TD9 : Cohomology of sheaves H^1

The main source is the book of Forster [For91].

Exercise 1. Suppose $p_1, ..., p_n$ are distinct point of $\mathbb{C}$ and let $X = \mathbb{C} \setminus \{p_1, ..., p_n\}$. Prove that

$$H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^{n+1};\ H^1(X, \mathbb{C}) \simeq \mathbb{C}^{n+1}.$$

Exercise 2. (a) Let X be a (smooth) manifold, $U \subset X$ open and $V \Subset U$. Show that V meets only a finite number of connected components of U.

(b) Let X be a compact manifold and $\mathfrak{U} = (U_i)_{i\in I}$, $\mathfrak{V} = (V_i)_{i\in I}$ be finite open covering of X such that $V_i \Subset U_i$ for every $i \in I$. Prove that the image of $Z^1(\mathfrak{U}, \mathbb{C})$ under the restriction map $t^{\mathfrak{U}}_{\mathfrak{V}}$ is a finite dimensional vector space.

(c) Let X be a compact Riemann manifold. Prove that $H^1(X, \mathbb{C})$ is a finite dimensional vector space.

Exercise 3. Let X be a compact Riemann manifold.

(a) Prove that the map $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})$ induced by the inclusion map is injective.

(b) Prove that $H^1(X, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.

(c) Prove that $H^1(X, \mathbb{Z})$ is a finite free $\mathbb{Z}$-module, i.e. it has finite generators which are linearly independent.

Exercise 4. Prove that $H^1(\mathbb{P}^1, \mathscr{O}) = 0$. How about $H^1(\mathbb{P}^2, \mathscr{O})$? $H^1(\mathbb{P}^1, \Omega_1)$? Here Ω_1 is the sheaf of holomorphic differential 1-forms on $\mathbb{P}^1$.

Exercise 5. Prove that $H^1(\mathbb{C}^2 \setminus \{0\}, \mathscr{O})$ is of infinite dimension.

Exercise 6. Let X be the open disk of radius $R \in (0, +\infty]$ centered at 0 in $\mathbb{C}$. Let $\mathscr{H}$ be the sheaf of complex valued harmonic functions on X. Prove that $H^1(X, \mathscr{H}) = 0$.

Exercise 7. Suppose g is a smooth function with compact support in $\mathbb{C}$. Prove that there exists a solution f (smooth) to $\bar{\partial} f = g$ having compact support iff

$$\int_{\mathbb{C}} z^n g(z) dz \wedge d\bar{z} = 0,\ \forall n \in \mathbb{N}.$$

Exercise 8. Let $X = \mathbb{C}/\Gamma$ be a torus. Prove that $H^1(X, \mathbb{C}) \simeq \mathbb{C}^2$.

Exercise 9. * Let X be a compact complex manifold.

(a) Prove that $H^2(X, \mathbb{R}) \simeq H^2_{DR}(X, \mathbb{R})$, the latter being De Rham cohomology group.

(b) Prove that the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O} \stackrel{\exp}{\longrightarrow} \mathscr{O}^* \longrightarrow 0$$

is exact. Hence it induces a coboundary map

$$H^1(X, \mathscr{O}^*) \stackrel{\delta}{\longrightarrow} H^2(X, \mathbb{Z}).$$

(c) A holomorphic line bundle L over X can be identified with an element in $H^1(X, \mathscr{O}^*)$. A smooth metric $h = \{h_U\}$ of L on X is a collection of smooth non-vanishing functions $h_U = e^{-\varphi_U} \in \mathcal{C}^\infty(U, \mathbb{R}^+)$ such that

$$\varphi_U = \varphi_V + \log |g_{UV}| \text{ in } U \cap V.$$

The curvature of (L, h) is defined to be the $(1, 1)$-form

$$\Theta_h := dd^c \varphi_U = dd^c \varphi_V.$$

Check that the curvature Θ_h makes sense globally on X. We can view δ as a map from $H^1(X, \mathscr{O}^*)$ to $H^2(X, \mathbb{R})$. Prove that $\delta(L) = \{\Theta_h\} \in H^2_{DR}(X, \mathbb{R})$.

References

[For91] Otto Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer-Verlag, New York, 1991, Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation. MR 1185074