TD6 : Cohenrence : Serre's theorem.

The main source is the book of Noguchi [Nog16]. Let X be a topological space.

Exercise 1. Let $\mathscr{R}_X$ be a sheaf of commutative unitary rings on X. Let $\mathscr{S}, \mathscr{T}$ be sheaves of $\mathscr{R}$-modules over X. Define tensor products of sheaves $\mathscr{S} \otimes \mathscr{T}$. Give an example showing that in general $(\mathscr{S} \otimes_{\mathscr{R}} \mathscr{T})(U)$ is not isomorphic to $\mathscr{S}(U) \otimes_{\mathscr{R}(U)} \mathscr{T}(U)$.

Exercise 2. Let A_1, A_2, A_3 and B be R-modules (R is a commutative unitary ring). Let

$$A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0$$

be an exact sequence of R-modules. Define $\tilde{\phi} := \phi \otimes 1$. Prove that the sequence

$$A_1 \otimes B \xrightarrow{\tilde{\phi}} A_2 \otimes B \xrightarrow{\tilde{\psi}} A_3 \otimes B \longrightarrow 0$$

is also exact.

Exercise 3. Let

$$\mathscr{A}_1 \xrightarrow{\phi} \mathscr{A}_2 \xrightarrow{\psi} \mathscr{A}_3 \longrightarrow 0$$

be an exact sequence of sheaves of $\mathscr{R}$-modules over X and let $\mathscr{B}$ be a sheaf of $\mathscr{R}$-modules. Then the sequence

$$\mathscr{A}_1 \otimes \mathscr{B} \xrightarrow{\phi \otimes 1} \mathscr{A}_2 \otimes \mathscr{B} \xrightarrow{\psi \otimes 1} \mathscr{A}_3 \otimes \mathscr{B} \longrightarrow 0$$

is also exact.

Exercise 4 (Serre's theorem)**.** Let $\mathscr{A}$ be a sheaf of unitary commutative rings over X. Assume that there is an exact sequence of sheaves of $\mathscr{A}$-modules

$$0 \longrightarrow \mathscr{R} \xrightarrow{\phi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0.$$

If any two of the sheaves above are coherent over $\mathscr{A}$ then the third one is also coherent over $\mathscr{A}$.

Exercise 5. Let $\Omega \subset \mathbb{C}^n$ be an open set. For a sheaf $\mathscr{S}$ of $\mathcal{O}_\Omega$-modules prove that the following are equivalent :

(i) $\mathscr{S}$ is coherent

(ii) For every $x \in \Omega$ there are neighborhood $U \subset \Omega$ and an exact sequence

$$\mathcal{O}_U^q \xrightarrow{\phi} \mathcal{O}_U^q \xrightarrow{\psi} \mathscr{S}\mid_U \longrightarrow 0.$$

Exercise 6. Let $\mathscr{S}$ and $\mathscr{T}$ be coherent sheaves over an open set $\Omega \subset \mathbb{C}^n$. Prove that the tensor product $\mathscr{S} \times \mathscr{T}$ is also coherent.

References

[Nog16] Junjiro Noguchi, *Analytic function theory of several variables*, Springer, Singapore, 2016, Elements of Oka's coherence. MR 3526579