# Hyperbolic configurations of roots and Hecke algebras

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**Abstract**

This paper deals with infinite Coxeter groups. We use geometric techniques to prove two main results. One is of Lie-theoretic nature; it shows the existence of many hyperbolic configurations of three pairwise disjoint roots in a given Coxeter complex, provided it is not a Euclidean tiling. The other is both of algebraic and measure-theoretic nature since it deals with Hecke algebras; it shows that for automorphism groups of buildings, convolution algebras of bi-invariant functions are never commutative, provided the building is not Euclidean. Proofs are of geometric nature: the main idea is to exhibit and use enough trees of valency $\geqslant 3$ inside a non-affine Coxeter complex.

*Keywords*: Coxeter groups; Buildings; Hecke algebras.

## Introduction

Finite and affine Coxeter groups have been known and understood for a long time: they are classified [Bou07]. Given the classification of finite Coxeter groups, the structure of the affine ones is quite simple: such a group is an extension of a finite Coxeter group by an abelian translation subgroup. On the other hand, the structure of general Coxeter groups remains mysterious, and less understood (see [Kra09] though, where many basic group-theoretic problems are nicely solved for arbitrary Coxeter groups); it is still an active domain of research.

**Infinite non-affine Coxeter groups.** There is an obvious dichotomy between finite and infinite Coxeter groups. In this paper we are only interested in infinite ones. In the study of the latter groups, a further sharp dichotomy has recently proven to be very useful, namely the distinction between affine and non-affine Coxeter groups. This difference concerning Coxeter groups has implied many other sharp alternatives concerning automorphism groups of buildings (recall that the Weyl group of a building, which reflects the shape of its apartments, is a Coxeter group). Indeed, for such an automorphism group $G$, the affineness of the Weyl group determines if $G$ as a matrix group [CR09], the existence or not of non-obvious quasi-characters in the sense of Gromov for $G$ (via techniques of bounded cohomology) [CF09]; and in the present paper, the existence or not of a Gelfand pair for $G$.

The original result in this spirit is the so-called "strong Tits alternative" for Coxeter groups [MV00, NV02], which is both a model and a source of inspiration for the techniques of proofs:

**Theorem.** *Let $W$ be an irreducible Coxeter group and $\Gamma$ a subgroup of $W$. Then either $\Gamma$ is virtually abelian, or $\Gamma$ virtually surjects onto a non-abelian free group.*

In particular, when $\Gamma = W$, we see that a non-affine Coxeter group virtually surjects onto a non abelian free group. The main tool to prove this is the existence of embeddings of thick trees in the Coxeter complex. In this paper we prove that, conversely, there are enough such trees to embed equivariantly the Coxeter complex itself in a suitable product of trees.

**Hyperbolic configuration of roots.** The first series of results in this paper is relevant to Lie theory, and more precisely to root systems of arbitrary Coxeter groups. In the following, we have a combinatorial point of view on roots: we see them as half-apartments, as in [Tit87]. We first provide a geometric proof of the existence of many triples of pairwise disjoint roots in a non-affine infinite Coxeter complex. We also disprove the existence of any non-trivial normal amenable subgroup of a non-affine Coxeter group. More precisely, the first result is the following.

**Theorem** (See Theorem 2.3)**.** *Let $W$ be an infinite irreducible, non affine Coxeter group. Then for every root $\alpha$, there exists two roots $\beta$ and $\gamma$ such that $\alpha$, $\beta$ and $\gamma$ are pairwise disjoint.*

This theorem —actually a slightly stronger version of it— was initially proved in [CR09, Theorem 14] by combinatorial arguments. Its main interest is that it rules out finite quotients for non-affine twin building lattices. This non-existence of finite index normal subgroup is the second half of the proof of simplicity of such lattices. So it turns out that, for twin building lattices, non-linearity and simplicity are more or less equivalent.

The search for a geometric proof of this statement led us to use the recently defined "combinatorial compactification" [CL09], which parametrizes amenable subgroups in automorphism groups of buildings: up to finite index, such a subgroup fixes a point in the combinatorial compactification. In return, we get the following non-existence statement:

**Proposition** (See Corollary 2.4)**.** *Let $W$ be an infinite irreducible, non affine Coxeter group. Then there is no non-trivial amenable normal subgroup of $W$.*

Note finally that by the strong Tits alternative, amenable subgroups of $W$ are virtually abelian, and abelian subgroups of Coxeter groups were thoroughly studied in [Kra09, Section 6.8]. Indeed, D. Krammer proved in particular that if $W$ contains some free abelian subgroup of rank $n \geqslant 2$, then this subgroup is contained in a parabolic subgroup of $W$ which is either affine or reducible. Combined with our result, this provides a fairly complete understanding of infinite amenable subgroups of Coxeter groups.

**Hecke and convolution algebras.** The second part of this paper is devoted to another application of the existence of a hyperbolic configuration of roots, namely the study of convolution algebras. Let $G$ be a group acting transitively enough (technically: strongly transitively, see Section 3.1) on a locally finite building, and let $U$ be a compact open subgroup of $G$ (typically: a facet stabilizer). We denote by $\mathcal{L}(G,U)$ the space of $\mathbb{C}$-valued bi-$U$-invariant functions with compact support. The vector space $\mathcal{L}(G,U)$ has a natural basis consisting of characteristic functions of classes in $U\backslash G/U$. When equipped with the convolution product, which we denote by $*$, the space $\mathcal{L}(G,U)$ becomes an associative $\mathbb{C}$-algebra which, by analogy with well-known classical cases, we call the *Hecke algebra* (with respect to $U$). Several choices of compact open subgroups $U$ are particularly relevant.

The first choice is when $U$ is equal to the stabilizer, say $\mathscr{B}$, of a given chamber of $X$. Then, by a well-known argument [Bou07, IV.2, Exercises 22 to 24], the convolution algebra $\mathcal{L}(G,\mathscr{B})$ is isomorphic to a deformation of the group algebra $\mathbb{C}[W]$. This algebra admits an algebraic presentation whose parameters can be made completely explicit by means of thicknesses of codimension 1 simplices in the building (called *panels*). In the classical case when $G$ is an algebraic $p$-adic semisimple group acting on its Bruhat-Tits building $X$, a fundamental paper of A. Borel [Bor76] uses the Hecke algebra $\mathcal{L}(G,\mathscr{B})$ in order to reduce the study of some representations of $G$ to an algebraic category of representations of $\mathcal{L}(G,\mathscr{B})$.

The second choice is when $U$ is equal to some maximal compact subgroup, say $K$, that is the stabilizer of some vertex of $X$. When $(\mathcal{L}(G,K),*)$ is commutative, the couple $(G,K)$ is called a *Gelfand pair*. The existence of a Gelfand pair in a locally compact group

$G$ is again of fundamental interest for the study of (spherical unitary) representations of $G$ and for the existence of a Plancherel formula on $G$, which gives an isomorphism between the Hilbert space of square integrable bi-$K$-invariant functions on $G$ and the Hilbert space $L^2(\Omega)$. Here $\Omega$ is the space of "zonal spherical functions", with a (slightly mysterious) measure $\mu$, see for example [Mac71, Theorem 1.5.1]. When $G$ is a real semi-simple Lie group, then for any maximal compact subgroup $K$, the pair $(G, K)$ is a Gelfand pair, and when $G$ is a $p$-adic Lie group, it is also true for some well-chosen maximal compact subgroups of $G$ [Sat63, Mac71], called *special* subgroups [BT72, 4.4.9]. The proof of the existence of such special subgroups crucially uses the geometry of $X$. In contrast, we prove the following complementary statement, which confirms the dichotomy between affine and non-affine Coxeter groups:

**Theorem** (See Theorem 3.5)**.** *Let $X$ be a non-affine building and $G$ be a closed, strongly transitive type-preserving subgroup of the group of automorphisms of $X$. Then, for any maximal compact subgroup $K$ of $G$, the convolution algebra $(\mathcal{L}(G, K), *)$ of continuous bi-$K$-invariant functions with compact support on $G$ is non-commutative.*

In the quite specific case when the building is right-angled Fuchsian (namely, apartments are tesselations of the Poincaré disk by regular right-angled polygons), the above result was proved by U. Bader and Y. Shalom [BS06, Section 5]. They disprove a well-known consequence of the existence of Gelfand pair on unitary representations. Our proof is constructive: we exhibit explicit functions which do not commute for the convolution. Note also that there exist affine buildings of dimension 2 not associated to any algebraic group. For such building, it is possible to define geometrically an algebra analogous to the algebra $(\mathcal{L}(G, K), *)$, and then to prove that this algebra is commutative [Par06]. Thus, this commutativity really depends on the geometry of the building, and not on the algebraic nature of the group.

The arguments of the previous parts of this introduction, based of the strong Tits alternative, provide a good intuition for the proof of the above theorem. Namely, $\mathcal{L}(G, K)$ is a sub-vector space of $\mathcal{L}(G, \mathscr{B})$; a canonical basis is given by characteristic functions of classes in $K\backslash G/K$, and each of these functions is a finite sum of characteristic functions in the canonical basis of classes in $\mathcal{L}(G, \mathscr{B})$. Recall that when $X$ is affine, the Weyl group $W$ is virtually abelian, the non-commutative part of $W$ being given by the linear part of $W$. This linear part, which is the spherical Weyl group, indexes the double classes mod $\mathscr{B}$ whose union provides the suitable maximal compact subgroup $K$. At the level of characteristic functions, this corresponds to making sums of double classes mod $\mathscr{B}$, and the restriction of the convolution product $*$ is commutative because the non-commutative part of $W$ is so to speak absorbed in these finite unions. Conversely, for the same reason we were led to expect the non-existence of any Gelfand pair in the non affine case, because by the strong Tits alternative it is precisely the case when $W$ is far from being virtually abelian. Intuitively, it is impossible to get a commutative subalgebra of $\mathcal{L}(G, \mathscr{B})$ by taking suitable finite sums of characteristic functions.

We finally mention that there does exist a wide family of topologically simple, locally pro-$p$ groups acting on non-affine buildings (and some other groups with even more general profinite facet stabilizers) [RR06]. These groups are provided by Kac-Moody, and even more general twin buildings. The analogy with algebraic groups over local fields suggest that they deserve to be studied from the point of view of harmonic analysis and unitary representations (see also the end of Section 3). The point of view of group cohomology linked to unitary representations has been initiated in [DJ02], generalizing to arbitrary building automorphism groups classical results due to A. Borel and N. Wallach [BW80].

**Structure of the paper.** The first two sections of the paper are devoted to the proof of Theorem 2.3. The first one explains how to embed a Coxeter complex into a product of trees. The second one uses this embedding to actually prove the theorem. The third section explains how one can deduce from it the non-commutativity of convolution algebras.

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# 1 Trees in Coxeter complexes

In this section, we explain how to construct trees in a general Coxeter complex, by techniques of dual trees which come from hyperbolic geometry. We prove that there are enough such trees to fully encode the Coxeter complex. The ideas that are presented here also appear in [DJ99], in [NV02].

## 1.1 Construction of the trees

Let $(W, S)$ be a Coxeter system, with $S$ finite. Let $\Sigma$ be the associated Coxeter complex [AB08, Chapter 3]. The complex $\Sigma$ has a geometric realization, called its Davis-Moussong realization. This construction is described in details for example in [Dav08] or in [AB08, 12.3]. In this realization, the chambers are replaced by copies of some suitable compact, piecewise Euclidean, metric spaces. We denote this geometric realization by $|\Sigma|$. It is a locally compact CAT(0) metric space. A *wall* in $\Sigma$ is a convex subspace of $|\Sigma|$ which divides $|\Sigma|$ into two connected components. Recall also that a *root* is a half-space, *i.e* a connected component of $|\Sigma|$ deprived of a wall. Combinatorially, it can be defined as a conjugate of a simple root $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$ (where $\ell$ is the length of an element of $W$ with respect to the canonical system of generators $S$).

*Example* 1.1. Let $W$ be the free product of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$. Then the Davis-Moussong realization $|\Sigma|$ of the associated Coxeter complex is a regular tree of valency $r$. A chamber in $|\Sigma|$ is a *star*, namely the union of a vertex and the $r$ half-edges adjacent to this vertex. A wall in $|\Sigma|$ is the middle of an edge. Thus the set of chambers of $|\Sigma|$ can be identified with the set of vertices, and the set of walls with the set of edges. A root is a connected component of the tree, deprived of the middle of an edge.

Since we know that there is an injection from $W$ to some $\mathrm{GL}_n(\mathbb{C})$ [Bou07, V. 4.4], Selberg's lemma [Alp87] allows us to find a normal torsion-free finite index subgroup $W_0$ of $W$. We will now see that the orbits of walls in $|\Sigma|$ can be identified to some trees. This remark has already be done in [NV02] or [DJ99]. The following lemma is well-known. A proof can be found in [Mil76] or [NV02, Lemma 3.3]:

**Lemma 1.2.** *Let $g \in W_0$, and let $H$ be a wall of $|\Sigma|$, with associated reflection $r$. Then either $gH = H$, or $gH \cap H = \varnothing$. In the first case, $r$ and $g$ commute.*

□

Consequently, the walls of the $W_0$-orbit of some given wall $H$ have pairwise empty intersection. They divide $|\Sigma|$ into connected components. Let us consider the graph $T_{W_0}(H)$ whose vertices are these components, which are linked by an edge if they are adjacent. Thus a wall in the $W_0$-orbit of $H$ is represented by an edge of $T_{W_0}(H)$.

It is clear that $T_{W_0}(H)$ is connected. Furthermore, removing an edge from $T_{W_0}(H)$ corresponds to removing a wall in $|\Sigma|$, and it turns it into a non-connected space. As the different walls in $W_0.H$ do not intersect, it follows that $T_{W_0}(H)$ is also divided into two connected components. Thus $T_{W_0}(H)$ is a tree. Note that this tree need not be locally finite. The tree $T_{W_0}(H)$ is, in other words, the dual tree of the tessellation of $|\Sigma|$ given by the walls $wH$, with $w \in W_0$.

By construction, $W_0$ acts transitively on the edges of the tree. Furthermore, since $W_0$ is normal in $W$, one can define a simplicial action of $W$ on the set of trees $T_{W_0}(H)$, with

$W_0$ fixed. This action is defined by $w.T_{W_0}(H) = T_{W_0}(wH)$. Indeed, for all $g \in W_0$, there exists $g' \in W_0$ such that $wgH = g'wH$.

Furthermore, since $W_0$ is of finite index in $W$, and there is a finite number of $W$-orbits of walls, the $W_0$-orbits of walls are also finite in number. Thus, there exists a finite number of walls $H_1, \ldots, H_l$, such that each wall of $\Sigma$ appears as an edge in exactly one of the $T_{W_0}(H_i)$, for each $1 \leqslant i \leqslant l$. Let us set $T_i = T_{W_0}(H_i)$.

## 1.2 Embedding of the Coxeter complex

In the preceding section, we defined a finite number of trees $T_i$, with $1 \leqslant i \leqslant l$. We also have seen that $W_0$ acts on each of these trees, and that $W$ acts on this set of trees by permuting them. More precisely, let $w \in W$. The image by $w$ of a wall in $\Sigma$ is another wall in $\Sigma$, and thus the image of some edge in some $T_i$ will be an edge in some $T_{\sigma(i)}$, for some permutation $\sigma$ associated to $w$. Furthermore, if two edges are adjacent in $T_i$, then their images will be again adjacent in $T_\sigma(j)$. So, from the action on the set of edges we get an action on the set of vertices on $T_1 \cup \cdots \cup T_l$. We define the action of $W$ on $T_1 \times \cdots \times T_l$ to be the diagonal action.

Let $r_i$ be the valency of the homogeneous tree $T_i$. Note that $r_i$ can be any integer, and can also be infinite (though countable). Still, it can not be equal to 1: this would mean that $W_0$ stabilizes a wall, which is impossible since it is infinite. We know that $T_i$ can be seen as the (Davis-Moussong realization) of the Coxeter group $W^{r_i} \simeq (\mathbb{Z}/2\mathbb{Z})^{*r_i}$. This allows us to speak of roots in $T_i$.

Using the arguments above, we can prove:

**Proposition 1.3.** *Let $W$ be a Coxeter group and $\Sigma$ its Coxeter complex. Then there exists a $W$-equivariant embedding*

$$\psi : \Sigma \to T_1 \times \cdots \times T_l.$$

*Proof.* Let $x$ be a chamber in $\Sigma$ and $H$ a wall in $\Sigma$. Since $x$ does not intersect any wall in the geometric realization $|\Sigma|$, it is contained in a unique connected component of $|\Sigma| \setminus W_0.H$. So we can associate to $x$ a unique vertex of $T_{W_0}(H)$. Thus, we get a map $\psi_H$ from the set of chambers of $\Sigma$ to the set of vertices of $T_{W_0}(H)$.

Let $e_H$ be the edge of $T_{W_0}(H)$ which corresponds to the wall $H$, and let $\sigma$ be a panel contained in $H$. We use the notion of *projection* of a chamber $C$ on a panel $\tau$: this projection, denoted $\mathrm{proj}_\tau$, is defined to be the unique chamber adjacent to $\tau$ at minimal distance of $x$. If $\mathrm{proj}_\sigma(x)$ is equal to some chamber $C$, then we have $\mathrm{proj}_{e_H}(\psi_H(x)) = \psi_H(C)$. In particular, if $\alpha$ is a root of $\Sigma$ with boundary wall $H$, then $\psi_H(\alpha)$ is a root of $T_{W_0}(H)$. Of course it is the same for any wall in the same $W_0$-orbit.

For each wall $H_1, \ldots, H_l$, we get some maps $\psi_{H_1}, \ldots, \psi_{H_l}$. Their product is a map $\psi : \Sigma \to T_1 \times \cdots \times T_l$, and we have to show that it is injective. Let $x, y \in \Sigma$ such that $\psi(x) = \psi(y)$. By considering the position of $\psi(x)$ and $\psi(y)$ in $T_1$, we can see that, if $\alpha$ is a root of $\Sigma$ whose boundary wall is in the $W_0$-orbit of $H_1$, then $x \in \alpha$ if and only if $y \in \alpha$. The same argument in each $T_i$ proves that this is valid for every root of $\Sigma$. Hence the set of roots containing $x$ is equal to the set of roots containing $y$. This implies that $x = y$ (indeed, $x$ is the unique chamber in the intersection of the root containing $x$ with boundary walls adjacent to $x$).

Now, we have to prove that $\psi$ is equivariant. Let $x \in \Sigma$ and $w \in W$. Let $(x_1, \ldots, x_l) = \psi(x)$ and $(y_1, \ldots, y_l) = \psi(w.x)$. Then $\psi(w.x)$ is uniquely determined by the roots in each tree containing $w.x$. These roots correspond to roots in $\Sigma$ containing $w.x$, *i.e.* to $\Phi(w.x) = w.\Phi(x)$. By definition of the action, $w\psi(x)$ is the unique point in $T_1 \times \cdots \times T_l$ which is contained in every root in each $T_i$ which corresponds to a root in $w.\Phi(x)$. So, we have $\psi(w.x) = w.\psi(x)$, and $\psi$ is equivariant. □

### 1.3 Combinatorial compactification

We refer to [CL09] for the definition and basic facts about the combinatorial compactification. We will use the compactification $\mathscr{C}_1(\Sigma)$, which is a compactification of the set of chambers $\mathrm{ch}(\Sigma)$.

In the case of a Coxeter group, this compactification can be defined in a simple way. Namely, if $C$ is a chamber, then $C$ is uniquely determined by the roots which contain $C$. Let $\Phi$ be the set of roots of $\Sigma$. This defines an injection $i : \mathrm{ch}(\Sigma) \to 2^\Phi$. The target space is equipped with the product topology, hence is compact. We define $\mathscr{C}_1(\Sigma)$ to be the closure of $i(\mathrm{ch}(\Sigma))$ in $2^\Phi$.

The combinatorial compactification of a tree is its usual compactification $\mathscr{C}_1(T) = T \cup \partial_\infty T$. The combinatorial compactification of a product of trees is then the product of their combinatorial compactifications: $\mathscr{C}_1(T_1 \times \cdots \times T_l) = \mathscr{C}_1(T_1) \times \cdots \times \mathscr{C}_1(T_l)$. Then, if $\Sigma$ is any Coxeter complex, it can be embedded as above in a product of trees $T_1 \times \cdots \times T_l$. The same proof as in Proposition 1.3 shows that the combinatorial compactification of $\Sigma$ is then the closure of $\Sigma$ in $\mathscr{C}_1(T_1 \times \cdots \times T_l)$.

A useful fact about this compactification is that it behaves well with respect to amenable subgroups of $W$. Namely, we have the following [CL09, Theorem 6.1]:

**Theorem 1.4.** *An amenable subgroup of $W$ virtually fixes a point in the combinatorial compactification $\mathscr{C}_1(\Sigma)$ of $W$.*

*Proof.* The precise statement of [CL09, Theorem 6.1] is that an amenable subgroup of $W$ virtually fixes a point in the compactification $\mathscr{C}_{\mathrm{sph}}(\Sigma)$ of spherical residues of $\Sigma$, which is larger than $\mathscr{C}_1(\Sigma)$. This compactification is defined in [CL09, Definition 2.1]. This means that this amenable subgroup fixes a spherical residue in some apartment $\Sigma_\xi$ of the stratification defined in [CL09, Theorem 5.5]. But, by definition, spherical residues in apartments are finite, so this amenable subgroup virtually fixes a chamber in $\Sigma_\xi$. These chambers are exactly the points of $\mathscr{C}_1(\Sigma)$. □

## 2 Consequences of the strong Tits alternative

In this section, we prove the existence of many hyperbolic configurations of roots in non-affine Coxeter groups. The key arguments are relevant to the geometric proof of the strong Tits alternative.

### 2.1 Existence of pairwise disjoint roots

The classical Tits alternative asserts that any finitely generated linear group either is virtually solvable or contains a non-abelian free group [Tit72]. We say that a class of groups satisfies the strong Tits alternative if any group in this class either is virtually abelian or virtually surjects onto a non-abelian free group. One of the main theorems of [MV00] is that Coxeter groups satisfy this strong Tits alternative. Later on, in [NV02], it was proved that subgroups of Coxeter groups also satisfy the strong Tits alternative. The strategy used in the latter is to give an interpretation of this fact in terms of the trees described above. More precisely, they prove the following:

**Proposition 2.1.** *Let $W$ be an infinite Coxeter group. Let $W_0$ be as above. If every tree $T_{W_0}(H)$ is a line, then $W_0$ (and hence $W$) is virtually abelian.*

□

Combined with the strong Tits alternative for the Coxeter group $W$, this yields the following:

**Proposition 2.2.** *In any infinite, non affine Coxeter group, there exists three roots which are pairwise disjoint.*

*Proof.* Let $W_0$ be as above. Since $W$ is not affine, it is not virtually abelian by the strong Tits alternative. So there is some wall $H$ for which $T_{W_0}(H)$ is not a line, and therefore is regular of valency greater or equal to 3. Thus there are three roots in $T_{W_0}(H)$ that are pairwise disjoint, and these roots are the image by $\psi$ of three disjoint roots of $W$. □

## 2.2 Uniformness

Our goal is to prove that the above proposition on the existence of pairwise disjoint roots can be refined, under an assumption of irreducibility, in order to prove that one of these three roots may be chosen arbitrarily.

**Theorem 2.3** (See [CR09, Theorem 14])**.** *Let $W$ be an infinite, irreducible, non affine Coxeter group. Let $\alpha$ be a root in $W$. Then there exists two roots $\beta$ and $\gamma$ which are disjoint and also disjoint from $\alpha$.*

*Proof.* Let $W_0$ be a normal, torsion-free, finite index subgroup of $W$. Since every root of $W$ appears as a root in some $T_{W_0}(H)$, we just have to prove that $T_{W_0}(H)$ can never be a line.

Let $r$ be a reflection in $W$, and $H$ its associated wall. If $w \in W$, then $wrw^{-1}$ is a reflection with associated wall $wH$. As we have seen above, $w$ sends the tree $T_{W_0}(H)$ onto $T_{W_0}(wH)$. So these two trees are isomorphic. Hence we only have to argue on conjugacy classes of reflections.

Let $R$ be the set of all reflections of $W$, and let $R'$ be the subset of $R$ consisting of reflections whose associated wall $H$ is such that $T_{W_0}(H)$ is a line. Let $R'' = R \setminus R'$. By a theorem of Deodhar [Deo89], we know that the groups $W'$ generated by $R'$ and $W''$ generated by $R''$ are Coxeter groups. Furthermore, the above argument proves that $R'$ is closed under conjugations by elements of $W$, so that $W'$ is normal in $W$. Similarly, $W''$ is normal in $W$. Note that, by Proposition 2.2, $R''$ is not empty, and hence $W'' \neq \{1\}$.

Recall the following theorem of L. Paris [Par07, Theorem 4.1]: an infinite, irreducible Coxeter group cannot be the product of two non-trivial subgroups. In view of this fact, we prove that $W$ is the direct product of $W'$ and $W''$. Since $W'' \neq \{1\}$, this will prove that $W'' = W$, hence $R'' = R$. As $W'$ and $W''$ are both normal, and altogether obviously generate $W$, what is left to show is that their intersection is trivial.

Let $A = W' \cap W''$. The group $A$ is normal both in $W'$ and $W''$. We prove first that $A$ is finite. If $W_0' = W' \cup W_0$, then we see that for every wall $H$ of $W'$, the tree $T_{W_0'}(H)$ is in fact a line. Hence, by Proposition 2.1, $W'$ is an virtually abelian, and therefore is amenable.

Let $T_1, \dots, T_l$ be the trees constructed in Section 1. Up to permutation, we can assume that the valency of $T_1, \cdots, T_k$ is greater or equal to 3, and that the valency of $T_{k+1}, \dots, T_l$ is equal to 2. By construction, $W''$ stabilizes the product $T_1 \times \cdots \times T_k$.

Furthermore, $W''$ contains all reflections with respect to the walls in these trees. So, if $T$ is one of these trees and is of valency $r$, then $W''$ contains the free Coxeter group $(\mathbb{Z}/2\mathbb{Z})^{*r}$, acting on $T$ in a natural way. Since this group is not amenable, it can not have any finite index subgroup fixing a point in $\partial T$. Therefore the orbits of $(\mathbb{Z}/2\mathbb{Z})^{*r}$, and hence of $W''$ acting on $\partial_\infty T$ are infinite.

Since $A$ is amenable, it has a finite index subgroup that fixes a point in the combinatorial compactification $\mathscr{C}_1(\Sigma)$. Up to passing again to a subgroup of finite index, we can assume that $A$ really fixes this point, and furthermore that it stabilizes every tree $T_i$, for $i \leqslant k$. Let $T$ be one of those trees. Since we know that $A$ fixes a point in $\mathscr{C}_1(\Sigma)$, we know that $A$ fixes a point in $T \cup \partial_\infty T$, denoted by $\xi$. Assume first that $\xi \in \partial_\infty T$. Now, for every $w \in W''$, $wAw^{-1}$ fixes the point $w\xi$ in $\partial(wT)$. Since $\mathrm{Stab}_{W''}(T)$ acting on $\partial T$ has infinite orbits, this means that $A$ fixes infinitely many points in $\partial T$. Since a hyperbolic translation in $T$ only fixes two points, this means that the actions of $A$ on $T$ is only by elliptic elements. Obviously if $\xi \in T$, then $A$ also acts on $T$ by elliptic elements.

Furthermore, all the reflections of $W''$ are associated to reflections of one of the tree $T_i$, for $i \leqslant k$. Thus, in a similar way to Proposition 1.3, we see that $W''$ is equivariantly embedded in $T_1 \times \cdots \times \cdots \times T_k$. Since $A$ acts via elliptic elements on each of these trees, the action of $A$ on $W''$ has finite orbits. Hence, $A$ is finite.

Thus we have proved that $W' \cap W''$ is finite. But this is a finite normal subgroup of $W$. By [Par07, Proposition 4.2], this implies that this subgroup is trivial. □

Theorem 2.3 was proved by Caprace and Rémy with an application in mind: the study of non-affine Kac-Moody groups. These groups were constructed by Tits [Tit87]. In the affine case, Tits' construction leads to particular lattices in semi-simple Lie groups over local fields; in particular, they act on Euclidean buildings. The non-affine Kac-Moody groups act on non-Euclidean buildings, in which the geometry of non-affine Coxeter groups appear. Hence, it is not surprising to see that there are some qualitative differences between affine and non-affine Kac-Moody groups. Using Theorem 2.3, Caprace and Rémy prove that Kac-Moody groups over finite fields with an infinite, irreducible and non-affine Weyl group are simple (up to taking a finite-index quotient of a finite-index subgroup) [CR09, Theorem 19]. It is quite the opposite to lattices in semi-simple Lie groups over local fields, which are residually finite, as finitely generated linear groups.

Using both the proof and the result of Theorem 2.3, it is easy to deduce the following fact, which has an interest of its own:

**Corollary 2.4.** *An irreducible, non-affine Coxeter group has no non-trivial amenable subgroup.*

*Proof.* We have seen in the course of the proof that if all the trees $T_{W_0}(H)$ are of valency $\geqslant 3$, then a normal, amenable subgroup of $W$ is in fact finite, hence trivial. By Theorem 2.3, it is the case of all irreducible, non-affine Coxeter group. □

# 3 Non-commutativity of convolution algebras

In this section, we prove that Hecke algebras of groups acting on a non-Euclidean building, with respect to any maximal compact subgroup, can never be commutative. The proof consists first in translating the expression of the convolution product in geometric and combinatorial terms: we prove that the commutativity of the convolution product can be interpreted as an equality between some sets of chambers. Then we prove that this equality never holds in the non-affine case, wich is in sharp contrast with harmonic analysis on Bruhat-Tits buildings [Mac71].

## 3.1 Notation and preliminaries on buildings

Let $W$ be a Coxeter group and $X$ be a locally finite building whose Weyl group is $W$. We denote the Davis realization of $X$, as defined in [Dav08], by $|X|$ ; it is a CAT(0) and complete metric space. We assume $X$ is of finite thickness and denote $\delta$ the $W$-distance between chambers of $X$ [AB08, Chapter 5]. Recall that the $W$-distance is a function $\delta : X \times X \to W$ which encodes the relative position of chambers. More precisely, it can be defined as the element of $W$ obtained by concatenation of the types of the panels crossed by a minimal gallery from one chamber to another.

If $a$ is a chamber in $X$, the set of chambers $c$ such that $\delta(a, c) = w$ is denoted by $\mathcal{C}_w(a)$.

We endow the group $\mathrm{Aut}(X)$ of automorphisms of $X$ with the compact open topology: it is the smallest topology which makes stabilizers of points in $|X|$ open; it is then easy to see that these stabilizers are also compact. Let $G$ be a closed subgroup of $\mathrm{Aut}(X)$. Recall that the action of $G$ on $X$ is said to be *strongly transitive* if it is transitive on the pairs of the form $(C, A)$, where $C$ is a chamber included in an apartment $A$. We assume that $G$ acts strongly transitively on $X$, and that $G$ is type-preserving.

Let $K$ be a maximal compact subgroup of $G$. By a well-known fixed-point lemma [BH99, Corollary 2.8], we know that $K$ is the stabilizer of some vertex $o$ of $|X|$. By transitivity of the action of $G$, the quotient $G/K$ can be identified with the set of vertices in $X$ of the same type as $o$. Let $W_o$ be the stabilizer of $o$ in $W$. It is a maximal spherical subgroup of $W$. Let $\mathbb{A}$ be an apartment of $X$ containing $o$.

Since $X$ is locally finite, $G$ is locally compact, and therefore endowed with a Haar measure $\mathrm{d}g$. We assume that $\mathrm{d}g$ is normalised so that $\mathrm{Vol}(K, \mathrm{d}g) = 1$, which is possible because $K$ is compact and open.

Let $\mathcal{L}(G, K)$ be the space of continuous compactly supported functions from $G$ to $\mathbb{C}$ which are bi-$K$-invariant, that is, such that

$$\forall k, k' \in K \ \forall g \in G \quad f(kgk') = f(g).$$

It is a vector space over $\mathbb{C}$, a basis of which being the set of characteristic functions of classes in $K\backslash G/K$. The convolution product

$$(f * f')(g) = \int_G f(gh)f'(h^{-1})\mathrm{d}h = \int_G f(h)f'(h^{-1}g)\mathrm{d}h$$

turns it into an algebra. We call it the *Hecke algebra* of $G$ with respect to $K$.

In the classical context, where $\mathbb{G}$ is a semisimple algebraic group over a local field $k$, the topological group $G$ is the group of $k$-rational points $\mathbb{G}(k)$, and $X$ is the Bruhat-Tits building of $\mathbb{G}$ over $k$. The algebra $\mathcal{L}(G, K)$ is then a basic object of study for the spherical harmonic analysis of $G$ [Sat63, Mac71].

Functions in $\mathcal{L}(G, K)$ can also be thought of as functions on $G/K$ which are left-invariant. Therefore, we can see them as functions on the set of vertices of the same type as $o$, which are constant on any $K$-orbit of vertices.

Let $x \in G$ and $\lambda \in G$. We set $V_\lambda(x) = xK\lambda K$. It is clear that $V_\lambda(x)$ only depends on the class of $\lambda$ in $K\backslash G/K$ and on the class of $x$ in $G/K$; hence, it makes sense to write $V_\mu(y)$ when $\mu$ and $y$ are two vertices of the same type as $o$ in the building. Furthermore, since $V_\lambda(x)$ is invariant by right-multiplication by an element of $K$, the set $V_\lambda(x)$ can be seen as a set of vertices in the building $X$. Geometrically, $V_\lambda(x)$ should be thought of as a kind of sphere in $X$ centred at $x$ and of "radius" $\lambda$. When $x = o$, it is exactly the $K$-orbit of $\lambda$ in $X$. Note that $y \in V_\lambda(x)$ if and only if $x \in V_{\lambda^{-1}}(y)$.

Since the action of $G$ is strongly transitive, for every $\lambda \in G/K$, there is always a $\mu$ in the apartment $\mathbb{A}$ such that $V_\lambda(x) = V_\mu(x)$. Note that a vertex in $\mathbb{A}$ of the same type as $o$ can also be thought of as an element of $W/W_o$. Thus, it also makes sense to write $V_\lambda(x)$, when $\lambda$ and $x$ are element of $W/W_o$, or even elements of $W$.

At last, a root $\alpha$ is understood to be a half-space in some apartment $A$. Its boundary wall is denoted by $\partial\alpha$. The *star* of a vertex $x$ is defined as the set of chambers whose boundary contains $x$. It is denoted by $\mathrm{st}(x)$.

## 3.2 Geometric expression of the convolution product

This section is devoted to translate the expression of the convolution product into a geometric counting.

**Lemma 3.1.** *Let $\lambda, \mu \in G$. Let $f$ and $g$ be the characteristic functions of $K\lambda K$ and $K\mu K$, respectively. Then, for all $x \in G/K$, we have:*

$$(f * g)(x) = |V_\lambda(0) \cap V_{\mu-1}(x)|.$$

*Proof.* Since $K$ is compact and open in $G$, the quotient space $G/K$ is discrete, so that the $G$-invariant measure on $G/K$ is the counting measure. Hence we get

$$(f * g)(x) = \int_G f(h)g(h^{-1}x)\mathrm{d}h = \sum_{h \in G/K} \int_K f(hk)g(k^{-1}h^{-1}x)\mathrm{d}k.$$

Since $f$ and $g$ are bi-$K$-invariant, using that $\mathrm{d}g$ is normalised so that $\mathrm{Vol}(K, \mathrm{d}g) = 1$, we deduce that $(f * g)(x) = \sum_{h \in G/K} f(h)g(h^{-1}x)$. Finally, we have:

$$\begin{aligned} (f * g)(x) &= \sum_{h \in G/K} f(h)g(h^{-1}x) \\ &= |V_\lambda(o) \cap \{h \in G/K \mid h^{-1}x \in K\bar{\mu}K\}| \\ &= |V_\lambda(o) \cap V_{\mu^{-1}}(x)|, \end{aligned}$$

which concludes the proof. □

Thus, checking the commutativity of $\mathcal{L}(G, K)$ amounts to checking whether, for all $\lambda, \mu \in G$, we have $|V_\lambda(o) \cap V_{\mu^{-1}}(x)| = |V_\mu(o) \cap V_{\lambda^{-1}}(x)|$.

To calculate this quantity, we will rather do the calculation on chambers instead of vertices, in the spirit of [Par06]: it is more adapted to the point of view of buildings as $W$-metric spaces [AB08, Chapter 5]. Let $N$ be the number of chambers in the star of $o$. By transitivity of the action of $G$, the integer $N$ is also the number of chambers in the star of any vertex $x \in G/K$. Recall that, for a chamber $a$ in $X$ and $w \in W$, we write $\mathcal{C}_w(a)$ for the set of chambers $c$ such that $\delta(a, c) = w$.

**Lemma 3.2.** *For all $x \in G/K$, and for all $\lambda, \mu \in W$, we have*

$$|V_\lambda(o) \cap V_{\mu^{-1}}(x)| \leqslant \frac{1}{N} \sum_{a \in \mathrm{st}(o), b \in \mathrm{st}(x)} \sum_{w_1 \in \lambda W_o} \sum_{w_2 \in \mu^{-1} W_o} |\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b)|.$$

*Proof.* Let $\pi$ be the map which associates to a chamber of $X$ its only adjacent vertex which is of the same type as $o$. Then for any $y \in G/K$, the preimage $\pi^{-1}(y)$ is equal to $\mathrm{st}(y)$. In particular, $\pi^{-1}(y)$ is of cardinality $N$. Therefore,we have the equality:

$$N|V_\lambda(o) \cap V_{\mu^{-1}}(x)| = |\pi^{-1}(V_\lambda(o) \cap V_{\mu^{-1}}(x))|.$$

Let us choose a chamber $C$ whose closure contains $o$. The set of chambers $a$ such that $\pi(a) = o$ is equal to $\mathrm{st}(o)$. It is also the set of chambers whose $W$-distance to $C$ is in $W_o$. Likewise, the chambers of $\mathbb{A}$ containing $\lambda.o$ are the chambers of $\mathbb{A}$ whose $W$-distance to $C$ lies in $\lambda W_o$. Chambers $c$ containing a vertex in $V_\lambda(o)$ are then $K$-transforms of these ones: there exists $g \in K$ such that $\pi(gc) = \lambda.o$. Letting $a = g^{-1}C$, we see that $\delta(a, c) = \delta(ga, gc) = \delta(C, gc) \in \lambda W_o$. Therefore, if a chamber $c$ contains a vertex in $V_\lambda(o)$, then there exists a chamber $a$ containing $o$ and there exists also $w_1 \in \lambda W_o$ such that $c \in \mathcal{C}_{w_1}(a)$.

Furthermore, for any $a \in \mathrm{st}(o)$ and $w_1 \in \lambda W_o$, using the strong transitivity of the action of $G$, any chamber $c \in \mathcal{C}_{w_1}(a)$ is a $K$-transform of a chamber in $\mathbb{A}$ such that $\delta(a, c) \in \lambda W_o$, thus is such that $\pi(c) \in V_\lambda(o)$. Therefore, we have:

$$\pi^{-1}(V_\lambda(o)) = \bigcup_{a \in \mathrm{st}(o)} \bigcup_{w_1 \in \lambda W_o} \mathcal{C}_{w_1}(a). \qquad (*)$$

Applying $(*)$ with $o$ replaced by $x$ and $\lambda$ replaced by $\mu^{-1}$, we see that:

$$\pi^{-1}(V_{\mu^{-1}}(x)) = \bigcup_{b\in \mathrm{st}(x)} \bigcup_{w_2\in\mu^{-1}W_o} \mathcal{C}_{w_2}(b). \qquad (**)$$

Consequently, we have

$$\pi^{-1}(V_\lambda(o)\cap V_{\mu^{-1}}(x)) = \bigcup_{a\in\mathrm{st}(o),b\in\mathrm{st}(x)} \bigcup_{w_1\in\lambda W_o} \bigcup_{w_2\in\mu^{-1}W_o} \mathcal{C}_{w_1}(a)\cap\mathcal{C}_{w_2}(b),$$

which, taking cardinality, proves the lemma. □

*Remark* 3.3. In the equality $(*)$, the union is not disjoint: for example, if $\lambda = o$, we get everything twice. However, it can easily made so, by fixing a $a \in \mathrm{st}(o)$ and taking the union over all $w_1 \in W_o\lambda W_o$. Similarly, we could get a disjoint union in the equality $(**)$, thus getting an equality in the lemma. We will not need this fact.

## 3.3 Non-commutativity in the non-affine case

We assume hereafter that $W$ is infinite, non-affine and irreducible. We prove the non commutativity of $\mathcal{L}(G,K)$. To do this, we will find $\lambda,\mu$ and $x$, chosen in $W$, such that $|V_\lambda(o)\cap V_{\mu^{-1}}(x)| = 0$ whereas $|V_\mu(o)\cap V_{\lambda^{-1}}(x)| \neq 0$.

Let us take $\mu = x\lambda^{-1}$. Then $V_\mu(o)\cap V_{\lambda^{-1}}(x)$ contains $\mu.o$, and therefore is not empty. Our goal is thus to find $x$ and $\lambda$ such that $V_\lambda(o)\cap V_{x\lambda^{-1}}(x)$ is empty.

Using Lemma 3.2, we have to find $x$ and $\lambda$ such that for all chambers $a$ and $b$ containing $o$ and $x$ respectively, and for all $w_1 \in \lambda W_o$ and $w_2 \in \mu^{-1}W_o$, we have $\mathcal{C}_{w_1}(a)\cap\mathcal{C}_{w_2}(b) = \varnothing$. To do this, we must give an estimate of the $W$-distance between $a$ and $b$. We will in fact choose a reflection for $\lambda$.

**Lemma 3.4.** *Let $s \in S$ be a simple reflection. Let $a$ and $b$ be two chambers. Let $\mu \in W$. Assume there exists $w_1 \in sW_o$ and $w_2 \in \mu^{-1}W_o$ such that $\mathcal{C}_{w_1}(a)\cap\mathcal{C}_{w_2}(b) \neq \varnothing$. Then the $W$-distance between $a$ and $b$ belongs to $W_o\mu \cup sW_o\mu$.*

*Proof.* As a preliminary remark, let us note that, since $G$ acts strongly transitively on $X$, it is transitive on the set of couples of chambers with the same $W$-distance [AB08, Corollary 6.12]. Thus, $|\mathcal{C}_{w_1}(a)\cap\mathcal{C}_{w_2}(b)|$ does not depend on $a$ and $b$, but only on the $W$-distance between $a$ and $b$. Therefore, we can assume that $b \in \mu\,\mathrm{st}(o)$.

Let $c \in \mathcal{C}_{w_1}(a)\cap\mathcal{C}_{w_2}(b)$. Because of the choice we made on $b$, we remark that

$$\mathrm{st}(o) = \bigcup_{w\in\mu^{-1}W_o} \mathcal{C}_w(b).$$

Consequently, $c \in \mathrm{st}(o)$. Furthermore, $\delta(a,c) = sw_0 \in sW_o$. Hence there exists a minimal gallery of type $sw_0$ from $a$ to $c$, with $w_0 \in W_o$. This gallery is obtained by the concatenation of a gallery of type $s$ from $a$ to some chamber, say $c_0$, and a gallery of type $w_0$ from $c_0$ to $c$. Thus, we get that $\delta(c_0,c) = w_0 \in W_o$ and therefore $c_0 \in \mathrm{st}(o)$, so that $c_0 \in \mathcal{C}_w(b)$ for some $w \in \mu^{-1}W_o$.

Thus, we have $\delta(a,c_0) = s$ and $\delta(c_0,b) \in W_o\mu$. By definition of the $W$-distance [AB08, Definition 5.1], this implies that $\delta(a,b) \in W_o\mu \cup sW_o\mu$. □

We can now conclude:

**Theorem 3.5.** *Let $X$ be a locally finite building and $G$ a closed subgroup of* $\mathrm{Aut}(X)$. *Assume that $G$ acts strongly transitively on $X$. Let $K$ be the stabilizer in $G$ of a vertex $o$ in $X$. Then the convolution algebra $\mathcal{L}(G,K)$ is not commutative.*

*Proof.* Let $x \in W$ and $s$ be a reflection in $W$. Let $a$ be a chamber containing $o$ and $b$ a chamber containing $x$. The $W$-distance $\delta(a,b)$ belongs to $xW_o$. Let $\mu = xs$, so that $V_\mu(o) \cap V_s(x)$ contains $\mu.o$.

As explained in the beginning of the section, and using Lemma 3.2, we disprove the commutativity of $\mathcal{L}(G,K)$ if we choose $s$ and $x$ such that for any $a \in \mathrm{st}(o)$ and $b \in \mathrm{st}(x)$, for any $w_1 \in sW_o$ and $w_2 \in sx^{-1}W_o$, we have $\mathcal{C}_{w_1}(a) \cap \mathcal{C}_{w_2}(b) = \varnothing$. By Lemma 3.4, this implies that for such $a$ and $b$ we have $\delta(a,b) \in W_oxs \cup sW_oxs$. Since $a \in \mathrm{st}(o)$ and $b \in \mathrm{st}(x)$, we have furthermore $\delta(a,b) \in xW_o$. Hence, we will prove that $\mathcal{L}(G,K)$ is not commutative if we prove that there exist $x \in W$ and a reflection $s$ such that $W_oxs \cap xW_o = \varnothing$ and $sW_oxs \cap xW_o = \varnothing$.

Assume that $\mathcal{L}(G,K)$ is commutative. It implies that for all $x$ and $s$ one of these intersections is not empty. The idea is to choose $x$ and $s$, using Proposition 2.2, in order to get a contradiction. Let $s$ be a simple reflection. Since $W$ is infinite, we may assume that $s$ does not fix $o$.

As in Proposition 1.3, the group $W$ embeds in a product of trees $T_1 \times \cdots \times T_l$, and we know that each $T_i$ is of valency $\geq 3$.

Let $\tau \in W$ be the product of two reflections across two distinct walls of some tree $T_i$. Since these two walls do not intersect each other, $\tau$ is a hyperbolic element of $\mathrm{Aut}(\Sigma)$. It is also a hyperbolic element of $\mathrm{Aut}(T_i)$. Thus, there exists $\xi_i \in \partial_\infty T_i$ such that for all $y \in T$, we have $\lim_{n\to+\infty} \tau^n y = \xi_i$. Similarly, there exists $\eta$ in the ray boundary $\partial_\infty\Sigma$ of the Davis complex $\Sigma$ such that we have $\lim_{n\to+\infty} \tau^n z = \eta$ for every $z \in \Sigma$.

Note that, for any point $\eta' \in \partial_\infty\Sigma$ such that no wall of $T_i$ tends to $\eta'$, there is a well-defined projection of $\eta'$ on $T_i$: indeed, for each wall of $T_i$, $\eta'$ is contained in (the boundary of) only one of the two associated roots. This projection can be either an edge or a point in the boundary of $T_i$ ; when it is defined, we will denote it $p_i(\eta')$. It is clear that, for the point $\eta$ defined above, we have $p_i(\eta) = \xi_i$.

Since we assumed $\mathcal{L}(G,K)$ to be commutative, we know that for any reflection $s$ and any $n \in \mathbb{N}$, one of the intersections $W_o\tau^{-n}s \cap \tau^{-n}W_o$ and $sW_o\tau^{-n}s \cap \tau^{-n}W_o$ is not empty. Since $W_o$ is finite, this implies that there exists $g \in W_o$ such that either $\tau^{-n}g \in W_o\tau^{-n}s$ for an infinite number of $n$'s, or $\tau^{-n}g \in sW_o\tau^{-n}s$ for an infinite number of $n$'s. We extract a subsequence from $(\tau^{-n})_n$ such that one of these two cases occur for every $n$ and a fixed $g$. The first case means that $\tau^{-n}gs\tau^n \in W_o$, hence $gs\tau^n o = \tau^n o$. In the second case, we get $gs\tau^n o = \tau^n so$. Passing to the limit in the equalities above, we get in both cases $gs\eta = \eta$.

Therefore, $\eta$ and $s\eta$ are in the same $W_o$-orbit. Thus, we will get the contradiction we want if we choose $\eta$ such that this is not true. Let us call a point $\eta \in \partial_\infty\Sigma$ admissible if it can be constructed as above, *i.e.* is the limit of a sequence of the form $\tau^n o$, with $\tau$ a product of two reflections which act on the same tree. Note that the set of admissible $\eta$ is infinite, and furthermore that there are infinitely many points in the boundary of each tree $T_i$ on which some admissible $\eta$ projects.

We assume that $\eta$ and $s\eta$ are always in the same $W_o$-orbit. This means that there exists some $h \in sW_o$ for which we have $h\eta = \eta$ for infinitely many different $\eta$ in $\partial_\infty\Sigma$, and we may assume that the set of $\eta$ such that $h\eta = \eta$ projects onto an infinity of different points in the boundary of each tree.

We know that $h$ acts either in a hyperbolic or elliptic way on $T_1 \times \cdots \times T_l$. Assume first that $h$ is hyperbolic. Then there is some $n > 0$ such that $h^n$ stabilises every tree $T_i$, and of course $h^n$ fixes the same set of points than $h$. But $h^n$ is still hyperbolic, which means that there is some tree $T_i$ on which it is hyperbolic. But a hyperbolic element of $T_i$ only fixes two points of the boundary, and we get a contradiction.

So $h$ acts in a elliptic way on $T_1 \times \cdots \times T_l$. In particular, it acts with bounded orbits on the Coxeter complex $\Sigma$. Therefore, it is also elliptic on $\Sigma$, an so it must fix a vertex, say $a$. Since we also know that $h$ fixes infinitely many admissible points $\eta$, we know that

$h$ fixes pointwise infinitely many geodesic rays $[a, \eta)$. In particular it fixes also the convex hull of all those rays. If this convex hull did not intersect some open chamber, this would mean that all those rays are contained in some fixed wall $H$ containing $a$. This wall is represented by a vertex in some tree $T_i$, and there is a point $\eta$ fixed by $h$ which projects onto the boundary of $T_i$, and therefore is not in the boundary of $H$.

Therefore, $h$ fixes an open chamber in $\Sigma$. Since $h \in W$, this implies that $h = 1$. Therefore $s \in W_o$, which is a contradiction.

□

**Corollary 3.6.** *With the same hypotheses, the group $G$ has no Gelfand pair.*

*Proof.* Let $K_0$ be a compact subgroup of $G$. By the Bruhat-Tits fixed point lemma [BH99, Corollary 2.8], $K_0$ fixes a point in $|X|$, and therefore fixes a closed facet in $X$. Hence it fixes some vertex $o$, which means that it is contained in the stabilizer $K$ of $o$. By Theorem 3.5, the algebra $\mathcal{L}(G, K)$ is not commutative. Since $\mathcal{L}(G, K)$ is a sub-algebra of $\mathcal{L}(G, K_0)$, this implies that $\mathcal{L}(G, K_0)$ is not commutative, hence $(G, K_0)$ is not a Gelfand pair. □

As previously mentioned, this theorem is proved in [BS06] for Fuchsian right-angled buildings. In fact, using a (suitable) tree in such a building, and considering its (self-normalized) stabilizer $H$, they obtain the unitary representation $\ell^2(G/H)$. This representation is irreducible because $H = N_G(H)$, but has infinitely many linearly independent $K$-fixed vectors, for any compact subgroup $K$. This prevents $K$ from being a Gelfand subgroup of $G$. Moreover, since $H$ is open, non-compact and of infinite index, the vanishing at infinity of matrix coefficients of unitary representations does not hold; this property, called *Howe-Moore property*, holds in the classical Euclidean case, and implies some mixing property for boundary actions.

General groups acting on buildings can also be exotic in another way, related to the Hecke algebra $\mathcal{L}(G, \mathscr{B})$. From an algebraic viewpoint, this algebra is completely understood: it has an explicit presentation, which depends on the Weyl group and on the thicknesses of the building [Bou07, IV.2, Exercises 22 to 24]. Namely, it is generated as a vector space by symbols $a_w$, with $w \in W$. The algebra structure is given by the relations $a_s.a_w = (q_s - 1)a_w + q_s a_{sw}$ if $\ell(sw) < \ell(w)$, and $a_s a_w = a_{sw}$ if $\ell(sw) > \ell(w)$, where $1 + q_s$ is the thickness of panels of type $s \in S$. For algebraic groups over local fields, there are strong constraints on the parameters $q_s$, since they must be a power of the cardinality of the residue field. There are new examples of buildings with strongly transitive automorphism groups and for which any family of independent parameters $\{1 + q_s\}_{s \in S}$ can occur as family of thicknesses, provided $q_s$ is a prime power or a prime power minus 1 [RR06].

To sum up, the Hecke-theoretic part of this paper brings a further element in the dichotomy between automorphism groups of Euclidean and non-Euclidean buildings (another one being the possibility to have simple lattices or not). Indeed, the previous theorems and examples show that, in order to develop harmonic analysis on sufficiently transitive automorphism groups of buildings, some new methods and ideas are required with respect to classical techniques such as Gelfand pairs, Hecke algebras, etc. Still, the analogy viewpoint on automorphism groups of buildings is already known to be fruitful and should be investigated further. For instance, J. Dymara and T. Januszkiewicz [DJ02] proved that many classical techniques and results, such as induction and relationship between continuous and discrete cohomology as developped in the book [BW80], hold for more general for automorphism groups of buildings. The latter work calls for a deeper investigation of the unitary representation theory of such groups.

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