Essential loops of Hamiltonian homeomorphisms

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Abstract

We initiate the study of the fundamental group of the group of Hamiltonian homeomorphisms denoted by $\overline{\mathrm{Ham}}(M,\omega)$, i.e. the C^0-closure of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ in $\mathrm{Homeo}(M)$. We prove that in some situations, namely complex projective spaces and rational Hirzebruch surfaces, certain Hamiltonian loops that were known to be non-trivial in $\pi_1\mathrm{Ham}(M,\omega)$ remain non-trivial in $\pi_1\overline{\mathrm{Ham}}(M,\omega)$. This yields in some cases, including $\mathbb{C}\mathrm{P}^2$ and $S^2\times S^2$, the injectivity of the map $\pi_1\mathrm{Ham}(M,\omega)\to\pi_1\overline{\mathrm{Ham}}(M,\omega)$ induced by the inclusion.

Our method relies on results from C^0 symplectic topology and on computations of the valuation of Seidel elements and hence of the spectral norm on $\pi_1\mathrm{Ham}(M,\omega)$. Some of these computations were known before, but we also present new ones which might be of independent interest.

1 Introduction

Let (M,ω) be a closed symplectic manifold. We denote by $\mathrm{Ham}(M,\omega)$ its group of Hamiltonian diffeomorphisms and by $\overline{\mathrm{Ham}}(M,\omega)$ the closure of $\mathrm{Ham}(M,\omega)$ with respect to the C^0-topology in the set of all homeomorphisms of M. The elements of $\overline{\mathrm{Ham}}(M,\omega)$ are called *Hamiltonian homeomorphisms*. Their behavior is quite well understood on surfaces, in particular thanks to Le Calvez's foliation techniques established in [LC05], see e.g. Le Calvez's work on the subject starting from [LC06a, LC06b]. In higher dimension, partial results were obtained very recently by Buhovsky, Seyfaddini, and the first author, see e.g. [BHS18, BHS21].

In this note, we study the natural map

$$\iota:\mathrm{Ham}(M,\omega)\to\overline{\mathrm{Ham}}(M,\omega)$$

starting at the level of fundamental groups, on some symplectic manifolds of arbitrary dimension. Indeed, while the homotopy type of $\mathrm{Ham}(M,\omega)$ has been *extensively* studied, see below for some references which are used here, absolutely nothing is known about $\overline{\mathrm{Ham}}(M,\omega)$ beyond the case of surfaces where the map ι is known to be a homotopy equivalence. Before getting to the heart of the matter, let us point out that the analogous map $\mathrm{Symp}(M,\omega) \to \overline{\mathrm{Symp}}(M,\omega)$, between the groups of symplectic diffeomorphisms and homeomorphisms, was studied at the π_0 level by the second author [Jan21, Jan22].

1.1 Main results

The upshot of this work is a method which detects non-trivial elements in the image of ι_*, i.e. non-trivial elements in $\pi_1\big(\mathrm{Ham}(M,\omega)\big)$ which survive in the fundamental group after taking the C^0-closure. For our method to work, all symplectic manifolds will be required to be *rational*, meaning that their group of periods $\langle \omega, \pi_2(M)\rangle$ is generated by a unique positive element, which will be denoted by Ω. In other words, $\langle \omega, \pi_2(M)\rangle = \Omega\mathbb{Z}$.

The first application of our method concerns Hamiltonian circle actions and relies on deep work by McDuff and Tolman [MT06]. Recall that a fixed point component of a circle action is *semifree* if it admits a neighborhood in which the stabilizer of every point is either trivial or the whole circle.

Theorem 1. *Consider a Hamiltonian circle action Λ on a compact, rational symplectic manifold (M,ω). Assume that its extremal fixed point components are semifree. Then $\iota_*([\Lambda])$ is non-trivial in $\pi_1(\overline{\mathrm{Ham}}(M,\omega))$.*

As a direct consequence, we deduce the injectivity of ι_* in two different specific situations.

Corollary 2. *The map $\iota_* : \pi_1\big(\mathrm{Ham}(M,\omega)\big) \to \pi_1\big(\overline{\mathrm{Ham}}(M,\omega)\big)$ is injective when (M,ω) is the monotone product $S^2 \times S^2$ and for $\mathbb{C}\mathrm{P}^2$ endowed with the Fubini–Study symplectic form.*

This consequence is straightforward since, in these two cases, all non-trivial elements of $\pi_1(\mathrm{Ham}(M,\omega))$ may be represented by a Hamiltonian circle action satisfying the conditions of Theorem 1. Whenever $\pi_1(\mathrm{Ham}(M,\omega))$ has an element which does not admit such a representative, our method needs some extra work to produce an essential loop of Hamiltonian homeomorphisms. Notice in particular that semifreeness is not preserved under taking non-trivial powers. Therefore, we cannot directly extract information from Theorem 1 about elements of the form h^k with $|k| \neq 1$, even when h can be represented by a circle action with semifree extremal components.

Example 3. The fundamental group of the Hamiltonian diffeomorphism group of $S^2 \times S^2$ endowed with a non-monotone product symplectic form is generated by two order-2 elements and the class of a loop Λ of infinite order. All three are ensured to survive in $\pi_1\big(\overline{\mathrm{Ham}}(M,\omega)\big)$ by Theorem 1, however our method does not detect the even powers of Λ so that, as far as we know, $\iota_*([\Lambda])$ might as well be of order 2. ◀

However, we also have results in these more interesting situations, namely for complex projective spaces of any dimension and for all rational 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$. First, recall that Seidel [Sei97] proved that the group $\pi_1(\mathrm{Ham}(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}}))$ admits a non-trivial element h_n of order $n+1$.

Corollary 4. *For any $n \geqslant 1$, the element $\iota_*(h_n)$ has order $n+1$ in the group $\pi_1(\overline{\mathrm{Ham}}(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}}))$.*

Note that the above corollary can also be obtained by non-symplectic methods, as was pointed out to us by Randal-William. Indeed, Sasao shows in [Sas74] that the action of $U(n+1)$ on $\mathbb{C}\mathrm{P}^n$ induces an isomorphism between $\mathbb{Z}/(n+1)\mathbb{Z}$ and the fundamental group of the group of degree-1 continuous maps from $\mathbb{C}\mathrm{P}^n$ to itself. This immediately implies that the class h_n, which is induced by the aforementioned action, is non-trivial in $\mathrm{Ham}(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$ and in its closure $\overline{\mathrm{Ham}}(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$.

Second, recall that the symplectic 1-point blow-ups $\mathbb{F}^\mu = (\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2, \omega'_\mu)$ of $\mathbb{C}\mathrm{P}^2$ admit a standard symplectic form ω'_μ parameterized by a positive real number μ. The fundamental group of their respective Hamiltonian diffeomorphism groups was computed by Abreu and McDuff [AM00]: it is generated by a unique Hamiltonian circle action of infinite order.

Theorem 5. *The map ι_* is injective on all rational 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$. In other words, whenever $\mu \in \mathbb{Q}$, the map $\iota_* : \pi_1\big(\mathrm{Ham}(\mathbb{F}^\mu)\big) \to \pi_1\big(\overline{\mathrm{Ham}}(\mathbb{F}^\mu)\big)$ is injective.*

Remark 6. Consider the completion $\widehat{\mathrm{Ham}}(M, \omega)$ of $\mathrm{Ham}(M, \omega)$ with respect to the spectral norm γ as in [Hum08, Vit22]. It turns out that all the above results still hold when $\overline{\mathrm{Ham}}(M, \omega)$ is replaced with $\widehat{\mathrm{Ham}}(M, \omega)$. This is thus also true when $\overline{\mathrm{Ham}}(M, \omega)$ is replaced by the completion with respect to Hofer's metric. Such variations are obtained via minor modifications of the proof; see Remark 12 below for more details. ◀

1.2 The method

Our method is based on the following proposition of independent interest. In order to state it, we need to briefly recall a few well-established notions (necessary preliminaries are given in Section 2). The quantum homology of (M, ω) is denoted by $QH(M, \omega)$. We let $\nu : QH(M, \omega) \to \Omega\mathbb{Z} \cup \{-\infty\}$ be the quantum valuation map, and $\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to QH(M, \omega)^\times$ be the Seidel morphism. We consider the map $\Gamma : \pi_1(\mathrm{Ham}(M, \omega)) \to \Omega\mathbb{Z}$ defined by $\Gamma(h) = \nu\big(\mathcal{S}(h)\big) + \nu\big(\mathcal{S}(h^{-1})\big)$.

Proposition 7. *Assume that (M, ω) is a rational symplectic manifold. Then, the map $\Gamma : \pi_1(\mathrm{Ham}(M, \omega)) \to \Omega\mathbb{Z}$ factors through $\pi_1(\overline{\mathrm{Ham}}(M, \omega))$, i.e.*

there exists a map $\overline{\Gamma} : \pi_1(\overline{\mathrm{Ham}}(M,\omega)) \to \Omega\mathbb{Z}$ *such that the following diagram commutes*

$$\begin{tikzcd}
\pi_1(\mathrm{Ham}(M,\omega)) \arrow[r, "\iota_*"] \arrow[d, "\Gamma"'] & \pi_1(\overline{\mathrm{Ham}}(M,\omega)) \arrow[dl, "\overline{\Gamma}"] \\
\Omega\mathbb{Z} &
\end{tikzcd} \tag{1}$$

Based on this proposition, the proof of Theorem 1, Corollary 4, and Theorem 5 boils down to computing Γ in order to show that it is non-zero on elements of $\pi_1\big(\mathrm{Ham}(M,\omega)\big)$. The diagram above then ensures that their image via ι_* in $\pi_1(\overline{\mathrm{Ham}}(M,\omega))$ cannot be trivial.

Remark 8. McDuff already used the map Γ to estimate the length of loops of Hamiltonian diffeomorphisms of 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$, see [McD02, Lemma 5.1]. ◀

Remark 9. All the examples appearing in this work are *strongly semi-positive* symplectic manifolds, which is the setting of Seidel's seminal paper [Sei97]. However, the method introduced in this note extends to a much more general setting, see e.g. [LMP99] and [McD00]. ◀

The proof of Proposition 7 has two ingredients. The first one is the action of the Seidel homomorphism on spectral invariants. Indeed, it is not hard to see that, when (M,ω) is rational, Γ coincides with the restriction of the spectral pseudo-norm $\widetilde{\gamma} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ to $\pi_1\big(\mathrm{Ham}(M,\omega)\big)$, see Section 2.4 for details.

The second ingredient is the following result of C^0-continuity of the spectral pseudo-norm up to the action of the group of periods $\langle\omega,\pi_2(M)\rangle = \Omega\mathbb{Z}$, due to Kawamoto [Kaw22, Theorem 1].

Theorem (Kawamoto)**.** *Let (M,ω) be a rational symplectic manifold. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\phi \in \mathrm{Ham}(M,\omega)$, if $d_{C^0}(\mathrm{id},\phi) < \delta$, then for any lift $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ of ϕ there exists an integer $\ell \in \mathbb{Z}$ such that*

$$|\widetilde{\gamma}(\widetilde{\phi}) - \ell \cdot \Omega| < \varepsilon.$$

Let us now see how these two ingredients are combined to give a proof.

Proof of Proposition 7. The map $\widetilde{\gamma} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ descends to a map $\mathrm{Ham}(M,\omega) \to \mathbb{R}/\Omega\mathbb{Z}$. It follows from Kawamoto's theorem that this map extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$, hence factors through the inclusion:

$$\mathrm{Ham}(M,\omega) \to \overline{\mathrm{Ham}}(M,\omega) \to \mathbb{R}/\Omega\mathbb{Z}\,.$$

Applying the functor π_1, this yields a factorisation

$$\pi_1(\mathrm{Ham}(M,\omega)) \to \pi_1(\overline{\mathrm{Ham}}(M,\omega)) \to \Omega\mathbb{Z}$$

of the restriction of $\widetilde{\gamma}$ to $\pi_1(\mathrm{Ham}(M,\omega))$, which is nothing but Γ. □

1.3 Computation of the spectral pseudo-norm

As mentioned above, the proof of our main results boils down to computing the valuation of the Seidel elements associated to the elements of the fundamental group of Hamiltonian diffeomorphism groups. These computations are based on work by Entov and Polterovich [EP03] in the case of complex projective spaces, and on works by McDuff [McD02], by Ostrover [Ost06], and by Anjos and the third author [AL18, AL17] in the case of the Hirzebruch surfaces ($S^2 \times S^2$ and the 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$).

Since the resulting function Γ coincides with the restriction of the spectral pseudo-norm $\widetilde{\gamma}$ to $\pi_1(\mathrm{Ham}(M, \omega))$ when the manifold is rational, we get explicit computations of the latter. We collect below some phenomena of independent interest concerning $\widetilde{\gamma}$.

Proposition 10. *Let $\widetilde{\gamma} : \pi_1\big(\mathrm{Ham}(M, \omega)\big) \to \mathbb{R}$ denote the restriction of the spectral pseudo-norm.*

(F1) *Let $(M, \omega) = (S^2 \times S^2, \omega_\mu)$ with ω_μ the product symplectic form with area μ on the first factor and 1 on the second.*

When $\mu \in \mathbb{Q}$ and $\mu \neq 1$, $\widetilde{\gamma}$ is degenerate*: $\widetilde{\gamma}^{-1}(\{0\}) = \{h^{2p} \,|\, p \in \mathbb{Z}\}$ where h is the generator of infinite order of $\pi_1\big(\mathrm{Ham}(M, \omega)\big)$.*

(F2) *Let (M, ω) be any symplectic 1-point blow-up $(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2, \omega'_\mu)$ of $\mathbb{C}\mathrm{P}^2$ or the n-dimensional complex projective space $(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}})$, endowed with the standard Fubini–Study symplectic form.*

When $\mu \in \mathbb{Q}$, $\widetilde{\gamma}$ is non-degenerate.

We now focus on 1-point blow-ups $(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2, \omega'_\mu)$ of $\mathbb{C}\mathrm{P}^2$. The symplectic ω'_μ has area $\mu > 0$ on the exceptional divisor, and 1 on the fiber[1]. Under these conventions, the only monotone such symplectic manifold is the one for which $\mu = \frac{1}{2}$.

(F3) *The spectral norm $\widetilde{\gamma}$ is* not bounded *on "small" rational 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$, for which $\mu < \frac{1}{2}$ and $\mu \in \mathbb{Q}$.*

(F4) *On the monotone 1-point blow-up, that is when $\mu = \frac{1}{2}$, we have $\widetilde{\gamma}(k) = 2$ for all non-trivial elements k of $\pi_1\big(\mathrm{Ham}(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2, \omega'_{\mathrm{mon}})\big)$.*

(F5) *The image of the spectral norm is* bounded *on "big" rational 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$, for which $\mu \geqslant \frac{1}{2}$ and $\mu \in \mathbb{Q}$.*

(F6) *Let h' be the generator of $\pi_1\big(\mathrm{Ham}(\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2, \omega'_\mu)\big)$ and $p \in \mathbb{Z}$, then the function $\mu \mapsto \Gamma(h'^p)$, whose restriction to $\mathbb{Q}$ is $\mu \mapsto \widetilde{\gamma}(h'^p)$, is continuous and piecewise linear on $\mathbb{R}$.*

[1] Recall that $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ is the total space of the only non-trivial Hamiltonian fibration over $\mathbb{C}\mathrm{P}^1$ with fiber $\mathbb{C}\mathrm{P}^1$.

Remark 11. Concerning (F1) above, it was proved in [AL17] that Seidel's morphism is injective for *all* Hirzebruch surfaces. The computations of Section 3.3 show that $\nu(\mathcal{S})$ is also injective. However, $\nu(\mathcal{S})$ does not factor through $\pi_1(\overline{\mathrm{Ham}})$ a priori and Γ is *not* injective.

The fact (F2) for 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$ was proved in [McD02], by showing that Γ is positive. In the second part of Section 3.3, we compute the specific values of Γ on the fundamental group of the Hamiltonian diffeomorphism group. These computations yield the facts (F2) to (F6). ◀

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2 Preliminaries

2.1 Quantum homology

Let $\Bbbk$ be a field (which will be chosen to be $\mathbb{Q}$ in general, except for the case of $\mathbb{C}\mathrm{P}^n$ in Section 3.2 for which $\Bbbk = \mathbb{C}$). The (small) *quantum homology* of a strongly semi-positive symplectic manifold (M, ω) is the $\mathbb{Z}$-graded algebra defined as $\mathrm{QH}_*(M; \Lambda) = \mathrm{H}_*(M; \Bbbk) \otimes_{\Bbbk} \Lambda$ where $\Lambda = \Lambda^{\mathrm{univ}}[q, q^{-1}]$ has coefficients in the ring of generalized Laurent series in the degree-0 variable t:

$$\Lambda^{\mathrm{univ}} = \Big\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \,\Big|\, r_\kappa \in \Bbbk \text{ s.t. } \forall c \in \mathbb{R},\ \#\{\kappa > c \,|\, r_\kappa \neq 0\} < \infty \Big\}$$

and q is a variable of degree 2. The grading of an element of the form $a \otimes q^d t^\kappa$ with $a \in \mathrm{H}_l(M; \Bbbk)$ is simply given by $\deg(a \otimes q^d t^\kappa) = l + 2d$.

The *quantum intersection product* on $\mathrm{QH}_*(M; \Lambda)$ is a deformation of the usual intersection product on $\mathrm{H}_*(M; \Bbbk)$ by counts of certain Gromov–Witten invariants. More precisely, for $a \in \mathrm{H}_k(M; \Bbbk)$ and $b \in \mathrm{H}_l(M; \Bbbk)$,

$$a * b = \sum_{B \in H_2^S(M;\mathbb{Z})} (a * b)_B \otimes q^{-c_1(B)} t^{-\omega(B)}$$

where the sum runs over all spherical homology classes B, i.e. classes B in the image $H_2^S(M; \mathbb{Z})$ of the Hurewicz map $\pi_2(M) \to H_2(M; \mathbb{Z})$.

The class $(a * b)_B \in \mathrm{H}_*(M;\Bbbk)$ has degree $k + l - \dim(M) + 2c_1(B)$ and is defined by requiring its usual intersection product with any class $c \in \mathrm{H}_*(M;\Bbbk)$ to be given by the Gromov–Witten invariant

$$(a * b)_B \cdot c = \mathrm{GW}^M_{B,3}(a,b,c) \in \Bbbk$$

which counts the number of spheres in M, in the class B, which meet cycles representing a, b and c. The specific definition of $\mathrm{GW}^M_{B,3}$ is not necessary in this note. Only the following facts will be of interest:

(i) As expected, $\deg(a * b) = \deg(a) + \deg(b) - \dim(M)$.

(ii) The quantum intersection product turns the ring $\mathrm{QH}_*(M;\Lambda)$ into a $\mathbb{Z}$-graded commutative unital algebra.

(iii) The unit of this algebra is the fundamental class $[M]$ of the symplectic manifold, seen as an element in $\mathrm{QH}_{2n}(M;\Lambda)$.

Quantum valuation The quantum homology algebra comes with a natural *valuation* $\nu : \mathrm{QH}_*(M;\Lambda) \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\nu\Big(\sum_{\kappa\in\mathbb{R}} a_\kappa \otimes q^{d_\kappa} t^\kappa\Big) = \max\{\kappa \,|\, a_\kappa \neq 0\}\,.$$

Notice that any non-zero class has finite valuation because of the finiteness condition required in the definition of Λ^{univ}. By convention, the zero class has valuation $-\infty$.

The monotone case Morally speaking, the variables q and t respectively remember the first Chern number and the symplectic area of classes of spheres in $\pi_2(M)$.

In the (positively) monotone case, namely when

$$\exists\lambda > 0 \;\text{ such that }\; \omega|_{\pi_2(M)} = \lambda \cdot c_1|_{\pi_2(M)}\,,$$

the morphisms ω and c_1 are linearly dependent on $\pi_2(M)$. Hence there is no need to carry around two variables: in this case, Λ is usually replaced by $\Bbbk[[s]$ where s is of degree $2N$, i.e. twice the *minimal Chern number* N of M which is the positive generator of $\langle c_1, \pi_2(M)\rangle = N\mathbb{Z}$.

An element of the form $a\otimes s^j$ thus corresponds to $a\otimes q^{jN}t^{j\Omega}$ in the above description, with Ω the positive generator of $\langle \omega, \pi_2(M)\rangle$, which satisfies by monotonicity $\Omega = \lambda N$. The definition of the valuation has to be adapted consequently to $\nu\big(\sum_{j\in\mathbb{Z}} a_j \otimes s^j\big) = \Omega \cdot \max\{j \,|\, a_j \neq 0\}$.

This is for example the setting of [EP03] whose results are used below.

2.2 Floer homology

In order to prove Arnold's conjecture, Floer developed at the end of the 80's [Flo88a, Flo88b, Flo89a, Flo89b] an infinite dimensional Morse–Bott-type homology for the symplectic action functional. As Gromov–Witten invariants, this construction is another striking consequence of Gromov's celebrated work on pseudo-holomorphic curves [Gro85]. (And as Gromov–Witten invariants, it has had countably many applications.)

The upshot of Floer's construction, as far as this note is concerned, is that with a generic pair (H, J) formed of a time-dependent, non-degenerate Hamiltonian function H and a ω-compatible almost complex structure J, one can define a $\mathbb{Z}$-graded complex $(\mathrm{CF}_*(M : H), \partial_{(H,J)})$ whose homology $\mathrm{HF}_*(M, \omega) = \mathrm{H}_*(\mathrm{CF}(M\!:\!H), \partial_{(H,J)})$ satisfies the following properties.

(i) There exist canonical *continuation isomorphisms* between Floer homologies built from different admissible pairs of Floer data (H, J) and (H', J').

(ii) The *pair-of-pants product* $\star$ turns Floer homology into a graded algebra, with unit $[M] \in \mathrm{HF}_{2n}(M, \omega)$, the fundamental class of M seen as a Floer homology class.

(iii) There exist isomorphisms $\mathrm{PSS} : \mathrm{QH}_*(M; \Lambda) \to \mathrm{HF}_*(M, \omega)$ of graded commutative unital algebras.

(iv) Floer complexes are *filtered*: any given $\alpha \in \mathbb{R}$ defines a subcomplex $(\mathrm{CF}^\alpha_*(M\!:\!H), \partial_{(H,J)})$. Its homology does not depend on the choice of J and is denoted by $\mathrm{HF}^\alpha_*(M\!:\!H)$.

2.3 The Seidel morphism

The *Seidel morphism* was described in [Sei97] in two quite different but equivalent ways. On one hand, it can be seen as a morphism

$$\mathcal{S} : \pi_1\big(\mathrm{Ham}(M, \omega)\big) \longrightarrow \mathrm{QH}_*(M; \Lambda)^\times , \qquad h \longmapsto \mathcal{S}(h)$$

where $\mathrm{QH}_*(M; \Lambda)^\times$ denotes the multiplicative group of invertible elements of $\mathrm{QH}_*(M; \Lambda)$. The quantum class $S(h)$ is called the *Seidel element* associated to h. It is defined by counting pseudo-holomorphic sections of a Hamiltonian fibration over S^2 with fibre M obtained from a loop $(\phi^t)_{t\in[0,1]} \in h$ via the clutching construction.

On the other hand, it can be seen as a *representation*

$$\mathcal{S} : \pi_1\big(\mathrm{Ham}(M, \omega)\big) \longrightarrow \mathrm{Aut}(\mathrm{HF}_*(M, \omega)) , \qquad h \longmapsto \mathcal{S}_h .$$

The relation between these two viewpoints is straightforward: the automorphism of Floer homology $\mathcal{S}_h$ is nothing but the pair-of-pants multiplication by $\mathcal{S}(h)$ seen as a Floer homology class via the PSS morphism, i.e.

$$\forall b \in \mathrm{HF}_*(M, \omega) , \quad \mathcal{S}_h(b) = \mathrm{PSS}(\mathcal{S}(h)) \star b .$$

2.4 The spectral norm

The spectral norm is a norm defined on the universal cover of Hamiltonian diffeomorphism groups. It is based on the theory of *spectral invariants*, introduced by Viterbo [Vit92] via generating functions, and adapted to the Floer-theoretic setting by Schwarz [Sch00] for symplectically aspherical manifolds and Oh [Oh05] for monotone manifolds. Since then, they have been defined in a wide range of situations (and also had countably many deep consequences).

Spectral invariants Let $a \in \mathrm{QH}_*(M;\Lambda)$ be a non-zero quantum homology class. For any non-degenerate Hamiltonian function H on M, the *spectral invariant associated to a with respect to H* is the real number

$$c(a\!:\!H) = \inf\{\alpha \in \mathbb{R} \,|\, \mathrm{PSS}(a) \text{ belongs to the image of } \iota^\alpha\}$$

where $\iota^\alpha : HF_*^\alpha(M\!:\!H) \to HF_*(M,\omega)$ is the map induced by the inclusion. These numbers enjoy a list of standard properties. First, they are "spectral" in the sense that $c(a\!:\!H)$ belongs to the set of critical values of the action functional associated to H; this property which explains their name will not be needed here. They are also Hofer continuous, namely

$$\int_0^1 \min_M (H_t - K_t)\,dt \leqslant c(a\!:\!H) - c(a\!:\!K) \leqslant \int_0^1 \max_M (H_t - K_t)\,dt$$

for any two Hamiltonians $H(t,x) = H_t(x)$, $K(t,x) = K_t(x)$. In particular, $c(a\!:\!H)$ continuously depends on H and the map $c(a\!:\cdot\,)$ extends continuously to all (possibly degenerate) Hamiltonians.

Recall that a smooth path of Hamiltonian diffeomorphisms $(h^t)_{t\in[0,1]}$ with $h^0 = \mathrm{id}$ is generated by a unique mean normalized Hamiltonian H, i.e. satisfying $\int_M H_t\,\omega^n = 0$ for all t. Thus we may define spectral invariants of the isotopy (h^t) by setting $c(a\!:\!(h^t)) = c(a\!:\!H)$. These invariants applied to isotopies have the nice feature that they are invariant by homotopy with fixed end points. Therefore, we obtain a well-defined map on the universal cover of $\mathrm{Ham}(M,\omega)$:

$$c(a\!:\cdot\,) : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}\,.$$

These maps satisfy the so-called *triangle inequality*:

$$c(a * b\!:\!\widetilde{h}\widetilde{k}) \leqslant c(a\!:\!\widetilde{k}) + c(b\!:\!\widetilde{h})$$

for any classes a, $b \in QH_*(M,\Lambda)$ such that $a * b \neq 0$, and any $\widetilde{h}$, $\widetilde{k} \in \widetilde{\mathrm{Ham}}(M,\omega)$.

We will mainly use the spectral invariants associated to the fundamental class. Therefore, it will be convenient to introduce the notation $c_+(H) = c([M]\!:\!H)$ and $c_+(\widetilde{h}) = c([M]\!:\!\widetilde{h})$. Note that the triangle inequality for c_+ takes the form

$$c_+(\widetilde{h}\widetilde{k}) \leqslant c_+(\widetilde{h}) + c_+(\widetilde{k})\,. \tag{2}$$

Relation between c, $\mathcal{S}$, and ν In the language of spectral invariants, the quantum valuation defined in Section 2.1 behaves as the spectral invariant computed with respect to the zero Hamiltonian / the identity in $\widetilde{\mathrm{Ham}}$. By this, we mean that for any non-zero quantum homology class $a \in \mathrm{QH}_*(M;\Lambda)$

$$c(a:\widetilde{\mathrm{id}}) = \nu(a)\,. \tag{3}$$

Moreover, because Seidel's representation provides automorphisms of *filtered* Floer complexes, the action of any $h \in \pi_1(\mathrm{Ham}(M,\omega))$ on $\widetilde{\mathrm{Ham}}(M,\omega)$ (or in other words the difference between lifts of $\phi \in \mathrm{Ham}(M,\omega)$ to the universal cover) yields, in terms of spectral invariants:

$$\forall a \in \mathrm{QH}_*(M;\Lambda)\,, \quad c(a:\widetilde{k}h) = c(\mathcal{S}(h) * a:\widetilde{k}) \tag{4}$$

for all $\widetilde{k} \in \widetilde{\mathrm{Ham}}(M,\omega)$. In the specific situation where $\widetilde{k} = \widetilde{\mathrm{id}}$ and $a = [M]$, we get

$$c_+(h) = c([M]:h) = c(\mathcal{S}(h) * [M]:\widetilde{\mathrm{id}}) = \nu(\mathcal{S}(h)) \tag{5}$$

respectively by definition of c_+, by (4), and finally by (3) and the fact that $[M]$ is the unit of the quantum homology algebra. If not litterally *well-known*, this equality has been successfully used before, see e.g. Proposition 4.1 in [EP03], Section 6.3 of [Ost06], and Property (2.4) of spectral numbers in [McD10].

The spectral pseudo-norm The spectral pseudo-norm is defined as a group pseudo-norm $\widetilde{\gamma} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ by

$$\widetilde{\gamma}(\widetilde{h}) = c_+(\widetilde{h}) + c_+(\widetilde{h}^{-1}) = c_+(H) + c_+(\overline{H})$$

with H generating $(\phi_H^t)_{t\in[0,1]} \in \widetilde{h}$ and $\overline{H}$ the Hamiltonian function defined by $\overline{H}_t(x) = -H_{1-t}(x)$. Standard computations show that $\overline{H}$ generates the isotopy $\phi_H^{1-t}(\phi_H^1)^{-1} \in \widetilde{h}^{-1}$. The triangle inequality for $\widetilde{\gamma}$ as well as its non-negativity follow from (2) and the fact that $\widetilde{\gamma}(\widetilde{\mathrm{id}}) = 0$.

Notice that (5) yields for any $h \in \pi_1(\mathrm{Ham}(M,\omega))$

$$\widetilde{\gamma}(h) = c_+(h) + c_+(h^{-1}) = \nu(\mathcal{S}(h)) + \nu(\mathcal{S}(h^{-1})) = \Gamma(h) \tag{6}$$

where Γ, defined by the last equality, is the central object of Proposition 7 from the introduction. Notice that this implies that

$$\widetilde{\gamma}(h) \in \langle \omega, \pi_2(M)\rangle, \quad \forall h \in \pi_1(\mathrm{Ham}(M,\omega)).$$

The pseudo-norm $\widetilde{\gamma}$ induces a genuine (non-degenerate) norm γ on the group $\mathrm{Ham}(M,\omega)$, called the *spectral norm*, by the formula

$$\gamma(\phi) = \inf\left\{\widetilde{\gamma}(\widetilde{h}) \,\middle|\, \widetilde{h} \in \widetilde{\mathrm{Ham}}(M,\omega), \pi(\widetilde{h}) = \phi\right\}.$$

REMARK 12. Assume (M, ω) is rational and let Ω be the positive generator of $\langle \omega, \pi_2(M) \rangle$. It is known that if $\widetilde{h}_1$, $\widetilde{h}_2$ are two lifts to $\widetilde{\mathrm{Ham}}(M, \omega)$ of a given $\phi \in \mathrm{Ham}(M, \omega)$, then the difference $\widetilde{\gamma}(\widetilde{h}_1) - \widetilde{\gamma}(\widetilde{h}_2)$ belongs to $\langle \omega, \pi_2(M) \rangle = \Omega\mathbb{Z}$. It follows that for any $\phi \in \mathrm{Ham}(M, \omega)$ and any lift $\widetilde{h}$ of ϕ to $\widetilde{\mathrm{Ham}}(M, \omega)$, there exists an integer $\ell \in \mathbb{Z}$ such that

$$\widetilde{\gamma}(\widetilde{h}) - \ell \cdot \Omega = \gamma(\phi).$$

Therefore, Kawamoto's theorem from Section 1.2 holds after replacing the C^0 distance d_{C^0} with the spectral distance γ. We then deduce that Proposition 7 similarly holds with the γ-completion $\widehat{\mathrm{Ham}}(M, \omega)$ replacing $\overline{\mathrm{Ham}}(M, \omega)$. We can then use this modified version of Proposition 7 to adapt our proofs of Theorems 1, 4, 5 as well as Corollary 2 to the case of $\widehat{\mathrm{Ham}}(M, \omega)$. This justifies the part of Remark 6 related to γ.

For the part related to Hofer's distance (which we denote here by δ), the Hofer continuity of spectral invariants yields $\gamma \leqslant \delta$ and thus a natural map $\widehat{\mathrm{Ham}}^{\delta}(M, \omega) \to \widehat{\mathrm{Ham}}^{\gamma}(M, \omega)$ between the respective completions, and hence a factorization

$$\pi_1(\mathrm{Ham}(M, \omega)) \to \pi_1(\widehat{\mathrm{Ham}}^{\delta}(M, \omega)) \to \pi_1(\widehat{\mathrm{Ham}}^{\gamma}(M, \omega))\,.$$

Thus, our claims from Remark 6 for γ yield the same claims for δ. ◀

3 Computations

3.1 Proof of Theorem 1

This is a simple remark based on the following deep result from McDuff and Tolman which is part of [MT06, Theorem 1.10].

Theorem (McDuff–Tolman)**.** *Let Λ be a Hamiltonian circle action on a compact symplectic manifold (M, ω), generated by the moment map $K : M \to \mathbb{R}$. Assume K to be normalized and let $K_{\max} = \max_M K$. Assume that the fixed point component $F_{\max} = K^{-1}(K_{\max})$ is semifree. Then there are classes $a_B \in H_*(M)$ so that*

$$\mathcal{S}(\Lambda) = [F_{\max}] \otimes q^{-m_{\max}} t^{K_{\max}} + \sum a_B \otimes q^{-m_{\max} - c_1(B)} t^{K_{\max} - \omega(B)} \tag{7}$$

where the sum runs over all spherical class $B \in \mathrm{H}_2^S(M)$ with $\omega(B) > 0$.

Recall that semifreeness means that the action acts semifreely: the stabilizer of each point is trivial or the whole circle in a neighbourhood of $F_{\max}$. The integer $m_{\max}$ will not be used below. Let us simply mention that it is determined by the degree of $\mathcal{S}(\Lambda)$ and that it corresponds to the sum of the weights at a point $x \in F_{\max}$. Note that McDuff and Tolman were also able to specify the structure of the lower order terms in exchange of additional requirements on ω-compatible almost complex structures.

For our purpose, it is enough to notice that, since $[F_{\max}]$ and the a_B's are honest classes in $\mathrm{H}_*(M)$, the valuation of $\mathcal{S}(\Lambda)$ is $K_{\max}$. Moreover, Λ^{-1} is generated by $-K$ whose maximal fixed point component is nothing but $(-K)^{-1}(\max_M(-K)) = K^{-1}(\min_M K) = F_{\min}$. Requiring both extremal fixed point components to be semifree and applying McDuff–Tolman's Theorem to K and $-K$, yield

$$\Gamma(\widetilde{\Lambda}) = \nu(\mathcal{S}(\Lambda)) + \nu(\mathcal{S}(\Lambda^{-1})) = K_{\max} - K_{\min}\,.$$

Since K is not constant, $\Gamma(\widetilde{\Lambda}) > 0$. Under the additional assumption that (M,ω) is rational, Proposition 7 ensures that $\iota_*(\widetilde{\Lambda})$ is not trivial in $\pi_1(\overline{\mathrm{Ham}}(M,\omega))$. □

3.2 Proof of Theorem 4

Recall from [Sei97] that Seidel's morphism detects an element h_n of order $n+1$ in $\pi_1(\mathrm{Ham}(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}}))$. We claim that Γ is positive on all classes detected by Seidel's morphism. Since $(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}})$ is monotone, it is rational and Proposition 7 ensures that $\iota_*(h_n)$ is of order $n+1$ in $\overline{\mathrm{Ham}}(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}})$.

We now proceed and show that

$$\forall\, h \in \pi_1(\mathrm{Ham}(\mathbb{C}\mathrm{P}^n,\omega_{\mathrm{FS}})),\quad h \notin \ker(\mathcal{S}) \implies \Gamma(h) = \Omega \tag{8}$$

where $\Omega = \omega_{\mathrm{FS}}([\mathbb{C}\mathrm{P}^1]) > 0$ is the generator of $\langle \omega_{\mathrm{FS}}, \pi_2(\mathbb{C}\mathrm{P}^n)\rangle$. This obviously implies our claim, hence concluding the proof.

We recall that the quantum homology algebra of $\mathbb{C}\mathrm{P}^n$ is isomorphic to the algebra $\mathrm{k}[A]/\{A^{n+1} = s^{-1}\}$ with $\mathrm{k} = \Bbbk[[s]$, see our comment on the monotone case at the end of Section 2.1. Here, A is the hyperplane class of degree $2n-2$ and s is of degree $2N = 2(n+1)$. Notice that for any $m \in \mathbb{Z}$, A^m has degree $2n-2m$.

Moreover, recall from Proposition 4.2 of [EP03], that all Seidel elements are monomials of the form $A^m s^\beta$. Since they belong to $\mathrm{QH}_{2n}(\mathbb{C}\mathrm{P}^n;\Lambda)$, $\beta = \frac{m}{n+1}$ for degree reasons. Notice that when m is a multiple of $n+1$, we get $A^m s^\beta = [\mathbb{C}\mathrm{P}^n]$ as expected (see Proposition 4.3 of [EP03]).

We now assume that m is not a multiple of $n+1$. We need to compute $\nu(A^m s^\beta) + \nu(A^{-m}s^{-\beta})$ so that we may assume that $m \geqslant 0$. Then

$$m = q\cdot(n+1) + r \quad \text{ and } \quad -m = -(q+1)\cdot(n+1) + (n+1-r)$$

with $q = \lfloor \frac{m}{n+1} \rfloor$, and r and $n+1-r$ are integers in $(0, n+1)$. Hence,

$$A^m s^{\frac{m}{n+1}} = (A^{n+1})^q \cdot A^r s^{\frac{m}{n+1}} = ([\mathbb{C}\mathrm{P}^n] s^{-1})^q \cdot A^r s^{\frac{m}{n+1}} = A^r s^{\frac{m}{n+1}-q}$$

whose valuation is $\Omega \cdot (\frac{m}{n+1} - q) = \Omega \cdot (\frac{m}{n+1} - \lfloor \frac{m}{n+1} \rfloor)$ since $A^r \in \mathrm{H}_*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$. Similarly, by replacing q by $-(q+1)$ and r by $n+1-r$, we get that $A^{-m}s^{\frac{-m}{n+1}}$ has valuation $\Omega \cdot (\lfloor \frac{m}{n+1} \rfloor + 1 - \frac{m}{n+1})$, so that $\nu(A^m s^\beta) + \nu(A^{-m}s^{-\beta}) = \Omega$.

Hence, $\Gamma(h) = \Omega$ whenever h has a non-trivial associated Seidel element which proves (8) and ends the proof of Theorem 4. $\square$

3.3 About Example 3 and the proof of Theorem 5

In this section, we compute the Seidel elements associated to all elements of the fundamental group of the group of Hamiltonian diffeomorphisms of all Hirzebruch surfaces. This completes partial computations from [McD02], [Ost06], and [AL17].

Recall that in [AL18], Anjos and the third author computed Seidel elements of many 4-dimensional toric manifolds starting from the fundamental aforementioned result from McDuff and Tolman. In [AL17], these computations were used to prove that Seidel's morphism detects all the generators of the fundamental group of the Hamiltonian diffeomorphism group of all Hirzebruch surfaces, namely all symplectic products of $S^2 \times S^2$ and 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$.

We now show that, when we consider Γ rather than the Seidel elements themselves, the situation is more subtle:

- Γ vanishes on certain essential loops of Hamiltonian diffeomorphisms of all products of spheres except the monotone one, this will justify Example 3 and Fact (F1) of Proposition 10 from the introduction ;
- Γ is positive on all essential loops of Hamiltonian diffeomorphisms of the 1-point blow-ups of $\mathbb{C}\mathrm{P}^2$, this will prove Theorem 5 and justify Fact (F2) of Proposition 10.

The computations done in the latter case will also clarify Facts (F3) to (F6) of Proposition 10.

Remark 13. Note that all the quantities $\mathcal{S}, \nu, \Gamma, \widetilde{\gamma}$ depend on the symplectic forms, themselves being parameterized by a real number μ in both cases. However, we omit μ in the notation for short throughout this section. ◀

Even Hirzebruch surfaces. Recall that $\mathbb{F}^\mu_{2k}$ is identified to $M = S^2 \times S^2$ endowed with the symplectic form ω_μ with area 1 on the first factor and μ on the second.

We denote by $u = [S^2 \times \{\mathrm{pt}\}] \otimes q$ and $v = [\{\mathrm{pt}\} \times S^2] \otimes q$ the degree 4 quantum homology classes induced by each component of the product $M = S^2 \times S^2$. The quantum homology algebra of (M, ω_μ) is

$$\mathrm{QH}_*(M,\omega_\mu) \simeq \Lambda^{\mathrm{univ}}[u,v]/\langle u^2 = t^{-1}, v^2 = t^{-\mu}\rangle \,.$$

The fundamental group of the group of Hamiltonian diffeomorphisms of $\mathbb{F}_{2k}^{\mu}$ was computed in [AM00].
• When $\mu = 1$, the manifold (M, ω_1) is monotone hence rational, and the fundamental group of $\mathrm{Ham}(M, \omega_1)$ is generated by two elements of order 2, each generated by one factor of the product. These generators are Hamiltonian circle actions which satisfy the assumption of Theorem 1 so that they induce non-trivial elements in $\overline{\mathrm{Ham}}(M, \omega_1)$. Since they are of order 2, there is nothing more to prove.
• When $\mu > 1$ and rational, (M, ω_μ) is rational and $\pi_1(\mathrm{Ham}(M, \omega_\mu))$ is generated by the same order-2 circle actions, together with a third Hamiltonian circle action of infinite order Λ (denoted $\Lambda^2_{e_1}$ in [AL17]). The Seidel element of the latter and of its inverse are given by

$$\mathcal{S}(\Lambda) = (u + v) \otimes t^{\frac{1}{2}-\epsilon} \quad \text{and} \quad \mathcal{S}(\Lambda)^{-1} = (u - v) \otimes \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}}$$

with $\epsilon = \frac{1}{6\mu}$. Hence, for any positive integer ℓ,

$$\mathcal{S}(\Lambda)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} u^k v^{\ell-k} t^{(\frac{1}{2}-\epsilon)\ell}\,.$$

Notice that, depending on the parity of k and ℓ and up to some power of t, $u^k v^{\ell-k}$ is $[M]$, u, v, or uv, so that no products of powers of u and v vanish. Moreover, for all k and ℓ,

$$\nu\big(u^k v^{\ell-k} t^{(\frac{1}{2}-\epsilon)\ell}\big) = -\Big\lfloor \frac{k}{2} \Big\rfloor - \mu \Big\lfloor \frac{\ell-k}{2} \Big\rfloor - \Big(\frac{1}{2} - \epsilon\Big)\ell\,.$$

Since $\mu > 1$, we get that $\nu(\mathcal{S}(\Lambda)^\ell) = -\lfloor \frac{\ell}{2} \rfloor + (\frac{1}{2} - \epsilon)\ell$.

The computation of $\nu(\mathcal{S}(\Lambda)^{-\ell})$ is similar. Only notice additionally that $\frac{1}{1-t^{1-\mu}} = 1 + t^{1-\mu} + t^{2(1-\mu)} + \ldots$ so that, since $1 - \mu < 0$, $\nu\big((\frac{1}{1-t^{1-\mu}})^\ell\big) = 0$. Hence, we get $\nu(\mathcal{S}(\Lambda)^{-\ell}) = -\lfloor \frac{\ell}{2} \rfloor + (\frac{1}{2} + \epsilon)\ell$ which yields

$$\Gamma(\Lambda^\ell) = \ell - 2\Big\lfloor \frac{\ell}{2} \Big\rfloor = \begin{cases} 0 \text{ if } \ell \text{ is even,} \\ 1 \text{ if } \ell \text{ is odd.} \end{cases}$$

Hence, we see that Γ detects only odd powers of Λ, it follows that $\iota_*([\Lambda]) \neq 0$ (but it could be of order 2). $\square$

Odd Hirzebruch surfaces. We follow the conventions from [AL17, Section 3.2]: $\mathbb{F}_{2k+1}^{\mu}$ is identified with $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}^2}$ endowed with the symplectic form ω'_μ such that

- the exceptional divisor B of self-intersection -1 has area $\mu > 0$,
- the fiber F has area 1,

- the projective line $B+F$ has area $\mu+1$.

As a vector space, its quantum homology is generated by: $u_0 = [\mathrm{pt}] \otimes q^2$, $u = F \otimes q$, $u_3 = B \otimes q$, and $\mathbb{1}$. It is also convenient to denote by $u_1 = (B+F) \otimes q = u + u_3$. These classes satisfy the following relations[2]:

$$u_1\, u_3 = t^{-1}\,, \quad u^2 = u_3\, t^{-\mu}\,, \quad u^{-1} = u_0\, t^{\mu+1}\,. \tag{9}$$

As an algebra, the quantum homology is given as the quotient

$$\mathrm{QH}_*(\mathbb{F}^{\mu}_{2k+1}) \simeq \Lambda^{\mathrm{univ}}[u]/\langle u^4 t^{2\mu} + u^3 t^{\mu} - t^{-1}\rangle\,.$$

Last (but not least!), recall from [AM00] that the fundamental group of the group of Hamiltonian diffeomorphisms of $\mathbb{F}^{\mu}_{2k+1}$ is generated by a single class of infinite order. This class is induced by a circle action Λ whose associated Seidel element is $\mathcal{S}(\Lambda) = u^{-1} t^{-\varepsilon}$ where $\varepsilon = \frac{3\mu^2+3\mu+1}{3(1+2\mu)}$. Hence we get, for all integers p,

$$\Gamma(\Lambda^p) = \nu(\mathcal{S}(\Lambda^p)) + \nu(\mathcal{S}(\Lambda^{-p})) = \nu(u^{-p}) + \nu(u^p)\,.$$

The following proposition collects the results of the computations of $\Gamma(\Lambda^p)$ for all p (greater than some p_0) for all possible values of the parameter $\mu > 0$.

Proposition 14. • *For* $0 < \mu \leqslant \frac{1}{2}$,

$$\forall p \geqslant 7, \quad \Gamma(\Lambda^p) = -2\Big(\Big\lfloor \frac{p-1}{3} \Big\rfloor - 1\Big) \cdot \mu + \Big(\Big\lfloor \frac{p-1}{3} \Big\rfloor + 1\Big)\,. \tag{10}$$

- *For $\mu > \frac{1}{2}$, the value of $\Gamma(\Lambda^p)$ depends on the rest ℓ of the Euclidean division of p by 4, and on the sign of $\mu - 1$. Namely, for all $p \geqslant 12$, $\Gamma(\Lambda^p)$ is given by the following table:*

ℓ	$\frac{1}{2} < \mu \leqslant 1$	$1 < \mu$
0	2	2
1	2	$\mu+1$
2	$-2\mu+3$	1
3	2	$\mu+1$

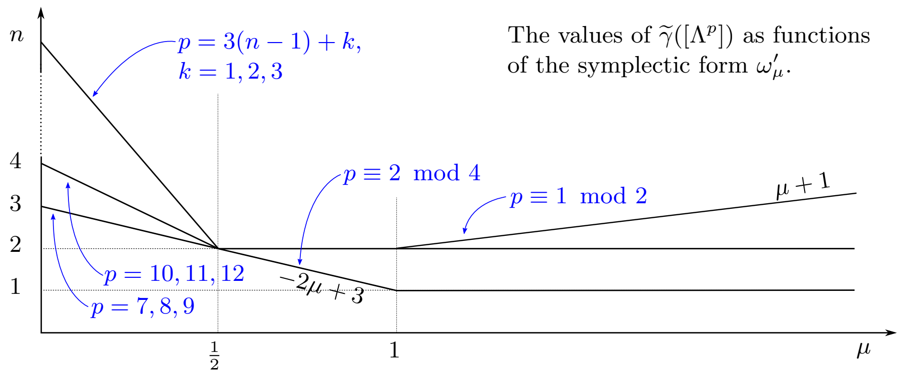
(11)

This proposition shows that, for any $\mu > 0$, $\Gamma(\Lambda^p) \neq 0$ for p big enough[3] which yields that $\iota_*([\Lambda])$ is of infinite order in $\pi_1\big(\overline{\mathrm{Ham}}(\mathbb{F}^{\mu}_{2k+1})\big)$ for any positive, *rational* number μ. Notice that Theorem 5 and the facts (F2) to (F6) from Proposition 10 follow from this proposition.

The graph below illustrates Proposition 14.

[2]The last one does not appear as such in [AL17] as it was not needed to get the expression of the quantum homology algebra. It is necessary here and can easily be computed with the methods used there. See also [MT06, Remark 5.6] and [Ost06, Section 3.3] in which it is explicitly computed *under different conventions* though.

[3]Computing the first few terms shows that this fact actually holds for all integers p. The formulas we present below, however, do not hold for small integers p.



Remark 15. It is interesting to compare the first item in Proposition 14 with the results of Ostrover [Ost06]. Indeed, he establishes that for $\mu > \frac{1}{2}$, the quantum homology $QH_4(\mathbb{F}^{\mu}_{2k+1})$ is a field, while it *splits* into field summands for $\mu < \frac{1}{2}$. Moreover, he proves that the restriction of Γ to any field summand is bounded. Our result shows that Γ is however not bounded on the whole quantum homology in the case $\mu < \frac{1}{2}$. ◀

It remains to prove Proposition 14 which follows from intermediate computations whose results are collected below, depending on the value of μ.

	$\nu(u^{-p})$	$\nu(u^p)$
$0 < \mu \leqslant \frac{1}{2}$	$\left(p - 2\lfloor\frac{p-1}{3}\rfloor\right)\mu + \left(\lfloor\frac{p-1}{3}\rfloor + 1\right)$	$-(p-2)\mu$
$\frac{1}{2} < \mu \leqslant 1$	$\left(p - 2\lfloor\frac{p+2}{4}\rfloor\right)\mu + \left(\lfloor\frac{p+2}{4}\rfloor + 1\right)$	$-\left(p - 2\lfloor\frac{p+1}{4}\rfloor\right)\mu - \left(\lfloor\frac{p+1}{4}\rfloor - 1\right)$
$1 < \mu$	$\lfloor\frac{p+1}{2}\rfloor\mu + \left(\lfloor\frac{p}{4}\rfloor + 1\right)$	$-\lfloor\frac{p}{2}\rfloor\mu - \lfloor\frac{p-1}{4}\rfloor$

These formulas are proved by induction. Let us first focus on the latter two cases.

$\boxed{\text{We assume } \mu > \frac{1}{2}.}$

We wish to show that for p big enough, $\nu(u^{-p})$ and $\nu(u^p)$ are of the form specified in the table above, depending on the sign of $\mu - 1$. The values of $\nu(u^{-p})$ will be extracted from the proof of the following lemma.

Lemma 16. *For all integers $q \geqslant 3$, there are polynomials P_i^q in t, so that*

$$u^{-4q} = u_0\, P_0^q(t) + u_1\, P_1^q(t) + u\, P_2^q(t) + \mathbb{1} P_3^q(t).$$

These polynomials are of the form

$$\begin{aligned}
P_0^q(t) &= \alpha_0^q\, t^{2q\mu+(q+1)} && + \textit{lower order terms}\\
P_1^q(t) &= \alpha_1^q\, t^{(2q-1)\mu+(q+1)} && + \textit{lower order terms}\\
P_2^q(t) &= \alpha_2^q\, t^{(2q-1)\mu+(q+1)} && + \textit{lower order terms}\\
P_3^q(t) &= \alpha_3^q\, t^{2q\mu+q} && + \textit{lower order terms}
\end{aligned}$$

where all coefficients α_i^q are positive.

Its proof consists of tedious but straigthforward computations. First, notice that the relations (9), give $u^{-1} = u_0\, t^{\mu+1}$ and the following multiplication table

$$\begin{array}{c||c|c|c|c} a & u_0 & u_1 & u & \mathbb{1} \\ \hline a * u^{-1} & u_1 & u\, t^{\mu} + \mathbb{1} & \mathbb{1} & u_0\, t^{\mu+1} \end{array} \tag{12}$$

Next, compute explicitly u^{-4q} for $q = 3$ as initialization:

$$u^{-12} = 3u_0\, t^{6\mu+4} + 3u_1\, t^{5\mu+4} + u\, t^{5\mu+4} + \mathbb{1}(t^{6\mu+3} + t^{4\mu+4})$$

which is of the form specified in Lemma 16 since $\mu \geqslant \frac{1}{2}$ (otherwise P_3^3 would have valuation $4\mu + 4$).

Finally, assume u^{-4q} has the expression given in the lemma and compute the 4 next powers of u. Up to lower order terms, we get (to ease the reading we denote $P_i^q(t)$ simply by P_i):

$$\begin{aligned} u^{-4q-1} &= u_0 P_3 t^{\mu+1} + u_1 P_0 + u P_1 t^{\mu} + \mathbb{1}(P_1 + P_2) \\ u^{-4q-2} &= u_0(P_1 + P_2)t^{\mu+1} + u_1 P_3 t^{\mu+1} + u P_0 t^{\mu} + \mathbb{1}(P_0 + P_1 t^{\mu}) \\ u^{-4q-3} &= u_0(P_0 + P_1 t^{\mu})t^{\mu+1} + u_1(P_1 + P_2)t^{\mu+1} + u P_3 t^{2\mu+1} + \mathbb{1}(P_0 + P_3 t)t^{\mu} \end{aligned} \tag{13}$$

This yields for $u^{-4(q+1)}$:

$$\begin{aligned} P_0^{q+1}(t) &= P_3^q(t) \cdot t^{2\mu+2} + P_0^q(t) \cdot t^{2\mu+1} && + \text{l.o.t.} \\ P_1^{q+1}(t) &= P_0^q(t) \cdot t^{\mu+1} + P_1^q(t) \cdot t^{2\mu+1} && + \text{l.o.t.} \\ P_2^{q+1}(t) &= (P_1^q(t) + P_2^q(t)) \cdot t^{2\mu+1} && + \text{l.o.t.} \\ P_3^{q+1}(t) &= P_3^q(t) \cdot t^{2\mu+1} + (P_1^q(t) + P_2^q(t)) \cdot t^{\mu+1} && + \text{l.o.t.} \end{aligned}$$

whose respective leading order terms are

$$\begin{aligned} P_0^{q+1} &: \quad (\alpha_0^q + \alpha_3^q)\, t^{2(q+1)\mu+(q+1)+1} \\ P_1^{q+1} &: \quad (\alpha_0^q + \alpha_1^q)\, t^{(2(q+1)-1)\mu+((q+1)+1)} \\ P_2^{q+1} &: \quad (\alpha_1^q + \alpha_2^q)\, t^{(2(q+1)-1)\mu+((q+1)+1)} \\ P_3^{q+1} &: \quad \alpha_3^q\, t^{2(q+1)\mu+(q+1)} \end{aligned}$$

since all the coefficients α_i^q are positive and since $\mu > \frac{1}{2}$ so that $(P_1^q(t) + P_2^q(t)) \cdot t^{\mu+1}$ only consists of lower order terms of P_3^{q+1}.

This proves the lemma. The formula giving $\nu(u^{-p})$ now follows from the lemma together with the valuation of intermediate powers of u^{-1} which easily follow from the computations (13). Indeed, depending on the sign of

$\mu - 1$, we get:

$$\begin{array}{c|c|c}
 & \frac{1}{2} < \mu \leqslant 1 & 1 < \mu \\
\hline
p = 4q & \multicolumn{2}{c}{2q\mu + (q+1)} \\
\hline
p = 4q+1 & \multicolumn{2}{c}{(2q+1)\mu + (q+1)} \\
\hline
p = 4q+2 & 2q\mu + (q+2) & (2q+1)\mu + (q+1) \\
\hline
p = 4q+3 & (2q+1)\mu + (q+2) & 2(q+1)\mu + (q+1)
\end{array} \tag{14}$$

We now turn to the valuation of positive powers of u. Symmetrically to the negative powers, $\nu(u^p)$ will be extracted from the proof of the following lemma.

Lemma 17. *For all integers $q \geqslant 2$, there are polynomials Q_i^q in t, so that*

$$u^{4q} = u_0\, Q_0^q(t) + u\, Q_2^q(t) + u_3\, Q_3^q(t) + \mathbb{1} Q_4^q(t).$$

These polynomials are of the form

$$\begin{aligned}
Q_0^q(t) &= -\beta_0^q\, t^{-2q\mu-(q-1)} && + \textit{lower order terms} \\
Q_2^q(t) &= -\beta_2^q\, t^{-(2q+1)\mu-(q-1)} && + \textit{lower order terms} \\
Q_3^q(t) &= \beta_3^q\, t^{-(2q+1)\mu-(q-1)} && + \textit{lower order terms} \\
Q_4^q(t) &= \beta_4^q\, t^{-2q\mu-q} && + \textit{lower order terms}
\end{aligned}$$

where all coefficients β_i^q are positive.

Notice that we changed the basis of the cohomology, this yields easier computations and a slight notational discrepancy. The relevant multiplication table now is

$$\begin{array}{c||c|c|c|c}
a & u_0 & u & u_3 & \mathbb{1} \\
\hline
a * u & \mathbb{1}\, t^{-\mu-1} & u_3\, t^{-\mu} & u_0 - u_3\, t^{-\mu} & u
\end{array} \tag{15}$$

We initialize the induction with u^{4q} for $q = 2$:

$$u^8 = -u_0\,(2t^{-4\mu-1} + t^{-6\mu}) - u\, t^{-5\mu-1} + u_3\,(3t^{-5\mu-1} + t^{-7\mu}) + \mathbb{1}(t^{-4\mu-2} + t^{-6\mu-1})$$

which is seen to be of the form specified by Lemma 17, again, because $\mu \geqslant \frac{1}{2}$.

We now assume u^{4q} has the expression given above and compute the 4 next powers of u. To ease the reading we denote $Q_i^q(t)$ simply by Q_i and we introduce the notation $Q_{2,3}$ for $Q_2 - Q_3$. We get, up to lower order terms:

$$\begin{aligned}
u^{4q+1} &= u_0 Q_3 + u Q_4 + u_3 Q_{2,3} t^{-\mu} + \mathbb{1} Q_0 t^{-\mu-1} \qquad (16) \\
u^{4q+2} &= u_0 Q_{2,3} t^{-\mu} + u Q_0 t^{-\mu-1} + u_3(Q_4 t^{-\mu} - Q_{2,3} t^{-2\mu}) + \mathbb{1} Q_3 t^{-\mu-1} \\
u^{4q+3} &= u_0(Q_4 t^{-\mu} - Q_{2,3} t^{-2\mu}) + u Q_3 t^{-\mu-1} \\
&\quad + u_3(Q_0 t^{-2\mu-1} - Q_4 t^{-2\mu} + Q_{2,3} t^{-3\mu}) + \mathbb{1} Q_{2,3} t^{-2\mu-1}
\end{aligned}$$

This yields for $u^{4(q+1)}$ (again, *up to lower order terms*):

$$
\begin{aligned}
&Q_0^{q+1}(t) = Q_0^q(t)\cdot t^{-2\mu-1} - Q_4^q(t)\cdot t^{-2\mu} + (Q_2^q(t) - Q_3^q(t))\cdot t^{-3\mu}\\
&Q_2^{q+1}(t) = (Q_2^q(t) - Q_3^q(t))\cdot t^{-2\mu-1}\\
&Q_3^{q+1}(t) = -Q_0^q(t)\cdot t^{-3\mu-1} + Q_3^q(t)\cdot t^{-2\mu-1} + Q_4^q(t)\cdot t^{-3\mu} - (Q_2^q(t) - Q_3^q(t))\cdot t^{-4\mu}\\
&Q_4^{q+1}(t) = Q_4^q(t)\cdot t^{-2\mu-1} - (Q_2^q(t) - Q_3^q(t))\cdot t^{-3\mu-1}
\end{aligned}
$$

whose respective leading order terms are

$$
\begin{aligned}
Q_0^{q+1} &: \quad -(\beta_0^q + \beta_4^q)\, t^{-2(q+1)\mu-((q+1)-1)}\\
Q_2^{q+1} &: \quad -(\beta_2^q + \beta_3^q)\, t^{-(2(q+1)+1)\mu-((q+1)-1)}\\
Q_3^{q+1} &: \quad (\beta_0^q + \beta_3^q + \beta_4^q)\, t^{-(2(q+1)+1)\mu-((q+1)-1)}\\
Q_4^{q+1} &: \quad \beta_4^q\, t^{-2(q+1)\mu-(q+1)}
\end{aligned}
$$

since all the coefficients β_i^q are positive and since $\mu > \frac{1}{2}$.

This proves Lemma 17. The expression of $\nu(u^p)$ then follows from it, together with the valuation of intermediate powers of u which easily follow from the computations (16). Namely, we get

	$\frac{1}{2} < \mu \leqslant 1$	$1 < \mu$
$p = 4q$	$-2q\mu - (q-1)$	
$p = 4q+1$	$-(2q+1)\mu - (q-1)$	$-2q\mu - q$
$p = 4q+2$	$-2(q+1)\mu - (q-1)$	$-(2q+1)\mu - q$
$p = 4q+3$	$-(2q+1)\mu - q$	

(17)

Together with the expression of $\nu(u^{-p})$ given by Table (14), this concludes the proof of Proposition 14 in the case $\mu > \frac{1}{2}$.

$\boxed{\text{We assume } 0 < \mu \leqslant \frac{1}{2}.}$

The sketch of the proof in this case is similar, the main lemmas only need a few adjustments.

For negative powers of u, the situation is perfectly similar to the previous cases, except that the results depend on the rest of the Euclidean division of p by 3 rather than 4.

Lemma 18. *For all integers $q \geqslant 3$, there are polynomials P_i^q in t, so that*

$$u^{-3q-1} = u_0\, P_0^q(t) + u_1\, P_1^q(t) + u\, P_2^q(t) + \mathbb{1} P_3^q(t).$$

These polynomials are of the form

$$\begin{aligned}
P_0^q(t) &= t^{(q+1)\mu+(q+1)} && + \textit{lower order terms}\\
P_1^q(t) &= \tfrac{(q-1)(q-2)}{2}\, t^{(q+2)\mu+q} && + \textit{lower order terms}\\
P_2^q(t) &= (q-1)\, t^{(q+2)\mu+q} && + \textit{lower order terms}\\
P_3^q(t) &= q\, t^{(q+1)\mu+q} && + \textit{lower order terms}\,.
\end{aligned}$$

We use the multiplication table (12) to compute

$$u^{-10} = u_0\, t^{4\mu+4} + u_1\, t^{5\mu+3} + 2u\, t^{5\mu+3} + 3\mathbb{1}\, t^{4\mu+3}$$

which is indeed of the expected form.

The expressions of the following 3 powers u^{-3q-2}, u^{-3q-3}, and $u^{-3(q+1)-1}$ are given from u^{-3q-1} by equation (13). From this we conclude that

$$\begin{aligned}
P_0^{q+1}:&\quad t^{(q+2)\mu+(q+2)} && + \text{l.o.t.}\\
P_1^{q+1}:&\quad \big(\tfrac{(q-1)(q-2)}{2} + (q-1)\big)\, t^{(q+3)\mu+(q+1)} && + \text{l.o.t.}\\
P_2^{q+1}:&\quad q\, t^{(q+3)\mu+(q+1)} && + \text{l.o.t.}\\
P_3^{q+1}:&\quad (q+1)\, t^{(q+1)\mu+(q+1)} && + \text{l.o.t.}
\end{aligned}$$

since $\mu \leqslant \frac{1}{2}$. Notice that this yields the expected expression of P_1^{q+1} since $\frac{(q-1)(q-2)}{2}$ is the sum of all integers between 1 and $q-2$.

This concludes the proof of the lemma, from which we immediately get that $\nu(u^{-3q-1}) = (q+1)\mu + (q+1)$. The other necessary valuations can be extracted from (13):

$$\nu(u^{-3q-2}) = (q+2)\mu + (q+1) \quad\text{and}\quad \nu(u^{-3q-3}) = (q+3)\mu + (q+1)\,. \tag{18}$$

For positive powers of u, the situation is somehow easier since we can directly compute the valuation of each power of u. The relevant lemma is the following.

Lemma 19. *For all integers $p \geqslant 5$, there are polynomials Q_i^p in t, so that*

$$u^p = u_0\, Q_0^p(t) + u\, Q_2^p(t) + u_3\, Q_3^p(t) + \mathbb{1} Q_4^p(t).$$

These polynomials are of the form

$$\begin{aligned}
Q_0^p(t) &= (-1)^{p+1}\, t^{-(p-2)\mu} && + \textit{lower order terms}\\
Q_2^p(t) &= (-1)^{p+1}\, t^{-(p-3)\mu-1} && + \textit{lower order terms}\\
Q_3^p(t) &= (-1)^{p}\, t^{-(p-1)\mu} && + \textit{lower order terms}\\
Q_4^p(t) &= (-1)^{p}\, t^{-(p-2)\mu-1} && + \textit{lower order terms}\,.
\end{aligned}$$

Using the multiplication table (15), we compute

$$u^5 = u_0 t^{-3\mu} + u t^{-2\mu-1} - u_3 t^{-4\mu} - \mathbb{1} t^{-3\mu-1}$$

and we compute u^{p+1} from u^p

$$u^{p+1} = u_0\, Q_3^p(t) + u\, Q_4^p(t) + u_3\, (Q_2^p(t) - Q_3^p(t))\, t^{-\mu} + \mathbb{1}\, Q_0^p(t)\, t^{-\mu-1} + \text{ l.o.t.}$$

Notice that Q_3^{p+1} is of the expected form because $\mu \leqslant \frac{1}{2}$.

This proves the lemma and, together with (18), end the proof of Proposition 14 in the last remaining case, for $0 < \mu \leqslant \frac{1}{2}$. $\square$

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