1. Historical sketch

1956:
• Cartier isomorphism
• Serre’s Witt vector cohomology,
• Dieudonné’s theory of Dieudonné modules

1963-65:
• Manin’s work on formal groups,
• Gauss-Manin connection

1967:
• Cartier et al.: big Witt vectors, Cartier modules
• Tate: \( p \)-divisible groups, Hodge-Tate decomposition
• Monsky-Washnitzer’s cohomology
• Grothendieck: crystalline cohomology

1967:
• Cartier et al.: big Witt vectors, Cartier modules
• Tate: \( p \)-divisible groups, Hodge-Tate decomposition
• Monsky-Washnitzer’s cohomology
• Grothendieck: crystalline cohomology

1970:
• Berthelot’s thesis
• Grothendieck’s crystalline Dieudonné theory, problem of the mysterious functor

• Mazur-Ogus: slopes of Frobenius (Katz inequality)

1974:
• Bloch: complex of typical curves on \( K \)-groups

1975:
• Deligne-Illusie: de Rham-Witt complex

1980:
• Fontaine’s \( p \)-adic period rings \( B_{\text{cris}}, B_{\text{dR}} \)

1980-85:
• Fine study of de Rham-Witt (Nygaard, Illusie-Raynaud, Ekedahl)
  • Bloch-Kato’s proof of Hodge-Tate decompositions (good ordinary case)
  • Fontaine-Messing’s proof of \( C_{\text{cris}} \) (dim \( X < p \), \( e \leq p - 1 \)), syntomic cohomology
  • Faltings’s almost étale theory, tentative proofs of \( C_{\text{cris}}, C_{\text{dR}} \) in general

1988:
• Fontaine-Jannsen’s \( C_{\text{st}} \) conjecture
• Fontaine-Illusie-Kato: log schemes
• Hyodo-Kato log crystalline cohomology, log de Rham-Witt complex
• Kato’s proof of \( C_{\text{st}} \) (2 dim \( X < p - 1 \))

1988 - ...:
• Berthelot’s rigid cohomology, arithmetic \( \mathcal{D} \)-modules

1997:
• Tsuji: proof of \( C_{\text{st}} \) in the general case
• Faltings: sketch of corrected proof of almost purity lemma and \( C_{\text{st}} \)
  (details worked out by Gabber-Ramero)

1998:
• Niziol’s proof of \( C_{\text{cris}} \) using \( K \)-theory
2000: • Fontaine, Colmez, André, Kedlaya, Christol-Mebkhout, .... : proofs of main conjectures on \( p \)-adic representations (weakly admissible ⇔ admissible, dR ⇔ pst, \( p \)-adic local monodromy conjecture, finiteness of rigid cohomology)

2004: • Hesselholt-Madsen’s absolute de Rham-Witt complex / \( \mathbb{Z}_p \)
• Langer-Zink’s relative de Rham-Witt complex / \( \mathbb{Z}_p \)
• Zink’s theory of displays

2007: • Olsson : stack theoretic variants of de Rham-Witt

2008: • Nizioł’s \( K \)-theoretic proof of \( C_{st} \)
• Davis-Langer-Zink : overconvergent de Rham-Witt complex

2011: • Beilinson : new proof of \( C_{dR} \) using derived de Rham complexes

\[
\text{Witt vector } H^* \\
\text{de Rham – Witt complex} \\
\text{de Rham and crystalline } H^* \\
\text{Hodge } H^* \\
p – \text{adic étale } H^*
\]

2. Witt vectors

2.1. Witt polynomials, ghost components

\( p = \text{prime number} \)

\[
w_n(X_0, \cdots, X_i, \cdots) := \sum_{0 \leq i \leq n} p^i X_{i}^{p^{n-i}} : \\
w_0 = X_0 \\
w_1 = X_0^p + pX_1 \\
w_2 = X_0^{p^2} + pX_1^p + p^2 X_2, \\
\vdots
\]

**Theorem 2.1.1.** For a set \( A \), let

\[
W(A) := A^\mathbb{N} = \{(a_0, \cdots, a_n, \cdots), a_i \in A\}.
\]

There exists a unique functor \( A \mapsto W(A) \) from rings to rings such that

\[
w : W(A) \to A^\mathbb{N}
\]

is a homomorphism of rings, where \( A^\mathbb{N} \) is equipped with the product structure.
Proof. [CL, II, §§ 5, 6]. Alternate proof : use Dwork’s lemma : If
\( f : A \to A, f(a) \equiv a^p \mod p, (x = (x_0, \cdots) \in w(A^N)) \iff (x_i = f(x_{i-1}) \mod p^i \forall i > 0). \) See also : [Demazure, III].

Ghost map, ghost components. \( 1 = (1, 0, \cdots, 0, \cdots), 0 = (0, \cdots, 0), \) \( S_n(a, b), P_n(a, b), S_0 = a_0 + b_0, S_1 = a_1 + b_1 - \sum_{0 < i < p} p^{-1}(p^i/i!(p-i)!)(a_0)^i, P_0 = a_0b_0, P_1 = b_0a_1 + b_1a_0^p + pa_1b_1. \)

2.2. Operators \( R, F, V \)
\( W_n(A), R, V, \) short exact sequences, \( [x] = (x, 0, \cdots) \)

There exists a unique \( F : W(A) \to W(A) \) functorial in \( A \) such that \( w(Fa) = (w_1(a), w_2(a), \cdots). \)

\( Fa = (f_0(a), \cdots, f_n(a), \cdots), f_n(a) = f_n(a_0, \cdots, a_{n+1}), f_0(a) = a_0^p + pa_1, \)
\( f_n(a) \equiv a_n^p \mod p \)
\( F : W_n(A) \to W_{n-1}(A) \)
\( FV = p, xVy = V((Fx)y), F[x] = [x^p], (VF = p) \iff (p = 0 \text{ in } A). \)

\( p = 0 \text{ in } A \Rightarrow Fa = (a_0^p, \cdots, a_n^p, \cdots). \)

\( m \in \mathbb{Z} \text{ invertible in } A \Rightarrow m \text{ invertible in } W_n(A) ; \) in particular, if \( A \) is a \( \mathbb{Z}_p \)-algebra, so is \( W_n(A). \)

2.3. Examples

- \( W_n(A), A \text{ perfect of char. } p \)
  \( V = pF^{-1}, W_n(A) = W(A)/p^nW(A), W(A) = \text{(the unique) strict } p \)-ring \( B \) of residual ring \( A \) \( (W(A) \xrightarrow{\sim} B, a \mapsto \sum r(a_n)p^rn, r : A \to B \text{ (the system of multiplicative representatives)} \)
  \( k \text{ perfect field of char. } p \Rightarrow W(k) = \text{(the) Cohen ring of } k ; W(F_p) = \mathbb{Z}_p. \)

- \( W_n(F_p[t]) \)

\[ W_n(F_p[t]) = E^0/V^nE^0, \]
where \( E^0 \subset Z_p/[p^{-\infty}] \) is the set of \( \sum_{k \in \mathbb{N}[1/p]} a_k t^k \) such that the denominator of \( k \) divides \( a_k \) for all \( k, \) with \( F, V \) induced by \( F, V \) on \( Q_p/[p^{-\infty}] \) given by \( Ft = t^p, V = pF^{-1}. \)

(see [DRW, I 2.3] : \( E^0 = \sum V^nZ_p[t] \); there’s a unique \( Z_p \)-algebra homomorphism \( E^0 \to W(F_p[t]) \) compatible with \( V, \) sending \( t \) to \( [t] \); it is injective and induces an isomorphism on \( gr_V. \)

Gives a decomposition

\[ W_n(F_p[t]) = \bigoplus_{k \text{ integral}} (Z/p^nZ)[t]^k \bigoplus \bigoplus_{k \text{ not integral}} V^{u(k)}(Z/p^{n-u(k)}Z)[t]p^{u(k)}k, \]

(\( p^{u(k)} \) being the denominator of \( k, \) and \( [t] \) the Teichmüller representative).

A similar description holds for \( F_p[t_1, \cdots, t_r] \) (loc. cit.).
• $W_n(Z(p))$

\[
W_n(Z(p)) = \prod_{0 \leq i \leq n-1} Z(p)V^i1
\]  

(as a $Z(p)$-module, with $V^i1.V^j1 = p^iV^j1$  $(0 \leq i \leq j < n$).

(see [Hesselholt-Madsen, 1.2.4] : $gr_V W_n(Z(p))$ free over $Z(p)$, $(V^i1)$ split the filtration : $\sum_{0 \leq i < n} V^i[a_i] = \sum_{0 \leq i < n} b_i V^i1$, with $a_i, b_i$ in $Z(p)$ (and the 1-1 correspondence $(a_i) \leftrightarrow (b_i)$ given by complicated functions))

2.4. Link with big Witt vectors

$W(A) := (1 + A[[t]])^*, \ u + w v := u v, \ (1 - at)^{-1} w (1 - bt)^{-1} := (1 - abt)^{-1}$

$A/Z(p) \Rightarrow W(A) \subset W(A), \ W(A) = \pi W(A), \ \pi x = E(t)x$.

$E(t) = \exp(\sum_{n \geq 0} t^n/n) = \prod_{n \in l(p)} (1 - t^n)^{-\mu(n)/n} \in W(Z(p))$ (Artin-Hasse exponential)

\[
a = (a_0, \cdots) \mapsto \prod_{n \geq 0} E(a_n t^n), \ W(A) \iso \pi W(A)
\]

(see [DRW 0 1.2], [Demazure], [Bloch]).

2.4. Sheafification

For $A$ a ring in a topos $T$, and $n \in \mathbb{N}$, $n > 0$, the presheaf $U \mapsto W(A(U))$ (resp. $U \mapsto W_n(A(U))$) is a sheaf of rings, denoted $W(A)$ (resp. $W_n(A)$).

If $X$ is a scheme, the underlying space of $X$ together with the sheaf $W_n(O_X)$ is a scheme, denoted $W_n(X)$ (LZ, Appendix). If $p$ is nilpotent in $A$, $VW_nA$ is nilpotent (since it’s a DP-ideal, see 3.2). If $p$ is nilpotent on $X$, $W_n(X)$ is a thickening of $X$.

3. Crystalline cohomology

3.1. Inputs from complex analytic geometry : Poincaré lemma, Gauss-Manin connection

• Poincaré lemma

analytic : $X/C$ smooth analytic space : $C \to \Omega_{X/C} = \text{quasi-isomorphism}$

formal : $k = \text{field of char. } 0, \ t = (t_1, \cdots, t_n) : k \to \Omega_{k[[t]]/k} = \text{quasi-isomorphism}$

algebraic : $k = \text{field of char. } 0, \ t = (t_1, \cdots, t_n) : k \to \Omega_{k[[t]]/k} = \text{quasi-isomorphism}$

\[(n = 1 : 0 \to k \to k[t] \to k[t]dt \to 0 \text{ exact, } t^i \mapsto it^i1dt (i \geq 1)\]

$\text{char}(k) = p > 0 \Rightarrow \Omega_{k[[t]]/k}$ quasi-isomorphic to $k[t^p] \otimes (k \otimes k[t^{-1}]dt[-1])$

(generalization : Cartier isomorphism)

• Gauss-Manin

relative Poincaré lemma : $f : X \to Y$ smooth morphism of complex analytic spaces $\Rightarrow f^{-1}\mathcal{O}_Y \to \Omega_{X/Y}$ quasi-isomorphism.
If \( f \) proper, then \( R^i f_* \mathcal{C} = \) local system, and
\[
\mathcal{H}_{dR}(X/Y) := R^i f_* \Omega_{X/Y} = \mathcal{O}_Y \otimes R^i f_* \mathcal{C}.
\]

\( \Rightarrow \) For \( Y/C \) smooth, get integrable connection \( \nabla = d \otimes \text{Id} : \mathcal{H}_{dR}(X/Y) \to \Omega^1_Y \otimes \mathcal{H}_{dR}(X/Y) \), with horizontal sections \( R^i f_* \mathcal{C} \).

If \( Y = \text{smooth } C\text{-scheme}, f : X \to Y \) proper smooth, by GAGA
\[
\mathcal{H}^i_{dR}(X/Y) = \mathcal{H}^i_{dR}(X^\text{an}/Y^\text{an}),
\]
and by Manin there exists a canonical integrable connection
\[
\nabla_{GM} : \mathcal{H}^i_{dR}(X/Y) \to \Omega^1_Y \otimes \mathcal{H}_{dR}(X/Y)
\]
such that \((\nabla_{GM})^\text{an} = \nabla\). Purely alg. construction. Variants : Katz-Oda, Grothendieck.

\( \Rightarrow \) Grothendieck’s observation : \( k = \text{perfect field of char. } p > 0, W = W(k), t = (t_1, \ldots, t_n), X/S = \text{Spec} W[[t]] \) proper smooth such that \( \mathcal{H}_{dR}(X/S) \) free of finite type \( \forall i \). Let \( u : \text{Spec} W \to S, v : \text{Spec} W \to S \) such that \( u \equiv v \mod p \). Get : \( X_u := u^* X, X_v := v^* X \) such that \( X_u \otimes k = X_v \otimes k = Y \), and \( H^i_{dR}(X_u/W) = u^* \mathcal{H}^i(X/S), H^i_{dR}(X_v/W) = v^* \mathcal{H}^i(X/S) \). By \( \nabla = \nabla_{GM} \), get isomorphism
\[
\chi(u, v) : H^i_{dR}(X_u/W) \cong H^i_{dR}(X_v/W),
\]
\[
u^*(x) \mapsto \sum_{m \geq 0} (1/m!)(u^*(t) - v^*(t))^m v^*(\nabla(D)^m x)
\]
\((x \in H^i_{dR}(X/S), D = (D_1, \ldots, D_n), D_i = \partial/\partial t_i)\), with \( \chi(v, w)\chi(u, v) = \chi(u, w), \chi(u, u) = \text{Id} \) (NB. \( (1/m!)(u^*(t) - v^*(t))^m \in W \) ; series converge \( p \)-adically : \( p > 2 \) easy, by Berthelot in general).

\( \Rightarrow \) question (Grothendieck) : for \( Y/k \) proper, smooth, \( X_1, X_2 \) proper smooth liftings \( /W \), can one hope for an isomorphism (generalizing \( \chi(u, v) \))
\[
\chi_{12} : H^i_{dR}(X_1/W) \cong H^i_{dR}(X_2/W)
\]
with \( \chi_{23}\chi_{12} = \chi_{13} \)? (Monsky-Washnitzer : analogue in the affine case OK)

Answer : Yes : solution : crystalline cohomology \( H^i(Y/W) \) (depending only on \( Y \), with no assumption of existence of lifting), providing can. iso :
\[
\chi : H^i(Y/W) \cong H^i_{dR}(X/W)
\]
for any proper smooth lifting \( X/W \) of \( Y \), such that for \( X_1, X_2 \) as above, \( \chi_2 = \chi_{12}\chi_1 \).
Berthelot-Grothendieck’s definition: \( H^i(Y/W) = \text{proj} \lim_n H^i(Y/W_n), \) 
\( H^i(Y/W_n) = H^i((Y/W_n)_{\text{cris}}, \mathcal{O}), (Y/W_n)_{\text{cris}}: \text{crystalline site, } \mathcal{O} = \text{structural sheaf of rings}. \)

Later: \( H^i(Y/W) = H^i(Y_{zar}, W\Omega_Y), W\Omega_Y = \text{de Rham-Witt complex}. \)

3.2. Divided powers

\( I \subset A = \text{ideal}; \text{divided powers on } I = \text{family } \gamma_n : I \to A, n \in \mathbb{N}, \)
satisfying formally the properties of \( x^n/n! : \)

\( \gamma_0(x) = 1, \gamma_i(x) = x, \gamma_n(x) \in I \) for \( n \geq 1, \)

\( \gamma_n(x + y) = \sum_{p+q=n} \gamma_p(x)\gamma_q(y), \)

\( \gamma_n(\lambda x) = \lambda^n \gamma_n(x), \)

\( \gamma_p(x)\gamma_q(x) = ((p+q)!/p!q!)\gamma_{p+q}(x) \)

\( \gamma_p(\gamma_q(x)) = (pq)!/p!(q!)^p \gamma_{pq}(x). \)

In particular,

\( n! \gamma_n(x) = x^n. \)

DP-ideal, DP-structure.

Examples

- \( I = pW \subset W (W = W(k), k \text{ perfect, char. } p > 0). \) Then: \( \forall n \in \mathbb{N}, \)

\( p^n/n! \in W. \)

Proof. \( v_p(n!) = (n - \sum_{0 \leq i \leq r} a_i)/(p - 1), \) with \( n = \sum_{0 \leq i \leq r} a_i p^i, 0 \leq a_i < p, \)
hence

\( v_p(p^n/n!) = (n(p-2) + \sum a_i)/(p-1) \geq 0, \)

and \( > 0 \) if \( n > 0). \)

Note: \( p > 2 \Rightarrow \lim_{n \to \infty} p^n/n! = 0 \)

\( p = 2: v_2(2^n/n!) = \sum a_i (= 1 \text{ for } n = 2^m) \)

Induced DP on \( W_m. \)

\( A/W \) finite totally ramified, \( [A : W] = e, \pi \in A \) uniformizing parameter, then (\( \pi A \) has a DP structure) \( \iff (e \leq p-1). \)

- \( M \) an \( A \)-module,

\( \Gamma M = \oplus_{n \geq 0} \Gamma^n M = A \oplus M \oplus \Gamma^2 M \oplus \cdots \)

the DP-algebra on \( M, \Gamma^+ M = \oplus_{n \geq 0} \Gamma^n M \) (if \( M \) is locally free of finite type, \( \Gamma^n M = (S^n(M^\vee))^{\vee} = TS^n M). \)

\( ^1 \) There exists a unique DP on \( \Gamma^+ M \) extending \( M \to \Gamma^n M, x \mapsto x^n. \)

\[ A < t_1, \ldots, t_r > = \Gamma(\oplus_{1 \leq i \leq r} A t_i) = \oplus_{k=(k_1, \ldots, k_r)} A t^{[k]}. \]

\(^1TS^n M = (M^{\otimes n})^{S_n} \) is the submodule of symmetric tensors of degree \( n. \)
Divided power Poincaré lemma. There exists a unique integrable connection $d$ on the $A[t]$ module $A < t >$ such that $dt_i^{[n]} = t_i^{[n]}dt$ and $d(xy) = dx.y + y.dx$, and $A \to A < t > \otimes \Omega A[t]/A$ is a quasi-isomorphism.

- $A$ a $\mathbf{Z}_{(p)}$-algebra $\Rightarrow (\gamma_n)_{n \geq 1}$ on $I$ is determined by $\gamma_p$ (or $(p - 1)! \gamma_p$).
- $R$ a $\mathbf{Z}_{(p)}$-algebra $\Rightarrow \gamma_n(Vx) = (p^{n-1}/n!)Vx^n$ is in $VW(R)$ for $x \in W(R)$, $n > 0$, and $(\gamma_n)$ $(n > 0)$, $\gamma_0 = 1$ is a DP on $VW(R)$, called canonical.

Divided power envelope (Berthelot’s construction). For $(B, J)$, $J$ an ideal in $B$, there exists a (unique) pair $(D_B(J), J)$, $J$, an ideal in $D_B(J)$ equipped with DP $\gamma$ and a morphism $(B, J) \to (D_B(J), J)$ universal for morphisms in $(C, K)$, with $K$ a DP-ideal. Called DP-envelope of $(B, J)$.

Variant for $B$ an $A$-algebra, with a PD-ideal $I$ in $A$, with $\gamma$ on $J$ made compatible with the DP on $I$ (i.e. PD of $I$ extend to $ID_B(J)$ and compatible with the DP of $J$ on the intersection). Case of interest : $A = W_n(k)$, $I = (p)$.

Example. $M = A$-module, $B = SM = \oplus_{n \in \mathbb{N}} S^nM$ the symmetric algebra on $M$, $J = S^+M \Rightarrow (D_B(J), J) = (\Gamma M, \Gamma^+M)$.

3.3. The crystalline site.

$X/W_n$, $W_n = W_n(k)$, $k$ perfect of char. $p > 0$

$\text{Crys}(X/W_n)$ crystalline site : objects : $(U, T, \gamma)$, $U$ Zariski open (or étale) in $X$, $U \to T$ closed immersion $/W_n$, with DP $\gamma$ on $I = \text{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$ compatible with the canonical DP on $pW_n$ (NB. $p^n = 0 \Rightarrow I = \text{nil ideal : } U \to T$ a thickening) ; morphisms : obvious ; covering families : $(U_i, T_i) \to (U, T)$ such that $(T_i \to T)$ covering (Zar or étale). Zariski (resp. étale) crystalline site.

Sheaf on Zar (resp. ét) $\text{Crys}(X/W_n) \leftrightarrow$ compatible family of Zar (resp. ét) sheaves $F_{(U,T)}$ and maps $a_f : f^*F_{(V,Z)} \to F_{(U,T)}$ for $f : (U, T) \to (V, Z)$ such that $a_f = \text{iso}$ if $f : T \to Z$ open (resp. étale). Topos of sheaves on $\text{Crys}(X/W_n)$ denoted $(X/W_n)_{\text{crys}}$. Functorial in $X/W_n$. In particular, the absolute Frobenius of $X$ and $\sigma : \text{Spec}W_n \to \text{Spec}W_n$, $\sigma(a_0, \cdots, a_{n-1}) = (a_0^p, \cdots, a_{n-1}^p)$, induce a morphism $F : (X/W_n)_{\text{crys}} \to (X/W_n)_{\text{crys}}$.

Example : $(U, T) \mapsto \mathcal{O}_T$ is a sheaf of rings, called structural sheaf, denoted $\mathcal{O}_{X/W_n}$.

Canonical maps.

\[
i : X \to (X/W_n)_{\text{crys}}\]

\footnote{S. Yasuda observes that in fact the datum of a dp-structure is equivalent to that of a single function $g = (p - 1)! \gamma_p$ satisfying $g(\lambda x) = \lambda^n g(x)$, $pg(x) = x^p$, and $g(x + y) = g(x) + g(y) + \sum_{0 < i < d}(1/p)(p!/(p - i)!)} x^i g^{p-i}.$
(X = X_{zar} or X_{et}), a closed immersion of ringed toposes,

\[ 0 \to J_{X/W_n} \to \mathcal{O}_{X/W_n} \to i_* \mathcal{O}_X \to 0, \]

and a morphism of toposes (ringed by the constant ring \( W_n \))

\[ u = u_{X/W_n} : (X/W_n)_{crys} \to X, \]

\[ \Gamma(U, u_* F) := \Gamma((U/W_n)_{crys}, F). \]

Crystalline cohomology

\[ H^i(X/W_n) := H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n}), \]

a \( W_n \)-module. In derived style

\[ R \Gamma(X/W_n) := R \Gamma((X/W_n)_{crys}, \mathcal{O}_{X/W_n}) = R \Gamma(X, Ru_* \mathcal{O}_{X/W_n}). \]

Remark. Crystalline site, topos, structural sheaf \( \mathcal{O} \), canonical map \( u \) generalize to \( X \to (S, I, \gamma) \), \( p \) nilpotent on \( S \), \( I \subset \mathcal{O}_S \) ideal with DP \( \gamma \) extendable to \( X \).

3.4. Calculation of \( H^*(X/W_n) \)

Assume we have a closed embedding \( i : X \to Z \), of ideal \( I \), with \( Z/W_n \) smooth. Let \( (\mathcal{O}_D, \mathcal{O}) \) be the DP-envelope of \( I \) (compatible with the DP on \( (p) \)), so that \( X \to Z \) factors as

\[ X \to D \to Z, \]

with \( X \to D \) a thickening. Then \( \mathcal{O}_D \) has a canonical integrable connection \( d : \mathcal{O}_D \to \mathcal{O}_D \otimes \Omega^1_{Z/W_n} \) such that \( d(x^m) = x^{[m-1]} dx \) for \( x \in I \). Consider the corresponding de Rham complex of \( Z/W_n \) with coefficients in \( \mathcal{O}_D \):

\[ \mathcal{O}_D \otimes \Omega_{Z/W_n}. \]

**Theorem 3.4.1.** (Berthelot-Grothendieck) There exists a canonical isomorphism

\[ Ru_* \mathcal{O}_{X/W_n} \sim \mathcal{O}_D \otimes \Omega_{Z/W_n} \]

in \( D(X, W_n) \).

(In fact, there is constructed a transitive system of isomorphisms for variable embeddings \( X \subset Z \).)

**Corollary 3.4.2.**

\[ H^*(X/W_n) \sim H^*(Z, \mathcal{O}_D \otimes \Omega_{Z/W_n}). \]
In particular, for $X/k$ smooth, $Z/W_n$ a smooth lifting,

$$H^*(X/W_n) \sim H^*_{dR}(Z/W_n).$$

**Proof of 3.4.1.** The (sheaf defined by the) single DP-thickening $X \subset D$ covers the final object of $(X/W_n)_{\text{crys}}$, its powers $D^r$ ($= \text{DP-envelope of } X$ diagonally embedded in $(Z/W_n)^r$) are acyclic for $u_*$, and $u_*(O_{X/W_n}|D^r) = O_{D^r}$. Therefore

$$R u_* O_{X/W_n} \sim \check{\mathcal{C}}(D, \mathcal{O})$$

with

$$\check{\mathcal{C}}(D, \mathcal{O}) = (O_D \to O_{D^2} \to \cdots O_{D^r} \to \cdots).$$

Using the DP-Poincaré lemma one shows that the above complex (called the Čech-Alexander complex) is isomorphic in $D(X,W_n)$ to the de Rham complex $\Omega_{Z/W_n}^\cdot$. 

**Remark.** Th. 3.4.1 generalizes to $X \to (S,I,\gamma)$, with an embedding $X \to Z$ into $Z$ smooth over $S$ (see [B], [BO]).

**3.5. Crystalline cohomology for $X/k$ proper and smooth**

For $X/k$ proper and smooth,

$$H^i(X/W) := \text{proj.lim}_n H^i(X/W_n)$$

is a finitely generated $W$-module for all $i$. In fact, $H^i(X/W) = H^i$ of the perfect complex $R\Gamma(X/W) := R\text{proj.lim}_n R\Gamma(X/W_n)$. If $Z/W$ is a proper, smooth lifting of $X/k$, then

$$R\Gamma(X/k) \sim R\Gamma_{dR}(Z/W) := R\Gamma(Z,\Omega_{Z/W}).$$

For $A/W$ finite, totally ramified, with $e = [A : W]$, and $Z/A$ a proper, smooth lifting of $X$ (i.e. $Z \otimes_A k = X$), one still has

$$H^*(X/W) \otimes_W A \sim H^*_{dR}(Z/A)$$

if $e \leq p - 1$; in general, only

$$H^*(X/W) \otimes_W K \sim H^*_{dR}(Z/A) \otimes_A K,$$

for $K = \text{Frac}(A)$ (Berthelot-Ogus).

For $X/k$ proper, smooth, $X \to H^*(X/W) \otimes K_0$ ($K_0 = \text{Frac}(W)$) is a Weil cohomology: Künneth, Poincaré duality, cycle class, with “correct” Betti numbers, i.e. $\dim H^i(X/W) \otimes K_0 = \dim H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ ($\overline{k}$ an algebraic closure of $k$, $\ell \neq p$), at least if $X/k$ is projective (Katz-Messing) or liftable to char. 0 (i.e. to $A$ as above) (Berthelot-Ogus + Artin-Grothendieck).
For $k = \mathbb{F}_q$, $q = p^n$, by Berthelot,

$$Z(X/\mathbb{F}_q, t) = \prod \det(1 - F^a t, H^i(X/W) \otimes K_0)^{(-1)^{i+1}},$$

with $\det(1 - F^a t, H^i(X/W)) \otimes K_0) = \det(1 - F^a t, H^i(X_{\mathbb{F}_q}, \mathbb{Q}_t))$ if $X/k$ is projective (Katz-Messing).  

3.6. Slopes of Frobenius

Assume $k$ algebraically closed, let $X/k$ be proper, smooth, fix $i \in \mathbb{Z}$, and let $H := H^i(X/W) \otimes K_0$. Let $\varphi : H \to H$ be the $\sigma$-linear endomorphism defined by $F : (X/W_n)_{\text{crys}} \to (X/W_n)_{\text{crys}}$. Poincaré duality $\Rightarrow \varphi$ is bijective, i.e. $H$ is an $F$-isocrystal. By Dieudonné-Manin,

$$H = \oplus H_\lambda,$$

with $H_\lambda$ pure of slope $\lambda$, i.e. a direct sum of $m_\lambda$ copies of $M_\lambda := K_0,\sigma[F]/(F^a - p^r)$, $\lambda = r/s \geq 0$, $(r, s) = 1$, $F_\lambda = \sigma(\lambda) F$ (the slopes $0 \leq \lambda_1 < \cdots < \lambda_r$ of $H$ are the $\lambda$ for which $m_\lambda \neq 0$) (= $p$-adic valuations of “eigenvalues” of $\varphi$).

**Newton polygon** $\text{Nwt}_i(X) = \text{Nwt}(H)$ : slope $\lambda_i$ with horizontal length $m_{\lambda_i}s$ ($r/s = \lambda_i$).

**Hodge polygon** $\text{Hdg}_i(X) = \text{slope} r$ with multiplicity the Hodge number $h^r,s(X, \Omega^s_{X/k})$. Basic inequality : 

**Theorem 3.6.2.** (Mazur-Ogus) $\text{Nwt}_i(X)$ lies above $\text{Hdg}_i(X)$. 

In particular, for $k = \mathbb{F}_q$, if $H^i(X, \mathcal{O}) = 0$, all eigenvalues of $F^a$ on $H^i(X/W)$ are divisible by $q$.

The proof of 3.6.2 uses the Cartier isomorphism as an essential tool. See 4.5.3 for a key lemma.

**Remark.** Assuming only $k$ perfect, $H$ decomposes as in (3.6.1) with $H_\lambda$ the largest sub-$F$-crystal such that the slopes of $H_\lambda \otimes K_0(\overline{k})$ are all $\lambda$, and 3.6.2 is still valid.

**Remark.** Suppose $X = Z \otimes_A k$, $Z/A$ proper, smooth as above. Then $h^{r,s}(X) \geq h^{r,s}(Z_K)$ ($Z_K = Z \otimes K$) (semi-continuity). Hence $\text{Hdg}_i(Z_K)$ is above $\text{Hdg}_i(X)$, $p$-adic Hodge theory ($C_{\text{cris}}$ theorem) implies : $\text{Nwt}_i(X)$ lies above $\text{Hdg}_i(Z_K)$.

4. The de Rham-Witt complex

4.1. Witt complexes : the Langer-Zink construction

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$^3$2011/3/14 : I just received a preprint by J. Suh, *Symmetry and parity of slopes of Frobenius on proper smooth varieties*, in which he shows that this result and the one above still hold in the proper smooth, not necessarily projective case.
Definitions. (1) Let \( A \) be an \( A \)-algebra (in some topos \( T \)), \( I \subset B \) an ideal with \( DP \gamma_n, M \) a \( B \)-module. An \( A \)-dp-derivation \( D : B \rightarrow M \) is an \( A \)-derivation such that \( D\gamma_n(x) = \gamma_{n-1}(x)Dx \) for \( x \in I \) (i.e. local section of \( I \)). Denote by \( d : B \rightarrow \tilde{\Omega}^1_{B/A,\gamma} \) (or \( \tilde{\Omega}^1_{B/A} \)) the universal \( A \)-dp-derivation.

\[
\text{Hom}(\tilde{\Omega}^1_{B/A}, M) = \text{Der}_{A,\gamma}(B, M).
\]

(2) A \( B/A \)-dga is a strictly anticommutative graded \( B \)-algebra \( P = \oplus_{n \in \mathbb{N}} P^n \), equipped with an \( A \)-linear map \( d : P^n \rightarrow P^{n+1} \) such that \( d^2 = 0 \) and \( d(xy) = dx.y + (-1)^i x.dy \) for \( x \in P^i, y \in P^j \). A \( B/A \)-dp-dga is a \( B/A \)-dga such that \( B \rightarrow P^0 \rightarrow P^1 \) is a dp-derivation. Initial \( B/A \)-dp-dga denoted \( \tilde{\Omega}_{B/A} \).

(3) For \( A \) a \( \mathbb{Z}(p) \)-algebra, a Witt complex over \( B/A \) is a projective system of \( W_n(B)/W_n(A) \)-dga \( P_n \) for \( n \geq 1 \)
\[
\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_1
\]
equipped with maps \( F : P_{n+1} \rightarrow P_n, V : P_n \rightarrow P_{n+1} \), satisfying:
- \( W_n B \rightarrow P^0_n \) compatible with \( F, V \);
- \( Fx.Fy = F(xy) \);
- \( xVy = V(Fx.y) \);
- \( FV = p \);
- \( FdV = d \);
- \( Fd[x] = [x^{1-p}d[x]] \) for \( x \in B \)
  (here \( [x] = [x].1_{p_0} \) by abuse).

A map of Witt complexes is a map of projective systems compatible with all the structures.

(NB. The terminology Witt complex is borrowed from [HM]; a Witt complex is called an \( F-V \)-procomplex) in [LZ].)

Standard formulas in any Witt complex:
- \( dF = pFd, Vd = pdV \),
- \( V(xdy_1 \cdots dy_r) = Vx.dV y_1 \cdots .dV y_r \),
- \( Vdx = VFdV x = V1.dV x = d(V1.V x) = d(V(FV x)) = pdV x \).

Theorem 4.1.1. (Langer-Zink). For \( A \) a \( \mathbb{Z}(p) \)-algebra, the category of Witt complexes over \( B/A \) admits an initial object, denoted \( W\Omega_{B/A} \),
called the de Rham-Witt (pro)-complex of \( B/A \). Moreover:
(a) $W_n\Omega^0_{B/A} = W_nB$ for all $n$;
(b) The de Rham-Witt complex of $B/A$ is a projective system of dp-dga, for the canonical DP structure on $VW_{n-1}B$. The (unique) map of dp-dga

$$\tilde{\Omega}^{W_nB/W_nA} \to W_n\Omega_{B/A}$$

is surjective, and an isomorphism for $n = 1$:

$$\Omega_{B/A} \xrightarrow{\sim} W_1\Omega_{B/A}.$$

(c) If $p = 0$ in $A$, then $VF = p$.

**Proof.** One first checks the following two key points:

(i) If $P$ is a Witt complex, then, for all $n$, $d : W_nB \to P^1_n$ is a dp-derivation (and hence $P_n$ is a dp-dga)

(e. g., for $x \in B$, $d\gamma_p([x])dV[x] = p^{-2}V[x]dV[x] = p^{-2}V[x]dV[x]$, and already $dV[x]^p = d([x]V1) = V1d[x] = VFD[x] = V([x]^{p-1}d[x]) = V[x]^{p-1}dV[x]$)

(ii) If $D : W_nA \to M$ is a dp-derivation into a $W_nA$-module $M$, then $FD : W_{n-1}A \to F\ast M$ defined by

$$FDx = [a^{p-1}]D[a] + DVb$$

for $x = [a] + Vb$, is a dp-derivation.

It follows from (ii) that the projective system $\tilde{\Omega}^{W_nB/W_nA}$ acquires maps (of graded algebras) $F : \tilde{\Omega}^{W_nB/W_nA} \to \tilde{\Omega}^{W_{n-1}B/W_{n-1}A}$ satisfying some of the formulas in (3) ($FdVx = dx$ for $x \in W_nB$, $Fd[x] = [x^{p-1}]d[x]$ for $x \in B$, $dFx = pFDx$, for $x \in W_{n+1}B$). The projective system $W.\Omega_{B/A}$ is then constructed inductively as a quotient of $\tilde{\Omega}^{W_nB/W_nA}$.

In (ii), the fact that $FD$ is a derivation (already is additive) makes crucial use of the fact that $D$ is a dp-derivation. Compare with the definition of the Cartier operator $C^{-1}$, sending $dx$ to the class of $x^{p-1}dx$, which is additive (modulo boundaries). For $A$ of char. $p$, $F : W_2\Omega^1_{B/A} \to \Omega^1_{B/A}$ lifts the Cartier operator $C^{-1} : \Omega^1_{B/A} \to \Omega^1_{B/A}/dB$.

For a morphism $f : X \to S$ of schemes over $\mathbb{Z}(p)$,

$$W.\Omega^{X/S} := W.\Omega^{\mathcal{O}_X/f^{-1}(\mathcal{O}_S)}$$

is called the *de Rham-Witt (pro)-complex of $X/S$. Obvious functoriality in $B/A$ and $X/S$. We are mainly interested in the case where $p$ is nilpotent in $S$, and even $S = \text{Spec} k$ a perfect field of char. $p$. 12
4.2. Other constructions

- If $A$ is a perfect ring of char. $p$, $W\Omega_{B/A}$ coincides with Illusie’s de Rham-Witt complex constructed in [DRW] (if $I$ is the latter, $I$ is a Witt complex over $B/A$, and the corresponding map $W\Omega_{B/A} \to I$ is an isomorphism, as the universal property of $I$ as a $V$-pro-complex yields an inverse to it). This isomorphism is compatible with $F, V$. Langer-Zink’s approach simplifies the construction of $F$ on $I$.

- For $k$ a perfect field of char. $p > 2$ and $X/k$ smooth of dim. $< p$, it is shown in [DRW] that $W\Omega_{X/k}$ coincides with Bloch’s complex of typical curves on $SK_{i+1}, \cdots \to C^iX \to \cdots$. (Kato [K1] sketched how to remove the restrictions $p > 2$ and dim$X < p$ in Bloch’s construction, and presumably the isomorphism extends.)

- For $X/k$ smooth as above, it is shown in [DRW] that

$$W\Omega_X := \text{proj.lim}_n W\Omega_{X/k}$$

is the quotient of proj.lim$\Omega_{W_nO_X}$ by the closure (for the canonical filtration) of the $p$-torsion, a quotient considered first by Lubkin.

- For $B$ a $\mathbb{Z}(p)$-algebra, Hesselholt-Madsen [HM] define a Witt complex over $B$ as a projective system of strictly anticommutative $W_nB$-graded algebras $E_n$, with operators $F, d, V$ as in (3) above, (with $d^2 = 0$ and $d(xy) = dx.y + (-1)^i x.dy$), forgetting the $W_nA$-linearity of $d$. They show that the category of Witt complexes over $B$ has an initial object, called the (absolute) de Rham-Witt complex of $B$,

$$W\Omega_B.$$

They study it for $p > 2$. The Langer-Zink complex $W\Omega_{B/A}$ is a quotient of $W\Omega_B$, studied in [He].

- Other variants : Olsson’s variant of the Langer-Zink construction for certain morphisms of algebraic stacks [O], Davis-Langer-Zink overconvergent de Rham-Witt complex for $X/k$ smooth [DLZ].

4.3. Local description of $W\Omega_{X/S}$ (smooth case)

- Étale extensions

  (1) For $X/S$, $W_n\Omega^i_{X/S}$ is quasi-coherent on $W_n(X)$ for all $i, n$.

  (2) Assume $p$ nilpotent on $S$. Then, for $Y$ an $S$-scheme and $X \to Y$ étale, $W_n(X) \to W_n(Y)$ is étale, and

$$W_nO_X \otimes_{W_nO_Y} W_n\Omega^i_{Y/S} \to W_n\Omega^i_{X/S}$$
is an isomorphism.

Proof. The main point is to show the first assertion of (2). See [LZ, appendix]. Much easier if $p = 0$ (cf. [DRW]). It is shown in [LZ] that (2) holds if, instead of assuming $p$ nilpotent on $S$, one assumes that $Y$ is $F$-finite, i.e. the absolute Frobenius of $Y \otimes F_p$ is finite.

- Canonical bases
  For $X/S$ smooth, the determination of the local structure of $W_n\Omega_{X/S}$ is reduced by (2) to that of $W_n\Omega_{B/A}$ for a polynomial algebra $B = A[T_1, \cdots, T_r]$.

  Case $A = F_p$. We have the following description of $W_n\Omega_B := W_n\Omega_{B/F_p}$, due to Deligne:
  
  $$W_n\Omega_B = E / (V^n E + dV^n E'),$$

  where $E'$ is the so-called complex of integral forms, defined by
  
  $$E' \subset \Omega_C/Q_p, \quad C = Q_p[T_1^{p^{-\infty}}, \cdots, T_r^{p^{-\infty}}],$$

  with
  
  $$V = pF^{-1}, FT_i = T_i^p,$$

  where $(\omega \in E') \Leftrightarrow (\omega$ and $d\omega$ integral) (i.e. coefficients in $Z_p$).

  Proof. As $E^0/V^n E^n = W_n(B)$, $E := (E' / (V^n E + dV^n E'))_{n \geq 1}$ is a Witt complex over $B/F_p$, so we have a natural map $W.\Omega_B/F_p \rightarrow E$ of Witt complexes. To show that it’s an isomorphism, one uses:

  As a complex of $Z_p$-modules, $E$ has a natural grading by the group

  $$\Gamma = (Z[1/p]_{\geq 0})^r,$$

  $$E = \bigoplus_{k \in \Gamma} kE,$$

  where $x = \sum a_i(T)d\log T_i$ belongs to $kE$, i.e. is of homogeneous of degree $k$, if and only if the polynomials $a_i(T)$ are (here $i = (i_1 < \cdots < i_m)$), $d\log T_i = d\log T_{i_1} \cdots d\log T_{i_m}$.

  Each $kE^m$ has a canonical basis consisting of elements $e_i(k)$ ($i = (i_1 < \cdots < i_m)$) sent to specific elements in the de Rham-Witt complex.

  Example: $r = 1$, $B = F_p[T], kE^0 = Z_p e_0(k), kE^1 = Z_p e_1(k)$, with $e_0(k) = p^{u(k)} T^k$ if $k \notin Z$ where $p^{u(k)}$ is the denominator of $k$, $e_0(k) = T^k$ otherwise, $e_1(k) = T^k d\log T$ ($k > 0$). Then $e_0(k)$ is sent to $[T]^k$ if $k \in Z$, to $V^{u(k)}[T]^{p^{u(k)}k}$ if $k \notin Z$, $e_1(k)$ to $[T]^{k} d\log[T] := [T]^{k-1} d[T]$ if $k \in Z$ ($k > 0$), $dV^{u(k)}[T]^{p^{u(k)}k}$ if $k \notin Z$. One gets direct sum decompositions

  $$W_n(B) = \bigoplus_{k \text{ integral}} (Z/p^n Z)[T]^k \oplus \bigoplus_{k \text{ not integral}} V^{u(k)}(Z/p^{u(k)} Z)[T]^{p^{u(k)}k},$$
\[ W_n \Omega^1_{B/F_p} = \bigoplus_{k>0, \ k \text{ integral}} (\mathbb{Z}/p^n\mathbb{Z})[T]^k \log[T] \]
\[ \bigoplus_{k \text{ not integral}} dV^{n(k)}(\mathbb{Z}/p^{n-u(k)}\mathbb{Z})[T]^{p^{u(k)}k}, \]
\[ W_n \Omega^i_{B/F_p} = 0, \ i > 1. \]

Key observation (Deligne) : \( W_n \Omega_{B/F_p} \) contains the de Rham complex \( \Omega_{(\mathbb{Z}/p^n\mathbb{Z})[T]} \) as a direct summand:
\[ W_n \Omega_{B/F_p} = \Omega_{(\mathbb{Z}/p^n\mathbb{Z})[T]} \oplus (W_n \Omega_{B/F_p})_{\text{not integral}}, \]
and the complement \((W_n \Omega_{B/F_p})_{\text{not integral}}\) is acyclic.

The limit \( W\Omega_B := \text{proj lim} W_n \Omega_B \),
can be described as
\[ WB = \{ \sum_{k \in \mathbb{N}[1/p]} a_k T^k, a_k \in \mathbb{Z}_p, \text{den}(k)|a_k \forall k, \lim_{k \to \infty} a_k = 0 \} \]
\[ W\Omega^1_B = \{ \sum_{k>0, k \in \mathbb{N}[1/p]} a_k T^k (dT/T), a_k \in \mathbb{Z}_p, \lim_{k \to \infty} \text{den}(k).a_k = 0 \} \]
\[ W\Omega^i_B = 0, \ i > 1. \]

All this is generalized to any \( r \) in [DRW] and to any \( A \) in [LZ]. In particular :
\[ W_n \Omega_{A[T_1,\ldots,T_r]/A} = \Omega_{W_n(A)[T_1,\ldots,T_r]/W_n(A)} \oplus (W_n \Omega_{A[T_1,\ldots,T_r]/A})_{\text{not integral}}, \]
with the not integral part acyclic. And for \( X/S \) smooth of relative dimension \( d \):
\[ W_n \Omega_{X/S} = (0 \to W_n \Omega_X \to W_n \Omega^1_{X/S} \to \cdots \to W_n \Omega^{d-1}_{X/S} \to W_n \Omega^d_{X/S} \to 0). \]

- The canonical filtration
  \[ W\Omega_{X/S} := \text{proj lim} W_n \Omega_{X/S}, \]
  \[ \text{Fil}^n W\Omega_{X/S} := \ker W\Omega_{X/S} \to W_n \Omega_{X/S} \]
  Then ([LZ]) : For \( X/S \) smooth,
  \[ \text{Fil}^n W\Omega^i_{X/S} = V^n W\Omega^i_{X/S} + dV^n W\Omega^{i-1}_{X/S}. \]

Moreover ([DRW] for \( S \) perfect, [BER] in general) : For \( S/F_p \), \( X/S \) smooth, \( gr^n W\Omega^i_{X/S} \) is an extension of \( \Omega^{i-1}_{X/S}/Z_n \Omega^{i-1}_{X/S} \) by \( \Omega^i_{X/S}/B_n \Omega^i_{X/S} \):
\[ 0 \to \Omega^i_{X/S}/B_n \Omega^i_{X/S} \to gr^n W\Omega^i_{X/S} \to \Omega^{i-1}_{X/S}/Z_n \Omega^{i-1}_{X/S} \to 0 \]
In particular, \(\text{gr}^n\) is locally free of finite type, of formation compatible with base change.

Here, \(Z_n\) and \(B_n\) are the iterated cycles and boundaries of \(\Omega_{X/S}\) defined inductively by the Cartier isomorphism, from \(Z_0 = \Omega_1, B_0 = 0, C^{-1} : B_n \Omega_{X/S} \sim B_{n+1} \Omega_{X/S}/B_1, C^{-1} : Z_n \Omega_{X/S} \sim Z_{n+1} \Omega_{X/S}/B_1\).

4.3. De Rham-Witt complex and crystalline cohomology

**Theorem 4.3.1.** \(k\) perfect field of char. \(p\), \(X/k\) smooth. There exists a canonical isomorphism of projective systems of \(D(X,W_n)\):

\[
R_u \ast O_{X/W_n} \overset{\sim}{\rightarrow} W_n \Omega_{X/k}
\]

(notations of 3.4.1).

This isomorphism is compatible with the multiplicative structures, and functorial in \(X/k\). It induces isomorphisms

\[
R \Gamma(X/W_n) \overset{\sim}{\rightarrow} R \Gamma(X,W_n \Omega_{X/k}),
\]

\[
H^*(X/W_n) \overset{\sim}{\rightarrow} H^*(X,W_n \Omega_{X/k}).
\]

**Proof.** First, suppose \(X\) affine. Choose an embedding \(i : X \rightarrow Z\) into a smooth \(W\)-scheme \(Z\). Let \(Z_n := Z \otimes W_n\). Construct inductively a compatible system of \(W_n\)-extensions \(u_n : W_n X \rightarrow Z_n\) of the inclusion \(i_n : X \hookrightarrow Z_n\). Let \(X \hookrightarrow D_n \rightarrow Z_n\) be the dp-envelope of \(i_n\). As the ideal of \(X \hookrightarrow W_n X\) has divided powers, \(u_n\) uniquely factors through \(D_n\). We get maps \(\Omega_{Z_n/W_n} \rightarrow \Omega_{W_n X/W_n} \rightarrow W_n \Omega_{X/k}\), whose composite factors through \(D_n \otimes \Omega_{Z_n/W_n} = \Omega_{D_n/W_n}\) as \(d : W_n O_X \rightarrow W_n \Omega_{X/k}\) is a dp-derivation. The resulting map

\[
R_u \ast O_{X/W_n} \overset{\sim}{\rightarrow} D_n \otimes \Omega_{Z_n/W_n} \rightarrow W_n \Omega_{X/k}
\]

does not depend on the choice of the embedding. To check it’s an isomorphism, we may assume \(Z_n\) lifts \(X\), and even reduce to \(X = \text{Spec}k[t_1, \cdots, t_r]\), \(Z_n = \text{Spec}W_n[t_1, \cdots, t_r]\). Then the result follows from the fact that the inclusion

\[
\Omega_{Z_n/W_n} \subset W_n \Omega_{X/k}
\]

is a quasi-isomorphism (cf. 4.3, end of Canonical bases).

General case : hypercover by open affines, use cohomological descent.

Comparison th. 4.3.1 extended by Langer-Zink to \(X/S\) smooth, \(p\) nilpotent on \(S\):

\[
R_u \ast O_{X/W_n(S)} \overset{\sim}{\rightarrow} W_n \Omega_{X/S}.
\]

Same proof.
Remark. The proof actually gives an isomorphism in the derived category of projective systems of $W_n$-modules over $X$ (this is finer, and needed to apply $R\lim$ functors).

4.4. The slope spectral sequence

4.4.1. Suppose now $X/k$ proper and smooth. Then 4.3.1 gives:

$$R\Gamma(X/W) \sim \to R\Gamma(X, W\Omega_{X/k})$$

and $R\Gamma(X/W)$ is a perfect complex, with $R\Gamma(X/W) \otimes^L_{W} k \to R\Gamma(X, \Omega_{X/k})$.

Moreover:

- The $(\sigma$-linear) endomorphism $\varphi$ of $R\Gamma(X/W)$ induced by the absolute Frobenius of $X$ is induced by the endomorphism $\Phi$ of $W\Omega_{X/k}$ such that $\Phi = p^i F$ in degree $i$.
- $F : W\Omega^d_{X/k} \to W\Omega^d_{X/k}$ is bijective, which yields a $\sigma^{-1}$-linear endomorphism $v$ of $R\Gamma(X/W)$ such that $\varphi v = v \varphi = p^d$.

The next result is deeper:

**Theorem 4.4.2.** For any $(i, j)$, the canonical map

$$H^j(X, W\Omega^i_{X/k}) \to \text{proj. lim}_n H^j(X, W_n\Omega^i_{X/k})$$

is an isomorphism, $H^j(X, W\Omega^i_{X/k})$ is separated and complete for the $V$-topology, its subgroup $T^{a,j}$ of $p$-torsion is killed by a power of $p$, and

$$H^j(X, W\Omega^i_{X/k})/T^{a,j}$$

is a free $W$-module of finite rank.

**Proof.** The argument in [DRW], imitated from Bloch, consists in studying $H^*(X, W\Omega^\leq i)$, with the operator $V_i$ given on $W\Omega^\leq i$ by $p^{i-j}V$ in degree $j$. Using the structure of gr$^nW\Omega$, one shows that $H^*(X, W\Omega^\leq i)$ is finitely generated over $W[[V]]$ and of finite length modulo $V$. Using $\Phi$ (with $\Phi V_i = V_i \Phi = p^{i+1}$, this implies that $H^*(X, W\Omega^\leq i)$ is sum of a free $W$-module of finite rank and a $p$-torsion module killed by a power of $p$, and 4.4.2 follows by dèvissage.

Remark. As observed in [BBE], the proof shows that the conclusion of 4.4.2 holds for $i = 0$ and $X/k$ proper, not necessarily smooth.

**Corollary 4.4.3.** $H^j(X, W\Omega^i_{X/k})/T^{a,j}$, with the operators $F$, $V$ induced by $F$, $V$ on $W\Omega^i$, is the Cartier module of a smooth formal $p$-divisible group. Equipped with the operator $p^i F$, it’s an $F$-crystal of slopes in $[i, i + 1]$.

**Corollary 4.4.4.** The $(\Phi$-equivariant) spectral sequence

$$E^{ij}_1 = H^j(X, W\Omega^i_{X/k}) \Rightarrow H^{i+j}(X, W\Omega_{X/k}) (= H^{i+j}(X/W))$$
degenerates at $E_1$ modulo torsion and gives isomorphisms

$$H^i(X, \Omega^i_{X/k}) \otimes K_0 \sim (H^{i+j}(X/W) \otimes K_0)_{[i, i+1]}$$

where $(H^{i+j}(X/W) \otimes K_0)_{[i, i+1]}$ is the part of the $F$-isocrystal $H^{i+j}(X/W) \otimes K_0$ of slopes in $[i, i+1]$

The spectral sequence of 4.4.4 is called the slope spectral sequence.

In particular:

**Corollary 4.4.5.** There is a natural isomorphism, for all $j$,

$$H^j(X, W\Omega_X) \otimes K_0 \sim (H^i(X/W) \otimes K_0)_{[0, 1]}$$

Remark. It was recently shown by Berthelot, Bloch and Esnault [BBE] that 4.4.5 extends to the proper, possibly singular case, provided that $H^i(X/W) \otimes K_0$ is replaced by Berthelot’s rigid cohomology $H^i_{\text{rigid}}(X/K_0)$.

Remark. The slope spectral sequence is studied in more detail in [DRW], [IR], and by Ekedahl [E]. See also the survey [I]. One application, described in [DRW, II 5.12], is the (refined) Igusa-Artin-Mazur inequality: if $k$ is algebraically closed, and $X/k$ projective, smooth, then

$$\rho = b_2 - 2h - r,$$

where $\rho = \text{rkNS}(X/k)$, $b_2 = \dim H^2(X/W) \otimes K_0$, $h = \dim (H^2(X/W) \otimes K_0)_{[0, 1]}$, and $r = \text{rk} T_p H^2(X, G_m)$. When Artin-Mazur’s formal Brauer group $\Phi^2$ of $X$ is representable by a smooth formal group, $h$ is the dimension of its $p$-divisible part. The projectiveness assumption is used in loc. cit. to ensure a symmetry property of slopes of Frobenius on $H^2$. This property has been shown by J. Suh to actually hold in the general proper smooth case as well (see footnote 2).

4.5. **Higher Cartier isomorphisms, alternate construction of the de Rham-Witt complex**

For $X/S$ smooth, $S/\mathbf{F}_p$, the Cartier isomorphism is an isomorphism of graded algebras

$$C^{-1}_{X/S} : \oplus \Omega^i_{X/(p)/S} \sim \oplus H^i F_* \Omega^i_{X/S},$$

where $X^{(p)} = \text{pull-back of } X \text{ by the absolute Frobenius of } S$, $F : X \to X^{(p)}$ the relative Frobenius, such that $C^{-1}$ sends $a \otimes 1 \in \mathcal{O}_{X(p)}$ to $a^p$ and $da \otimes 1$ to the class of $a^{p-1} da$.

Suppose $S = \text{Spec } k$, $k$ perfect of char. $p$. Then $F : W_2 \Omega^1_X \to \Omega^1_X$ lifts the absolute Cartier isomorphism $C^{-1}$ (composed of $C^{-1}_{X/S}$ and the canonical
isomorphism $\Omega_X^i \sim \Omega_{X^{(p)}}$ (cf. 4.1.1 (ii)). (We drop $/k$ for short.) More
generally:

**Theorem 4.5.1.** For $n \geq 1$, $F^n : W_{2n}\Omega^i_X \to W_n\Omega^i_X$ induces an isomorphism

$$W_n\Omega^i_X \sim \mathcal{H}W_n\Omega_X,$$

compatible with products, and equal to $C^{-1}$ for $n = 1$.

**Proof.** Main point: show: $F^n W_{2n}\Omega^i_X = ZW_n\Omega^i_X$. The proof given in
[DRW] is insufficient, corrected in [IR]. Makes crucial use of the description
of $W_n\Omega_X$ for $X = \text{Spec} [t_1, \cdots, t_r]$ in terms of the complex of integral forms
(4.3) and, of course, of the Cartier isomorphism.

By 4.3.1, $F^n$ induces $W_n$-linear isomorphisms

$$(4.5.2) \quad W_n\Omega^i_X \sim \sigma^n_{\mathcal{H}}(X/W_n),$$

where $\mathcal{H}(X/W_n) = R^i u_* \mathcal{O}_{X/W_n}$.

Assume $X$ lifted to formal smooth $Z/W$, let $Z_n := Z \otimes W_n$. Then
$\mathcal{H}(X/W_n) = \mathcal{H}_{dR}(Z_n/W_n)$ (3.4.1), and (4.5.2), for $i = 0$ and $i = 1$ are
given by:

$i = 0$: $a = (a_0, \cdots, a_{n-1}) \in W_n\mathcal{O}_X$ sent to $b_0^p + pb_1^p + \cdots + p^{n-1}b_{n-1}$
in $\mathcal{H}_{dR}(Z_n/W_n)$, where $b_i$ in $\mathcal{O}_Z$ lifts $a_i$,

$i = 1$: $d(a_0, \cdots, a_{n-1})$ in $W_n\Omega^1_X$ sent to $\sum b_i^p d b_i$ in $\mathcal{H}_{dR}(Z_n/W_n)$.

For $i = 0$, (4.5.2) factors the $n$-th ghost component $w_n : W_{n+1}(\mathcal{O}_{Z_{n+1}}) \to \mathcal{O}_{Z_{n+1}}$, and, for $i = 1$, the composite map $(4.5.2) dR : W_{n+1}\mathcal{O}_X \to \Omega^1_{Z_n}/d\mathcal{O}_{Z_n}$
lifts $F^n d : W_{n+1}\mathcal{O} \to \Omega^1_X/d\mathcal{O}_X$.

$\Rightarrow$ reconstruction of $W_i\Omega_X$ (suggested by Katz):

$$W_n\Omega^i_X := \sigma^n_{\mathcal{H}}(X/W_n),$$

$$F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$$
given by the restriction $\mathcal{H}(X/W_{n+1}) \to \mathcal{H}(X/W_n)$,

$$d : W_n\Omega^i_X \to W_{n+1}\Omega^{i+1}_X$$
given locally by the Bockstein operator associated with the exact sequence

$$0 \to \Omega_{Z_n/W_n} \to \Omega_{Z_{2n}/W_{2n}} \to \Omega_{Z_n/W_n} \to 0,$$

where the first map is multiplication by $p^n$,

$$V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X.$$
induced by multiplication by $p$ on $\Omega_{Z_{n+1}/W_{n+1}}$.

To reconstruct $R : W_{n+1}Ω_{X} \rightarrow W_{n}Ω_{X}$, suppose $Z/W$ admits a formal lifting $Φ$ of Frobenius (exists if $X/k$ affine). Then, $Φ^*$ is divisible by $p^i$ on $Ω^i_{Z/W}$, let $f = p^{-i}Φ$ on $Ω^i_{Z/W}$. For $x \in \mathcal{H}^i(X/W_{n+1}) = \mathcal{H}_{dR}^i(Z_{n+1}/W_{n+1})$, there exists $y \in Ω^i_{Z/W}$, unique modulo $p^iΩ^i_{Z/W} + dΩ^{i-1}_{Z/W}$, such that $x = fy$ mod $p^{i+1}Ω^i_{Z/W} + dΩ^{i-1}_{Z/W}$. Then, for $y_n$ the image of $y$ in $Ω^i_{Z_{n+1}/W_{n+1}}$, $dy_n = 0$, and $x \mapsto$ class of $y_n$ in $\mathcal{H}_{dR}^i(Z_{n}/W_n)$ defines $R$.

Existence and uniqueness of $y$ rely on the following key lemma:

**Lemma 4.5.3.** (Ogus). With the above notations, let $L \subset Ω_{Z/W}$ be the subcomplex defined by

$$L^i = \{ x \in p^iΩ^i_{Z/W} | dx \in p^{i+1}Ω^{i+1}_{Z/W} \}.$$

Then $Φ^* : Ω_{Z/W} \rightarrow Ω_{Z/W}$ factors through $L$ and induces, for each $n \geq 1$, a quasi-isomorphism

$$Ω_{Z_{n}/W_n} \rightarrow L_n := L \otimes W_n.$$

(To get $y$ from $x$, apply 4.5.3 to the class of $p^i\tilde{x}$ in $\mathcal{H}^i(L_n)$, for $\tilde{x} \in Ω^i_{Z/W}$ lifting $x$.)

**Proof.** [BO, 8.8]: dévissage, reducing to Cartier isomorphism. Lemma 4.5.3 is the crucial ingredient in the proof of the Mazur-Ogus theorem 3.6.2.

**Applications.**

- Structure (for $X/W$ proper and smooth) of the conjugate spectral sequence

$$E_2^{ij} = \text{proj. lim } H^j(X, \mathcal{H}^i(X/W_n)) \Rightarrow H^{i+j}(X/W)$$

(degenerates at $E_2$ modulo torsion), and analysis of the log-Hodge-Witt groups

$$H^j(X, WΩ^i_{X,\log}) := \text{proj. lim } H^j(X, W_nΩ^i_{X,\log}),$$

where $W_nΩ^i_{X,\log} \subset W_nΩ^i_X$ is the additive subsheaf étale locally generated by the forms $d\log[x_1] \cdots d\log[x_i]$, for $x_m \in \mathcal{O}_X^*$, $1 \leq m \leq i$.

- Construction of $WΩ_X$ via (4.5.2) works in the log context, see §6 (Hyodo-Kato).

### 5. Review of log schemes

Pre-log structure, log structure, log scheme

Examples : trivial log str., $\mathcal{O}_X \cap j_*\mathcal{O}_U$

Morphisms : $\{\text{schemes}\} \subset \{\text{log schemes}\}$

Associated log structure $M^\alpha :$ push-out of

$$\mathcal{O}^* \xleftarrow{\alpha^{-1}} (\mathcal{O}^*) \rightarrow M$$
\[(u, a) \equiv (v, b) \iff \exists c, d \in \alpha^{-1}(\mathcal{O}^*) \mid ad = bc, cu = dv \text{ for } (u, a) \text{ and } (v, b) \text{ in } (\mathcal{O}^*, M))\], universal property

\[f^*M := (f^{-1}M)^a, \text{ strict morphism}\]

Chart \(P \to M, X \to \text{Spec}\mathbb{Z}[P]\); chart of a morphism

Examples: \(\text{Spec}\mathcal{O}_S[T_1, \cdots, T_r], (t_1 \cdots t_r = 0) \subset \text{Spec}A, A \text{ regular local, } (t_i) \text{ regular parameters}; \text{ trait, standard log point } (N \to k, 1 \to 0)^a, \text{ semistable reduction}\)

\(P \to P^{op}, \text{ integral, fine, fs monoid (resp. log scheme)}\)

Examples: dnc, affine toric variety, toric variety (torus embedding), toroidal embedding

Fiber products, base change, strict case

\[\Omega^{1}_{(X,M)/(S,N)}, d, \text{dlog, } \alpha(a) \text{dlog} = d\alpha(a)\]

\[\Omega^{1}_{(X,M)/(S,N)} = (\Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes \mathbb{Z}M^{gp}) / \langle d\alpha(a), 0 \rangle - (0, \alpha(a) \otimes a), (0, 1 \otimes b) \rangle > (a \in M^{gp}, b \in N^{gp})\]

\[\omega^1_{X/S}, \Omega^1_{X/S}, \Omega^1_{(X,M)/(S,L)}, \text{ log dR complex } \Omega_{(X,M)/(S,L)} \text{ (or } \omega^1_{X/S}, \text{ or } \Omega^1_{X/S}, \text{ or } \Omega^1_{X/S})\]

Examples: relative dnc: \(\Omega^1_{X/S}(\log D), \text{ semistable reduction: } \Omega^1_{X/S}(\log(D/E))\), toric varieties

Exact closed immersion, log thickening

Log smooth, log étale; strict case; chart characterization

Examples: toroidal embeddings, relative dnc, semistable reduction, \(\text{Spec} k[x, y/x] \to \text{Spec} k[x, y], \text{ log blow-up}\)

Cartier isomorphism:

- **semistable type:** \((s = \text{Spec} k, L) \text{ standard log point, } (X, M) \text{ of semistable type over } (s, L) : \text{ étale loc. } X = \text{Spec} k[t_1, \cdots, t_d]/(t_1 \cdots t_r), \text{ with charts}\)

\[
k[t_1, \cdots, t_d]/(t_1 \cdots t_r) \leftarrow \mathbb{N}^r
\]

\[
\begin{array}{c}
k \leftarrow 1 \rightarrow 0 \\
1 \rightarrow (1, \cdots, 1) \\
\end{array}
\]

(e. g. special fiber of semistable scheme over trait).

- **more generally, log smooth Cartier type:** \(f : (X, M) \to (S, L), S/\mathbb{F}_p, \text{ log smooth and saturated morphism of fs log schemes (saturated = (log integral + reduced geometric fibers). (} \leftrightarrow \text{ (log integral and in the Frobenius diagram (with cartesian square) }\)

\[
\begin{array}{c}
(X, M) \leftarrow (X', M') \leftarrow X, M \\
\downarrow f \\
(S, L) \leftarrow F_{abs} (S, L)
\end{array}
\]
the relative Frobenius $F$ is exact, see [K2], [Ts, II 3.1]) ($F_{\text{abs}} : a \mapsto a^p$ on $O_S$ and on $L$). Examples: (poly) semistable reduction, log smooth saturated toric morphism $\text{Spec} A[P] \rightarrow \text{Spec} A[Q]$; Kummer étale (e. g. $x^n = t$, $(n, p) = 1$): not Cartier type.

log smooth, Cartier type ⇒ Cartier isomorphism

$$C^{-1} : \Omega^i_{(X', M')/\text{Spec} \ k} \sim F^* \Omega^i_{(X, M)/\text{Spec} \ k},$$

$(a \otimes 1) d\log x_1 \cdots d\log x_r \mapsto a^p d\log x_1 \cdots d\log x_r, a \in O_X, x_i \in M.$

$(\Rightarrow)$ decompositions of Deligne-Illusie type of $F^* \Omega^i_{(X, M)/\text{Spec} \ k}$ in situations lifted mod $p^2$ and $\dim f < p$. Applications to (classical) Hodge theory (e. g. [IKN]).

Definitions of integral and exact: $P, Q$ fine monoids, $h : Q \rightarrow P$ integral if $Z[Q] \rightarrow Z[P]$ flat; $h$ exact if $Q = (h^{gp})^{-1}(P)$ in $Q^{gp}$; $f : (X, M) \rightarrow (Y, N)$ integral (resp. exact) if $(f^* N)_x \rightarrow M_x$ integral (resp. exact) $\forall x \in X$.

6. De Rham-Witt complex and log crystalline cohomology

See slides.

7. The Hyodo-Kato isomorphism

See [HK] and slides Illusie-Sapporo-Hyodo-Kato.pdf. See also [Nak, §7] for complements and corrections to [HK]. For a new approach to the Hyodo-Kato isomorphism, see [Be].

8. Rational points over finite fields for regular models of algebraic varieties of Hodge type $\geq 1$, after P. Berthelot, H. Esnault and K. Rülling

8.1. Slopes of Frobenius and rational points

Recall: For $q = p^a$, $k = \mathbf{F}_q$, $Y/k$ separated, finite type,

$$Z(Y, t) = \exp(\sum_{n \geq 1} |Y(\mathbf{F}_q^n)| t^n/n) = \prod(1 - t^{\deg(x)})^{-1} \in (1 + t\mathbb{Z}[t]) \cap \mathbb{Q}(t),$$

(Dwork), hence

$$Z(Y, t) = \prod (1 - \alpha_i t) / \prod (1 - \beta_j t),$$

$\alpha_i, \beta_j$ algebraic integers, $\alpha_i \neq \beta_j$ for all $(i, j)$. By Grothendieck,

$$Z(Y, t) = \prod \det(1 - F^a t, H^i_c(Y_\mathbb{F}_t, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$
with inverse roots of $\det(1 - F^n t, H^i_c(Y_k, \mathbb{Q}_\ell))$ algebraic integers (Deligne), but we won’t use these results in this section. The next statement is an easy consequence of the slope spectral sequence:

**Proposition 8.1.1.** Assume:

(i) $Y/k$ geometrically connected,

(ii) $Y/k$ proper and smooth,

(iii) $H^i(Y, W\mathcal{O}_Y) \otimes \mathbb{Q} = 0$ for all $i > 0$.

Then:

(iv) For all finite extensions $k' = \mathbb{F}_{q^n}$ of $k$, $|Y(k')| \equiv 1 \mod q^n$.

**Proof.** Recall Berthelot’s formula

$$Z(Y, t) = \prod P_i(t)^{-1 + i},$$

$$P_i(t) = \det(1 - F^a t, H^i(Y/W)).$$

As $H^i(Y, W\mathcal{O}_Y) \otimes \mathbb{Q} = (H^i(Y/W) \otimes \mathbb{Q})_{0,1}$; (iii) ⇒ all slopes of Frobenius on $H^m(Y/W)$ for $m > 0$ are $\geq 1$, hence (Dieudonné-Manin) all $\alpha_i, \beta_j$ above appearing in $P_m$, $m > 0$ are divisible by $q$. As $P_0(t) = 1 - t$ by (i),

$$Z' / Z = \sum_{n \geq 1} |Y(\mathbb{F}_{q^n})| t^{n-1} = \sum_{n \geq 1} a_n t^{n-1},$$

with $a_n = |Y(\mathbb{F}_{q^n})| \equiv 1 \mod q^n$.

In [BBE], Berthelot, Bloch and Esnault show that (i) and (iii) suffice for (iv) to hold. By Étesse-Le Stum, Berthelot’s formula (*) holds with crystalline cohomology replaced by Berthelot’s compactly supported rigid cohomology $H^i_{c,\text{rig}}(Y/K_0)$, and it is proven in [BBE] that a suitably defined cohomology group with compact supports $H^i_{c,\text{rig}}(Y, W\mathcal{O}) \otimes \mathbb{Q}$ is finite dimensional and, again, calculates the part of $H^i_{c,\text{rig}}(Y/K_0)$ of slope $< 1$.

8.2. **Berthelot-Esnault-Rülling’s theorem**

Suppose now that $Y = X_k$ is the special fibre of a scheme $X$ over a dvr $R$ of mixed char. $(0, p)$, with perfect residue field $k$ and fraction field $K$.

**Theorem 8.2.2.** ([BER]) Assume:

(i) $X$ regular, and proper and flat over $R$;

(ii) $X_K$ geometrically connected;

(iii) $H^i(X_K, \mathcal{O}_{X_K}) = 0$ for all $i > 0$.

Then, if $k = \mathbb{F}_q$, $|X_k(\mathbb{F}_{q^n})| \equiv 1 \mod q^n$ for all $n \geq 1$.

**Remarks.**

(1) Esnault proved the conclusion of 8.2.2 assuming (i), (ii), and instead of (iii), that $X_K$ is of coniveau $\geq 1$ in degree $> 0$, i.e. for each $i > 0$, there
exists a dense open $U$ in $X_K$ such that the restriction map $H^i(X_K, \mathbb{Q}_\ell) \to H^i(U_K, \mathbb{Q}_\ell)$ is zero. By mixed Hodge theory this condition implies (iii), and should be equivalent to it according to Grothendieck’s generalized Hodge conjecture.

(2) By Zariski connectedness theorem (i) and (ii) in 8.2.2 imply $Y = X_k$ is geometrically connected. Therefore, by [BBE] 8.2.2 follows from :

**Theorem 8.2.3.** ([BER]) Under the assumptions (i), (ii), (iii) of 8.2.2 one has (for $Y = X_k$) :

(iv) $H^i(Y, W\mathcal{O}_Y) \otimes \mathbb{Q} = 0$ for all $i > 0$.

Actually, an even stronger result is proven in [BER] :

**Theorem 8.2.4.** ([BER]) Let $X$ be regular and proper and flat over $R$. If, for one $q \in \mathbb{Z}$, $H^q(X_K, \mathcal{O}) = 0$, then (for $Y = X_k$) $H^q(Y, W\mathcal{O}_Y) \otimes \mathbb{Q} = 0$.

Note : base changing by Spec$\hat{R}$ changes neither assumptions nor conclusions so we may and will assume $R$ complete.

**Particular cases.**

(a) Assume $X/R$ smooth. Then the conclusion of 8.2.4 means that the slopes of Frobenius on $H^q(Y/W)$ are $\geq 1$. Assume furthermore :

(a1) $H^q(X, \mathcal{O}) = H^{q+1}(X, \mathcal{O}) = 0$.

Then, by base change, $H^q(Y, \mathcal{O}) = 0$, so, by the Mazur-Ogus inequality, the slopes of $H^q(Y/W)$ are $\geq 1$ (One can also show by induction $H^q(Y, W_n\mathcal{O}) = 0$, hence $H^q(Y, W\mathcal{O}) = 0$.)

Without the assumption (a1), it may happen that $H^q(Y, \mathcal{O}) \neq 0$ (Serre’s examples of failure of Hodge symmetry in char. $p$). In this case, the Mazur-Ogus inequality says nothing. However, as observed in 3.6.2, $p$-adic Hodge theory (the $C_{cris}$ theorem) implies that the Newton polygon of $H^q(Y/W)$ is above the Hodge polygon of $H^q_{Hdg}(X_K)$, hence the slopes of $H^q(Y/W)$ are $\geq 1$.

(b) Assume $X/R$ has semistable reduction. By the slope spectral sequence for the log de Rham-Witt complex, the conclusion of 8.2.4 still means that the slopes of Frobenius on $H^q(Y/(W, W(L)))$ ((Speck, $L$) the standard log point) are $\geq 1$, and this is true by the $C_{st}$ theorem.

8.3. **Strategy of proof of 8.2.4.**

The general idea is to reduce to the semistable case by using de Jong alterations and cohomological descent.

- **Use of de Jong alterations**
**Starting point**: because \( X \) is integral and flat over \( R \), by de Jong, there exists a finite extension \( K_1 \) of \( K \), with ring of integers \( R_1 \), and a commutative diagram

\[
\begin{array}{c}
X & \xrightarrow{f} & Z \\
\downarrow & & \downarrow \\
\text{Spec}R & \xrightarrow{f} & \text{Spec}R_1 \\
\end{array}
\]

with \( Z \) integral, semistable over \( R_1 \), and \( Z \to X \) a projective alteration. The morphism \( Z_{K_1} \to X_{K_1} \) may not be surjective, but passing to a Galois extension \( K' \) of \( K \) containing \( K_1 \) and taking a disjoint sum \( X_0 \) of translated by the Galois group of pull-backs of \( Z/\text{Spec}X_{R_1} \) to \( \text{Spec}R' \), \( (X_0)_{K'} \to X_{K'} \) is surjective.

**Iteration**: Fix \( m > q \). Iterating the process, one constructs an augmented \( m \)-truncated simplicial scheme

\[
\varepsilon : X_* \to X_{R'}
\]

(\( R' \) the ring of integers of a suitable extension \( K' \) of \( K \)), such that:
- each \( X_n \) is a sum of pull-backs of semistable schemes over rings of integers of subextensions of \( K' \)
- \( \varepsilon_{K'} : (X_*)_{K'} \to X_{K'} \) is a proper \( m \)-truncated hypercovering
- \( X_0 \) is, as above, the disjoint sum of base changes of a semistable \( Z/R_1 \), with \( f : Z \to X \) a projective alteration, \( Z \) integral.

- **Use of cohomological descent and classical Hodge theory**

Since \( q < m \), as each \( (X_n)_{K'} \) is smooth over \( K' \) and \( \varepsilon_{K'} \) is a proper \( m \)-truncated hypercovering, it follows from Deligne’s mixed Hodge theory that

\[
H^q(X_{K'}, \Omega_{X_{K'}/K'}) \to H^q((X_*)_{K'}, \Omega_{(X_*)_{K'}/K'})
\]

is an isomorphism of filtered spaces (for the Hodge filtration). In particular, \( H^q((X_*)_{K'}, \mathcal{O}) = 0 \).

- **Use of \( p \)-adic Hodge theory**

By the \( C_{st} \) theorem for truncated simplicial semistable schemes (Tsuji), it follows that the slopes of Frobenius on \( H^q((X_*)_{K'}/(W(k'), W(L))) \) are \( \geq 1 \). By a generalization of de Rham-Witt theory to the truncated simplicial semistable case, this means that

\[
H^q((X_*)_{K'}, W\mathcal{O}) \otimes \mathbb{Q} = 0.
\]

- **A trace argument**
If the map
\[ \varepsilon_{k'} : (X_\bullet)_{k'} \rightarrow X_{k'} \]
was a truncated proper hypercovering, cohomological descent for rigid cohomology (Tsuzuki) - and its compatibility with slopes - would give the vanishing of \( H^q(X_{k'}, W\mathcal{O}) \otimes \mathbb{Q} \), hence that of \( H^q(X_k, W\mathcal{O}) \otimes \mathbb{Q} \). However, \( \varepsilon_{k'} \) is not in general a truncated proper hypercovering. Still, the functoriality map
\[ H^q(X_k, W\mathcal{O}) \otimes \mathbb{Q} \rightarrow H^q((X_0)_{k'}, W\mathcal{O}) \otimes \mathbb{Q} \]
is zero, as it factors through \( H^q((X_\bullet)_{k'}, W\mathcal{O}) \otimes \mathbb{Q} = 0 \). Therefore it’s enough to show that (8.3.2) is injective. By the construction of \( X_0 \) as a sum of pull-backs of \( Z \), it’s enough to show that
\[ f^*_k : H^q(X_k, W\mathcal{O}) \otimes \mathbb{Q} \rightarrow H^q(Z_k, W\mathcal{O}) \otimes \mathbb{Q} \]
is injective. This is achieved by a trace argument. One constructs a trace map
\[ \tau_{f_k} : H^q(Z_k, W\mathcal{O}) \otimes \mathbb{Q} \rightarrow H^q(X_k, W\mathcal{O}) \otimes \mathbb{Q} \]
such that
\[ (8.3.4) \quad \tau_{f_k} f_k^* = r \cdot \text{Id}, \]
where \( r \) is the generic degree of the alteration \( f \).

8.4. The trace map

As \( X \) and \( Z \) are regular, integral, with \( \dim Z = \dim X \), \( f : Z \rightarrow X \) is a complete intersection morphism of virtual relative dimension zero (i.e. locally defined by a regular immersion of codimension \( d \) in a smooth \( X \)-scheme of relative dimension \( d \)). Moreover, \( f \) is projective (in the sense that \( Z \) is a closed subscheme of some projective space \( \mathbb{P}_{X}^d \)). The construction of \( \tau_{f_k} \) and the proof of (8.3.4) uses essentially only these facts. There are three steps. Denote by \((-)_n\) the reduction mod \( p^{n+1} \).

- **Step 1**

Construction of (compatible) trace maps
\[ \text{Tr}_{f_n} : Rf_{n*}\mathcal{O}_Z \rightarrow \mathcal{O}_{X_n} \]
with
\[ (8.4.1) \quad \text{Tr}_{f_n} f_n^* = r \cdot \text{Id} \]

26
(where \( f_n^* = O_{X_n} \to Rf_n_*O_{Z_n} \) is the adjunction map).

This is more or less standard Grothendieck duality [Ha] (with signs made precise by Conrad [C]). In terms of a factorization

\[
\begin{align*}
\begin{array}{c}
Z \\
\downarrow i \\
\downarrow f \\
\end{array} & \quad \begin{array}{c}
P = P^d_X \\
\downarrow \pi \\
X \\
\end{array}
\end{align*}
\]

(with \( i \) a regular immersion of codimension \( d \)), \( \text{Tr}_{f_n} \) is the composition

\[
\text{Tr}_{f_n} = \text{Tr}_{\pi_n} \text{Tr}_{i_n},
\]

with \( \text{Tr}_{\pi_n} \) given by the canonical isomorphism \( R^d\pi_n*\Omega_{P_n/X}^d \iso O_{X_n} \), and \( \text{Tr}_{i_n} \) by the cohomology class of \( i_n \).

- **Step 2**

Construction of (compatible) trace maps, for \( n \geq 1 \),

\[
(\tau_{f_0})_n : R(f_0)_*W_nO_{Z_0} \to W_nO_{X_0}.
\]

This is a new construction, similar to the previous one, but using the de Rham-Witt complex (of Langer-Zink) of \( P_0/X_0 \).

- **Step 3**

Comparison of trace morphisms and proof of the key formula

\[
(8.4.2) \quad (\tau_{f_0})_n(f_0)^*_n = r.\text{Id},
\]

where \( (f_0)_n : W_nO_{X_0} \to R(f_0)_*W_nO_{Z_0} \) is the adjunction map. (This formula implies (8.3.4) because \( Z_k \subset Z_0 \), \( X_k \subset X_0 \) are nilpotent immersions, and (by a result of [BBE]) the restriction maps \( H^q(X_0,W\mathcal{O})\otimes \mathbb{Q} \to H^q(X_k,W\mathcal{O})\otimes \mathbb{Q} \), \( H^q(Z_0,W\mathcal{O})\otimes \mathbb{Q} \to H^q(Z_k,W\mathcal{O})\otimes \mathbb{Q} \) are isomorphisms.)

This is the most ingenious part of the proof of 8.2.4. The basic tool is the unique factorization of the \( n \)-th phantom map

\[
w_n = F^n : W_{n+1}(O_{X_{n-1}}) \to O_{X_{n-1}},
\]

\[
w_n(b_0, \cdots, b_n) = b_0^p + \cdots + p^{n-1}b_{n-1}^p + p^n b_n = b_0^p + \cdots + p^{n-1}b_{n-1},
\]

into

\[
\begin{align*}
\begin{array}{c}
W_{n+1}(O_{X_{n-1}}) \\
\downarrow F^n \\
W_n(O_{X_0}) \\
\end{array}
\end{align*}
\]
Comparing cohomology classes of a regular immersion in both theories, one shows the commutativity of the diagram

\[
\begin{array}{c}
\longrightarrow \\
W_n(O_{Z_0}) \\
\longrightarrow
\end{array}
\begin{array}{c}
\longrightarrow \\
O_{X_{n-1}}
\end{array}
\]

where the vertical maps are given by \( \tilde{F}^n \). It follows that \( (\tau_{f_0})_n(f_0)_n^* \) is the multiplication by a class \( c_n \in H^0(X_0, W_n(O_{X_0})) \) such that \( c := \text{proj.lim} c_n \in H^0(X_0, W\mathcal{O}_{X_0}) \) has the following two properties:

(i) \( Fc = c \),

(ii) \( \tilde{F}^n(c - r) = 0 \) for all \( n \geq 1 \).

One shows that this implies that \( c - r = 0 \), hence \( c_n = r \). One shows more generally that \( \text{Ker}(F - 1) \cap \bigcap_{n \geq 1} \text{Ker}(\tilde{F}^n : W\mathcal{O}_{X_0} \to O_{X_{n-1}}) = 0 \).

References


