Mini Workshop on Algebraic Geometry

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Vanishing theorems and Shimura varieties, after K.-W. Lan and J. Suh

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THE GOAL

$M/\mathbf{C}$: Shimura variety of (general) PEL type

G : associated reductive group

V : lisse, torsionfree, Betti $\mathbf{Z}$-sheaf $/M$

associated to irreducible representation of G

p : prime number

Give criteria in terms of p, highest weight of V for :

- $H^i_{\text{int}}(M, V) = 0$ for $i \neq \dim M$
- $H^{\dim M}_{\text{int}}(M, V)$ p-torsionfree

($H^i_{\text{int}} = H^i$ if M compact,
interior cohomology (= Im $H^i_c \to H^i$) in general)

Bott (1957), Griffiths-Schmid (1969), Faltings (1982),
Mokrane-Tilouine ('02), Li-Schwermer ('04), ...,
Lan-Suh ('10)

THE INGREDIENTS

- representation theory

$\Rightarrow$ modular interpretation of V

- integral models of toroidal compactifications

of M, A^n/M ($A =$ univ. ab. scheme $/M$)

$$+$$

- p-adic comparison theorems

$\Rightarrow$ reduction to vanishing for

(log) de Rham cohomology groups

• (weak) positivity of automorphic line bundle

$L = e^*(\mathrm{det}\Omega^1_{A/S})$

• vanishing theorems in char. p for

$H^i(Y, L^{-1} \otimes K)$, $i < \mathrm{dim} Y$

(Y = "reduction mod p" of (compactification of) M,

K = suitable "Kodaira-Spencer complex")

$$+$$

• Faltings-BGG weight decomposition of K

$\Rightarrow$ goal

PLAN

1. Review of some decomposition and vanishing theorems in char. p
2. Lan-Suh's vanishing theorem
3. The ample case
4. Esnault-Viehweg's cyclic covers revisited
5. The general case
6. Applications to Shimura varieties

1. REVIEW OF SOME DECOMPOSITION AND VANISHING THEOREMS IN CHAR p

(a) Absolute vanishing

THEOREM 1 (Esnault-Viehweg,1992)

X/k projective, smooth, dim(X) $= d$;

$D \subset X$ sncd ; L line bundle on X

Assume :

- (*) $\exists$ effective D', supp$(D') \subset D$, and $\nu_0 \geq 0$

s. t. $L^\nu(-D')$ ample $\forall \nu \geq \nu_0$

- k perfect, char$(k) = p > 0$,

$d \leq p$, and (X, D), and L lift to $W_2(k)$.

Then :

$H^j(X, L^{-1} \otimes \Omega^i_X(\log D)) = 0 \ i + j < d$.

Remarks

- ample $\Rightarrow$(*) $\Rightarrow$ nef and big
- example of (*) : $L = \pi^*$(ample),

$\pi : X = \mathrm{Bl}_I(Y) \to Y$, $I.\mathcal{O}_X = \mathcal{O}(-D')$

- L ample : Deligne-I. (1987)
- basic ingredient : decomposition th. (D-I)

$\bigoplus \Omega^i_{X/k}(\log D)[-i] \xrightarrow{\sim} F_* \Omega^{\cdot}_{X/k}(\log D)$

(in $D(X)$, $F : X \to X$ = Frobenius)

(b) Relative vanishing
(semistable reduction case)

k perfect, $\mathrm{char}(k) = p > 0$

X/k, Y/k proper, smooth

$E = \sum E_i \subset Y$: sncd

$f : X \to Y$; $D := f^{-1}(E)$

Assume : $f : (X, D) \to (Y, E)$ semistable along E

(étale locally on X : f = external product of

copies of $x_1 \cdots x_r = t$)

($\Rightarrow$ $D \subset X$ = ncd, f flat, smooth / $Y - E$)

$\Omega^{\cdot}_{X/Y}(\log(D/E))$: relative log de Rham complex

$H := \oplus_i R^i f_*(\Omega^{\cdot}_{X/Y}(\log(D/E))$

$\nabla : H \to \Omega^1_{Y/k}(\log E) \otimes H$: Gauss-Manin connection

$\Omega^{\cdot}_{Y/k}(\log E)(H) := (H \to \Omega^1_{Y/k}(\log E) \otimes H \to \cdots)$:

log DR complex of H, with Hodge filtration

$F^i \Omega^{\cdot}_{Y/k}(\log E)(H) =$

$(F^i H \to F^{i-1} H \otimes \Omega^1_{Y/k}(\log E) \to \cdots)$

(Griffiths transversality)

Define

$$K = \oplus_i \mathrm{gr}^i \Omega^{\cdot}_{Y/k}(\log E)(H)$$

(total) log Kodaira-Spencer complex of H :

$$K = (\mathrm{gr}^{\cdot} H \to \mathrm{gr}^{\cdot -1} H \otimes \Omega^1_{Y/k}(\log E) \to \cdots)$$

Note : K is $\mathcal{O}_Y$-linear

THEOREM 2 (I., 1990)

dim$(Y) = e$, dim$(X) = d$,

Assume $d < p$, $(X, D) \to (Y, E)$ lifts to $W_2(k)$.

Then :

(i) $H^q = \oplus R^q f_*(\Omega^{\cdot}_{X/Y}(\log(D/E))$, $R^j f_* \Omega^i_{X/Y}(\log(D/E))$

locally free of finite type $\forall q, j, i$,

$E_1^{ij} = R^j f_* \Omega^i_{X/Y}(\log(D/E)) \Rightarrow R^{i+j} f_* \Omega^{\cdot}_{X/Y}(\log(D/E))$

degenerates at E_1

(ii)

$$K \xrightarrow{\sim} F_*\Omega^{\cdot}_{Y/k}(\mathrm{log} E)(H)$$

in $D(Y)$

($F : Y \to Y$ = Frobenius,

$H = \oplus H^q$,

$K = \oplus_i \mathrm{gr}^i \Omega^{\cdot}_{Y/k}(\mathrm{log} E)(H)$

= Kodaira-Spencer complex)

COROLLARY 1 (cf. D-I-Raynaud)

If $L =$ line bundle on Y, then :

$$h^q(Y, L \otimes K) \leq h^q(Y, L^p \otimes K) \ \forall q.$$

COROLLARY 2

L ample. Then :

(1) $H^q(Y, L \otimes K) = 0$ for $q > e$

(2) $H^q(Y, L^{-1} \otimes K) = 0$ for $q < e$

2. LAN- SUH'S THEOREM

Common generalization of th. 1 (Esnault-Viehweg)

and

(2) $H^q(Y, L^{-1} \otimes K) = 0$ for $q < e$

of cor. 2 of th. 2

under stronger liftability assumptions

(plus additional technical hypotheses

verified in Shimura case)

THEOREM 3 (Lan-Suh, 2010)

Data :

$m \in \mathbf{N}$

k perfect, char$(k) = p > 0$, $W = W(k)$

$f : (X, D) \to (Y, E)$ /W, (notations changed !)

X, Y proper, smooth /W, $d =$ dim$X$, $e =$ dim$Y$

$D \subset X$, $E \subset Y$ relative sncd

$L =$ line bundle on Y

Assumptions :

- (regularity) f log smooth, integral, vertical

(i. e. étale locally on X : $f =$ external product of copies of $x_1^{a_1} \cdots x_r^{a_r} = t$, plus $f^{-1}(E)_{\mathrm{red}} = D$)

- (Hodge to de Rham degeneration)

$E_1^{ij} = R^j f_* \Omega^i_{X/Y}(\log(D/E)) \Rightarrow R^* f_* \Omega^\cdot_{X/Y}(\log(D/E))$ degenerates at E_1, and E_1 locally free

• (Poincaré duality)

Put : $\mathcal{H}^i = R^i f_* \Omega^{\cdot}_{X/Y}(\log(D/E))$, $n = d{-}e = \dim(X/Y)$

(i) trace map $\mathrm{Tr} : \mathcal{H}^{2n} \xrightarrow{\sim} \mathcal{O}_Y$,

with $\mathrm{Tr}(Fx) = p^n \sigma(\mathrm{Tr}(x))$,

(F = crystalline Frobenius,

σ = local lifting of Frobenius of $Y \otimes k$)

(ii) cup-product

$\mathcal{H}^i \otimes \mathcal{H}^{2n-i} \to \mathcal{H}^{2n}$

perfect duality

• (nilpotence of residues)

$\forall i$, $\text{Res}_{E_i}(\nabla)$ is nilpotent

$(E = \sum E_i$, $\nabla : \mathcal{H}^* \to \Omega^1_{Y/W}(\log E) \otimes \mathcal{H}^*$

$=$ Gauss-Manin connection)

• (positivity)

(*) $\exists$ effective E', $\text{Supp}(E') \subset E_1$, and $\nu_0 \geq 0$

s. t. $L_1^\nu(-E')$ ample $\forall \nu \geq \nu_0$,

($(-)_1$ reduction mod p)

Conclusion :

Set $K(\mathcal{H}^m) := \mathrm{gr}\Omega^{\cdot}_{Y/W}(\log E)(\mathcal{H}^m)$,

$K_1(\mathcal{H}^m) = K(\mathcal{H}^m) \otimes_W k$

(Kodaira-Spencer complex)

Then :

(1) If $m + e < p$,

$H^q(Y_1, L_1^{-1} \otimes K_1(\mathcal{H}^m)) = 0$ for $q < e$

(2) If $m + e < p$ or $2n - m + e < p$,

$H^q(Y_1, L_1(-E_1) \otimes K_1(\mathcal{H}^m)) = 0$ for $q > e$

($(-)_1$ reduction mod p).

Remarks

- f semistable ($\Leftrightarrow$ saturated), $d < p$

$\Rightarrow$ Hodge to de Rham degeneration (th. 2),

Poincaré duality (Tsuji), nilpotence of residues (Katz) satisfied

- L_1 ample $\Rightarrow$ nilpotence of residues unnecessary

3. THE AMPLE CASE

Key ingredient in proof of th. 3 is

following decomposition theorem

(a variant of th. 3):

THEOREM 4 (Ogus-Lan-Suh)

Let $f : (X, D) \to (Y, E)$

satisfying hypotheses of th. 3 of regularity,

Hodge to de Rham degeneration, Poincaré duality.

Assume : $m+e<p$ or $2n-m+e<p$

($e = \dim(Y/W)$, $n = d-e = \dim(X/Y)$),

$K_1(\mathcal{H}^m) := (\bigoplus \mathrm{gr}^i \Omega^{\cdot}_{Y/W}(\log E)(\mathcal{H}^m))_1$

= Kodaira-Spencer complex, ($(-)_1$ reduction mod p).

Then :

$$K_1(\mathcal{H}^m) \xrightarrow{\sim} F_* \Omega^{\cdot}_{Y_1/k}(\log E_1)(\mathcal{H}^m_1)$$

in $D(Y_1)$.

Proof. Uses Ogus's theory of F-T-crystals (Astérisque 221) : generalization of classical relations (Katz-Mazur-Ogus) between Hodge filtration on $\mathcal{H}^i_1$ and p-divisibility of crystalline Frobenius Φ on $\mathcal{H}^i$

Main points :

- hypotheses $\Rightarrow$ $\Phi = p$-isogeny
- restriction of f to $Y - E$ proper, smooth, E transverse to p

COROLLARY 1

If L_1 = line bundle on Y_1, then :

$$h^q(Y_1, L_1 \otimes K_1(\mathcal{H}^m)) \leq h^q(Y_1, L_1^p \otimes K_1(\mathcal{H}^m)) \ \forall q.$$

COROLLARY 2

L_1 ample. Then :

(1) $H^q(Y_1, L_1 \otimes K_1(\mathcal{H}^m)) = 0$ for $q > e$

(2) $H^q(Y_1, L_1^{-1} \otimes K_1(\mathcal{H}^m)) = 0$ for $q < e$

Remark (Suh) (1) may fail for L_1 satisfying (*) in th. 3, not ample

From ample case to general case :

• induction on $e = \dim(Y/W)$ reduces to vanishing for integral parts of $\mathbf{Q}$-divisors $L_1^{(i)}$ sitting between L_1 and ample $L_1^\nu(-E' + E'_{\mathrm{red}})$, $\nu >> 0$

• desired vanishing proved by Esnault-Viehweg's method : Frobenius interpolation, using nilpotence of residues for $\mathcal{H}^m$

4. ESNAULT-VIEHWEG'S CYCLIC COVERS REVISITED

change notations :

Y/k smooth, $E' = \sum_{1 \leq i \leq r} a_i E_i$, $a_i \geq 0$,

$E = \sum_{1 \leq i \leq r} E_i$ sncd ;

$N \geq 1$ invertible in k ; assume $\mu_N \subset k$

L line bundle on Y s. t. $L^N = \mathcal{O}_Y(E')$.

Esnault-Viehweg : $(Y, E', L, N) \mapsto \mu_N$-cover

$$g : C = C(L, N, E') \to Y$$

ramified along E : C = normalization of Spec$A$,

$A = \mathcal{O}_Y \oplus L^{-1} \oplus \cdots \oplus L^{-(N-1)}$, $L^{-N} = \mathcal{O}_Y(-E') \hookrightarrow \mathcal{O}_Y$.

μ_N acts on C via action of $\mu_N \subset \mathcal{O}^*$ on L

Properties

- g finite, flat, Galois étale $/Y - E$ of group μ_N ;

C = normalization of Y in $C|Y - E$

- Put log structure on Y defined by E. Then :

$\exists$ unique log structure M on C s. t.

$(C, M) \to (Y, E)$ = μ_N-Kummer étale cover of Y

extending $C|Y - E$

locally on Y :

$C \to Y$ = pull-back of Spec$\mathbf{Z}[P] \to$ Spec$\mathbf{Z}[\mathbf{N}^r]$

where P = saturated amalgamated sum :

$$\begin{array}{ccc} \mathbf{N} & \longrightarrow & P \\ \uparrow & & \uparrow \\ \mathbf{N} & \longrightarrow & \mathbf{N}^r \end{array},$$

$\mathbf{N} \to \mathbf{N}$ by $x \mapsto Nx$, $\mathbf{N} \to \mathbf{N}^r$ by $x \mapsto (a_1 x, \cdots, a_r x)$.

• μ_N-equivariant decomposition into eigen bundles :

$g_* \mathcal{O}_C = \oplus_{0 \le i \le N-1} (L^{(i)})^{-1}$

$L^{(i)} := L^i \otimes \mathcal{O}_Y(-[iE'/N])$, $L^{(1)} = L$ if $N > a_i \ \forall i$

action of μ_N on $L^{(i)}$ via χ^i,

$\chi : \mu_N \hookrightarrow \mathcal{O}^*$ canonical character :

$(L^{(i)})^{-1} = g_* \mathcal{O}_C(\chi^{-i})$.

g log étale $\Rightarrow$

$$g^*\Omega^1_{Y/k}(\log E) = \Omega^1_{C/k}(\log M)$$

$$g_*\Omega^{\cdot}_{C/k}(\log M) = \Omega^{\cdot}_{Y/k}(\log E)(g_*\mathcal{O}_Y)$$

$$= (g_*\mathcal{O}_C \to \Omega^1_{Y/k}(\log E) \otimes g_*\mathcal{O}_C \to \cdots)$$

$g_*\mathcal{O}_C$ has μ_N-equivariant integrable log connection :

$$\nabla = \oplus\nabla_i : \oplus(L^{(i)})^{-1} \to \oplus\Omega^1_{Y/k}(\log E) \otimes (L^{(i)})^{-1},$$

local calculation $\Rightarrow$

Proposition (Esnault-Viehweg)

$$\mathrm{Res}_{E_j}(\nabla_i) = (ia_j/N - [ia_j/N]).Id$$

5. THE GENERAL CASE

Change notations (avoid sbscripts $(-)_1$) :

$f:(X,D)\to(Y,E)\mapsto\tilde f:(\tilde X,\tilde D)\to(\tilde Y,\tilde E)$,

$L\mapsto\tilde L$, $L_1\mapsto L$

$f_1:(X_1,D_1)\to(Y_1,E_1)\mapsto f:(X,D)\to(Y,E)$

$\mathcal{H}^i_1=(R^if_*\Omega^{\cdot}_{X/Y}(\log(D/E)))_1\mapsto H^i$

K_1 (Kodaira-Spencer complex $/Y_1$) $\mapsto K$.

Recall

$m + e < p$ (or $2n - m + e < p$) ($n = d - e = \dim(X/Y)$)

$L^\nu(-E')$ ample $\forall \nu \geq \nu_0$, $E'_{\text{red}} \subset E$

local freeness $\Rightarrow$ $H^m = \oplus R^m f_* \Omega^{\cdot}_{X/Y}(\log(D/E))$

$K = K(H^m) := \text{gr}\Omega^{\cdot}_Y(\log E)(H^m)$

(Kodaira-Spencer complex of H^m)

Have to show :

(2) $H^q(Y, L^{-1} \otimes K) = 0$ for $q < e = \dim(Y)$

Recall : (2) known if L ample (Cor. 2 of th. 4 = decomposition th. of Ogus-Lan-Suh)

Induction on $e = \dim(Y)$; WMA $k = \overline{k}$

Step 1 : Use of a hyperplane section

Write $E' = \sum c_i E_i$; up to increasing c_i and ν_0 WMA $\forall \nu \geq \nu_0$

$L^\nu(-E')$, $L^\nu(-E' + E'_{\text{red}})$ very ample and

$H^i(Y, L^\nu(-E')) = 0 \ \forall i > 0$

(Esnault-Viehweg, uses th. 2, Cor. 2 for $f = \text{Id}$)

Choose $s \geq 1$ s. t.

$$N = p^s + 1 > \nu_0,$$

and $N > c_i \ \forall i$ (recall $E' = \sum c_i E_i$) ($\Rightarrow [E'/N] = 0$)
Write $\tilde{E}' = \sum c_i \tilde{E}_i$, take sufficiently general

$$t \in H^0(\tilde{Y}, \tilde{L}^N(-\tilde{E}'))$$

s. t. $\tilde{Z} := V(t)$ smooth $/W$, transversal to E

Then : $\tilde{E} + \tilde{Z}$, $\tilde{D} + \tilde{f}^{-1}(\tilde{Z})_{\text{red}}$ sncd $/W$,

$\tilde{f} : (\tilde{X}, \tilde{D} + \tilde{f}^{-1}(\tilde{Z})_{\text{red}}) \to (\tilde{Y}, \tilde{E} + \tilde{Z})$,

$\tilde{f}|\tilde{Z}$ satisfy regularity, degeneration, nilpotence assumptions

(along $\tilde{E}$, $\tilde{E} \cap \tilde{Z}$)

And :

$$L^N = \mathcal{O}_Y(E' + Z),$$

$$\Omega^{\cdot}_{X/Y}(\log(D/E)) = \Omega^{\cdot}_{X/Y}(\log((D+(f^{-1}Z))_{\text{red}}/(E+Z))).$$

Local freeness, base change compatibility of $R^q f_*(\Omega^\cdot_{X/Y}(\log(-/-))$ and $R^j f_*(\Omega^i_{X/Y}(\log(-/-)) \Rightarrow$

$$0 \to \mathrm{gr}^\cdot(\Omega^\cdot_{Y/k}(\log E)\otimes H^m) \to \mathrm{gr}^\cdot(\Omega^\cdot_{Y/k}(\log E+Z)\otimes H^m) \to \mathrm{gr}^{\cdot -1}(\Omega^\cdot_{Z/k}(\log(E\cap Z))\otimes H^m)[-1] \to 0.$$

Inductive hypothesis $\Rightarrow$ enough to show :

$$(*)_1 \quad H^q(Y, L^{-1}\otimes \mathrm{gr}(\Omega^\cdot_{Y/k}(\log(E+Z))\otimes H^m) = 0$$

for $q < e$.

Step 2 : Enters cyclic cover

$$g : C := C(L, E' + Z, N) \to Y$$

cyclic cover associated with $L^N = \mathcal{O}_Y(E' + Z)$,

$g_*\mathcal{O}_C = \oplus(L^{(i)})^{-1}$, $L^{(i)} = L^i(-[i(E' + Z)/N])$.

$N > \mathrm{sup}(1, (c_i)) \Rightarrow L^{(1)} = L$

$L^{(N-1)} = L^{N-1}(-E' + E'_{\mathrm{red}})$

Recall : $N - 1 = p^s \geq \nu_0$,

$L^{(p^s)} = L^{p^s}(-E' + E'_{\text{red}})$ ample

$\Rightarrow$ we know (cor. 2 of th. 4) that, for $q < e$,

$$H^q(Y, (L^{(p^s)})^{-1} \otimes \text{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H^m)) = 0$$

Want to show :

$$(*)_1 \quad H^q(Y, L^{-1} \otimes \text{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H^m)) = 0$$

Will show by descending induction $(i = s, \cdots, 0)$

$$(*)_i \;\; H^q(Y, (L^{(p^i)})^{-1} \otimes \text{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H^m)) = 0$$

Step 3 : Frobenius interpolation

$(*)_{i+1} \Rightarrow (*)_i$ follows from

analogue of cor. 1 of th. 4 :

Key lemma.

For $0 < a < pa < N$, $m \geq 0$,

$$\dim H^q(Y, (L^{(a)})^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H^m))$$
$$\leq \dim H^q(Y, (L^{(pa)})^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H)).$$

Proof.

th. 4 $\Rightarrow$

$F_*(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H^m)$

$= \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H^m)$

$\Rightarrow$ (projection formula)

$H^q(Y, (L^{(a)})^{-1} \otimes K) = H^q(Y, F^*((L^{(a)})^{-1}) \otimes \Omega \otimes H^m)$

$(K := \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H^m)$,

$\Omega := \Omega^{\cdot}_{Y/k}(\log(E+Z))$ for short)

Key point

The inclusion :

$F^*((L^{(a)})^{-1}) = L^{-pa}(p[a(E' + Z)/N]) \hookrightarrow (L^{(pa)})^{-1}$

(i) is compatible with connections $1 \otimes d_{Y/k}$ and ∇_{pa}

(ii) induces quasi-isomorphism

$F^*((L^{(a)})^{-1}) \otimes \Omega \otimes H^m \to (L^{(pa)})^{-1} \otimes \Omega \otimes H^m$

Key point $\Rightarrow$

$H^q(Y, F^*((L^{(a)})^{-1}) \otimes \Omega \otimes H^m)$

$\xrightarrow{\sim} H^q(Y, (L^{(pa)})^{-1} \otimes \Omega \otimes H^m)$

$\Rightarrow$ key lemma, as

$H^q(Y, (L^{(a)})^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H^m)) =$

$\mathrm{H}^q(Y, F^*((L^{(a)})^{-1}) \otimes \Omega \otimes H^m)$, and

$\dim H^q(Y, (L^{(pa)})^{-1} \otimes \Omega \otimes H^m) \leq \dim H^q(Y, (L^{(pa)})^{-1} \otimes$

$\mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H^m)$

(abutment $\leq$ initial term)

Proof of key point

(i) (Esnault-Viehweg) : seen on Frobenius diagram :

$$\begin{array}{ccccc} C & \leftarrow & C' & \xleftarrow{F} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} & \swarrow{\scriptstyle g} & \\ Y & \xleftarrow{F} & Y & & \end{array}$$

cartesian square, log étale vertical maps :

inclusion $= (g'_*\mathcal{O}_{C'}(\chi^{-pa}) \hookrightarrow g_*\mathcal{O}_C(\chi^{-pa}))$

(ii) (core of the proof) :

$F^*((L^{(a)})^{-1}) \otimes H^m = (L^{(pa)})^{-1}(-B) \otimes H^m$

$\hookrightarrow (L^{(pa)})^{-1} \otimes H^m$,

$(B = [pa(E' + Z)/N] - p[a(E' + Z)/N] = \sum b_i E_i$,

$b_i = [pac_i/N] - p[ac_i/N]$, $0 \leq b_i < p$, $E' = \sum c_i E_i)$

Look at residues :

By (Prop. 1)(Esnault-Viehweg)

$\mathrm{Res}_{E_i}(L^{(pa)})^{-1} = pac_i/N - [pac_i/N] = -b_i \bmod p$

$b_i \neq 0 \Rightarrow 0 < b_i < p \Rightarrow \mathrm{Res}_{E_i}(L^{(pa)})^{-1}$ invertible

$R_i := \mathrm{Res}_{E_i}(H^m)$ nilpotent

$\Rightarrow S_i := \mathrm{Res}_{E_i}((L^{(pa)})^{-1} \otimes H^m) = -b_i \otimes Id + Id \otimes R_i$

$\Rightarrow S_i$ invertible

$\Rightarrow$ (by Esnault-Viehweg's lemma below)

$$\Omega(-B_i) \otimes (L^{(pa)})^{-1} \otimes H^m \to \Omega \otimes (L^{(pa)})^{-1} \otimes H^m$$

= quasi-isomorphism

$$\Rightarrow F^*((L^{(a)})^{-1}) \otimes \Omega \otimes H \to (L^{(pa)})^{-1} \otimes \Omega \otimes H$$

= quasi-isomorphism, as

$$F^*((L^{(a)})^{-1}) \otimes H^m = (L^{(pa)})^{-1}(-B) \otimes H^m$$

Lemma (Esnault-Viehweg)

X/k smooth, $D = D_1 + \cdots + D_r$ ncd on X,

$\nabla : V \to \Omega^1_{X/k}(\log D) \otimes V$

vector bundle with integrable log connection .

Assume :

$\mathrm{Res}_{D_1}(\nabla) : V \otimes \mathcal{O}_{D_1} \to V \otimes \mathcal{O}_{D_1} =$ isomorphism.

Then, for $a \geq 0$:

$\Omega^{\cdot}_{X/k}(\log D)(-aD_1) \otimes V \to \Omega^{\cdot}_{X/k}(\log D) \otimes V$

$=$ quasi-isomorphism.

6. APPLICATIONS TO SHIMURA VARIETIES

6.1. The geometric set-up

Given integral PEL datum $D = (\mathcal{B}, *, L, \langle , \rangle, h_0)$,

$\mathcal{B}$ = order in finite dim. semisimple algebra $/\mathbf{Q}$

with positive involution $*$ ($\mathrm{Tr}(bb^*) > 0$)

L symplectic $\mathcal{B}$-lattice ($L = \mathbf{Z}^{2g}$ + action of $\mathcal{B}$ +

$\langle , \rangle$ alternating, non degenerate $\mathcal{B}$-pairing)

$h_0 : \mathbf{C} \to \mathrm{End}_{\mathcal{B}\otimes_{\mathbf{Z}}\mathbf{R}}(L \otimes_{\mathbf{Z}} \mathbf{R})$ = polarization :

$(1/2\pi i)\langle x, h_0(i)y\rangle | L \otimes \mathbf{R} > 0$, $\langle h_0(z)x, y\rangle = \langle x, h_0(z^c)y\rangle$

associated reductive group G,

$G(R) = \{(g, r) \in \mathbf{GL}_{\mathcal{B}\otimes R}(L \otimes R) \times R^*,$

$\langle gx, gy \rangle = r\langle x, y \rangle\}$

Hodge decomposition

$L \otimes \mathbf{C} = V_0 \oplus V_0^c,$

$h(z)(x \oplus y) = (1 \otimes z)x \oplus (1 \otimes z^c)y$

reflex field

$F_0 = \{\mathrm{Tr}_{\mathbf{C}}(b|V_0), b \in \mathcal{B}\} \subset \mathbf{C}$

$=$ field of def. of V_0 as $\mathcal{B} \otimes \mathbf{C}$-module

good prime p, (unramified in $\mathcal{B}$, $\langle , \rangle \otimes \mathbf{Z}_p$ self-dual)

neat, prime to p level $H \subset G(\prod_{\ell \neq p} \mathbf{Z}_\ell)$

(neat : e. g. $\subset \{g \equiv 1 \bmod n\}$, $n \geq 3$) $\Rightarrow$ smooth, quasi-projective moduli scheme

$$M_H/S_0$$

(S_0 = localization at p of ring of integers of F_0)
($M_H = \{A/S$+PEL structure of type $(D, H)\}$),

Shimura variety $/F_0$

$$\mathsf{Sh}_H \subset M_H \otimes F_0,$$

$$(\mathsf{Sh}_H \otimes_{F_0} \mathbf{C})^{\mathsf{an}} = G(\mathbf{Q}) \setminus \mathcal{X} \times G(\mathbf{A}^f)/(H \times G(\mathbf{Z}_p))$$

$$\mathcal{X} = G(\mathbf{R})h_0 =$$

finite union of hermitian symmetric domains and compactifications : minimal (Satake-Baily-Borel), toroidal (Chai-Faltings et al.)

$$\begin{array}{ccc} A & \subset & A^{\text{tor}} \;, \\ \downarrow & & \downarrow \\ M_H & \subset & M^{\text{tor}}_{H,\Sigma} \\ & \searrow & \downarrow \pi \\ & & M^{\text{min}}_H \end{array}$$

$Y = M^{\text{tor}}_{H,\Sigma}$ proper, smooth $/S_0$,

$E = M^{\text{tor}}_{H,\Sigma} - M_H$ sncd$/S_0$

A universal abelian scheme,

A^{tor} toroidal compactification of A

basic automorphic line bundle on Y

$$\omega := \det(e^*\Omega^1_{\tilde{A}/Y})$$

($\tilde{A}$ semi-abelian extension of A, acts on A^{tor})

ω not ample in general

($= \pi^*$(ample line bundle on M^{min}),

$\pi : M^{\text{tor}} \to M^{\text{min}}$ = normalized blow-up of I,

$I.\mathcal{O}_Y = \mathcal{O}_Y(-E')$, $E'_{\text{red}} \subset E$)

but satisfies Esnault-Viehweg condition (*) :

$\exists \nu_0 \geq 0$ s. t. $\omega^\nu(-E')$ ample $\forall \nu \geq \nu_0$

final adjustments :

• replace M, M^{tor}

by schematic closure of Sh_H ($\hookrightarrow M \otimes F_0$) in M, M^{tor},

• pull-back to suitable $S = \text{Spec}\, W(k)/S_0$,

k perfect, char(k) = p

• keep same notations : $A^{\text{tor}} \to M^{\text{tor}}$, $E \subset M^{\text{tor}}$.

6.2. Vanishing and p-torsionfreeness results

$\mathcal{V}_\mu$ = bundle $/Y$ with integrable log connection (log poles $/E$) associated with irreducible representation $G \to GL(V)$, highest weight μ

$\mathcal{V}_\mu$ = direct summand of $R^m f_* \Omega^\cdot_{X/Y}(\log(D/E))$,

$f = f_m : X \to Y$ = suitable toroidal compactification of

$A^m \to M_H$,

$m = |\mu|$

(e. g. $|\mu| = \sum \mu_i$ for $\mu = (\mu_1 \geq \mu_2 \geq \cdots \mu_r \geq 0)$)

THEOREM 5 (Lan-Suh, '10)

Assume μ to be p-small, sufficiently regular.

Then :

(i) $H^i(Y, \Omega^{\cdot}_{Y/W}(\log E)(\mathcal{V}_\mu^\vee)) = 0$ for $i < e$,

($e = \dim(Y/S) = \dim(\mathrm{Sh}_H)$)

(ii) $H^i(Y, \Omega^{\cdot}_{Y/W}(\log E)(-E)(\mathcal{V}_\mu^\vee)) = 0$ for $i > e$

(iii) $H^i_{dR,\mathrm{int}}(Y, \mathcal{V}_\mu^\vee) = 0$ for $i \neq e$ ($H^i_{\mathrm{int}} = \mathrm{Im}(ii) \to (i)$)

(iv) $H^e_{dR,\mathrm{int}}(Y, \mathcal{V}_\mu^\vee)$ = free, finite type $/W$.

Remark. Conditions on μ independent of H.

"sufficiently regular" means

"far enough from the walls of

the fundamental Weyl chamber", roughly :

$(\mu, \alpha^\vee) \geq C(\alpha) > 0 \ \forall \alpha \in \Phi_G^+$

$G \supset P \supset M \supset T$, $B \supset T.R_u(M)$,

$\Phi_G^+ \subset \Phi_G$, $X_G = \mathrm{Hom}(T, \mathbf{G}_m)$

"p-small" means roughly :

$|\mu| + e < p$, $(\mu + \rho, \alpha^\vee) \leq p \ \forall \alpha \in \Phi_G$

$\rho = (1/2) \sum \alpha, \alpha \in \Phi_G^+$

COROLLARY (Lan-Suh, '10)

$\mathcal{V}_{Betti}$ = lisse $\mathbf{Z}$-sheaf on $Y_{\mathbf{C}}$ associated with μ

Then :

(i) If μ sufficiently regular,

$H^i_{\text{int}}(Y_{\mathbf{C}}, \mathcal{V}^{\vee}_{Betti}) = 0$ for $i \neq e$, ($e = \dim(Y_{\mathbf{C}})$)

(ii) If moreover μ p-small,

$H^e_{\text{int}}(Y_{\mathbf{C}}, \mathcal{V}^{\vee}_{Betti})$ p-torsion free

NB. (i) $\Rightarrow$ Faltings's theorem (1982)

Note : In general, no semistable model f_m exists.

But : suitable log smooth, integral models f_m exist,
local freeness of $\mathcal{H}^*_{DR}$, Poincaré duality,
nilpotence of residues OK (Lan)
(Chai-Faltings in Siegel case)