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Glimpses on vanishing cycles, from Riemann to today

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1. The origins

Vanishing cycles in Riemann?

No, but ...
Riemann (1857) studied the hypergeometric equation \( E(\alpha, \beta, \gamma) \)

\[
t(1 - t)f'' + (\gamma - (\alpha + \beta + 1)t)f' - \alpha\beta = 0
\]

(\(\alpha, \beta, \gamma \in \mathbb{C}\)), and the monodromy of its solutions around its singular points (0, 1, \(\infty\)).

\( E(\alpha, \beta, \gamma) \) has regular singularities at these points (moderate growth of solutions).

The hypergeometric function

\[
F(\alpha, \beta, \gamma, t) = \sum_{n \geq 0} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)} \frac{t^n}{n!}
\]

(|\(t\)| < 1), where \((u, n) = \prod_{0 \leq i \leq n-1} (u + i)\), is the unique solution which is holomorphic at 0 with value 1.
Solutions form a complex local system $\mathcal{H}_C$ of rank 2 over $S = \mathbb{P}^1_C - \{0, 1, \infty\}$. For a chosen base-point $t_0 \in S$, it is given by

$$\rho : \pi_1(S, t_0) \to \text{GL}((\mathcal{H}_C)_{t_0}) \cong \text{GL}_2(\mathbb{C}).$$

Suitable standard loops around $s = 0, 1, \infty$ give local monodromy operators $T_s \in \text{GL}_2(\mathbb{C})$, satisfying $T_0 T_1 T_\infty = 1$, generating the global monodromy group

$$\Gamma := \rho(\pi_1(S, t_0))) \subset \text{GL}_2(\mathbb{C}).$$

What are the $T_s$’s? What is $\Gamma$?
An example: the Legendre family

Consider the family $X/S$ of elliptic curves on $S = \mathbb{P}^1_\mathbb{C} - \{0, 1, \infty\}$:

$$X_t : y^2 = x(x - 1)(x - t).$$

For $\alpha = \beta = 1/2, \gamma = 1$,

$$E(1/2, 1/2, 1) : t(1 - t)f'' + (1 - 2t)f' - \frac{f}{4} = 0$$

is the DE satisfied by the periods of holomorphic differential forms on $X_t$.

The relative de Rham cohomology group $H_{dR} := H^1_{dR}(X/S)$ is a free $\mathcal{O}_S$-module of rank 2, equipped with the Gauss-Manin connection $\nabla$. 

\[ \mathcal{H}_{\text{dR}} = \mathcal{O}_S e_1 \oplus \mathcal{O}_S e_2, \]
\[ e_1 = [dx/y], \ e_2 = \nabla(d/dt)(e_1), \]
with
\[ \nabla(d/dt)e_2 = \frac{(2t - 1)e_2}{t(1 - t)} + \frac{e_1}{4t(1 - t)}. \]

Horizontal solutions \( f_1 e_1^\vee + f_2 e_2^\vee \) of the dual of \( \mathcal{H}_{\text{dR}} \) are given by \( f_1 = f \), \( f_2 = f_1' \), where \( f \), a local section of \( \mathcal{O}_S \), satisfies
\[ E(1/2, 1/2, 1) : t(1 - t)f'' + (1 - 2t)f' - \frac{f}{4} = 0. \]

We have
\[ \mathcal{H}_{\text{dR}}^{\nabla=0} = \mathcal{H}_Z \otimes \mathbb{C}, \]
where \( \mathcal{H}_Z := \mathcal{H}^1(X/S, \mathbb{Z}) \), a rank 2 \( \mathbb{Z} \)-local system, equipped with the (symplectic, unimodular) intersection form \( \langle , \rangle \).
If $\gamma$ is a local horizontal section of $\mathcal{H}^\vee = \mathcal{H}_1(X/S, \mathbb{Z})$, the period
\[ \int_{\gamma} \frac{dx}{y} \]
is a solution of $E(1/2, 1/2, 1)$. For example, the hypergeometric function
\[ F(1/2, 1/2, 1, t) = \frac{1}{\pi} \int_{1}^{\infty} \frac{dx}{y} \]
is a solution.

The representation $\rho : \pi_1(S, t_0) \to \text{GL}((\mathcal{H}_C)_{t_0})$ is deduced from
\[ \rho : \pi_1(S, t_0) \to \text{Sp}((\mathcal{H}_C)_{t_0}) \simeq \text{SL}_2(\mathbb{Z}). \]

Local monodromies around 0 and 1 can be calculated by choosing suitable symplectic bases $(\gamma, \delta)$ of $(\mathcal{H}_Z)_t$, using the description of $X_t$ as a 2-sheeted cover of $\mathbb{P}^1_\mathbb{C}$. 
• In a suitable symplectic base, $T_0$ and $T_1$ are given by

\[ T_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \]

• The global monodromy group is conjugate in $\text{SL}_2(\mathbb{Z})$ to the subgroup $\Gamma = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}$ of index 2 of the congruence subgroup $\Gamma(2)$ defined by $a \equiv d \equiv 1 \mod 4$. It acts freely on the Poincaré upper half plane $D = \{ \text{Im} z > 0 \}$.

• Riemann’s period mapping $t \mapsto (\int_\gamma \omega, \int_\delta \omega)$, where $\omega = \frac{dx}{y} \in H^0(X_t, \Omega^1)$, induces an isomorphism

\[ S = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\} \simeq D/\Gamma. \]

which extends to an isomorphism

\[ \mathbb{P}^1_{\mathbb{C}} \simeq M_2 \ (= (D \cup \mathbb{P}^1(\mathbb{Q}))/\tilde{\Gamma}(2)) \]

sending $0, 1, \infty$ to the 3 cusps of $M_2$ ($\tilde{\Gamma}(2)$ = image of $\Gamma(2)$ in $\text{PSL}_2(\mathbb{Z})$).
In particular, as \( \chi(S) = -1 \), and \([\mathbb{SL}_2(\mathbb{Z}) : \Gamma] = 12\), the Galois cover

\[ D \to S = D / \Gamma \]

implies that \( S = B\Gamma \), hence \( \chi(\Gamma) = -1 \), and

\[ \chi(\mathbb{SL}_2(\mathbb{Z})) = -\frac{1}{12}, \]

as is well known.

It was discovered by Picard (around 1880) that the form of \( T_0 \) is "explained" by the fact that \( \delta \) vanishes when \( t \to 0 \), and that the singularity of the surface \( X \) at \( (x = 0, y = 0) \) is equivalent to \( u^2 + v^2 = t^2 \) (Picard-Lefschetz formula).
2. The Milnor fibration

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function having an isolated critical point at $0$ with $f(0) = 0$.

Milnor (1967) proved that, for $\varepsilon > 0$ small, and $0 < \eta << \varepsilon$, if $B = \{z| \sum_{0}^{n} |z_i|^2 \leq \varepsilon\}$, $D = \{|t| \leq \eta\}$, the restriction of $f$ to $B \cap f^{-1}(D)$,

$$f : B \cap f^{-1}(D) \to D,$$

induces over $D - \{0\}$ a locally trivial $C^\infty$ fibration in (real) $2n$-dimensional manifolds with boundary

$$M_t = f^{-1}(t) \cap B,$$

trivial along the boundary $\partial M_t$.

This is now called the Milnor fibration, and $M_t$ is called a Milnor fiber.
Moreover, Milnor proved:

- \( M_t \) has the homotopy type of a bouquet of \( \mu \ n \)-dimensional spheres:

\[
S^n \vee \cdots \vee S^n \ (\mu \ \text{terms}),
\]

hence, if \( \tilde{H}^i = \text{Coker}(H^i(\text{pt}) \to H^i) \),

\[
\tilde{H}^i(M_t, \mathbb{Z}) = \begin{cases} 
\mathbb{Z}^\mu & \text{if } i = n \\
0 & \text{if } i \neq n.
\end{cases}
\]

- The Milnor number \( \mu = \mu(f) \) is given by

\[
\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_0, \cdots, z_n\}/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n).
\]
Letting $t$ turn once around zero clockwise in $D$ gives an automorphism of $H^n(M_t, \mathbb{Z})$, the monodromy automorphism

$$T \in \text{Aut}(H^n(M_t, \mathbb{Z})).$$

Milnor conjectured:

- The eigenvalues of $T$ are roots of unity (i.e., $T$ is quasi-unipotent).

Grothendieck proved it, using Hironaka’s resolution of singularities and his theory of $R\Psi$ and $R\Phi$. 
3. Grothendieck and Deligne

Given a 1-parameter family \((X_t)_{t \in S}\) of (algebraic, or analytic varieties), and a point \(s \in S\), Grothendieck (1967) constructed in SGA 7 a complex of sheaves on \(X_s\), called complex of vanishing cycles, measuring the difference between \(H^*(X_s)\) and \(H^*(X_t)\) for \(t\) “close" to \(s\) (special fiber \(X_s\) vs general fiber \(X_t\)), and a closely related one, called nowadays complex of nearby cycles.

Set-up : complex analytic, or étale.

Will discuss only the étale one.
Étale set-up

$S = (S, s, \eta)$, a strictly local trait

$\eta$: the generic point

$\bar{\eta}$: a separable closure of $\eta$.

For $f : X \to S$, get cartesian squares

$$
\begin{array}{ccc}
X_s & \xrightarrow{i} & X & \xleftarrow{j} & X_{\bar{\eta}} \\
\downarrow & & \downarrow f & & \downarrow \\
S & \xrightarrow{s} & S & \xleftarrow{\eta}
\end{array}
$$

Work with coefficients ring $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ ($\ell$ prime, invertible on $S$) (or $\mathbb{Z}_{\ell}$, $\mathbb{Q}_{\ell}$, $\overline{\mathbb{Q}}_{\ell}$, $\ell$ prime, invertible on $S$), write $D(\cdot)$ for $D(\cdot, \Lambda)$.

For $K \in D^+(X_{\bar{\eta}})$, the complex of nearby cycles is:

$$
R\Psi_f(K) := i^*Rj_*(K|X_{\bar{\eta}}) \in D^+(X_s).
$$

Comes equipped with an action of the inertia group $I = \text{Gal}(\bar{\eta}/\eta)$ (complex of sheaves of $I$-modules on $X_s$).
For $K \in D^+(X)$, get an ($I$-equivariant) exact triangle

$$K|_{X_s} \to R\Psi_f(K|_{X_\eta}) \to R\Phi_f(K) \to,$$

where $R\Phi_f(K)$ is called the complex of vanishing cycles.

A generalization

$S = (S, s, \eta)$ henselian trait, not necessarily strictly local. Take strict localization of $S$ at a separable closure $\tilde{s}$ of $s$:

$$\tilde{S} = (\tilde{S}, \tilde{s}, \tilde{\eta}) \to (S, s, \eta).$$

For $f : X \to S$, base changed $\tilde{f} : \tilde{X} \to \tilde{S}$, and $K \in D^+(X_\eta)$ (resp. $K \in D^+(X)$), define

$$R\Psi_{\tilde{f}}K \ (\text{resp. } R\Phi_{\tilde{f}}K) \in D^+(X_{\tilde{s}})$$

as $R\Psi_{\tilde{f}}(K|_{\tilde{X}_{\tilde{\eta}}}) \ (\text{resp. } R\Phi_{\tilde{f}}(K|_{\tilde{X}}))$. Get action of full Galois group $\text{Gal}(\tilde{\eta}/\eta) \ (\tilde{\eta} \to \tilde{\eta})$, not just of inertia $I = \text{Gal}(\tilde{\eta}/\eta) \subset \text{Gal}(\tilde{\eta}/\eta)$.
General properties

- **Functoriality** Consider a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{f} & & \downarrow{g} \\
S & \xleftarrow{\text{g}} & \\
\end{array}
\]

If \( h \) is smooth, the natural map

\[ h^* R\Psi_Y \rightarrow R\Psi_X h^* \]

is an isomorphism. In particular, if \( f \) is smooth, \( R\Phi_f(\Lambda) = 0 \).

If \( h \) is proper, the natural map

\[ Rh_* R\Psi_X \rightarrow R\Psi_Y Rh_* \]

is an isomorphism. In particular (taking \( Y = S \)), if \( f \) is proper, for \( K \in D^+(X_\eta) \), we have a canonical isomorphism

(compatible with the Galois actions)

\[ R\Gamma(X_\tilde{s}, R\Psi_X K) \cong R\Gamma(X_{\eta}, K). \]
For $X/S$ proper, the triangle $K|_{X_\tilde{S}} \to R\Psi_f(K|_{X_\eta}) \to R\Phi_f(K) \to$ gives an exact sequence

$$\cdots \to H^{i-1}(X_\tilde{S}, R\Phi_X(K)) \to H^i(X_\tilde{S}, K) \xrightarrow{sp} H^i(X_{\eta}, K)$$

$$\to H^i(X_\tilde{S}, R\Phi_X(K)) \to \cdots,$$

where $sp$ is the specialization map:

$$sp : H^i(X_\tilde{S}, K) \simeq H^i(X_\tilde{S}, K) \to H^i(X_{\eta}, K).$$

When $K = \Lambda$, $R\Phi_X(\Lambda)$ is concentrated on the points $x \in X_\tilde{S}$ where $X/S$ is not smooth.
• **Finiteness** (Deligne, 1974) Nearby cycles are constructible: $R\psi_X$ induces

$$R\psi_X : D_c^b(X_\eta) \to D_c^b(X_{\tilde{s}}).$$

• **Perversity** (Gabber, 1981) $R\psi$ commutes with Grothendieck-Verdier duality:

$$R\psi(D_{X_\eta}K) \sim D_{X_{\tilde{s}}} R\psi K,$$

induces $\text{Per}(X_\eta) \to \text{Per}(X_{\tilde{s}}).$
In the **analytic setup**, there are analogous definitions and properties, and a comparison theorem (Deligne, 1968) between the étale $R\Psi$ and the analytic $R\Psi$, similar to Artin-Grothendieck’s comparison theorem Betti vs étale.

Over $\mathbb{C}$, nearby cycles have been extensively studied in connection with **Hodge theory** (Steenbrink et al.), and the **theory of $\mathcal{D}$-modules** (M. Saito et al.).
Let $X/S$ be as above, with $S$ strictly local, and $x \to X_s$ be a geometric point.

For $K \in D^+(X)$, by general nonsense on étale cohomology, the stalk of $R\Psi(K)$ ($:= R\Psi_X K$) at $x$ is given by

$$(R\Psi K)_x = R\Gamma((X_{(x)})_{\overline{\eta}}, K).$$

Here $X_{(x)}$ is the strict localization of $X$ at $x$ (a kind of Milnor ball), and $(X_{(x)})_{\overline{\eta}}$ its geometric generic fiber (a kind of Milnor fiber).
But \((R^q\Psi K)_x\) is difficult to calculate!

Known for \(K = \Lambda\) (constant sheaf), when \(X\) has **semistable reduction** at \(x\), i.e., étale locally at \(x\),

\[
X \xrightarrow{\sim} S[t_1, \cdots, t_n]/(t_1 \cdots t_r - \pi)
\]

(\(\pi\) a uniformizing parameter in \(S\)). Then:

- \[(R^1\Psi \Lambda)_x = \operatorname{Ker}(\mathbb{Z}^r \xrightarrow{\text{sum}} \mathbb{Z}) \otimes \Lambda(-1)\]
- \[(R^q\Psi \Lambda)_x = \Lambda^q(R^1\Psi \Lambda)_x\]

(\(\Lambda^q = q\)-th exterior power, \(\Lambda(m) = m\)-th Tate twist).

- The inertia group \(I\) acts trivially on \((R^q\Psi \Lambda)_x\).
For \( X = S[t_1, \cdots, t_r]/(t_1 \cdots t_r - \pi) \), topological model of \((X_{(x)})_{\eta} : \text{fiber of} \)

\[
(S^1)^r \to S^1, (z_1, \cdots, z_r) \mapsto z_1 \cdots z_r.
\]

Proof combines:

- Grothendieck’s calculation of tame nearby cycles \((R^q \psi \Lambda)_t := (R^q \psi \Lambda)^P (P \subset I \text{ the wild inertia}), \) modulo validity of Grothendieck’s absolute purity conjecture for components of \((X_{(x)})_s\)

- validity OK and \((R^q \psi \Lambda)_t = R^q \psi \Lambda \) (Rapoport-Zink, 1982).
Recall Grothendieck’s absolute purity conjecture:

For regular divisor $D \subset X$, $X$ regular, $\Lambda = \mathbb{Z}/\ell^\nu \mathbb{Z}$, \ldots as above, $\ell$ invertible on $X$, $\mathcal{H}^q_D(X, \Lambda) = \begin{cases} \Lambda_D(-1) & \text{if } q = 2 \\ 0 & \text{if } q \neq 2. \end{cases}$
Modulo absolute purity conjecture (OK if $S/Q$, and now in general by Gabber (1994)), Grothendieck calculated tame nearby cycles for $X$ étale locally of the form $S[t_1, \cdots, t_n]/(ut_1^{n_1} \cdots t_r^{n_r} - \pi)$ ($u$ a unit):

$$R^q\psi\Lambda_{t,x} = \mathbb{Z}[\mu_{\ell^m}] \otimes \Lambda^q(\text{Ker}(\mathbb{Z}^r \xrightarrow{\sum n_ix_i} \mathbb{Z})) \otimes \Lambda(-q)$$

where $\gcd(n_1, \cdots, n_r) = \ell^md$, $(\ell, d) = 1$.

Here $I$ acts on $\mathbb{Z}[\mu_{\ell^m}]$ by permutation through its tame quotient $\mathbb{Z}_{\ell}(1)$, in particular, acts on $R^q\psi\Lambda_{t,x}$ through a finite quotient, hence quasi-unipotently on $R\psi\Lambda_{t,x}$.

Combined with Hironaka’s resolution of singularities, and functoriality of $R\psi$ for proper maps, calculation yields a proof of Milnor’s conjecture on the monodromy of isolated singularities.
4. Grothendieck’s local monodromy theorems

Grothendieck’s arithmetic local monodromy theorem is the following:

**Theorem**

$S = (S, s, η)$ henselian, $k = k(s)$, $\ell$ prime different from $p = \text{char}(k)$. Assume that no finite extension of $k$ contains all roots of unity of order a power of $\ell$ (e. g., $k$ finite). Let

$$\rho : \text{Gal}(\bar{η}/η) \rightarrow \text{GL}(V)$$

be a continuous representation into a finite dimensional $\mathbb{Q}_\ell$-vector space $V$. Then, there exists an open subgroup $I_1 \subset I$, such that, for all $g \in I_1$, $\rho(g)$ is unipotent.

**Proof.**

Exercise ! (Use strong action of $\text{Gal}(\bar{k}/k)$ on tame inertia $I_t$: $g\sigma g^{-1} = \sigma^{\chi(g)}$, $\chi = \text{cyclotomic character}.)$
A corollary is that there exists a unique nilpotent morphism

\[ N : V(1) \to V, \]

called the monodromy operator, such that, for all \( \sigma \in I_1 \) and \( x \in V \),

\[ \sigma x = \exp(N(t_\ell(\sigma)x)), \]

where \( t_\ell : I \to \mathbb{Z}_\ell(1) \) is the \( \ell \)-component of the tame character.

The operator \( N \) is \( \text{Gal}(\bar{\eta}/\eta) \)-equivariant. In particular, for \( k = \mathbb{F}_q \), if \( F \in \text{Gal}(\bar{\eta}/\eta) \) is a lifting of the geometric Frobenius \((a \to a^{1/q})\), then

\[ NF = qFN. \]

Led to the Weil-Deligne representation.
The geometric local monodromy theorem is the following result, due to Grothendieck in a weaker form, later improved by various authors:

**Theorem**

Let $S$ be an (arbitrary) henselian trait. Let $X_\eta$ be separated and of finite type over $\eta$. Then, there exists an open subgroup $I_1 \subset I$, independent of $\ell$, such that for all $i \in \mathbb{Z}$ and all $g \in I_1$,

$$(g - 1)^{i+1} = 0$$

on $H^i(X_\eta, \Lambda)$ (resp. $H^i_c(X_\eta, \Lambda)$).

**History**

- Existence of $I_1$ (**a priori** $\ell$-dependent) for $H^i_c$ with $i + 1$ replaced by uncontrolled bound, proved by Grothendieck, as a consequence of the arithmetic local monodromy theorem (reduction to $k$ small). Method generalized to $H^i$ once finiteness of $H^i$ was proved (Deligne, 1974).
- Existence of $l_1$ (\textit{a priori} $\ell$-dependent), with bound $i + 1$, proved by Grothendieck for $X_\eta/\eta$ proper and smooth, modulo validity of absolute purity and resolution of singularities, as a consequence of local calculation of $R^q\Psi Z_\ell$ in the (quasi-) semistable case. Unconditional for $i \leq 1$, or $p = 0$.

- Existence of $l_1$, independent of $\ell$, but with $i + 1$ replaced by uncontrolled bound, proved by Deligne (1996), using Rapoport-Zink’s calculation of $R\Psi Z_\ell$ in the semistable case, and de Jong’s alterations. Final result obtained by refinement of this method (Gabber - I., 2014).
Why care for exponent $i + 1$?

Grothendieck’s motivation: for $i = 1$, exponent 2 is a crucial ingredient in his proof of the semistable reduction theorem for abelian varieties:

**Theorem**

*With* $S$ *as before, let* $A_\eta$ *be an abelian variety over* $\eta$. *There exists a finite extension* $\eta_1$ *of* $\eta$ *such that* $A_{\eta_1}$ *acquires semistable reduction over the normalization* $(S_1, s_1, \eta_1)$ *of* $S$ *in* $\eta_1$, *i.e., the connected component* $A^0_{s_1}$ *of the special fiber of the Néron model of* $A_{\eta_1}$ *is an extension of an abelian variety by a torus:*

$$0 \to \text{(torus)} \to A^0_{s_1} \to \text{(abelian variety)} \to 0.$$
Deligne-Mumford (1969) deduced from it the **semistable reduction theorem** for curves:

**Corollary**

*Let $X_{\eta}$ be a proper, smooth curve over $\eta$. There exists a finite extension $\eta_1$ of $\eta$ such that $X_{\eta_1}$ has semistable reduction over the normalization $S_1$ of $S$ in $\eta_1$, i.e., is the generic fiber of a proper, flat $X_1/S_1$, with $X_1$ regular, and special fiber $(X_1)_{s_1}$ a reduced curve having simple nodes.*
• Corollary is the key tool in Deligne-Mumford’s proof of the irreducibility of the coarse moduli space $M_g$ (over any algebraically closed field $k$).

• Proofs of corollary independent of theorem found later (Artin-Winters, 1971; T. Saito, 1987).

• For $\text{char}(k) = 0$, a generalization of corollary to arbitrary dimension proved by Mumford et al. (1973).

5. The Deligne-Milnor conjecture

At the opposite of semistable reduction, we have isolated singularities.

Let $S = (S, s, \eta)$ be a strictly local trait, with $k = k(s)$ algebraically closed. Assume $X$ regular, flat, finite type over $S$, relative dimension $n$, smooth outside closed point $x \in X_s$. Then $R\Phi\Lambda$ is concentrated at $x$, and in cohomological degree $n$:

$$(R\Phi^q\Lambda)_x = \begin{cases} 
0 & \text{if } q \neq n \\
\Lambda^r & \text{if } q = n
\end{cases}$$

The coherent module $\mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{O}_X)$ is concentrated at $x$, its length

$$\mu := \lg(\mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{O}_X))$$

generalizes the classical Milnor number.
The action of $I$ on $R^n\Phi\Lambda$ has a Swan conductor $Sw(R^n\Phi\Lambda) \in \mathbb{Z}$, measuring wild ramification ($= 0$ if $S$ of char. 0).

Deligne conjectured (SGA 7 XVI, 1972):

$$\mu = r + Sw(R^n\Phi\Lambda).$$

Generalizes Milnor formula over $\mathbb{C}$.

Conjecture proved:

- if $X/S$ finite, or $x$ is an ordinary quadratic singularity, or $S$ is of equal characteristic (Deligne, loc. cit.)
- if $n = 1$ (Bloch, 1987 + Orgogozo, 2003)

6. The Picard-Lefschetz formula

Let $X/S$ as before, with relative dimension $n$. Assume $x$ is an ordinary quadratic singularity of $X/S$, i.e., étale locally at $x$, $X/S$ is of the form ($\pi$ a uniformizing parameter):

$$
\sum_{1 \leq i \leq m+1} x_i x_{i+m+1} = \pi
$$

$(n = 2m + 1)$,

$$
\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 = \pi
$$

$(n = 2m, \; p > 2)$,

$$
\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + ax_{2m+1} + \pi = 0
$$

with $a^2 - 4\pi \neq 0 \; (n = 2m, \; p = 2)$. 
Then

$$(R^n \Phi \Lambda)_x \sim \Lambda,$$

with action of inertia $I$ trivial is $n$ odd, through a character of order 2 if $n$ even, tame if $p > 2$.

Assume now $X/S$ proper, flat, of relative dimension $n > 0$, smooth outside $\Sigma \subset X_s$ finite, and each $x \in \Sigma$ is an ordinary quadratic singularity.
Then the monodromy of $H^*(X_{\overline{\eta}})$ is described as follows (Deligne, SGA 7 XV, 1972):

- For $i \neq n, n + 1$, $H^i(X_s) \rightarrow H^i(X_{\overline{\eta}})$.
- For each $x \in \Sigma$, there exists $\delta_x \in H^n(X_{\overline{\eta}})(m)$ ($n = 2m$ or $2m + 1$), well defined up to sign, called the vanishing cycle at $x$, and the sequence

$$0 \rightarrow H^n(X_s) \xrightarrow{\text{sp}} H^n(X_{\overline{\eta}}) \xrightarrow{(-,\delta_x)} \sum_{x \in \Sigma} \Lambda(m - n) \rightarrow H^{n+1}(X_s)$$

$$\xrightarrow{\text{sp}} H^{n+1}(X_{\overline{\eta}}) \rightarrow 0.$$ is exact. One has $(\delta_x, \delta_y) = 0$ for $x \neq y$, $(\delta_x, \delta_x) = 0$ for $n$ odd, and $(\delta_x, \delta_x) = (-1)^m \cdot 2$ for $n = 2m$. Here $(a, b) = \text{Tr}(ab)$, where $\text{Tr} : H^{2n} \rightarrow \Lambda(-n)$. 
The inertia \( I \) acts trivially on \( H^i(X_\eta) \) for \( i \neq n \), and on \( H^n(X_\eta) \) through orthogonal (resp. symplectic) transformations for \( n = 2m \) (resp. \( n = 2m + 1 \)), given by the Picard-Lefschetz formula:

For \( \sigma \in I \), \( a \in H^n(X_\eta) \),

\[
\sigma a - a = \begin{cases} 
(−1)^m \sum_{x \in \Sigma} \frac{\varepsilon_x(\sigma)}{2} (a, \delta_x) \delta_x & \text{if } n = 2m \\
(−1)^{m+1} \sum_{x \in \Sigma} t_\ell(\sigma) (a, \delta_x) \delta_x & \text{if } n = 2m + 1.
\end{cases}
\]

Here \( t_\ell : I \to \mathbb{Z}_\ell(1) \) is the tame character, and \( \varepsilon_x : I \to \pm 1 \) is the unique character of order 2 if \( p > 2 \) and that defined by the quadratic extension \( t^2 + at + \pi = 0 \) for \( X \) locally at \( x \) of the form

\[
\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + ax_{2m+1} + \pi = 0.
\]
Difficult case in the proof: \( n \) odd, \( n = 2m + 1 \). Use factorization:

\[
\begin{array}{c}
H^n(X_{\overline{\eta}}) \rightarrow \bigoplus_{x \in \Sigma} (R^n\Phi\Lambda)_x, \\
\sigma - 1 \downarrow \downarrow \Var(\sigma)_x \\
H^n(X_{\overline{\eta}}) \leftarrow \bigoplus_{x \in \Sigma} H^n_x(X_s, R\Psi\Lambda)
\end{array}
\]

where:

- Top row is part of specialization sequence
- Bottom row = composition of \( H^n_x \rightarrow H^n \) and \( H^n(X_s, R\Psi\Lambda) = H^n(X_{\overline{\eta}}) \).
- \((R^n\Phi\Lambda(m + 1))_x\) and \( H^n_x(X_s, R\Psi\Lambda)(m) \) are isomorphic to \( \Lambda \), with respective generators \( \delta'_x, \delta_x \) defined up to sign, with \( \langle \delta'_x, \delta_x \rangle = 1 \), for a perfect pairing with values in \( \text{Tr} \)

\[
H^{2n}_x(X_s, R\Psi\Lambda(n)) \xrightarrow{\sim} \Lambda. \text{ We have } \delta_x \mapsto \delta_x \in H^n(X_{\overline{\eta}}).
\]
The map $\text{Var}(\sigma)_x$, called variation, is given by the local Picard-Lefschetz formula:

$$\text{Var}(\sigma)_x(\delta'_x) = (-1)^{m+1} t_\ell(\sigma) \delta_x,$$

which is the crux of the matter.

- Original proof (Deligne) required lifting to char. 0 and a transcendental argument.
- Purely algebraic proof given later (I., 2000), as a corollary of Rapoport-Zink’s theory of nearby cycles in the semistable case.
Over $\mathbb{C}$, Milnor fiber $M_t$ of $f : (x_1, \cdots, x_{2m+2}) \mapsto \sum x_i^2$ is fiber bundle in unit balls of tangent bundle to sphere $S^n = \{ x \in \mathbb{R}^{n+1} | \sum x_i^2 = 1 \}$.

- $R^n \Phi_x$ corresponds to $\tilde{H}^n(M_t)$,
- $H^n_x(X_s, R\Psi)$ corresponds to $H^n_c(M_t - \partial M_t)$,
- $\delta_x$ dual to $\delta^\vee_x \in H_n(M_t, \partial M_t)$ given by one fiber of $M_t$ over $S^n$,
- $\delta'_x$ dual to $(\delta'_x)^\vee \in \tilde{H}_n(M_t)$ given by $S^n \subset M_t$.

Next slide: picture, for $n = 1$ ($m = 0$) of the dual variation map ($T$ the positive generator of $\pi_1(S^1)$)

$$\text{Var}(T)^\vee : H_1(M_t, \partial M_t) \rightarrow \tilde{H}_1(M_t),$$

$$\delta^\vee_x \mapsto -(\delta'_x)^\vee.$$
Back to the Legendre family:

\[ X_t : y^2 = x(x - 1)(x - t). \]

Locally at \( x = y = t = 0 \), \( X/S \) is \( x_1^2 + x_2^2 = t^2 \), instead of \( x_1^2 + x_2^2 = t \), hence variation is doubled, and get

\[ T(\delta) = \delta, \quad T(\gamma) = \gamma \pm 2\delta \]
Arithmetic applications

- Grothendieck used the PL formula in his theory of the monodromy pairing for abelian varieties having semistable reduction (SGA 7 IX), with a formula for calculating the group of connected components of the special fiber of the Néron model. Variants, generalizations, and arithmetic applications by Raynaud, Deligne-Rapoport, Mazur, Ribet.

- Most importantly, the PL formula was the key to the cohomological study (by Deligne and Katz, SGA 7 XVIII) of Lefschetz pencils, which led to the first proof, by Deligne, of the Weil conjecture (Weil I).
Variants and generalizations

• Tame variation

Recall the case of isolated singularities: $X$ regular, flat, finite type over $S$, relative dimension $n$, smooth outside closed point $x \in X_s$. Then $R\Phi\Lambda$ is concentrated at $x$, and in cohomological degree $n$:

$$(R\Phi^q\Lambda)_x = \begin{cases} 
0 & \text{if } q \neq n \\
\Lambda^r & \text{if } q = n
\end{cases}$$

Moreover,

$$H^n_{\{x\}}(X_s, R\Psi\Lambda) = \Lambda^r,$$

with a perfect intersection pairing

$$R^n\Phi(\Lambda)_x \otimes H^n_{\{x\}}(X_s, R\Psi\Lambda) \to H^{2n}_{\{x\}}(X_s, R\Psi\Lambda) = \Lambda(-n).$$
Finally, if $l$ acts tamely on $R\psi \Lambda$, i.e., through its quotient $\mathbb{Z}_\ell(1)$, and if $\sigma$ is a topological generator of it, then $\sigma - 1$ induces an isomorphism

$$\text{Var}(\sigma) : R^n\Phi(\Lambda)_x \xrightarrow{\sim} H^n_{\{x\}}(X_s, R\psi \Lambda),$$

called the variation at $x$ (I., 2003), a (weak) generalization of the local Picard-Lefschetz formula. The analogue over $\mathbb{C}$ had been known since the 1970’s (Brieskorn).
• **Thom-Sebastiani theorems**
  The Picard-Lefschetz theory describes vanishing cycles, monodromy and variation at the isolated critical point \( \{0\} \) of the function
  \[ x_1^2 + \cdots + x_m^2. \]
  The classical Thom-Sebastiani theorem \((/\mathbb{C})\) describes the same invariants at the isolated critical point \( \{0\} \) of a function of the form
  \[ f(x_1, \cdots, x_m) = f_1(x_1) + \cdots + f_m(x_m), \]
  where the \( x_i \) are independent packs of \( n_i + 1 \) variables, and \( f_i : \mathbb{C}^{n_i+1} \rightarrow \mathbb{C} \) has an isolated critical point at \( \{0\} \).
If \( n = \sum n_i \) (= rel. dim. of \( f \)), then (for coefficients \( \mathbf{Z} \))

\[
R^n \Phi_f = \bigotimes_{1 \leq i \leq m} R^{n_i} \Phi_{f_i},
\]

with monodromy

\[
T = \bigotimes_{1 \leq i \leq m} T_i,
\]

and variation

\[
Var = \bigotimes_{1 \leq i \leq m} Var_i.
\]

Algebraic analogues?
(over an alg. closed field \( k \), in the étale set-up)
Deligne’s observation: analogue wrong in general, tensor product must be replaced by

local convolution product $\ast$

of Deligne-Laumon.

Quite recently, T. Saito, in conjunction with Beilinson’s construction of a singular support
\[ SS(\mathcal{F}) \subset T^*X \]
for a constructible sheaf \( \mathcal{F} \) on a smooth \( X/k \) (an equidimensional conic closed subset of \( T^*X \), of dimension \( = \dim(X) \)), defined a characteristic cycle supported on \( SS(\mathcal{F}) \), with coefficients in \( \mathbb{Z}[1/p] \) (actually, in \( \mathbb{Z} \) (Beilinson)):
\[ CC(\mathcal{F}) \in \mathbb{Z}_{\dim(X)}(T^*X), \]
proved a generalization of the Deligne-Milnor formula (equal characteristic case), and as a corollary, a global index formula for the Euler number of \( \mathcal{F} \).
The **global index formula** reads:

For $X/k$ proper and smooth, $k$ alg. closed, $\Lambda = \mathbb{Q}_\ell$,

$$\chi(X, \mathcal{F}) = (CC(\mathcal{F}), T_X^* X).$$

Here $\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim H^i(X, \mathcal{F})$, $T_X^*(X) = 0$-section of $T^* X$.

This work was inspired by Kashiwara-Schapira’s analogous theory over $\mathbb{C}$, and various conjectures of Deligne.

**Ingredients**

- Radon and Legendre transforms (Brylinski), geometric theory of Lefschetz pencils (Katz, SGA 7 XVII)
- Ramification theory for imperfect residue fields (Abbes, T. Saito)
- Deligne’s theory of vanishing cycles over general bases (Deligne, Gabber, Orgogozo) (also used in generalized Thom-Sebastiani theorems).
Thank you!