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On the restriction of holomorphic forms

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Abstract. In this paper the author improved a proposition of Sommese on the restriction of holomorphic forms. By studying several examples, it turns out that this improvement is more or less optimal.

1. Introduction

Sommese [8, Proposition 2.3] proved the following:

Proposition 1. *Suppose $X \hookrightarrow Y$ is an holomorphic embedding of a complex projective manifold X into a connected complex manifold Y, X is of dimension n and codimension e. Assume that the normal bundle $N_{X|Y}$ is k-ample and globally generated. Then the restriction map:*

$$H^0(Y, \Omega_Y^q) \to H^0(X, \Omega_X^q)$$

is injective for $0 \leq q \leq n - e - k$.

Remark 2. We recall from [8] the definition of *k-ample* vector bundles. Let X be a compact complex space. An holomorphic line bundle L over X is *k-ample* if $L^{\otimes t}$ is spanned by global holomorphic sections for some $t \geq 0$ and the maximum of the dimension of the fibres of the associated map $X \to \mathbb{P}(H^0(X, L^{\otimes t}))$ is less than or equal to k. A vector bundle E on X is *k-ample* if $\mathscr{O}_{\mathbb{P}(E)}(1)$ is *k-ample* on $\mathbb{P}(E)$. In [8], general properties of *k-ample* vector bundles are studied. It turns out that *k-ampleness* is a quite natural generalization of ampleness.

Remark 3. Indeed Sommese proved the above result under a weaker condition: the normal bundle $N_{X|Y}$ just needs to be *k-ample* and *generically* spanned by its global sections.

There are several variants of this proposition. In the same paper, Sommese proved another theorem: if X is the zero-locus of a section of a *k-ample* vector bundle E over Y, then the restriction map

$$H^q(Y, \mathbb{Z}) \to H^q(X, \mathbb{Z})$$

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is an isomorphism for $0 \le q \le n - e - k$ and injective for $q = n - e - k + 1$. Schneider and Zintl [7] proved that if Y is the projective space and X is an arbitrary submanifold of Y, where the normal bundle is automatically ample, then

$$H^q(Y, \mathbb{C}) \to H^q(X, \mathbb{C})$$

is an isomorphism for $0 \le q \le n - e$ and injective for $q = n - e + 1$. This is exactly a cohomological proof of the Barth–Lefschetz theorem. Along the same line, Debarre [3] proved a similar result when Y is an abelian variety and $N_{X|Y}$ is *k-ample*. In this case, the assumption on the normal bundle is also not too restrictive: when Y is a simple abelian variety, the normal bundle is always ample.

In this paper we improve Proposition 1 by considering the restriction of closed forms:

Theorem 4. *Under the assumption of Proposition* 1*, let* $H^0(Y, \Omega_Y^q)^c$ *be the set of closed forms; then the restriction map:*

$$H^0(Y, \Omega_Y^{n-e-k+1})^c \to H^0(X, \Omega_X^{n-e-k+1})$$

is injective.

Corollary 5. *With the same assumptions and notations as in Proposition* 9*, if Y is a compact Kähler manifold then the restriction map:*

$$H^0(Y, \Omega_Y^q) \to H^0(X, \Omega_X^q)$$

is injective for $0 \le q \le n - e - k + 1$.

Proof. Since Y is a compact Kähler manifold, all holomorphic forms are closed. Then the corollary follows from Proposition 1 and Theorem 4. □

2. Proof of the main theorem

The strategy follows from that of Schneider and Zintl. The main ingredient is the Le Potier vanishing theorem and its variant. For the proof of Le Potier's vanishing theorem, one can read the book of Lazarsfeld [6].

Theorem 6. (*Le Potier's vanishing theorem*) *Assume that X is a smooth irreducible complex projective variety of dimension n and E is a vector bundle of rank e over X which is k-ample. Then*

$$H^i(X, \Omega_X^j \otimes E) = 0,$$

for $i + j \ge n + e + k$.

Theorem 7. (*Variant*) *Under the same assumption of Theorem 6 and in addition E is globally generated, then*

$$H^0(X, \Omega_X^j \otimes \wedge^s E^* \otimes S^t E^*) = 0,$$

for $0 \le j \le n - e - k$, $s, t \ge 0$ *and* $s + t \ge 1$.

Proof. By Theorem 6, we know that $H^0(X, \Omega_X^j \otimes E^*) = 0$, for $0 \le j \le n-e-k$. Let $V = H^0(X, (E^*)^{\otimes(s+t-1)})$. As E is globally generated, there is an injection

$$\Omega_X^j \otimes (E^*)^{\otimes(s+t)} \subset \Omega_X^j \otimes E^* \otimes V^*.$$

Hence $H^0(X, \Omega_X^j \otimes (E^*)^{\otimes s+t} = 0$ and then $H^0(X, \Omega_X^j \otimes \wedge^s E^* \otimes S^t E^*) = 0$, for $\wedge^s E^* \otimes S^t E^*$ is a sub-bundle of $(E^*)^{\otimes s+t}$. □

Now Sommese's idea is that since the following composition of maps

$$H^0(Y, \Omega_Y^q) \to H^0(X, \Omega_Y^q \mid_X) \to H^0(X, \Omega_X^q)$$

is just the restriction map, we just need to prove: (1) The morphism $H^0(Y, \Omega_Y^q) \to H^0(X, \Omega_Y^q \mid_X)$ is injective for $0 \le q \le n-e-k$;

(2) The The morphism $H^0(X, \Omega_Y^q \mid_X) \to H^0(X, \Omega_X^q)$ is injective for $0 \le q \le n-e-k$.

Let us first prove (2). We have an exact sequence of vector bundles:

$$0 \to N_{X|Y}^* \to \Omega_Y \mid_X \to \Omega_X \to 0.$$

Taking the exterior powers, we have

$$0 \to M_q \to \Omega_Y^q \mid_X \to \Omega_X^q \to 0.$$

We need to prove that $H^0(X, M_q) = 0$. The point is that M_q has a filtration $L^i M_q = \wedge^i N_{X|Y}^* \wedge (\Omega_Y^{q-i} \mid_X)$ with $Gr^i M_q = L^i M_q / L^{i-1} M_q = \wedge^i N_{X|Y}^* \otimes \Omega_X^{q-i}$, for $1 \le i \le q$. Since $N_{X|Y}$ is ample and globally generated, we apply Theorem 7 to deduce that $H^0(X, \wedge^i N_{X|Y}^* \otimes \Omega_X^{q-i}) = 0$, for $q-i \le n-e-k$, so $H^0(X, M_q) = 0$, for $0 \le q \le n-e-k+1$. Hence we obtain that the map

$$H^0(X, \Omega_Y^q \mid_X) \to H^0(X, \Omega_X^q)$$

is injective for $0 \le q \le n-e-k+1$. Here the conclusion is slightly stronger than (2) and this gives a reason to improve Proposition 1.

Now we turn to the proof of (1). Denoting by $\mathscr{I}_X$ the ideal sheaf of X, we have the exact sequence

$$0 \to \Omega_Y^q \otimes \mathscr{I}_X \to \Omega_Y^q \to \Omega_Y^q \mid_X \to 0.$$

We need to prove that $H^0(Y, \Omega_Y^q \otimes \mathscr{I}_X) = 0$ for $0 \le q \le n-e-k$. Since we have

$$0 \to \Omega_Y^q \otimes \mathscr{I}_X^{l+1} \to \Omega_Y^q \otimes \mathscr{I}_X^l \to \Omega_Y^q \mid_X \otimes (\mathscr{I}_X^l / \mathscr{I}_X^{l+1}) \to 0,$$

for $q, l \ge 1$. Once we could prove that $H^0(X, \Omega_Y^q \mid_X \otimes (\mathscr{I}_X^l / \mathscr{I}_X^{l+1})) = 0$, for $l \ge 1$ and $0 \le q \le n-e-k$, then

$$H^0(Y, \Omega_Y^q \otimes \mathscr{I}_X^{l+1}) \to H^0(Y, \Omega_Y^q \otimes \mathscr{I}_X^l)$$

is an isomorphism for $l \ge 1$ and $0 \le q \le n-e-k$. Since Y is connected, we conclude $H^0(Y, \Omega_Y^q \otimes \mathscr{I}_X) = 0$, for $0 \le q \le n-e-k$. In all, we just need the following:

Claim. Under the above assumption, we have $H^0(X, \Omega_Y^q\mid_X \otimes(\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) = 0$, for $l \geq 1$ and $0 \leq q \leq n-e-k$.

Proof. We note that $\mathscr{I}_X^l/\mathscr{I}_X^{l+1} \simeq S^l N_{X|Y}^*$ as $\mathscr{O}_X$ modules and that $\Omega_Y^q\mid_X \otimes(\mathscr{I}_X^l/\mathscr{I}_X^{l+1})$ has a filtration

$$L^i(\Omega_Y^q\mid_X) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1}) = (\Omega_Y^{q-i}\mid_X \wedge(\wedge^i N_{X|Y}^*)) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1}),$$

with $Gr^i(\Omega_Y^q\mid_X)\otimes(\mathscr{I}_X^l/\mathscr{I}_X^{l+1}) \simeq \Omega_X^{q-i} \otimes \wedge^i N_{X|Y}^* \otimes S^l N_{X|Y}^*$ for $i \geq 0$ and $l \geq 1$. By theorem 7, $H^0(X, Gr^i(\Omega_Y^q\mid_X) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) = 0$ for $i \geq 0$ and $0 \leq q \leq n-e-k$, so $H^0(X, \Omega_Y^q\mid_X \otimes(\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) = 0$ for $l \geq 1$ and $0 \leq q \leq n-e-k$. □

Now it is time to prove Theorem 4. Recall that in the proof of (2) we have already got the injectivity of the map $H^0(X, \Omega_Y^q\mid_X) \to H^0(X, \Omega_X^q)$ for $0 \leq q \leq n-e-k+1$. Thus in order to prove Theorem 4 we just need to prove that

$$H^0(Y, \Omega_Y^{n-e-k+1})^c \to H^0(X, \Omega_Y^{n-e-k+1}\mid_X)$$

is injective. As the same argument in the proof of (1), we just need to prove that

$$H^0(Y, \Omega_Y^{n-e-k+1} \otimes \mathscr{I}_X^l)^c \to H^0(Y, \Omega_Y^{n-e-k+1} \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1}))$$

is zero for $l \geq 1$. Since $H^0(Y, Gr^i(\Omega_Y^{n-e-k+1}\mid_X) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) = H^0(X, \Omega_X^{n-e-k+1-i} \otimes \wedge^i N_{X|Y}^* \otimes S^l N_{X|Y}^*) = 0$ for $i \geq 1$ and $l \geq 1$, it suffices to prove that the following composition of maps is zero:

$$\begin{aligned} H^0(Y, \Omega_Y^{n-e-k+1} \otimes \mathscr{I}_X^l)^c &\to H^0(Y, \Omega_Y^{n-e-k+1} \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) \\ &\to H^0(Y, Gr^0(\Omega_Y^{n-e-k+1}) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})), \end{aligned}$$

for $l \geq 1$, where $Gr^0(\Omega_Y^{n-e-k+1}) \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})$ is isomorphic to $\Omega_X^{n-e-k+1} \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})$. Hence the following lemma is sufficient for our purpose.

Lemma 8. *The composition of maps* $H^0(Y, \Omega_Y^{n-e-k+1}\otimes\mathscr{I}_X^l)^c \to H^0(Y, \Omega_Y^{n-e-k+1}\otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1})) \to H^0(X, \Omega_X^{n-e-k+1} \otimes (\mathscr{I}_X^l/\mathscr{I}_X^{l+1}))$ *is zero for* $l \geq 1$.

Proof. We will prove it by a local calculation. Suppose that locally in Y we have coordinates $z_1, z_2, \ldots, z_{n+e}$ and X is defined by $z_{n+1} = \cdots = z_{n+e} = 0$. For $w \in H^0(Y, \Omega_Y^{n-e-k+1} \otimes \mathscr{I}_X^l)^c$, we may write $w = \sum_{\substack{|I|=n-e-k+1 \\ I\subseteq\{1,2,\ldots,n\}}} f_I dz_I + \sum_{\substack{|J|=n-e-k+1 \\ J\nsubseteq\{1,2,\ldots,n\}}} g_J dz_J$. Then the image of w is $\sum_{\substack{|I|=n-e-k+1 \\ I\subseteq\{1,2,\ldots,n\}}} \overline{f_I} dz_I$, here we denote by $\overline{f_I}$ the image of f_I in $\mathscr{I}_X^l/\mathscr{I}_X^{l+1}$. As w is closed, for all I and $i \geq n+1$, we have

$$\frac{\partial f_I}{\partial z_i} = \sum_{j\in\{1,2,\ldots,n\}} \pm\frac{\partial g_J}{\partial z_j},$$

where $J = I \cup \{i\} \setminus \{j\}$. As $g_J \in \mathscr{I}_X^l$ and $1 \leq j \leq n$, we have $\frac{\partial g_J}{\partial z_j} \in \mathscr{I}_X^l$, thus we conclude that $\frac{\partial f_I}{\partial z_i} \in \mathscr{I}_X^l$ for any $i \geq n+1$, which implies that $f_I \in \mathscr{I}_X^{l+1}$. □

3. Examples

With the same assumption of Proposition 1, one may ask whether $q = n - e - k + 1$ is optimal for the injectivity of the restriction map. In this section we study some explicit cases and it seems like that it is reasonable to expect the $q = n - e - k + 1$ to be optimal except when $q = 1$ and $k = 0$.

3.1.

Proposition 9. *Suppose $X \hookrightarrow Y$ is a holomorphic embedding from a compact complex manifold X to a connected complex manifold Y. If $N_{X|Y}$ is ample, then the restriction map $H^0(Y, \Omega_Y)^c \to H^0(X, \Omega_X)$ is injective.*

Proof. One can prove this proposition using the same method above. Here we give another argument with additional assumption that Y is a compact Kähler manifold. We consider the Albanese morphism:

$$alb_Y : Y \to Alb(Y) = H^0(Y, \Omega_Y)^*/H_1(Y, \mathbb{Z}).$$

We will identify $H^0(Alb(Y), \Omega_{Alb_Y})$ with $H^0(Y, \Omega_Y)$. If the restriction map is not injective, then $R = ker(H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z}))$ is a nontrivial sub-Hodge structure with (1, 0) part $T = ker(H^0(Y, \Omega_Y) \to H^0(X, \Omega_X))$. Hence there is a quotient map $Alb(Y) \to B = T^*/R^*$. By the very definition of B, the composition of map $X \hookrightarrow Y \to Alb(Y) \to B$ is a constant morphism $X \to x \in B$. Taking a nonconstant holomorphic function f in a neighborhood of x on B satisfying the condition that its pull-back to Y is not constant and $f(x) = 0$. Hence the pull-back of f yields a global section of $H^0(X, \mathscr{I}_X^l/\mathscr{I}_X^{l+1})$, for some $l \geq 1$. This contradicts the condition that $N_{X|Y}$ is ample. □

3.2.

When $q = n - e + 2 = 2$, (and so $n = e$ and $k = 0$), there is already a simple example showing that the restriction map for $H^0(Y, \Omega_Y^2)^c$ is generally not injective. Let X be a smooth manifold and $T = T^*X$ be the total space of the cotangent bundle. We have the projection map $Y \xrightarrow{\pi} X$ and X identifies naturally the 0-*section* of π. There exists a canonical one form θ on Y, namely at each point $\alpha \in \pi^{-1}(x)$, $\theta \mid \alpha$ is just $\pi^*\alpha$. We set $w = d\theta$. Then the restriction of θ on X vanishes by the very definition and so is w. We then take X to be a variety with ample cotangent bundle, then $N_{X|Y} \simeq \Omega_X$ is ample, but $w \in H^0(Y, \Omega_Y^2)^c$ and $\theta \in H^0(Y, \Omega_Y)$ both vanish in X. Debarre [4] proved that the intersection of at least $d/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension d has ample cotangent bundle. Hence this construction works for any dimension n, $n \geq 1$.

Remark 10. The vanishing of θ shows that in order to improve Sommese's result, we should consider the restriction of closed forms.

3.3.

The next example comes from projective geometry. We assume $Y \subset \mathbb{P}^N$ is a smooth hypersurface and $X \subset Y$ is a smooth hyperplane section in Y. Let $F(X)$ and $F(Y)$ be their Fano varieties of lines, namely the variety of lines contained in X and Y. Let $I(X)$ and $I(Y)$ be their incidence varieties, i.e. $I(X) = \{(x,l) \mid x \in l \subset X\} \subset X \times F(X)$ and $I(Y) = \{(x,l) \mid x \in l \subset Y\} \subset Y \times F(Y)$. Such varieties have been studied in [2, 5]. We will need the following theorem, which is a small part of Théorème 2.1 in [5]:

Theorem 11. *Suppose X is a subvariety of $\mathbb{P}^N$, $N \geq 3$, defined by a equation of degree d and $F(X)$ is its Fano variety of lines.*

1. *If $2N-d-3 \geq 0$, then $F(X)$ is nonempty and $F(X)$ is smooth and of dimension $2N-d-3$ if X is generic.*
2. *If $2N-d-3 \geq 1$, then $F(X)$ is connected.*

Let $Y \subset \mathbb{P}^{n+3}$ be a generic hypersurface of degree $n+1$ and X is a smooth hyperplane section, $n \geq 2$. According to Theorem 11, $F(X)$ and $F(Y)$ are smooth connected varieties of dimension n and $n+2$. It is easy to see that $F(X)$ and $F(Y)$ are naturally subvarieties of the Grassmannian $G(1, \mathbb{P}^{n+3})$. We denote by $\sum$ the tautological sub-bundle of the Grassmannian, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{F(X)} & \longrightarrow & T_{G(1,\mathbb{P}^{n+2})} & \longrightarrow & Sym^{n+1}\Sigma^* \mid F(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T_{F(Y)} \mid F(X) & \longrightarrow & T_{G(1,\mathbb{P}^{n+3})} \mid F(X) & \longrightarrow & Sym^{n+1}\Sigma^* \mid F(X) & \longrightarrow & 0
\end{array}$$

so $N_{F(X)|F(Y)} \simeq N_{G(1,\mathbb{P}^{n+2})|G(1,\mathbb{P}^{n+3})} \simeq \Sigma^*$ is nef and globally generated. We consider the relative hyperplane morphism $\gamma : \mathbb{P}(\Sigma^* \mid_{F(X)}) = I_X \to X$. If we could prove that γ is finite, then it would be clear that $\Sigma^* \mid_{F(X)}$ is ample. We need the following lemma:

Lemma 12. *Let V be a vector space of dimension $n+1$, $n \geq 3$ and $X(F)$ be a hypersurface of $\mathbb{P}(V)$ defined by a polynomial F of degree d, $d \leq n+1$, then through any point of $X(F)$ there is a line contained in $X(F)$. Moreover if F is generic, then for every $x \in X(F)$, the variety of lines in $X(F)$ passing through x has dimension $n-d-1$. In particular $\sum^* \mid_{F(X)}$ is $n-d-1$ ample.*

Proof. Suppose x is a point of $X(F)$ with coordinates $(1 : 0 \cdots : 0)$. We denote by W the vector space $V/x\mathbb{C}$. There is a line contained in $X(F)$ joining x and another point $(0 : x_1 \cdots : x_n)$ if and only if $F(t, x_1, \ldots, x_n) = 0$ for any $t \in \mathbb{C}$. We may write $F(t, x_1, \ldots, x_n) = t^{d-1}G_1(x_1, \ldots, x_n) + \cdots + G_d(x_1, \ldots, x_n)$, where G_i is a polynomial of degree i or 0. We just need to prove that the variety defined by $G_1, \ldots, G_d$ in $\mathbb{P}(W)$ is nonempty. Since $d \leq n-1$, this is trivial. The second assertion is a problem of dimension counting. We define

$$I = \{(x, f) \mid f(x = 0)\} \subset \mathbb{P}(V) \times Sym^d V^*,$$

and let $\mathscr{X} \subset I$ be a closed subset consisting of (x, f) such that the variety of lines in $X(f)$ passing through x has dimension greater than $n-d-1$. Let $pr_1 : \mathscr{X} \to \mathbb{P}(V)$ and $\mathscr{X} \to Sym^d V^*$ be the projections, then it suffices to prove that pr_2 is not dominant. We will prove that $dim \mathscr{X} < dim(Sym^d V^*) = \binom{n+d}{d}$. For $x \in \mathbb{P}(V)$ could be supposed to be $(1 : 0 \cdots : 0)$, then $pr_1^{-1}(x)$ is isomorphic to

$$C = \{(G_1, \ldots, G_d) \in Sym^1 W^* \times \cdots \times Sym^d W^* \mid (G_1, \ldots, G_d) \text{ is not a regular sequence}\}.$$

If $G_1, \ldots, G_d$ is not a regular sequence, then there exists a natural number l such that G_{l+1} vanishes on some irreducible component Γ of the variety defined by the regular sequence $G_1, \ldots, G_l$, $0 \leq l \leq d-1$. We denote by C_l the component of C consisting of those sequences satisfying the above condition. Then $C \subset \cup_{0 \leq l \leq d-1} C_l$. Denote by $pr_1^l : C_l \to Sym^1 W^* \times \cdots \times Sym^l W^*$ and $pr_1^2 : C_l \to Sym^{l+1} W^*$ the canonical projections. Fixing $(f_1, \ldots, f_l) \in Sym^1 W^* \times \cdots \times Sym^l W^*$ to be regular, then

$$pr_2^l((pr_1^l)^{-1}(f_1, \ldots, f_l)) = \cup_\Gamma U_\Gamma,$$

where $U_\Gamma = \{h \in Sym^{l+1} W^* \mid h \; vanishes \; in \; \Gamma\}$ and Γ goes through all the irreducible components of the variety defined by $f_1, \ldots, f_l$. Since Γ is of dimension $n-l-1$, there exists a linear space of dimension $l-1$ which doesn't intersect Γ. Taking projection of this space, we get a dominant morphism from Γ to $\mathbb{P}^{n-l-1}$. Hence we have

$$dim H^0(\Gamma, \mathscr{O}_\Gamma(k)) \geq dim H^0(\mathbb{P}^{n-l-1}, \mathscr{O}_{\mathbb{P}^{n-l-1}}(k)),$$

for $k \geq 0$. We deduce that

$$codim(U_\Gamma \subset Sym^{l+1} W^*) \geq \binom{n}{l+1}.$$

Hence $codim(C_l \subset Sym^1 W^* \times \cdots \times Sym^d W^*) \geq \binom{n}{l+1} \geq n$. Hence we conclude that

$$dim(C) \leq n + \binom{n+d}{d} - 1 - n \leq \binom{n+d}{d}$$

We are done. □

Now in our cases $F(X) \subset F(Y)$ is a subvariety of dimension n and codimension 2 and $N_{F(X)|F(Y)}$ is ample and globally generated. But we have the following proposition:

Proposition 13. *Let $Y \subset \mathbb{P}^{n+3}$ be a generic hypersurface of degree $n+1$, $n \geq 2$, defined by f and X is a smooth hyperplane section and $F(X)$ and $F(Y)$ are smooth varieties of dimension n and $n+2$. then the restriction map $Res : H^0(F(Y), \Omega^n_{F(Y)}) \to H^0(F(X), \Omega^n_{F(X)})$ is not injective.*

Remark 14. The above proposition shows that when $k = 0$ and $e = 2$, $n \geq 2$, $q = n - e - k + 1$ is already optimal.

Proof. The idea of the proof is to find a class of *Dolbeault* cohomology w of Y,and then we can construct a holomorphic n-form θ by applying the Abel–Jacobi map to w. We will see easily that θ vanishes on $F(X)$. The existence of w is guaranteed by two theorems due to Griffiths and Macaulay respectively, which is explained in Voisin's book [9].

Let $S = \oplus_{k=0}^{\infty} S^k V$ be the graded symmetric algebra of a vector space V which has coordinates $(x_0, x_1, \ldots, x_m)$. Assume that $f \in S^d V$ defines a smooth hypersurface $X(f)$ of $\mathbb{P}(V)$. Denote by J_f the homogeneous ideal generated by the homogeneous polynomial $\frac{\partial f}{\partial x_i}$ for $0 \leq i \leq m$, which is called the Jacobian ideal of f. Let R_f denote S/J_f.

Consider the Gysin map $l_* : H^{m-1}(X(f), \mathbb{C}) \to H^{m-1}(\mathbb{P}(V), \mathbb{C})$. We will denote by $H^{m-1}(X(f))_{prim}$ the kernel of the Gysin map.

Theorem 15. (Griffiths) *There is a natural isomorphism, which depends holomorphically on f,*

$$\alpha_p : R_f^{pd-m-1} \to H^{m-p,p-1}(X(f))_{prim}.$$

Definition 16. A sequence of polynomials $G_i \in S^{d_i} V$, $i = 0, \ldots, m$ and $d_i \geq 0$ is called a regular sequence if $G_1, \ldots, G_m$ don't have any common zero.

Given a regular sequence $G = (G_1, \ldots, G_m)$, we denote by R_G the quotient ring $S/(G_1, \ldots, G_m)S$.

Theorem 17. (Macaulay) *The ring R_G satisfies the following properties: for $N = \sum_{i=0}^{m} d_i - m - 1$, we have $rk(R_G^N) = 1$ and for any integer $0 \leq k \leq N$, the pairing induced by multiplication:*

$$R_G^k \times R_G^{N-k} \to R_G^N$$

is non-degenerate. In particular, $R_G^k \neq 0$, for $0 \leq k \leq N$.

The following lemma due to Beauville and Donagi [1] would be crucial.

Lemma 18. *Suppose X is a complex manifold with a holomorphic vector bundle Σ of rank k. Denote by $p : \mathbb{P}(\Sigma) \to X$ the associated projective bundle and let $\xi = c_1(\mathcal{O}_{\mathbb{P}(\Sigma)}(1))$. Suppose that there exists a holomorphic map $q : \mathbb{P}(\Sigma) \to Y$, with $h \in H^2(Y)$ such that $q^*(h) = \xi$. Then for all $w \in H^*(Y)$ satisfying $wh = 0$ and $q^*(w) \neq 0$, we have $p_* q^*(w) \neq 0$.*

Proof. It is well know that $H^*(\mathbb{P}(\Sigma)) = H^*(X)[\xi]/(\xi^k + c_1(\Sigma)\xi^{k-1} + \cdots + c_k(\Sigma))$. We may assume that $q^*(w) = \xi^{k-1} p^* w_1 + \cdots + p^* w_k \neq 0$, where $w_1 = p_*^*(w)$. Since

$$q^*(wh) = q^*(w)\xi = -c_k(\Sigma) p^* w_1 + \sum_{i=1}^{k-1} \xi^i (-p^* w_1 c_{k-i}(\Sigma) + p^* w_{k-i+1}) = 0,$$

we deduce that $-p^* w_1 c_{k-i}(\Sigma) + p^* w_{k-i+1} = 0$ for $1 \leq i \leq k-1$ and $c_k(\Sigma) p^* w_1 = 0$. As $q^* w \neq 0$, $p_* q^* w = w_1$ should not be zero. □

Since Y is a hypersurface of degree $n+1$ of $\mathbb{P}^{n+3}$. By the theorems of Griffiths and Macaulay, $H^{n+1,1}(Y) = H^{n+1,1}(Y)_{prim} \simeq R_f^{n-2} \neq 0$. According to lemma 12, the morphism $q_Y : \mathbb{P}(\Sigma^* |_{F(Y)}) = I_Y \to Y$ is dominant and hence $q_Y^* : H^*(Y) \to H^*(I_Y)$ is injective. Then we can apply Lemma 18 to conclude that $p_{Y*}q_Y^* : H^{n+1,1}(Y) \to H^{n,0}(F(Y))$ is injective.
Furthermore, we claim that the composition of maps

$$H^{n+1,1}(Y) \xrightarrow{p_{Y*}q_Y^*} H^{n,0}(F(Y)) \xrightarrow{Res} H^{n,0}(F(X))$$

is zero. In fact we have the commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow[q_X]{} & I_X & \xrightarrow{p_X} & F(X) \\ {\scriptstyle l_X}\downarrow & & \downarrow & & \downarrow \\ Y & \xleftarrow[q_Y]{} & I_Y & \xrightarrow{p_Y} & F(Y). \end{array}$$

Hence the above map is just $H^{n+1,1}(Y) \xrightarrow{l_X^*} H^{n+1,1}(X) \xrightarrow{p_{X*}q_X^*} H^{n,o}(F(X))$. Since $H^{n+1,1}(X) \simeq H^{n,0}(X) = 0$, the claim is trivial. It follows that $H^0(F(Y), \Omega^n_{F(Y)}) \xrightarrow{Res} H^0(F(X), \Omega^n_{F(X)})$ is not injective. □

Indeed Proposition 13 can be generalized easily to cases where $k \geq 0$. Concretely:

Proposition 19. *Fixed $k \geq 1$ and $n \geq 3k+2$, let Y be a smooth generic hypersurface of $\mathbb{P}^{n-k+3}$ of degree $n+1-2k$ and X is a smooth hyperplane of Y. Both $F(X)$ and $F(Y)$ are smooth and of dimension n and $n+2$, respectively, and the normal bundle k-ample and globally generated. But the restriction map $Res : H^0(F(Y), \Omega^{n-k}_{F(Y)}) \to H^0(F(X), \Omega^{n-k}_{F(X)})$ is not injective.*

We omit the proof since almost the same argument in the proof of Proposition 13 works here also.

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