Erratum to: Scaling limits for the uniform infinite quadrangulation

by Jean-François Le Gall and Laurent Ménard

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Nicolas Curien has pointed to us that the constants in Proposition 5 of [1] are incorrect. This proposition should be replaced by the following statement, where we recall that $\mathcal{I}$ is the random measure on $\mathbb{R}_+$ which is the limit in distribution of the (suitably rescaled) profile of distances from the root in the uniform infinite planar quadrangulation (see Theorem 6 in [1]).

Proposition 1. *For every nonnegative measurable function g on $\mathbb{R}_+$,*

$$E[\langle \mathcal{I}, g\rangle] = \frac{4}{21}\int_0^\infty dr\, r^3\, g(r).$$

In particular, for every $r > 0$,

$$E[\mathcal{I}([0,r])] = \frac{1}{21}\, r^4.$$

Proof. From the definition of $\mathcal{I}$ and the construction of the eternal conditioned Brownian snake, we get

$$E[\langle \mathcal{I}, g\rangle] = 2\, E\left[\int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s)\right)\right]$$

(note that the similar formula on p.413 of [1] incorrectly contains a factor 4 instead of 2 above). For every $z > 0$, let

$$\varphi_g(z) = \mathbb{N}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\int_0^\sigma \mathrm{d}s\, g(\widehat{W}_s).\right)$$

Let $(\xi_t)_{t\geq 0}$ denote a linear Brownian motion that starts from z under the probability measure P_z. Then, by the case $p = 1$ of Theorem 2.2 in [2], we have

$$\begin{aligned}
\varphi_g(z) &= \int_0^\infty \mathrm{d}a\, E_z\left[g(\xi_a)\, \exp\left(-4\int_0^a \mathrm{d}s\, \mathbb{N}_{\xi_s}\left(\mathcal{R}\cap]0,\infty[\neq \emptyset\right)\right)\right] \\
&= \int_0^\infty \mathrm{d}a\, E_z\left[g(\xi_a)\, \exp\left(-6\int_0^a \frac{\mathrm{d}s}{\xi_s^2}\right)\right] \\
&= \int_0^\infty \mathrm{d}a\, z^4\, E_z\left[Z_a^{-4} g(Z_a)\right],
\end{aligned}$$

where the nine-dimensional Bessel process Z starts from z under the probability measure P_z. In the second equality we used Lemma 2.1 of [2], and in the third one we applied the absolute continuity properties of laws of Bessel processes (see e.g. Proposition 2.6 in [2]).

Recall that the nine-dimensional Bessel process has the same distribution as the Euclidean norm of a nine-dimensional Brownian motion. Using the explicit form of the Green function of Brownian motion in $\mathbb{R}^9$, we get

$$\varphi_g(z) = \frac{\Gamma(\frac{7}{2})}{2\pi^{9/2}}\, z^4 \int_{\mathbb{R}^9} \mathrm{d}y\, |y - x_z|^{-7}\, |y|^{-4}\, g(|y|),$$

where x_z is an arbitrary point of $\mathbb{R}^9$ such that $|x_z| = z$. For every $r > 0$, let $\sigma_r(dy)$ be the uniform probability measure on the sphere of radius r centered at the origin in $\mathbb{R}^9$. Since the function $y \mapsto |y - x_z|^{-7}$ is harmonic, an easy argument gives

$$\int \sigma_r(dy)\, |y - x_z|^{-7} = (r \vee z)^{-7}.$$

We can then integrate in polar coordinates in the previous formula for $\varphi_g(z)$, and recalling that the volume of the unit sphere in $\mathbb{R}^9$ is $2\pi^{9/2}/\Gamma(\frac{9}{2})$, we get

$$\varphi_g(z) = \frac{2}{7}\, z^4 \int_0^\infty dr\, r^4\, (r \vee z)^{-7}\, g(r).$$

By substituting this in the first display of the proof, and arguing in a similar way as above, we obtain

$$\begin{aligned}
E[\langle \mathcal{I}, g\rangle] &= 2\, E\Big[\int_0^\infty \mathrm{d}t\, \varphi_g(Z_t)\Big]\\
&= 2\Big(\frac{2}{7}\Big)^2 \int_0^\infty \mathrm{d}z\, z^5 \int_0^\infty \mathrm{d}r\, r^4\, (r \vee z)^{-7}\, g(r)\\
&= 2\Big(\frac{2}{7}\Big)^2 \int_0^\infty dr\, r^4\, g(r)\Big(r^{-7}\int_0^r \mathrm{d}z\, z^5 + \int_r^\infty \mathrm{d}z\, z^{-2}\Big)\\
&= \frac{4}{21} \int_0^\infty \mathrm{d}r\, r^3\, g(r).
\end{aligned}$$

This completes the proof of Proposition 1. $\square$

References

[1] Le Gall, J.-F., Ménard, L. Scaling limits for the uniform infinite quadrangulation. *Illinois J. Math.* **54**, 1163-1203 (2010)

[2] Le Gall, J.-F., Weill, M. Conditioned Brownian trees. *Annales Inst. H. Poincaré Probab. Stat.* **42**, 455-489 (2006)