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$p$-adic cohomologies: history, and new developments

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Plan

0. Introduction

1. Review of crystalline cohomology

2. Old and new on the de Rham-Witt complex

3. Liftings mod $p^2$, de Rham-Witt and derived de Rham complexes
0. Introduction

$X/\mathbb{C}$ proper, smooth scheme (proper $\iff X^{\text{an}} = X(\mathbb{C})$ compact).

Example: $X = V(f) \subset \mathbb{P}^n_\mathbb{C}$, $f \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous, $\deg(f) > 0$, with $z \in \mathbb{C}^{n+1} - \{0\} \Rightarrow \exists i, \partial f / \partial x_i(z) \neq 0$.

For $n \in \mathbb{Z}$:

**Betti cohomology:** $H^n(X^{\text{an}}, \mathbb{C})$

**de Rham cohomology:** $H^n_{dR}(X/\mathbb{C}) := H^n(X, \Omega^\bullet_{X/\mathbb{C}})$,

$$\Omega^\bullet_{X/\mathbb{C}} := (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N_{X/\mathbb{C}})$$

$(N = \dim(X))$ the de Rham complex

**Poincaré lemma:** exactness of the sequence

$$0 \to \mathbb{C}_{X^{\text{an}}} \to \mathcal{O}_{X^{\text{an}}} \xrightarrow{d} \Omega^1_{X^{\text{an}}/\mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N_{X^{\text{an}}/\mathbb{C}} \to 0$$

and **Serre’s GAGA**

$$H^j(X, \Omega^i_{X/\mathbb{C}}) \sim H^j(X^{\text{an}}, \Omega^i_{X^{\text{an}}/\mathbb{C}})$$
imply comparison isomorphism

\[(1) \quad H^n(X^{an}, \mathbb{C}) \sim \rightarrow H^n_{dR}(X/\mathbb{C}).\]

Moreover: Hodge decomposition (Deligne, 1968, for \(X\) proper, not necessarily projective):

\[H^n(X^{an}, \mathbb{C}) = \bigoplus_{i+j=n} H^{i,j} = \bigoplus_{i+j=n} H^j(X, \Omega^i).\]

In particular, Hodge-to-de Rham spectral sequence degenerates at \(E_1\):

\[E_1^{ij} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{dR}(X/\mathbb{C}).\]

Remark. Liftings mod \(p^2 + \) Cartier isomorphism \(\Rightarrow\) algebraic proof (Deligne-I, 1987).
Assume now that $X/C$ has a proper smooth model $\mathcal{X}$ over $S = \text{Spec}(\mathbb{Z}[1/m])$, for some $m \geq 1$: $X = \mathcal{X}_C$.

For $p \in \text{Spec}(\mathbb{Z}[1/m])$ (i.e. $p \nmid m$), $\mathcal{X}_p := \mathcal{X} \otimes F_p$ is smooth: good reduction of $\mathcal{X}_Q$ at $p$.

Example: $X = V(f) \subset P^n_C$ as above, with $f \in \mathbb{Z}[1/m][x_0, \ldots, x_n]$ (s. t. for all $p \nmid m$, and all geometric points $z \neq 0$ of $\mathcal{X}_p$, $\exists i, \partial f / \partial x_i(z) \neq 0$).

By

$$H^n(X^{\text{an}}, \mathbb{C}) = H^n(X^{\text{an}}, \mathbb{Q}) \otimes \mathbb{C},$$

$$H_{dR}^n(X/C) = H_{dR}^n(\mathcal{X}_Q/Q) \otimes \mathbb{C},$$

and the comparison isomorphism (1)

$$H^n(X^{\text{an}}, \mathbb{C}) \xrightarrow{\sim} H_{dR}^n(X/C)$$

get the (highly transcendental) period isomorphism

(2) $H_{dR}^n(\mathcal{X}_Q/Q) \otimes \mathbb{C} \xrightarrow{\sim} H^n(X^{\text{an}}, \mathbb{Q}) \otimes \mathbb{C}$. 
Problems:

(a) How about replacing
- \( \mathbb{Q} \) by \( \mathbb{Q}_\ell \) (\( \ell \) prime)?
- \( \mathcal{X} \) by \( \mathcal{X}_{\mathbb{Q}_p} \) (\( p \) prime)?

(b) How about integral variants of (2)?
Problem (a): replacing $\mathbb{Q}$ by $\mathbb{Q}_\ell$, $\mathcal{X}$ by $\mathcal{X}_{\mathbb{Q}_p}$

$\mathbb{Q} \mapsto \mathbb{Q}_\ell$: inputs from étale cohomology:

$$H^n(X^{\text{an}}, \mathbb{Q}) \otimes \mathbb{Q}_\ell = H^n(X_{\text{et}}, \mathbb{Q}_\ell) = H^n(X_{\mathbb{Q},\text{et}}, \mathbb{Q}_\ell),$$

Galois group $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on

$$H^n(X_{\mathbb{Q}^c}, \mathbb{Q}_\ell) := H^n(X_{\mathbb{Q},\text{et}}, \mathbb{Q}_\ell).$$

For $\overline{\mathbb{Q}} \mapsto \overline{\mathbb{Q}}_p$ chosen, decomposition group $G_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts through

$$H^n(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell) \sim H^n(X_{\mathbb{Q}}, \mathbb{Q}_\ell).$$

Recall

$$\mathbb{Q}_p \rightarrow \mathbb{Q}_p,\text{ur} \rightarrow \overline{\mathbb{Q}}_p,$$

Inertia $I_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p,\text{ur}) \subset G_{\mathbb{Q}_p}$

$$\text{Gal}(\mathbb{Q}_p,\text{ur}/\mathbb{Q}_p) \sim \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}$$

(generator: arithmetic Frobenius $\sigma : x \mapsto x^p$).
• Assume first $p \nmid m$ (hence $\mathcal{X}_p$ smooth).

Two cases:

(i) $\ell \neq p$. Then, we have a Galois equivariant isomorphism (wrt $G_{Q_p} \rightarrow G_{Q_p}/I_p = \text{Gal}(\overline{F}_p/F_p)(= \hat{\mathbb{Z}})$)

$$H^n(\mathcal{X}_{Q_p}, Q_\ell) \sim H^n(\mathcal{X}_p \otimes \overline{F}_p, Q_\ell),$$

which implies that $I_p$ acts trivially, i.e. $G_{Q_p}$ acts through $\text{Gal}(\overline{F}_p/F_p)$

Action well understood by the Weil conjectures (Grothendieck, Deligne). In particular:
Zeta function

\[ Z(\mathcal{X}_p, t) := \exp\left( \sum_{n \geq 1} \#(\mathcal{X}_p(F_p^n)) \frac{t^n}{n} \right), \]

given by

\[ Z(\mathcal{X}_p, t) = \prod_{0 \leq i \leq 2d} P_i(t)(-1)^{i+1}, \]

\[ P_i(t) := \det(1 - Ft, H^i(\mathcal{X}_p \otimes \overline{F}_p, \mathbb{Q}_\ell)) \]

\((d = \dim(\mathcal{X}), F = \sigma^{-1} \text{ the geometric Frobenius}), \text{ and} \)

\[ P_i(t) \in \mathbb{Z}[t], \]

independent of \(\ell\), with inverse roots Weil numbers of weight \(i\).
(ii) \( \ell = p \). \( I_p \) no longer acts trivially on \( H^n(X_{Q_p}, Q_{\ell}) \).

Example: \( X = \mathbb{P}^1_C \).
Here, can take \( m = 1, \chi := \mathbb{P}^1_Z \) (good reduction of \( \mathbb{P}^1_Q \) everywhere).

Recall:

\[
Q_{\ell}(1) := Q_{\ell} \otimes Z_{\ell}(1), \quad Z_{\ell}(1) := \varprojlim \mu_{\ell^n}(Q_p)
\]

\( (\sim Q_{\ell} \text{ non canonically}) \), with Galois action: \( g z = z \chi(g) \),
\( \chi : \text{Gal}(Q_p/Q_p) \rightarrow Z_\ell^* \) the cyclotomic character, and for \( r \in Z \)

\[
Q_{\ell}(r) := (Q_{\ell}(1)) \otimes r
\]

Whether \( \ell \neq p \) or \( \ell = p \), we have

\[
H^2(P^1_{Q_p}, Q_{\ell}) = Q_{\ell}(-1).
\]

But: \( \bullet \) If \( \ell \neq p \), \( \mu_{\ell^n}(Q_p) \subset Q_{p,ur} \), \( I_p \) acts trivially, and

\[
\det(1 - Ft, H^2(P^1_{F_p}, Q_{\ell})) = \det(1 - Ft, Q_{\ell}(-1)) = 1 - pt.
\]

\( \bullet \) If \( \ell = p \), \( \mu_{\ell^n}(Q_p) \not\subset Q_{p,ur} \), \( I_p \) acts nontrivially (with wild ramification).
Miracle: action of $G_{Q_p}$ on $H^n(\mathcal{X}_{\overline{Q_p}}, Q_p)$ related to de Rham cohomology:

$$H^n_{dR}(\mathcal{X}_{Q_p}/Q_p) \leftrightarrow H^n(\mathcal{X}_{\overline{Q_p}}, Q_p),$$

by $p$-adic Hodge theory. How can it be? NO Galois action on LHS!

LHS has hidden extra structure, especially an action of Frobenius, coming from crystalline cohomology:

$$H^n_{dR}(\mathcal{X}_{Q_p}/Q_p) = H^n_{dR}(\mathcal{X}_{Z_p}/Z_p) \otimes Q_p,$$

and, actually, $H^n_{dR}(\mathcal{X}_{Z_p}/Z_p)$ is uniquely determined by the special fiber $\mathcal{X}_p$:

$$H^n_{dR}(\mathcal{X}_{Z_p}/Z_p) = H^*(\mathcal{X}_p/Z_p) \ (\text{RHS} := \text{crystalline cohomology})$$
The (absolute) Frobenius endomorphism $F$ of $X_p$, though it does not, in general, lift to $X_{\mathbb{Z}_p}$ defines an isogeny

$$\varphi : H_{dR}^n(X_{\mathbb{Z}_p}/\mathbb{Z}_p) \to H_{dR}^n(X_{\mathbb{Z}_p}/\mathbb{Z}_p)$$

(i.e., $\varphi \otimes \mathbb{Q}_p$ an isomorphism), which, together with the Hodge filtration on $H_{dR}^n(X_{\mathbb{Q}_p}/\mathbb{Q}_p)$, enables to recover $H^n(X_{\overline{\mathbb{Q}_p,et}}, \mathbb{Q}_p)$ via Fontaine’s rings and $p$-adic period isomorphisms (Fontaine’s $C_{cris}$ conjecture).

Study of action of Frobenius on crystalline cohomology of proper, smooth varieties over perfect fields led to the theory of de Rham-Witt complexes.
Problem (a) (replacing $\mathbb{Q}$ by $\mathbb{Q}_\ell$, $\mathcal{X}$ by $\mathcal{X}_{\mathbb{Q}_p}$), cont’d.

• Assume now $p \mid m$, i.e. $p \notin \text{Spec}(\mathbb{Z}[1/m])$ (possibly bad reduction of $\mathcal{X}_{\mathbb{Q}}$ at $p$).

(i) $\ell \neq p$. Far less well understood than in the good reduction case. Big open problems: weight monodromy conjecture (despite great advances by Scholze), Serre’s and Serre-Tate’s conjectures of independence of $\ell$ for characteristic polynomials of Frobenius (or elements of the Weil group) on various $\ell$-adic cohomology groups of the geometric generic fiber $\mathcal{X}_{\mathbb{Q}_p}$.

(ii) $\ell = p$. No obvious crystalline cohomology groups in sight (except in the case of semistable reduction, using log geometry), but still get comparison theorems ($p$-adic period isomorphisms) (Hodge-Tate conjecture, and Fontaine’s $C_{dR}$ conjecture).
Problem (b): Integral comparison.

Assume $p \nmid m$ (hence $X_p$ smooth). Relations between

$$H^n_{dR}(X_{Z^p/Z^p}) \leftrightarrow H^n(X_{Q_p}, \mathbb{Z}_p)(\sim H^n(X_{an}, \mathbb{Z}) \otimes \mathbb{Z}_p)$$

are the object of the (recent) integral $p$-adic Hodge theory and prismatic cohomology (Bhatt-Morrow-Scholze, Bhatt-Scholze). In particular,

$$\lg(H^n_{dR}(X_{Z^p/Z^p})_{tors}) \geq \lg(H^n(X_{an}, \mathbb{Z})_{p-tors})$$

Variants for $p | m$, semistable reduction case, due to Cesnavicius-Koshikawa.
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3. Liftings mod $p^2$, de Rham-Witt and derived de Rham complexes
1. Review of crystalline cohomology

May 1966: Grothendieck’s letter to Tate: crystals, crystalline site.
Sources of inspiration:

- **Dieudonné theory** ($p$-divisible groups) (Tate, Serre-Tate, Oda)
- **de Rham cohomology**: - Gauss-Manin connection
  - Monsky-Washnitzer’s work on formal cohomology

Talks at IHES, fall of 1966; notes by Coates-Jussila

Crystalline cohomology developed in Berthelot’s thesis (SLN 407, 604 pp.), and later by many authors (Katz, Mazur, Ogus, etc.)
Notation

$k$: perfect field of characteristic $p > 0$, $W = W(k)$,
$W_n = W_n(k) = W/p^nW$, $K = \text{Frac}(W)$
$\sigma : W \xrightarrow{\sim} W$ the Frobenius automorphism
$(\sigma(a_0, \cdots, a_n, \cdots) = (a_0^p, \cdots, a_n^p, \cdots))$

Definitions

- For $X/k$, crystalline site $(X/W_n)_{\text{crys}}$ with objects: thickenings $X \supset U \hookrightarrow U'$,
  $U$ open in $X$, $U'/W_n$, with PD (= divided powers) on the ideal of $U \hookrightarrow U'$, compatible with that on $(p)$,
morphisms: obvious, covering families: $(U_i \leftrightarrow U'_i) \rightarrow (U \leftrightarrow U')$
such that $(U'_i) = \text{Zariski cover of } U'$.
structural sheaf: $\mathcal{O} = \mathcal{O}_{X/W_n}: (U \leftrightarrow U') \mapsto \mathcal{O}_{U'}$.

PD: \textit{a priori} mysterious additional structure, motivated by acyclicity of dR complex of PD-algebras $W < t_1, \cdots, t_d >$. 
• crystalline cohomology

\[ H^i(X/W_n) := H^i((X/W_n)_{\text{crys}}, \mathcal{O}), \]
\[ R\Gamma(X/W) := R \lim_{\leftarrow} R\Gamma(X/W_n), \quad H^i(X/W) = H^i R\Gamma(X/W) \]

Main properties

• Weil cohomology

\( X/k \) proper smooth \( \Rightarrow \quad H^*(X/W) = \bigoplus H^i(X/W) \) is a finitely generated \( W \)-algebra

\( X/k \mapsto H^*(X/W) \otimes K \quad (K = \text{Frac}(W)) \) is a Weil cohomology theory filling the gap at \( p \) among \( \ell \)-adic cohomology theories

\( H^*(X_{\overline{k}}, Q_\ell) \quad (\ell \neq p, \overline{k} = \text{algebraic closure of} \ k) \)

(i.e.: Künneth, Poincaré duality, cycle class). Moreover:

\[ \dim_K(H^i(X/W) \otimes K) = \dim_{Q_\ell}(H^i(X_{\overline{k}}, Q_\ell)). \]

• Relation with de Rham cohomology

\( X/k \) lifted to \( Z/W \) proper and smooth \( \Rightarrow \) canonical iso

\[ H^*(X/W) \sim H^*_{\text{dR}}(Z/W). \]
Slopes of Frobenius
For $X/k$, by functoriality, absolute Frobenius $F_X$ of $X$ ($a \mapsto a^p$ on $\mathcal{O}_X$) defines $\sigma$-linear endomorphism

$$\varphi : H^*(X/W) \to H^*(X/W).$$

Assume $X/k$ proper, smooth. Then $H^*(X/W)$ is finitely generated over $W$, and $\varphi$ is an isogeny, i.e., $\varphi \otimes K$ is an isomorphism (if $X$ is of pure dimension $d$, there exists a $\sigma^{-1}$-linear endomorphism $\nu$ of $H^*(X/W)$ such that $\varphi \nu = \nu \varphi = p^d$). Hence, for each $n \in \mathbb{Z}$,

$$(H^n(X/W), \varphi)$$

is an $F$-crystal.

For $k = F_q$, $q = p^\nu$, Katz-Messing:

$$\det(1 - \varphi^\nu t, H^*(X/W) \otimes K) = \det(1 - \text{Frob} t, H^*(X_k, \mathbb{Q}_\ell))$$

($\ell \neq p$, Frob = relative Frobenius of $X/k$).
In general, **Dieudonné-Manin decomposition** into slopes:

\[ H^n(X/W) \otimes K = \bigoplus_{\lambda \in \mathbb{Q}} H^n_{\lambda}, \]

where \( H^n_{\lambda} \) is isoclinic component of pure slope \( \lambda \) (i.e., over \( K(\overline{k}) := \text{Frac}(W(\overline{k})) \), \( H^n_{\lambda} \) decomposes into \( \bigoplus K(\overline{k})_\sigma[F]/(F^s - p^r) \), \( \lambda = r/s \)).

**Newton polygon** \( N_{\text{wt}_n} \): slope \( \lambda \), horizontal length \( \dim(H^n_{\lambda}) \).

**Main result (solution of Katz conjecture):**

**Theorem (Mazur-Ogus, 1972)** \( X/k \) proper, smooth \( \Rightarrow \)

\[ N_{\text{wt}_n} \geq \text{Hdg}_n, \]

where \( \text{Hdg}_n = \text{Hodge polygon} \), slope \( i \), horizontal length \( h^{i,n-i}, h^{i,j} := \dim_k H^j(X, \Omega^i_{X/k}). \)
Remarks: (a) by Katz-Messing, for $k = \mathbb{F}_q$, $\text{Nwt}_n \geq \text{Hdg}_n$ implies estimates on $p$-adic valuations of eigenvalues of Frob on $H^n(X_k, \mathbb{Q}_\ell)$, and Chevalley-Warning type congruences on $\# X(\mathbb{F}_{q^m})$ for smooth complete intersections in the projective space: for $X = V(a_1, \cdots, a_r) \subset \mathbb{P}_k^{d+r}$,

$$
\# X(\mathbb{F}_{q^m}) \equiv \# \mathbb{P}^d(\mathbb{F}_{q^m}) \mod q^{cm},
$$

where

$$
c = \sup(0, \text{ceiling}(\frac{d + r + 1 - \sum a_i}{\sup(a_i)})).
$$

(generalizations by Katz (1971), Esnault-Katz (2005)).

(b) If $X = X_k$ for a proper, smooth $X/\mathcal{O}_L$, $[L : K] < \infty$, then, stronger inequality:

$$
\text{Nwt}_n \geq \text{Hdg}_n(X_L).
$$

Follows from Berthelot-Ogus comparison th. (1983):

$$
H^n(X/W) \otimes L \overset{\sim}{\to} H^n_{dR}(X_L/L),
$$

and $C_{\text{cris}}$ comparison th. ($\Rightarrow$ weak admissibility of $H^n_{dR}(X_L/L)$) (Tsuji et al., 1997 - ...).
Towards the de Rham-Witt complex

Grothendieck’s questions in letter to Barsotti, May, 1970.

Rewrite $H^n(X/W) \otimes K$ as

$$H^n(X/W) \otimes K = \bigoplus_{i \in \mathbb{Z}} (H^n(X/W) \otimes K)[i, i+1],$$

where

$$(H^n(X/W) \otimes K)[i, i+1] = \bigoplus_{\lambda \in [i, i+1]} H^n_\lambda.$$

• Cohomological interpretation for $(H^n(X/W) \otimes K)[i, i+1]$?

• $(H^n(X/W) \otimes K)[i, i+1] \otimes (K, p^{-i} \sigma) = F$-(iso)crystal of slopes $\in [0, 1)$

$\leftrightarrow$ (by Cartier theory) formal $p$-divisible group $G^{ij}$.

Dimension of $G^{ij}$? Height of $G^{ij}$?

dRW theory brings answers to these questions.
2. Old and new on the de Rham-Witt complex

2.1. Old

$X/k$ smooth. For $\dim(X) < p$, $p > 2$, Bloch (1974) constructs the so-called complex of typical curves

$$C = (C^0 \to C^1 \to \cdots \to C^{\dim(X)} \to 0),$$

a complex of abelian sheaves on $X_{zar}$, with $C^0 = W\mathcal{O}_X$, $C^i$ equipped with $F$, $V$ satisfying $FV = VF = p$, $dF = pFd$, and extending the usual $F$, $V$ on $W\mathcal{O}_X$, with the property that, for $X/k$ proper and smooth, there is a natural isomorphism

$$H^*(X, C) \sim H^*(X/W),$$

with $\varphi$ on $H^*(X/W)$ given by the endomorphism $p^\bullet F$ of $C$. Moreover he shows that $\dim H^i(X, C^i) \otimes K < \infty$, and

$$H^i(X, C^i) \otimes K = (H^{i+j}(X/W) \otimes K)_{[i,i+1)},$$

solving one of Grothendieck’s questions in this case.
In particular,

\[ H^n(X, WO) \otimes K = (H^n(X/W) \otimes K)_{[0,1)}. \]

Construction inspired by Artin-Mazur formal groups \( \Phi^i \), with Cartier modules \( H^i(X, WO) \): \( WO = TC(G_m), \ C^i = TC(SK_{i+1}) \) (\( TC = \) typical curves), \( SK_{i+1} = \) symbolic part of Quillen’s K group.

1975: Deligne sketches differential geometric approach to Bloch’s construction, with no K-groups, working without restrictions of dimension or characteristic.
Inspired by work of Lubkin (1970) on de Rham complexes of Witt vectors.
Carried out in [I, Complexe de de Rham-Witt et cohomologie cristalline, Ann. ENS, 4ème série, 12, 1979, 501-661].
What is the de Rham-Witt complex?

Let $k$ as before, and $X$ a $k$-scheme. The de Rham-Witt complex of $X/k$ is a strictly commutative differential graded algebra on the Zariski site of $X$:

$$
W\Omega^\bullet_X = (W\Omega^0_X \xrightarrow{d} W\Omega^1_X \xrightarrow{d} \cdots),
$$

with $W\Omega^0_X = W\mathcal{O}_X$, and each component $W\Omega^i_X$ is equipped with additive operators $F$, $V$, extending the usual ones on $W\mathcal{O}_X$, satisfying the following relations

$$
FV = VF = p, \quad xVy = V(Fx.y), \quad Fx.Fy = F(xy), \quad Fd[a] = [a]^{p^{-1}}d[a]
$$

(for $[a] = (a, 0, \cdots, 0, \cdots)$ the Teichmüller representative of $a \in \mathcal{O}_X$), and

$$
FdV = d.
$$
\[ FV = VF = p, \ xVy = V(Fx.y), \ Fx.Fy = F(xy), \ Fd[a] = [a]^{p-1}d[a] \]

(for \([a] = (a, 0, \cdots, 0, \cdots)\) the Teichmüller representative of \(a \in \mathcal{O}_X\)), and

\[ FdV = d. \]

Implies: for \(n \geq 1\), \(V^nW\Omega_X^\bullet + dV^nW\Omega_X^\bullet\) is a differential graded ideal. Let

\[ W_n\Omega_X^\bullet := W\Omega_X^\bullet/(V^nW\Omega_X^\bullet + dV^nW\Omega_X^\bullet) = (W_n\mathcal{O}_X \to \cdots) \]

be the quotient. The projective system

\[ W\Omega_X^\bullet = (\cdots \to W_{n+1}\Omega_X^\bullet \to W_n\Omega_X^\bullet \to \cdots \to W_1\Omega_X^\bullet), \]

together with the induced operators \(F : W_{n+1}\Omega_X^i \to W_n\Omega_X^i\), \(V : W_n\Omega_X^i \to W_{n+1}\Omega_X^i\), is characterized by a certain universal property: **universal** \(F-V\)-pro-complex over \(W\mathcal{O}_X\), in Langer-Zink’s terminology.
\[ W_\bullet \Omega^\bullet_X = (\cdots \to W_{n+1} \Omega^\bullet_X \to W_n \Omega^\bullet_X \to \cdots \to W_1 \Omega^\bullet_X), \]
Moreover, the sheaves \( W_n \Omega^i_X \) are quasi-coherent on \( W_n(X) \), the canonical map
\[
\Omega^\bullet_{W_n \mathcal{O}_X/W_n} \to W_n \Omega^\bullet_X
\]
is surjective, and an isomorphism for \( n = 1 \). For \( X = \text{Spec}(k) \), \( W_n \Omega^\bullet_k = W_n(k) \).
Short definition for $X/k$ smooth

$$W\Omega_X^\bullet = \hat{\Omega}_{W\mathcal{O}_X/W}/\overline{T},$$

where $\overline{T}$ = closure of $p$-torsion $T = \Omega_{W\mathcal{O}_X/W}[p^\infty]$ for topology given by

$$\hat{\Omega}_{W\mathcal{O}_X/W} = \lim_{\leftarrow} \Omega_{W_n\mathcal{O}_X/W_n}^\bullet,$$

i.e.,

$$W\Omega_X^\bullet = \lim_{\leftarrow} \Omega_{W\mathcal{O}_X/W}/(T + K_n),$$

where $K_n := \text{Ker}(\Omega_{W\mathcal{O}_X}^\bullet \to \Omega_{W_n\mathcal{O}_X}^\bullet)$.

**Remark.** $T^0 = 0$, but $T^1 \neq 0$ if $\dim(X) > 0$, e.g., if $x^0 := [t], x_1 = Vx_0$, then $x_1^p = p^p x_0 \Rightarrow$

if $y = x_1^{p-1}dx_1 - p^{p-1}dx_0 \in \Omega^1_{W(F_p[t])}$, $y \neq 0$, $py = 0$.

**Operators** $F$, $V$, etc.: Use $Fa \equiv a^p \mod p$ for $a \in W\mathcal{O}_X$, $F$ induces $\varphi$ on $\Omega^1$, uniquely divisible by $p$ on $\Omega^1/T$, thus $\varphi = p^i F$ on $\Omega^i/T$; $V$, relations follow.
Additional properties for $X/k$ smooth.

- $W\Omega^i_X$ $p$-torsion free for all $i$, in particular, $F$, $V$ injective
- $W\Omega^i_X = 0$ for $i > \dim(X)$
- (saturation) $d^{-1}(pW\Omega_X^{i+1}) = FW\Omega_X^i$ (in particular, $F$ bijective on $W\Omega_X^{\dim(X)}$).

This property was the motivation for BLM (Bhatt-Lurie-Mathew) approach (see below).

- $W\Omega^\bullet_X/pW\Omega^\bullet_X \rightarrow W_1\Omega^\bullet_X = \Omega^\bullet_X$ is a quasi-isomorphism.

Main theorems

- Comparison with crystalline cohomology
- Structure of slope spectral sequence, and applications
Comparison with crystalline cohomology

Theorem 1. For $X/k$ smooth, there is a canonical, functorial isomorphism of graded algebras

$$H^*(X/W) \sim H^*(X, W\Omega^\bullet),$$

with Frobenius $\varphi$ on $H^*(X/W)$ given by endomorphism $p^\bullet F$ of the complex $W\Omega^\bullet_X$.

Comes from refined, local theorem:
Theorem 1’. There exist a compatible system of (canonical, functorial) isomorphisms

$$Ru_\ast \mathcal{O}_{X/W_n} \sim \mathcal{W}_n \Omega^\bullet_X,$$

where

$$u : (X/W_n)_{\text{crys}} \to X_{\text{zar}}$$

is Berthelot’s canonical map ($u^{-1}(U) = (U/W_n)_{\text{crys}}$). In particular, if $R \Gamma(X/W) := R \lim_{\leftarrow} R \Gamma(X, Ru_\ast \mathcal{O}_{X/W_n})$,

Corollary. For $X/k$ proper and smooth,

$$R \Gamma(X/W) \sim R \Gamma(X, \mathcal{W} \Omega^\bullet_X)$$

is a perfect complex of $\mathcal{W}$-modules, and, for $X$ of pure dimension $d$, the $\sigma^{-1}$-linear endomorphism $\nu$ of $R \Gamma(X/W)$ induced by the endomorphism of $\mathcal{W} \Omega^\bullet_X$ given by $p^{d-1-i} V$ in degree $i$ (and $F^{-1}$ in degree $d$) satisfies

$$\varphi \nu = \nu \varphi = p^d.$$
Structure of slope spectral sequence and applications

Theorem 2. For $X/k$ proper and smooth, the spectral sequence

$$E_1^{ij} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}(X, W\Omega^i_X)(\sim H^{i+j}(X/W))$$

(called slope spectral sequence) degenerates at $E_1$ modulo $p$-torsion, $H^j(X, W\Omega^i_X)/H^j(X, W\Omega^i_X)[p^{\infty}]$ is finitely generated over $W$, with $V$ topologically nilpotent, $H^j(X, W\Omega^i_X)[p^{\infty}]$ is killed by a power of $p$, and the degeneration induces an isomorphism

$$H^i(X, W\Omega^i_X) \otimes K \cong (H^{i+j}(X/W) \otimes K)_{[i,i+1]}.$$

In particular,

$$H^i(X, W\mathcal{O}_X) \otimes K \cong (H^i(X/W) \otimes K)_{[0,1]}.$$

When Artin-Mazur’s functor $\Phi^j$ is representable by a smooth formal group, then $H^j(W\mathcal{O})$ is its Cartier module of typical curves, and $\dim_K(H^i(X, W\mathcal{O}_X) \otimes K)$ is the height of its largest $p$-divisible quotient.
Refinements and complements

by I., I.-Raynaud, Nygaard, Ekedahl, especially, theory of coherent graded modules over the Raynaud ring (see slide 49)

In particular:

• (Nygaard) New (simpler) proofs of Rudakov-Shafarevich theorem on K3’s ($H^0(X, T_X) = 0$), and Ogus’ th. ($\text{Nwt}_n \geq \text{Hdg}_n$) (via Nygaard filtration).

• (I.) (generalization of Igusa-Artin-Mazur inequality) For $k = \overline{k}$,

$$\text{rk}(\text{NS}(X)) = b_2 - 2\dim(H^2(W\mathcal{O}) \otimes K) - \text{rk} T_p(\text{Br}(X)).$$

• (answers to Grothendieck’s question) (I., Ekedahl)

$H^j(X, W\Omega^i)/V$-torsion = Cartier module of smooth formal group $G^{ij}$; dimensions of $p$-divisible quotient and unipotent part of $G^{ij}$ are interesting numerical invariants of slope spectral sequence (Ekedahl’s theory of Hodge-Witt numbers). If $H^j(X, W\Omega^i)$ is $p$-torsion free, $G^{ij}$ is $p$-divisible, and

$$\dim(G^{ij}) = \dim_k H^j(X, W\Omega^i)/V, \quad \text{ht}(G^{ij}) = \dim_k H^j(X, W\Omega^i)/p.$$
• (I.) Study of torsion of $H^2(X/W)$ (discovery of exotic torsion)
• (I. I.-Raynaud, Ekedahl, Bloch, Gabber, Kato, Kerz, Morrow) Logarithmic Hodge-Witt sheaves, relation with Milnor $K$-groups, and fixed points of $F$ on $H^*(X, W\Omega^*)$

Further developments

• Hyodo-Kato’s theory of log crystalline cohomology and log de Rham-Witt complexes (used in formulation - and proof - of Fontaine-Jannsen’s comparison conjecture $C_{st}$)
• Relative variants (Langer-Zink) (application to theory of displays), overconvergent variants (Davis-Langer-Zink), arithmetic variants (Hesselholt-Madsen), recent links with THH and cyclotomic spectra (BMS2), connections with integral $p$-adic Hodge theory and prismatic cohomology (BMS1, BS).
2.2. New approach to de Rham-Witt (BLM)

For $X/k$ proper, but singular, $H^*(X/W)$ bad (Bhatt’s examples with $\dim_K H^*(X/W) \otimes K = \infty$), but Berthelot’s rigid cohomology $H^*_{\text{rig}}(X/K)$ good (finite dimensional over $K$, and $\varphi$ an isomorphism) (NB. vast generalizations by Kedlaya et al.). In particular, slope decomposition:

$$H^n_{\text{rig}}(X/K) = \bigoplus_{i \in \mathbb{Z}} H^n_{\text{rig}}(X/K)[i, i+1),$$

BLM approach yield new dRW complexes $W\Omega^\bullet_X$ with:

- $W\Omega^\bullet_X = W\Omega^\bullet_X$ for $X/k$ smooth,

- for certain singular $X/k$ (conjecturally all proper $X/k$):

$$H^*(X, W\Omega_X^\bullet)$$ finitely generated over $W$

$$H^*(X, W\Omega_X^\bullet) \otimes K \overset{\sim}{\to} H^*_{\text{rig}}(X/K)$$

Slope spectral sequence and cohomological interpretation of slope decomposition as in the proper smooth case.
Construction of BLM’s de Rham-Witt complexes

A Dieudonné algebra is a strict cdga

\[ A = (A^0 \xrightarrow{d} A^1 \xrightarrow{d} \ldots) \]

with each \( A^i \) equipped with an additive endomorphism \( F \) satisfying:

\[ (Fx)(Fy) = F(xy), \quad dF = pFd, \quad Fa \equiv a^p \mod pA^0 \]

for \( a \in A^0 \).

The Dieudonné algebra \( A \) is called saturated if, for all \( i \), \( A^i \) is \( p \)-torsion free, \( F : A^i \rightarrow A^i \) is injective, and \( d^{-1}(pA^{i+1}) = FA^i \).

If \( A \) is saturated, there exists a unique additive \( V : A^i \rightarrow A^i \) such that \( FV = p \). It follows that:

\[ VF = p, \quad FdV = d, \quad xVy = V(Fx.y). \]

Then, for all \( n \geq 1 \), \( V^nA + dV^nA \) is a dg ideal in \( A \). One sets

\[ \mathcal{W}_nA := A/(V^nA + dV^nA). \]
Examples.

(1) $A = \Omega^\bullet_{\mathbb{Z}_p[t_1, \ldots, t_r]/\mathbb{Z}_p}$; lift $t_i \mapsto t_i^p$ of Frobenius gives endomorphism $\varphi$ of $\Omega^\bullet$, with $\varphi = p^i F$ on $\Omega^i$; $(A, d, F)$ is a Dieudonné algebra; not saturated.

Variant: $R/k$ smooth; $B/W$ smooth formal lifting, with lifting $F$ of Frobenius; then $A := \widehat{\Omega}^\bullet_{B/W}$, with $F = p^{-i} \varphi$ on $\widehat{\Omega}^i$ ($\varphi =$ endomorphism of $A$ induced by $F : B \to B$) is a Dieudonné algebra; not saturated.

(2) $R/k$ smooth; $(W\Omega^\bullet_R, d, F)$ is a saturated Dieudonné algebra (cf. p. 30, Additional properties).
Let $A$ be a saturated Dieudonné algebra. Recall:

$$VF = p, \ FdV = d, \ xVy = V(Fx.y).$$

$$\mathcal{W}_nA := A/(V^nA + dV^nA).$$

Define

$$\mathcal{W}A := \lim_{\leftarrow} \mathcal{W}_nA.$$  

It’s again a saturated Dieudonné algebra. One says that $A$ is strict if the canonical map

$$A \rightarrow \mathcal{W}A$$

is an isomorphism. For example, $W\Omega^\bullet_R$ in Example (2) is strict.

Note: $A$ strict $\Rightarrow$ $A^0/VA^0$ reduced, and $A^0 \sim W(A^0/VA^0)$. 
**Theorem 1** (BLM). The functor

\[ \text{DA}_{\text{str}} \rightarrow \text{F}_p-\text{alg}, \ A \mapsto A^0/VA^0 \]

has a left adjoint \( R \mapsto W\Omega^\bullet_R : \):

\[ \text{Hom}_{\text{F}_p-\text{alg}}(R, A^0/VA^0) \xrightarrow{\sim} \text{Hom}_{\text{DA}_{\text{str}}}(W\Omega^\bullet_R, A). \]

The strict Dieudonné algebra \( W\Omega^\bullet_R \) is called the saturated de Rham-Witt complex of \( R \).

Functorial in \( R \). For \( R \) perfect, \( W\Omega^\bullet_R = W(R) \).

**Proof of Theorem 1.** Easy. Uses Deligne-Ogus décalage functor \( \eta_p \).
For a complex $M$ of $p$-torsion free abelian groups, define the subcomplex

$$\eta_p M \subset M[1/p], \ (\eta_p M)^i = p^i M^i \cap d^{-1}(p^{i+1} M^{i+1}).$$

A $p$-torsion free Dieudonné algebra $A$ is saturated if and only if

$$\alpha_F : A \to \eta_p A, \ a \in A^i \mapsto p^i Fa$$

is an isomorphism. The saturation functor

$$\text{Sat} : \text{DA} \to \text{DA}_{\text{sat}}$$

defined by

$$\text{Sat}(A) := \lim_{\alpha_F} (\eta_p)^n (A/A[p^{\infty}])$$

is left adjoint to the inclusion. For $A = \Omega_{\mathbb{Z}[t_1,\ldots,t_r]/\mathbb{Z}_p}$ as in Example (1), $\text{Sat}(A)$ played crucial role in study of classical dRW in smooth case (Deligne’s complex of integral forms).
Construction of saturated dRW

Put

\[ \mathcal{W}\Omega^\bullet_{R} := \mathcal{W}\text{Sat}(\Omega^\bullet_{\mathcal{W}(R_{\text{red}})}). \]

There is indeed a unique structure of Dieudonné algebra on \( \Omega^\bullet_{\mathcal{W}(R_{\text{red}})} \) inducing \( F \) in degree 0.

Construction globalizes on schemes, yielding, for \( X/F_p \), the saturated de Rham-Witt complex

\[ \mathcal{W}\Omega^\bullet_X = (\mathcal{W}\Omega^0_X \xrightarrow{d} \mathcal{W}\Omega^1_X \xrightarrow{d} \cdots) \]

a (strictly commutative) dga on \( X_{\text{zar}} \) (over \( \mathcal{W}(k) \) if \( X/k, k \) as before), with \( F, V \) satisfying same formulas as for the (classical) dRW complex, except that:

\[ \mathcal{W}\Omega^0_X = \mathcal{W}(\mathcal{W}\Omega^0_X/V\mathcal{W}\Omega^0_X), \]

and adjunction map

\[ \mathcal{O}_X \to \mathcal{W}\Omega^0_X/V\mathcal{W}\Omega^0_X \]

not necessarily an isomorphism.
Basic properties

In what follows, we assume $X/k$, $k$ as before. First of all,

$$\mathcal{W}_X = \lim_{\leftarrow} \mathcal{W}_n \Omega^\bullet_X,$$

with

$$\mathcal{W}_n \Omega^\bullet_X := \mathcal{W}_X \omega^\bullet / (V^n + dV^n),$$

and $\mathcal{W}_n \Omega^i_X$ quasi-coherent over $\mathcal{W}_n(X)$, and compatible with étale localization. For $U = \text{Spec}(R)$ open in $X$,

$$\Gamma(U, \mathcal{W}_n \Omega^i) = \mathcal{W}_n \Omega^i_R.$$

The inverse system

$$\mathcal{W}_\bullet \Omega^\bullet_X = (\mathcal{W}_n \Omega^\bullet_X)_{n \geq 1}, F, V)$$

is a Langer-Zink $F$-$V$-pro-complex over $\mathcal{W}_\bullet \mathcal{O}_X$, hence a canonical map (of $F$-$V$-pro-complexes)

$$\text{can} : \mathcal{W}_\bullet \Omega^\bullet_X \rightarrow \mathcal{W}_\bullet \Omega^\bullet_X.$$
Theorem 2. (BLM). For $X/k$ smooth, the canonical map

$$can : \mathcal{W} \Omega_X^\bullet \rightarrow \mathcal{W} \Omega_X^\bullet$$

is an isomorphism, hence so is the resulting map

$$can : \mathcal{W} \Omega^\bullet_X \rightarrow \mathcal{W} \Omega^\bullet_X.$$

Proof. Formal consequence of known structure of $\mathcal{W} \Omega^\bullet_R$ for $R = \mathbb{F}_p[t_1, \cdots, t_r].$

Bonus.
● Independent, budget proofs for main results of I. on $\mathcal{W} \Omega^\bullet_X$ in smooth case, including comparison with crystalline cohomology ("laborious calculations" of I. avoided), and Ogus’ key lemma in proof of $\text{Ntw}_n \geq \text{Hdg}_n$, namely $\varphi : Ru_* \mathcal{O}_X/W \rightarrow Ru_* \mathcal{O}_X/W$ induces isomorphism $Ru_* \mathcal{O}_X/W \sim L\eta_p Ru_* \mathcal{O}_X/W.$
Bonus (cont’d)

• Abstract formulation of I.-Katz-Raynaud’s reconstruction of $\mathcal{W}_\cdot \Omega_X^\bullet$ from $Ru_*\mathcal{O}_{X/W}$ (in smooth case), in terms of a general fixed point theorem for $L\eta_p$.

New features
(a)

$$\mathcal{W}_\cdot \Omega_X^\bullet \sim \mathcal{W}_\cdot \Omega_{X_{\text{red}}}^\bullet.$$ More generally: if $X \to Y$ is a morphism of $k$-schemes which is a universal homeomorphism with trivial residue extensions,

$$\mathcal{W}_\cdot \Omega_Y^\bullet \to \mathcal{W}_\cdot \Omega_X^\bullet$$ is an isomorphism. In particular, if $R^{\text{sn}}$ is the Swan seminormalization of an $F_p$-algebra $R$, then

$$\mathcal{W}_\cdot \Omega_R^\bullet \to \mathcal{W}_\cdot \Omega_{R^{\text{sn}}}^\bullet$$ is an isomorphism, which gives the following formula for $R^{\text{sn}}$:

$$R^{\text{sn}} = \mathcal{W}_\cdot \Omega^0_R / \mathcal{W} \mathcal{W}_\cdot \Omega^0_R.$$
(b) If $X \to Y$ is a universal homeomorphism (with possibly non-trivial residue extensions), then

$$\mathcal{W}_Y \Omega^\bullet \otimes K \to \mathcal{W}_X \Omega^\bullet \otimes K$$

is an isomorphism. In particular, the Frobenius endomorphism $\varphi : \mathcal{W}_X \Omega^\bullet \to \mathcal{W}_X \Omega^\bullet$ is an isogeny, i.e.

$$\varphi \otimes K : \mathcal{W}_X \Omega^\bullet \otimes K \to \mathcal{W}_X \Omega^\bullet \otimes K$$

is an isomorphism. More precisely, if $X/k$ is of finite type, and there is an integer $d$ such that $\Omega^1_{X/k}$ is generated by at most $d$ elements, then

$$\mathcal{W}_X \Omega_i^\bullet = 0$$

for $i > d$. As in the smooth case (but $d > \dim(X)$ if $X$ is singular), this implies that $F : \mathcal{W}_X \Omega^d \to \mathcal{W}_X \Omega^d$ is bijective, hence $\nu : \mathcal{W}_X \Omega^\bullet \to \mathcal{W}_X \Omega^\bullet$ defined by $p^{d-1-i} \nu$ in degree $i$ (and $F^{-1}$ for $i = d$) satisfies

$$\varphi \nu = \nu \varphi = p^{d}.$$
Remark. In contrast with the simplicity of the proofs of the general properties of $\mathcal{W}\Omega^\bullet$, the results in (a) and (b) require delicate arguments of commutative (and homological) algebra.

In particular, the proof of (b) uses the theory of derived de Rham-Witt complex $L\mathcal{W}\Omega_X^\bullet$ and its (curious) relation with the saturated one (which, roughly, says that $\mathcal{W}\Omega_X^\bullet$ is the derived $p$-completion of the saturation of $L\mathcal{W}\Omega_X^\bullet$).
Questions and prospects

(1) The finiteness problem.
For the classical dRW complex, the canonical map
\( \Omega^\bullet_{W_n\mathcal{O}_X} \to \mathcal{W}_n\Omega^\bullet_X \quad (n \geq 1) \) is surjective, in particular, for \( X/k \) of finite type, \( \mathcal{W}_n\Omega^i_X \) is coherent over \( \mathcal{W}_n(X) \) for all \( i \).
The similar map
\[ \Omega^\bullet_{W_n\mathcal{O}_X} \to \mathcal{W}_n\Omega^\bullet_X \]
is not surjective in general, as (a) above shows (e.g. if
\( R = k[x, y]/(x^2 - y^3) \), \( \mathcal{W}_1\Omega^0_R = k[t] \)). That raises:

**Question 1.** For \( X/k \) of finite type, is \( \mathcal{W}_1\Omega^i_X \) coherent on \( X \)?
Remarks.

(i) One can show that if $\mathcal{W}_i^1 \Omega^i_X$ is coherent on $X$ for all $i$, then $\mathcal{W}_n^i \Omega^i_X$ is coherent on $W_n(X)$ for all $i$.

(ii) Let

$$R = W(k)_\sigma[F, V; d]/(FV = VF = p, FdV = d, d^2 = 0) = R^0 \oplus R^1$$

be the Raynaud ring of $k$, a non-commutative graded ring, where $R^0 = W_\sigma[F, V]/(FV = VF = p)$ is the Cartier-Dieudonné ring, and $d$ is placed in degree 1. (Left) graded modules $M$ over $R$ correspond to complexes $M$ of $W$-modules, where each component $M^i$ is equipped with (semi-linear) operators $F, V$, satisfying

$$FV = VF = p, FdV = d,$$

e.g., a saturated Dieudonné complex (over $W$) is a graded $R$-module.
Derived category $D(R)$ extensively studied by I.-Raynaud, Ekedahl, especially the full subcategory $D_{bc}^b(R)$ of $D^b(R)$ consisting of complexes of (graded) $R$-modules $M$, with coherent cohomology complexes $H^i(M)$, characterized by the fact that

$$M \to R \lim_{\leftarrow} R/(V^nR + dV^nR) \otimes^L_R M$$

is an isomorphism and each $H^i(R/(VR + dVR) \otimes^L_R M)$ (a complex of $k$-vector spaces) has finite dimensional cohomology.

Can show (I.): if $X/k$ is proper and Question 1 has a positive answer for $X$, then one has $R\Gamma(X, \mathcal{W}\Omega^\bullet) \in D_{bc}^b(R)$, and, in particular (by I.-Raynaud, Ekedahl):

- $H^*(X, \mathcal{W}\Omega^\bullet)$ is finitely generated over $W$. 
• As in the smooth case, the \textit{slope spectral sequence}

\[ E_1^{ij} = H^j(X, \mathcal{W}\Omega^i) \Rightarrow H^{i+j}(X, \mathcal{W}\Omega^\bullet) \]

degenerates at \( E_1 \) modulo torsion, \( H^j(X, \mathcal{W}\Omega^i)/H^j(X, \mathcal{W}\Omega^i)[p^{\infty}] \)
is finitely generated over \( \mathcal{W} \), with \( V \) topologically nilpotent, \( H^j(X, \mathcal{W}\Omega^i)[p^{\infty}] \) is killed by a power of \( p \), and the degeneration induces an isomorphism

\[ H^j(X, \mathcal{W}\Omega^i) \otimes K \xrightarrow{\sim} (H^{i+j}(X, \mathcal{W}\Omega^\bullet) \otimes K)_{[i,i+1)}. \]

In particular, we have

\[ H^j(X, \mathcal{W}\Omega^0) \otimes K \xrightarrow{\sim} (H^j(X, \mathcal{W}\Omega^\bullet) \otimes K)_{[0,1)}. \]
Status of Question 1 (finiteness problem)

Positive answer known in the following cases:

(a) (I.) $X$ has normal crossing singularities, i.e. is locally smooth over $Y = \text{Spec}(k[t_1, \cdots, t_r]/(t_1 \cdots t_r))$.

If $D_i = V(t_i) \in \text{Spec}(W[t_1, \cdots, t_r])$, then $\mathcal{W}\Omega_Y$ is a du Bois type complex:

$$\mathcal{W}\Omega_Y = \text{Ker}(\oplus_i \mathcal{W}\Omega_{D_i/W} \to \oplus_{i<j} \mathcal{W}\Omega_{D_i \cap D_j}/W)$$

(and similarly for $\mathcal{W}_n\Omega_Y$). In particular,

$$\mathcal{W}_1\Omega_Y = \text{Ker}(\oplus_i \Omega_{D_i \otimes k/k} \to \oplus_{i<j} \Omega_{D_i \cap D_j \otimes k/k}).$$

(b) (I.) $X/k$ is a curve. Follows from invariance of $\mathcal{W}\Omega$ by passing to the seminormalization ($\mathcal{W}\Omega_R \sim \mathcal{W}\Omega_{R_{sn}}$) and local calculation in seminormal case, using canonical factorization

$$X^n \to X^{sn} \to X$$

where $X^n$ (resp. $X^{sn}$) is the normalization (resp. seminormalization) of $X$. 
(c) (Ogus, work in progress) $X/k$ is an affine, toric scheme, i.e.,
$X = \text{Spec}(k[P])$ where $P$ is a fine, saturated, torsion free monoid.

Proof uses lifting of Frobenius on $W[P]$, $F = a \mapsto pa$ on $P$.

Hope: affine, toric can be relaxed to toroidal singularities.

(d) First unknown cases: conic singularities not of the above type, e.g. for $p > 2$, $\sum_{1 \leq i \leq 5} x_i^2 = 0$. 
(2) **Comparison with rigid cohomology**

Let $X/k$ be proper. Recall Berthelot’s rigid cohomology $H^\text{rig}_*(X/K)$ is finite dimensional over $K$, and has slope decomposition

$$H^n_{\text{rig}}(X/K) = \bigoplus_{i \in \mathbb{Z}} H^n_{\text{rig}}(X/K)[i,i+1].$$

**Question 2.** Can we construct a (functorial, $\varphi$-compatible) isomorphism

$$H^\text{rig}_*(X/K) \sim \rightarrow H^*(X, \mathcal{W}\Omega^\bullet) \otimes K?$$

**Remarks.** (i) A positive answer would imply that

$$\varphi : R\Gamma(X, \mathcal{W}\Omega^\bullet) \rightarrow R\Gamma(X, \mathcal{W}\Omega^\bullet)$$

is an isogeny. This is a theorem by BLM: in fact,

$$\varphi \otimes K : \mathcal{W}\Omega^\bullet_X \otimes K \rightarrow \mathcal{W}\Omega^\bullet_X \otimes K$$

is already an isomorphism (see slide 46).
(ii) By a result of Berthelot-Bloch-Esnault, there is a 
(\(\varphi\)-compatible) canonical isomorphism

\[ H^*_{\text{rig}}(X/K)[0,1) \rightarrow H^*(X, W\mathcal{O}) \otimes K. \]

By BLM, \( W\Omega^0_X = W(O^s_X) \), and, as \( O_X \rightarrow O^s_X \) is a universal homeomorphism,

\[ H^*(X, W\mathcal{O}) \otimes K \rightarrow H^*(X, W(O^s_X)) \otimes K \]

is an isomorphism (by elementary properties of \( H^*(W\mathcal{O}) \)).
Therefore, when Question 1 (finiteness) has a positive answer, hence (see slide 46)

\[ H^j(X, W\Omega^0) \otimes K \cong (H^j(X, W\Omega^\bullet) \otimes K)[0,1), \]

a positive answer to Question 2 yields Berthelot-Bloch-Esnault’s result.
Status of Question 2.

Positive answer known only in cases (a) (snc singularities) and (b) (curve) above, thanks to du Bois type property of $\mathcal{W}\Omega^\bullet$, and Tsuzuki’s proper cohomological descent for $H^\ast_{\text{rig}}(X/K)$.

Remark. $H^\ast(X, \mathcal{W}\Omega^\bullet)$ does not satisfy proper cohomological descent (as Frobenius shows), and does not satisfy cdh descent either (as an example of Bhatt shows).
Further developments and problems

- (Zijian Yao) Log variants of BLM constructed.
- Relative variants (comparison with Langer-Zink)? Variants with $F$-crystal coefficients? Relation with Ekedahl’s theory of $F$-gauges?
- Relation with prismatic cohomology?
3. Liftings mod $p^2$, dRW and derived dR complexes

1. Review of Deligne-I. and statement of main result

$k$: perfect field, $\text{char}(k) = p > 0$, $W = W(k)$, $W_n = W_n(k)$

For $X/k$, relative Frobenius $F$, sitting in

\[
\begin{array}{ccc}
X & \xleftarrow{F} & X' \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xleftarrow{F_k} & \text{Spec}(k)
\end{array}
\]

\[
F_*\Omega^\bullet_{X/k} \in D^b(X', \mathcal{O}_{X'}).\]

For $X/k$ smooth, Cartier isomorphism

\[
C^{-1} : \Omega^i_{X'/k} \sim \mathcal{H}^i F_*\Omega^\bullet_{X/k}.
\]
Theorem 1 (Deligne-I.). Assume $X/k$ smooth and $\dim(X) < p$. With any smooth lifting of $X$ to $W_2$ is associated a decomposition in $D(X')$ ($= D(X', \mathcal{O}_{X'})$):

$$\bigoplus \Omega^i_{X'} [-i] \xrightarrow{\sim} F_* \Omega^\bullet_X$$

($/k$ omitted for brevity), inducing $C^{-1}$ on $\mathcal{H}^i$.

Recall: implies various Hodge degeneration and Kodaira vanishing theorems.

Thanks to the multiplicative structures on both sides, Th. 1 follows from:

Theorem 1' (Deligne-I.). Assume $X/k$ smooth. With any smooth lifting of $X$ to $W_2$ is associated a decomposition in $D(X')$:

$$\mathcal{O}_{X'} \bigoplus \Omega^1_{X'} [-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^\bullet_X,$$

where $\tau_{\leq i}$ denotes a canonical truncation.
Goal of this talk: sketch a proof of the following stronger result:

**Theorem 2** (I., 2019). For any smooth $X/k$ there is a canonical, functorial isomorphism in $D(X')$: 

$$
(\tau_{\geq -1} L_{X'/W_2})[-1] \sim \tau_{\leq 1} F_* \Omega^\bullet_X,
$$

where $\tau_{\geq i}$ denotes a canonical truncation, and $L_{X'/W_2}$ is the cotangent complex of $X'/W_2$.

**Proof of Th. 2 $\Rightarrow$ Th. 1'.**

Let $\tilde{X}$ be a smooth lifting of $X$ to $W_2$. Then

$$
L_{X/W_2} = L_{X/k} \oplus L_{X/\tilde{X}}.
$$

But

$$
L_{X/k} = \Omega^1_X, \quad \tau_{\geq -1} L_{X/\tilde{X}} = \mathcal{O}_X[1].
$$
Remarks. (a) A generalization of Th. 2 to the prismatic set-up has recently been obtained (independently) by Bhatt, private communication. See comments at the end.

(b) If $L = (L^0 \to L^1)$ is a complex of $\mathcal{O}$-modules over a locally ringed space, with $\mathcal{H}^1(L)$ locally free of finite type, a splitting of $L$ is a section of $L^1 \to \mathcal{H}^1(L)$. Splittings exist locally, and two splittings $s_1, s_2$ are locally related by an $h : \mathcal{H}^1(L) \to L^0$ such that $s_2 - s_1 = dh$. The sheaf of automorphisms of a splitting $s$ is isomorphic to $\mathcal{H}^0(L)$.

Local splittings of $L$ form a gerbe

$$\text{Split}(L).$$

Similarly, smooth liftings of $X$ to $W_2$ exist locally, and two such are locally isomorphic, the sheaf of automorphisms of a fixed lifting is the tangent sheaf $T_X$. Local liftings form a gerbe

$$\text{Lift}(X/W_2).$$
Th. 2 implies the $S = \text{Spec}(k)$ special case of a theorem of Deligne-I:

**Theorem 1”**. There is a natural equivalence of gerbes:

$$\text{Lift}(X'/W_2) \sim \rightarrow \text{Split}(\tau_{\leq 1} F_* \Omega^\bullet_X).$$

(Note: here $\text{Lift}(X'/W_2) \sim \rightarrow \text{Lift}(X/W_2)$.)

Indeed, elementary theory of cotangent complex and deformations yield:

$$\text{Lift}(X/W_2) \sim \rightarrow \text{Split}(\tau_{\geq -1} L_{X/W_2})$$

$$\tilde{U} \mapsto (\tau_{\geq -1} L_{U/W_2} \sim \rightarrow L_{U/k} \oplus \tau_{\geq -1} L_{U/\tilde{U}} = \Omega^1_U \oplus \mathcal{O}_U[1])$$
2. Sketch of proof of Th. 2

Preliminary observation: $L_{X/W}$ is a perfect complex, of perfect amplitude in $[-1, 0]$, and

$$L_{X/W} \sim \tau_{\geq -1} L_{X/W_2}.$$ 

If $X \hookrightarrow Y$ is a closed embedding in $Y/W$ smooth, with ideal $J$, then

$$L_{X/W} \sim (J/J^2 \rightarrow \mathcal{O}_X \otimes \Omega^1_Y)$$

(placed in degrees -1, 0).

Main ingredient of proof: the de Rham-Witt complex $W\Omega_X^\bullet$ and its Nygaard filtration.

A priori, no dRW complex in sight. How can it come on the scene?
It comes as a by-product of the construction of the comparison map from crystalline cohomology to de Rham-Witt.

For simplicity, assume there exists a closed embedding \( X \hookrightarrow Y \), with \( Y/W \) smooth, and endowed with a lifting \( F \) of Frobenius. By Cartier, there exists a unique \( F \)-compatible section

\[
s_F : \mathcal{O}_Y \to \mathcal{W}(\mathcal{O}_Y)
\]

of the canonical projection. Composing with \( \mathcal{W}(\mathcal{O}_Y) \to \mathcal{W}(\mathcal{O}_X) \), get a map \( \Omega_Y^* \to \Omega_W^*(\mathcal{O}_X)/\mathcal{W} \), hence (composing with \( \Omega^*_W(\mathcal{O}_X)/\mathcal{W} \to \mathcal{W}\Omega^*_X \)) a map (of dga)

\[
c : \Omega_Y^* \to \mathcal{W}\Omega_X^*
\]

which, in degree 0, is \( \mathcal{O}_Y \to \mathcal{W}\mathcal{O}_X \), sending \( J \subset \mathcal{O}_Y \) into \( \mathcal{VW}\mathcal{O}_X \subset \mathcal{W}\Omega_X^* \).
Therefore, for $r \geq 0$, 

$$c(J^r \Omega_Y^\bullet) \subset \mathcal{N}^r W\Omega_X^\bullet;$$

Here:

$$\Omega_Y^\bullet = J^0 \Omega_Y^\bullet \supset J\Omega_Y^\bullet \supset \cdots \supset J^r \Omega_Y^\bullet \supset \cdots$$

is the $J$-adic filtration of $\Omega_Y^\bullet$, defined by 

$$J^r \Omega_Y^\bullet = (J^r \xrightarrow{d} J^{r-1} \Omega_Y^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_Y \xrightarrow{d} \Omega_Y^{r} \xrightarrow{d} \cdots),$$

and

$$W\Omega_X^\bullet = \mathcal{N}^0 W\Omega_X^\bullet \supset \mathcal{N}^1 W\Omega_X^\bullet \supset \cdots \supset \mathcal{N}^r W\Omega_X^\bullet \supset \cdots$$

is the Nygaard filtration defined, for $r \geq 1$, by
$\nabla^r W\Omega^\bullet_X = (p^{r-1} VW\mathcal{O}_X \overset{d}{\to} p^{r-2} VW\Omega^1_X \overset{d}{\to} \cdots \overset{d}{\to} VW\Omega^{r-1}_X \overset{d}{\to} W\Omega^r_X \overset{d}{\to} W\Omega^{r+1}_X \overset{d}{\to} \cdots ).$

Thus, $c$ induces a map

$$\text{gr}^r J\Omega^\bullet_Y \to \text{gr}^r \nabla W\Omega^\bullet_X.$$ 

For $r = 1$,

$$\text{gr}^1 J\Omega^\bullet_Y = (J/J^2 \overset{d}{\to} \mathcal{O}_X \otimes \Omega^1_Y),$$

with $J/J^2$ placed in degree 0. Composing with the canonical isomorphism $L_{X/W} \sim (J/J^2 \to \mathcal{O}_X \otimes \Omega^1_Y)$ recalled above, we get a map (in $D(X)$)

$$(1) \quad L_{X/W}[-1] \to \text{gr}^1 \nabla W\Omega^\bullet_X,$$

which a diagonal argument shows to be independent of the choice of the embedding in $(Y, F)$. What is the RHS?
For $r \geq 1$, consider the map

$$\mathcal{N}^r W\Omega_{X'}^i \rightarrow F_*\tau_{\leq r}\Omega_X^.$$ 

sending $p^{r-1-i}V_x$ to $x$ for $i \leq r - 1$, $F_x$ for $i = r$ (and 0 for $i > r$). It induces a map (of complexes of $O_{X'}$-modules)

$$\text{gr}_N^r W\Omega_X^ \rightarrow \tau_{\leq r}F_*\Omega_X^.$$ 

A basic result is:

**Lemma** (Nygaard, 1981). The map (2) is a quasi-isomorphism.

Composing (2) with (1) (for $X'$)

$$L_{X'/W}[-1] \rightarrow \text{gr}_N^1 W\Omega_{X'},$$

and recalling the isomorphism $L_{X'/W} \sim \tau_{\geq -1}L_{X'/W_2}$, we get the map announced in Th. 2:

$$\tau_{\geq -1}L_{X'/W_2}[-1] \rightarrow \tau_{\leq 1}F_*\Omega_X^.$$
To show that

\[(\tau_{\geq -1} L_{X'/W_2})[-1] \to \tau_{\leq 1} F_* \Omega^\bullet_X.\]

is an isomorphism, we may assume that \(X\) has a formal smooth lifting \((Y, F)\) over \(W\). Then (3) boils down to the map

\[O_{X'} \oplus \Omega^1_{X'}[-1] \to F_*(O_X \to Z\Omega^1_X)\]

induced by \(F: Y \to Y\), which is a quasi-isomorphism, inducing the Cartier isomorphism \(C^{-1}\) on \(H^*\).

This disposes of the case where there exists a closed embedding \(X \hookrightarrow Y\), with \(Y/W\) smooth, and endowed with a lifting \(F\) of Frobenius. The general case is reduced to this one by cohomological descent for an open Zariski cover of \(X\).
Remarks. (a) Let \( M \) be a saturated Dieudonné complex in the sense of BLM: \( M \) is a complex of abelian groups, endowed with \( F : M^i \to M^i \) satisfying \( dF = pF \) \( d \), and such that \( M \) is \( p \)-torsion free, and

\[
p^\bullet F : M \to \eta_p M
\]

is an isomorphism (saturation condition). Then \( M^i \) is endowed with \( V \) such that \( VF = FV = p \) and \( FdV = d \). The Nygaard filtration

\[
M \supset \cdots \supset \mathcal{N}^r M \supset \mathcal{N}^{r+1} M \supset \cdots
\]

is defined, for \( r \in \mathbb{Z} \), similarly to the case of \( \mathcal{W}_X \Omega^\bullet \), by

\[
\mathcal{N}^r M^i = p^{r-1-i} VM^i \text{ for } i < r \text{ and } \mathcal{N}^r M^i = M^i \text{ for } i \geq r.
\]

We have

\[
\mathcal{N}^r M = (p^\bullet F)^{-1}(p^r M \cap \eta_p M)
\]

Then, as \( \mathcal{W}_X \Omega^\bullet / p\mathcal{W}_X \Omega^\bullet \to \Omega^\bullet_X \) is a quasi-isomorphism, the following (easy lemma) generalizes Nygaard’s lemma:

Lemma (BLM). The map \( p^\bullet F \) induces an isomorphism

\[
\text{gr}_\mathcal{N}^r M \sim \tau_{\leq r}(M/pM).
\]
(b) The Nygaard filtration has deep relations (see BMS2, BS3) with
- topological Hochschild homology,
- integral $p$-adic Hodge theory,
- prismatic cohomology.
3. Variants and generalizations

1. Let $S$ be a $k$-scheme, $\tilde{S}$ a flat lifting of $S$ to $W_2$, $X/S$ smooth, $X' = X \times_{(S,F_S)} S$, $F : X \to X'$ the relative Frobenius. The following generalization of Th. 1” is proved in Deligne-I.:

**Theorem 3.1.** There is a natural equivalence of gerbes:

$$\text{Lift}(X'/\tilde{S}) \sim \text{Split}(\tau_{\leq 1} F_* \Omega^\bullet_X).$$

(Note: here Lift$(X/\tilde{S})$ and Lift$(X'/\tilde{S})$ are in general not equivalent.)

Implies a canonical isomorphism

$$(\tau_{\geq -1} L_{X'/\tilde{S}})[-1] \sim \tau_{\leq 1} F_* \Omega^\bullet_X/S.$$

An independent direct proof can be given, though no dRW complex is available.

Log variants of the above for log smooth morphisms of Cartier type (Kato, 1989).
2. **Prismatic variant (Bhatt)**

Let \( (T = \text{Spec}(A), S : V(I) \subset T, \delta) \) be a prism. By definition:

- \( (A, \delta) \) is a \( \delta \)-ring, with associated Frobenius lift \( \varphi : a \mapsto a^p + p\delta(a) \)
- \( I \) is an ideal in \( A \) defining a Cartier divisor \( S \) in \( T \)
- \( A \) is derived \((I, p)\)-complete (e.g. \( p \)-complete and \( f \)-complete if \( I = (f) \))
- \( S \cap \varphi^{-1}(S) \subset V(p) \).

Assume in addition \((T, I, \delta)\) bounded, i.e., \( A/I \) has bounded \( p^\infty \)-torsion.
Let $X/S$ formally smooth. Let $(X/T)^\Delta$ be the prismatic site, and

$$\nu : (X/T)^\Delta \to X_{zar}$$

be the canonical projection (similar to the Berthelot map $u : (X/W)_{\text{crys}} \to X_{zar}$). Then:

**Theorem 3.2 (Bhatt).** There is a canonical isomorphism (in $D(X_{zar}, \mathcal{O}_X)$)

$$L_{X/T}[-1] \sim \tau_{\leq 1}(R\nu_*(\mathcal{O}_{X/T}^\Delta) \otimes_{\mathcal{A}} A/I) \otimes (I/I^2)$$

**Recall 3.1. Th. 2 (I., 2019).** For any smooth $X/k$ there is a canonical, functorial isomorphism in $D(X')$:

$$(\tau_{\geq -1}L_{X'/W_2})[-1] \sim \tau_{\leq 1}F_*\Omega_X^\cdot,$$

where $\tau_{\geq i}$ denotes a canonical truncation, and $L_{X'/W_2}$ is the cotangent complex of $X'/W_2$. 
Remark. Th. 3.2 (Bhatt) ⇒ Th. 2:

Take $A = W(k)$, $I = (p)$, $\varphi(a) = \sigma^*(a)$. Then $A/I = k$. Use crystalline comparison theorem (BS):

$$\sigma^* R\nu_* \mathcal{O}_{X/T}'^{\Delta} \sim R\mu_* \mathcal{O}_{X/W},$$

hence

$$R\nu_* \mathcal{O}_{X'/T}^{\Delta} \otimes^{L}_A A/I \sim F_* \Omega^\bullet_{X/k}.$$ 

and

$$L_{X/T}[-1] \sim \tau_{\leq 1} (R\nu_* (\mathcal{O}_{X/T}^{\Delta}) \otimes^{L}_A A/I) \otimes (I/I^2)$$

gives

$$(\tau_{\geq -1} L_{X'/W_2})[-1] = L_{X'/W}[-1] \sim \tau_{\leq 1} F_* \Omega^\bullet_{X/k}.$$ 

Question: Common generalization of Th. 3.1 and Th. 3.2 ?
4. Inputs from derived de Rham complexes

Review of $L\Omega^\bullet$.

For an $A$-algebra $R$,

$$L\Omega^\bullet_{R/A} := \text{Tot}(\Omega^\bullet_{P_\bullet(R)/A}),$$

where $P_\bullet(R)$ = standard simplicial resolution of $R/A$ by polynomial algebras.

Comes with Hodge filtration $\text{Fil}^i_{\text{Hdg}}L\Omega^\bullet_{R/A}$ deduced from $\Omega^{\geq i}$, with

$$\text{gr}^i L\Omega^\bullet_{R/A} \sim L\Omega^i_{R/A}[-i](:= L\Lambda^i L_{R/A}[-i]).$$

Globalizes on schemes: $L\Omega^\bullet_{X/S}$, $\text{Fil}^i_{\text{Hdg}}$, $\text{gr}^i = L\Omega^i[-i]$. 
Back to the Nygaard filtration

**Theorem 4.1 (I., 2019).** Let $X/k$ be smooth. There exists a canonical filtered isomorphism:

$$c : L\Omega^\bullet_X/W/\text{Fil}^p_{\text{Hdg}} \sim W\Omega^\bullet_X/\mathcal{N}^p,$$

where $\mathcal{N}^i = \mathcal{N}^i W\Omega^\bullet_X$ is the Nygaard filtration, with filtrations induced by the Hodge and the Nygaard filtration.

**Corollary 4.2.** For $i < p$,

$$\text{gr}^i L\Omega^\bullet_{X'/W} \sim \tau_{\leq i} F_* \Omega^\bullet_X.$$

($i = 1$: Cor 4.2 = Th. 2 (I.))
Corollary 4.3. A lifting $\tilde{X}$ of $X$ to $W_2$ gives a DI-decomposition

$$\bigoplus_i \Omega^i_X, [-i] \sim \tau_{<p} F_* \Omega^\bullet_X.$$ 

Apply Cor. 4.2 for $i = p - 1$, using the decomposition

$$L_{X/W} = \mathcal{O}_X[1] \oplus \Omega^1_X$$

given by $\tilde{X}$.

Remarks. (a) The isomorphism

$$c : L\Omega^\bullet_{X/W}/\text{Fil}_\text{Hdg}^p \sim \sim W\Omega^\bullet_X/\mathcal{N}^p,$$

of 4.1 does not extend to an isomorphism

$$L\Omega^\bullet_{X/W} \sim W\Omega^\bullet_X.$$
Example (Bhatt). Take $X = \text{Spec}(k)$. Then

$$R \lim_{\leftarrow n} (L\Omega^\bullet_{k/W_n}) \sim (\lim_{\leftarrow n} W_n\langle x \rangle)/(x - p),$$

where $W_n\langle x \rangle$ means the PD-algebra on $x$. RHS has $p$-torsion (e.g. $(x - p)^{[p]}$).
(b) Generalization to prisms.

**Theorem (Bhatt)** Notation as in Th. 3.2: $(T = \text{Spec}(A), S = \text{Spec}(A/I), \varphi : A \to A)$ a prism, $X/S$ formal smooth, $\nu : (X/T)^\Delta \to X_{zar}$ the canonical projection.

There exists a canonical filtered isomorphism

$$L\Omega_{X/T}^\bullet / \text{Fil}^p_{\text{Hdg}} \simto (\varphi^* R\nu_* \mathcal{O}_{X/T}^\Delta) / \mathcal{N}^p$$

Here $\mathcal{N}^i$ is the **Nygaard filtration** defined in such a way that the basic isomorphism of prismatic cohomology, factorizing $\varphi$:

$$\tilde{\varphi} : \varphi^* R\nu_* \mathcal{O}_{X/T}^\Delta \simto L\eta_{I^r} R\nu_* \mathcal{O}_{X/T}^\Delta$$

be a filtered isomorphism, where the RHS is equipped with the filtration induced by the $I$-adic filtration ($I^r \cap \eta I$).
Remark. In the case \((A, I) = (W, p)\) one recovers the isomorphism \(c\) of 4.1. Indeed, in this case the basic isomorphism \(\tilde{\varphi}\) boils down to the isomorphism

\[
p^\bullet F : W\Omega^\bullet_X \rightarrow \eta_p W\Omega^\bullet_X,
\]

expressing the saturation of \(dRW\), and (cf. Remark p. 64) \(N^r\) becomes the Nygaard filtration previously defined.
Lci variants

For $X/k$, replace smooth by locally complete intersection. Partial results by Bhatt: decompositions (in presence of liftings), and partial degeneration of Hodge to de Rham spectral sequences, both in char. $p > 0$ and in char. 0. Work in progress.