic separately is holomorphic.

Exercise 3 (Hartogs' theorem on separately holomorphic functions)**.** Let Ω be an open subset of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ be a function which is holomorphic in each variable whenever the others are fixed. Then f is holomorphic in Ω (i.e. it is of class $\mathcal{C}^\infty$ in all variables).

References

[Hör73] Lars Hörmander, *An introduction to complex analysis in several variables*, revised ed., North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, North-Holland Mathematical Library, Vol. 7. MR 0344507

[Mer15] Joël Merker, *Holomorphie séparée*, François de Marçay, 2015, Notes de cours M2.