

Grothendieck and vanishing cycles

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Grothendieck's first mention of vanishing cycles is in a letter to Serre, dated Oct. 30, 1964 ([3], p. 214). He considers a regular, proper, and flat curve X over a strictly local trait $S = (S, s, \eta)$, whose generic fiber is smooth and reduced special fiber is a divisor with normal crossings. He analyses the difference between the (étale) cohomology of the special fiber $H^*(X_s)$ and that of the generic geometric fiber $H^*(X_{\bar{\eta}})$, the coefficients ring being \mathbf{Z}_ℓ , ℓ a prime number invertible on S . A little more precisely, assuming that the action of the inertia group I on $H^*(X_{\bar{\eta}})$ is tame, he shows that the defect of the *specialization map* $H^*(X_s) \rightarrow H^*(X_{\bar{\eta}})$ ² to be an isomorphism is controlled by certain groups (the vanishing cycles groups), that he estimates. He deduces that there exists an open subgroup I_1 of I such that, for all $g \in I_1$, $(g - 1)^2$ acts trivially on $H^*(X_{\bar{\eta}})$, a key step in his proof of the semistable reduction theorem for abelian varieties.

We will recall this proof in §3, after a quick review, in §§1, 2, of the definition and basic properties of nearby and vanishing cycles, and Grothendieck's geometric local monodromy theorem. In §4 we discuss Grothendieck's monodromy pairing for abelian varieties over local fields, a complement to his semistable reduction theorem. In §5 we say a few words about the developments that arose from Grothendieck's work and questions. In §6 we give a quick update on some of the topics of §§1 – 4.

1 Nearby and vanishing cycles

1.1

In [2] Grothendieck introduced and studied the functors $R\Psi$ and $R\Phi$, both in the context of Betti cohomology and that of étale cohomology. He called them functors of vanishing cycles, but in the 1970's it became customary to call the former one the functor of *nearby cycles*, the name *vanishing cycles* being reserved to the latter one. Let me recall their definition in the étale setup, as described by Deligne in ([2], XIII).

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²This is the composition of the inverse $H^*(X_s) \xrightarrow{\sim} H^*(X)$ of the proper base change isomorphism and the restriction map $H^*(X) \rightarrow H^*(X_{\bar{\eta}})$.

Let $S = (S, s, \eta)$ be a henselian trait, with closed point s , and generic point η . Let \bar{s} be a geometric point of S over s , and $\bar{\eta}$ a separable closure of the generic point $\tilde{\eta}$ of the strict localization $\tilde{S} = S_{(\bar{s})}$ of S at \bar{s} . We have a commutative diagram with cartesian squares

$$(1.1.1) \quad \begin{array}{ccccc} & & & & \bar{\eta} \\ & & & \nearrow \bar{j} & \downarrow \\ & & \tilde{S} & \xleftarrow{\tilde{j}} & \tilde{\eta} \\ \tilde{s} \xrightarrow{\tilde{i}} & & \downarrow & & \downarrow \\ s \longrightarrow & S & \longleftarrow & \eta \end{array}$$

where \tilde{s} , the closed point of \tilde{S} , is a separable closure of s in \bar{s} . For a morphism $f : X \rightarrow S$, we get morphisms deduced by base change

$$X_{\tilde{s}} \xrightarrow{\tilde{i}} X_{\tilde{S}} \xleftarrow{\tilde{j}} X_{\tilde{\eta}}.$$

Let us choose a ring of coefficients $\Lambda = \mathbf{Z}/\ell^\nu \mathbf{Z}$, $\nu \geq 1$, with ℓ a prime number invertible on S . Other choices are possible, e.g., \mathbf{Z}_ℓ , \mathbf{Q}_ℓ , or $\overline{\mathbf{Q}}_\ell$ (when one works with schemes of finite type over S ³). We will write $D(-)$ for $D(-, \Lambda)$. For $K \in D^+(X_\eta)$, the nearby cycles complex of K is

$$(1.1.2) \quad R\Psi_f(K) := \tilde{i}^* R\bar{j}_*(K|X_{\tilde{\eta}}).$$

This is an object of $D^+(X_{\tilde{s}})$, more precisely an object of the derived category of sheaves of Λ -modules on $X_{\tilde{s}}$ equipped with a continuous action of the Galois group $\text{Gal}(\bar{\eta}/\eta)$, compatible with that on $X_{\tilde{s}}$. For $K \in D^+(X)$, the complex of vanishing cycles $R\Phi_f(K)$ is the cone of the natural morphism $\tilde{i}^* K \rightarrow R\Psi_f(K|X_\eta)$, i.e., we have a distinguished triangle in the category just mentioned,

$$(1.1.3) \quad \tilde{i}^* K \rightarrow R\Psi_f(K|X_\eta) \rightarrow R\Phi_f(K) \rightarrow .$$

If \bar{x} is a geometric point of X over \bar{s} , the stalk of $R\Psi_f(K)$ at \bar{x} is

$$(1.1.4) \quad R\Psi_f(K)_{\bar{x}} = R\Gamma((X_{(\bar{x})})_{\bar{\eta}}, K).$$

The scheme $(X_{(\bar{x})})_{\bar{\eta}}$, geometric generic fiber of the strict localization of X at \bar{x} , plays the role of a *Milnor fiber* of f at \bar{x} . We sometimes write $R\Psi_X$ (resp. $R\Phi_X$) for $R\Psi_f$ (resp. $R\Phi_f$), and drop the subscript X when no confusion can arise.

³This is to ensure that D_c^b is stable under the six operations.

1.2

The main *functoriality properties* of these functors are the following. Consider a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \downarrow & & \swarrow g \\ S & & \end{array}$$

(a) If h is *smooth*, the natural map

$$(1.2.1) \quad h^* R\Psi_Y \rightarrow R\Psi_X h^*$$

is an isomorphism. In particular (taking $Y = S$), if f is smooth, then $R\Phi_X(\Lambda) = 0$. So, in general, $R\Phi_X(\Lambda)$ is concentrated on the non-smoothness locus of X/S .

(b) If h is *proper*, the natural map⁴

$$(1.2.2) \quad Rh_* R\Psi_X \rightarrow R\Psi_Y Rh_*$$

is an isomorphism. In particular (taking $Y = S$), if f is proper, for $K \in D^+(X_\eta)$, we have a canonical isomorphism (compatible with the Galois actions)

$$(1.2.3) \quad R\Gamma(X_{\bar{s}}, R\Psi_X K) \xrightarrow{\sim} R\Gamma(X_{\bar{\eta}}, K).$$

The triangle (1.1.3) thus gives rise to a long exact sequence

$$(1.2.4) \quad \cdots \rightarrow H^{i-1}(X_{\bar{s}}, R\Phi_X(K)) \rightarrow H^i(X_{\bar{s}}, K) \xrightarrow{\text{sp}} H^i(X_{\bar{\eta}}, K) \rightarrow H^i(X_{\bar{s}}, R\Phi_X(K)) \rightarrow \cdots,$$

where sp is the specialization map, generalizing that considered in the introduction.

It was later proved by Deligne ([20], Th. finitude) that, for X of finite type over S and $K \in D_c^b(X_\eta)$, $R\Psi_f K$ is in $D_c^b(X_{\bar{s}})$ (where $D_c^b(-)$ means the full subcategory of $D(-)$ consisting of complexes with bounded and constructible cohomology).

2 The geometric local monodromy theorem

In SGA 7 Grothendieck proved the following theorem:

⁴In (a) and (b), there are obvious abuses of notation for h^* and h_* .

Theorem 2.1. *Let $S = (S, s, \eta)$ be as in 1.1, and X_η be separated and of finite type over η . Let $I = \text{Gal}(\bar{\eta}/\tilde{\eta}) \subset \text{Gal}(\bar{\eta}/\eta)$ be the inertia group, and Λ be as in 1.1. Then there exists an open subgroup $I_1 \subset I$ such that, for all $g \in I_1$ and all $i \in \mathbf{Z}$, g acts unipotently on $H_c^i(X_{\bar{\eta}}, \Lambda)$.*

The main ingredient in his proof was his *arithmetic local monodromy theorem* ([60], Appendix):

Theorem 2.2. *Assume that no finite extension of $k(s)$ contains all roots of unity of order a power of ℓ . Let $\rho : \text{Gal}(\bar{\eta}/\eta) \rightarrow \text{GL}(V)$ be a continuous representation into a finite dimensional \mathbf{Q}_ℓ -vector space V . Then there exists an open subgroup I_1 of I such that, for all $g \in I_1$, $\rho(g)$ is unipotent.*

The proof of 2.2 is an elegant, elementary exercise. Once we have reduced to the case where the image of ρ is contained in $1 + \ell^2 M_n(\mathbf{Z}_\ell)$, the whole inertia group I acts unipotently. Indeed, I acts through its ℓ -primary tame quotient $t_\ell : I \rightarrow \mathbf{Z}_\ell(1)$, and Grothendieck exploits the strong action of the arithmetic Galois group $G_k = \text{Gal}(\tilde{s}/s)$ on $\mathbf{Z}_\ell(1)$ by conjugation, given by $g\sigma g^{-1} = \sigma^{\chi(g)}$, where $\chi : G_k \rightarrow \mathbf{Z}_\ell^*$ is the cyclotomic character (ℓ^2 ensures that exponential and logarithm are defined and inverse to each other).

The deduction of 2.1 from 2.2 is more difficult. It uses Néron's desingularization, and a spreading out argument to reduce to the case where the residue field is radicial over a finite type extension of \mathbf{F}_p , see ([2], I 1.3). Once the finite generation of the groups $H^i(X_{\bar{\eta}}, \Lambda)$ (and generic constructibility of direct images) was known ([20], Th. finitude), the same reduction worked — hence the conclusion of 2.1 held — for $H^i(X_{\bar{\eta}}, \Lambda)$ as well.

Grothendieck gave a conditional, alternate proof of 2.1, based on the formalism of §1. It assumed the validity (in certain degrees and dimensions) of resolution of singularities and of his absolute purity conjecture ([1] I)⁵. This was the case for s of characteristic zero, or $\dim(X_\eta) \leq 1$, or $i \leq 1$. The advantage of the method is that it gave bounds on the *exponent of unipotence* $n(g)$ of $g \in I_1$ acting on H^i , i.e., the smallest integer $n \geq 0$ such that $(g - 1)^{n+1} = 0$. For example, for $i = 1$, one gets $n(g) \leq 1$. Resolution is still an open problem, but the absolute purity conjecture was proved by Rapoport-Zink in the situation arising from a semistable reduction [49]⁶, and thanks to de Jong's alteration theorems [40], it was possible to make Grothendieck's argument work in general. As Deligne observed, a by-product of this argument was that, at least for $\Lambda = \mathbf{Q}_\ell$, the open subgroup I_1

⁵One form of this conjecture is that if $j : U = X - D \hookrightarrow X$ is the inclusion of the complement of a regular divisor D in a regular scheme X , and if ℓ is invertible on X , then $R^q j_* \Lambda$ is Λ for $q = 0$, $\Lambda_D(-1)$ for $q = 1$, and 0 for $q > 1$.

⁶It was later proved by Gabber in general [22], but the semistable reduction case sufficed.

in 2.1 (and its variant for H^i) can be chosen *independent of ℓ* (see Berthelot's Bourbaki exposé [12]). With a little more work, one can also get a general bound for $n(g)$, namely, one has the following result:

Theorem 2.3. *With the assumptions and notation of 2.1, there exists an open subgroup $I_1 \subset I$, independent of ℓ , such that, for all $g \in I_1$ and all $i \in \mathbf{Z}$, $(g - 1)^{i+1} = 0$ on $H_c^i(X_{\bar{\eta}}, \Lambda)$ (resp. $H^i(X_{\bar{\eta}}, \Lambda)$).*

This result is essentially due to Gabber. See §7 for the proof.

Remark 2.4. For smooth, projective, geometrically connected X_η/η , explicit uniform bounds for I_1 in terms of the Betti numbers of X_η and numerical invariants associated with a very ample line bundle on X_η were obtained by Umezaki [66].

Remark 2.5. Suppose X_η/η is proper and equidimensional of dimension d . Let $IH^i(X_{\bar{\eta}}, \Lambda) := H^i(X_{\bar{\eta}}, \underline{IC}[-d])$, where \underline{IC} is the pull-back to $X_{\bar{\eta}}$ of the intersection complex $\underline{IC}_{X_\eta} := j_{!*}(\Lambda_U[d])$ (where $j : U \hookrightarrow X_\eta$ is the inclusion of a dense open subscheme such that $(U_{\bar{\eta}})_{\text{red}}$ is smooth). One can ask whether there exists an open subgroup I_1 of I , independent of ℓ , such that, for all $g \in I_1$ and all $i \in \mathbf{Z}$, $(g - 1)^{i+1} = 0$. The answer is yes for $\Lambda = \mathbf{Q}_\ell$ or $\overline{\mathbf{Q}}_\ell$, as, by de Jong [40], after replacing η by a finite extension, one can find an alteration $h : Z \rightarrow X_\eta$, with Z proper and smooth over η (and purely of dimension d), and, by the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber ([7], 5.3, 5.4)⁷, \underline{IC} is a direct summand of $Rh_*\Lambda_{Z_{\bar{\eta}}}[d]$. I don't know the answer in general.

2.6

The main step in Grothendieck's geometric proof is a calculation of the stalks of the *tame* nearby cycles groups $R^q\Psi_X(\Lambda)_t$, in a situation of quasi-semistable reduction (assuming that absolute purity is available — which is the case today). Let me recall the definition of these groups. In (1.1.1), $k(\tilde{\eta})$ is the maximal unramified extension of $k(\eta)$ contained in $k(\bar{\eta})$. Let $k(\eta_t)$ be the maximal tame extension of $k(\eta)$ contained in $k(\bar{\eta})$, i.e., $k(\eta_t) = \varinjlim k(\tilde{\eta})[\pi^{1/n}]$, where π is a uniformizing parameter of \tilde{S} , and n runs through the integers ≥ 1 prime to the characteristic exponent p of $k(s)$. Then $P := \text{Gal}(\bar{\eta}/\eta_t)$ is the wild inertia subgroup of $I = \text{Gal}(\bar{\eta}/\tilde{\eta})$, and

$$I_t = \text{Gal}(\eta_t/\tilde{\eta}) = \hat{\mathbf{Z}}'(1) := \varprojlim_{(n,p)=1} \mathbf{Z}/n(1)$$

⁷See e. g. ([62], 3.2.2) for the generalization used here.

its tame quotient. Replacing the upper part of (1.1.1) by

$$(2.6.1) \quad \begin{array}{ccc} & & \eta_t \\ & \nearrow^{j_t} & \downarrow \\ \tilde{s} & \xrightarrow{\tilde{i}} \tilde{S} & \xleftarrow{\tilde{j}} \tilde{\eta} \end{array}$$

one defines, for X over η , and $K \in D^+(X)$, the *tame nearby cycles complex*

$$(2.6.2) \quad R\Psi_f(K)_t := \tilde{i}^* Rj_{t*}(K|X_{\eta_t})$$

As P is a pro- p -group, the functor $(-)^P$ (invariants under P) is exact, and one has

$$(2.6.3) \quad R\Psi_f(K)_t \xrightarrow{\sim} R\Psi_f(K)^P.$$

For X/S and $K \in D^+(X)$, one defines the tame vanishing cycles complex $R\Phi_f(K)_t$ similarly to $R\Phi_f(K)$. One has a variant of 1.1.4:

$$(2.6.4) \quad R\Psi_f(K)_{t,\bar{x}} = R\Gamma((X_{(\bar{x})})_{\eta_t}, K),$$

with the Milnor fiber replaced by the tame one $(X_{(\bar{x})})_{\eta_t}$.

2.7

Assume now that X is regular, flat and of finite type over S , the generic fiber X_η is smooth, and the reduced special fiber $(X_s)_{\text{red}}$ is a divisor with normal crossings. Let \bar{x} be a geometric point of X over \bar{s} , let $(D_i)_{1 \leq i \leq r}$ be the branches of $(X_s)_{\text{red}}$ passing through \bar{x} , and let n_i be the multiplicity of D_i , i.e., X is locally defined near \bar{x} by an equation of the form $u \prod_{1 \leq i \leq r} t_i^{n_i} = \pi$, where π is a uniformizing parameter of S , the t_i 's are part of a system of regular parameters at the strict localization of X at \bar{x} , and u is a unit at \bar{x} . Then ([2], I 3.3) the stalks of the groups $(R^q\Psi\Lambda)_t$ at \bar{x} are given by

$$(2.7.1) \quad (R^q\Psi\Lambda)_{t,\bar{x}} = \Lambda[(\mathbf{Z}/d\mathbf{Z})(1)] \otimes_{\mathbf{Z}} \Lambda^q(C(-1)),$$

where

$$C = \text{Ker}((n_1, \dots, n_r) : \mathbf{Z}^r \rightarrow \mathbf{Z}),$$

and $\text{gcd}(n_i) = dp^m$, with $(d, p) = 1$. The inertia group I acts on them via its permutation action on $(\mathbf{Z}/d\mathbf{Z})(1)$ through the composition $I \twoheadrightarrow I_t \twoheadrightarrow (\mathbf{Z}/d\mathbf{Z})(1)$. The reason for this is that, because of absolute purity, on the cohomological level the tame Milnor fiber $(X_{(\bar{x})})_{\eta_t}$ (2.6.4) behaves like the (prime to p) *homotopy fiber* of the homomorphism $(S^1)^r \rightarrow S^1$, $(x_1, \dots, x_r) \rightarrow \prod x_i^{n_i}$, i.e., the wreath product $\text{Ker}((S^1)^r \rightarrow S^1) \rtimes (\mathbf{Z}/d\mathbf{Z})(1)$. An immediate consequence is:

Corollary 2.8. *Under the assumptions of 2.7, there exists an open subgroup I_1 of I such that, for all $q \in \mathbf{Z}$, I_1 acts trivially on $(R^q\Psi\Lambda)_t$.*

At the time of SGA 7, the proof of (2.7.1) was conditional to the validity of the absolute purity conjecture, which was known only in certain cases (e.g., in equal characteristic zero, and for $q \leq 1$ in the notation of footnote 1). Nevertheless, this, together with the cases where resolution of singularities was known, enabled him to show that 2.3 holds if X_η is proper and smooth and either S is the localization of a smooth curve over \mathbf{C} , or $i \leq 1$. Using the formalism of 1.2 (or rather its analogue in the complex case), he also deduced from 2.8 a positive answer to Milnor's question ([44], footnote p. 72)⁸ on the quasi-unipotency of the monodromy of isolated singularities (a question that had been one of the motivations for his theory of the functors $R\Psi$ and $R\Phi$).

Here is a sketch of his proof. As we have an isolated critical point, by a theorem of Arnol'd-Artin-Mather-Tougeron (see [4] for references), $\tilde{H}^n(M_f)$ depends only on a sufficiently high order jet of f , so, instead of the original analytic situation, we can consider an algebraic one, namely a smooth curve S over \mathbf{C} , with a closed point s , a smooth scheme X/\mathbf{C} , and a morphism $f : X \rightarrow S$, smooth outside a closed point x of the special fiber X_s . By Hironaka, we can find a proper map $h : X' \rightarrow X$, with X'/\mathbf{C} smooth, inducing an isomorphism outside X_s and such that $(X'_s)_{\text{red}}$ is a divisor with normal crossings. Let $f' = fh$. By 2.8, the action of a generator T of the local fundamental group of S at s on $R\Psi_{f'}\mathbf{Z}$ is quasi-unipotent. By (the complex analytic analogue of) (1.2.2), $R\Psi_f\mathbf{Z} = Rh_*R\Psi_{f'}\mathbf{Z}$, so T acts quasi-unipotently on $R\Psi_f\mathbf{Z}$ and $R\Phi_f\mathbf{Z}$, but, by (1.2.1), $R\Phi_f(\mathbf{Z})_x$ is a direct summand of $R\Phi_f(\mathbf{Z})$, hence the conclusion.

3 The semistable reduction theorem for abelian varieties

As an abelian variety over an algebraically closed field is a quotient of a Jacobian, 2.3 for $i \leq 1$ implies:

Theorem 3.1. *With the notation of 2.1, let A_η be an abelian variety over η . Then there exists an open subgroup I_1 of I such that, for all $g \in I_1$, $(g - 1)^2 = 0$ on $H^1(A_{\bar{\eta}})$.*

As mentioned in the introduction, this was the crucial tool enabling

⁸i.e., for a holomorphic germ $f : (\mathbf{C}^{n+1}, 0) \rightarrow (\mathbf{C}, 0)$ having an *isolated* critical point at 0, the eigenvalues of monodromy T on $\tilde{H}^n(M_f)$ are roots of 1 (M_f the *Milnor fiber* at 0, where $\tilde{H}^i = \text{Coker}(H^i(\text{pt}) \rightarrow H^i)$). Here $\tilde{H}^n(M_f) = R^n\Phi_f(\mathbf{Z})_0$.

Grothendieck to prove the following theorem (*semistable reduction theorem for abelian varieties*) ([2], IX 3.6):

Theorem 3.2. *With the notation of 3.1, there exists a finite extension η_1 of η such that A_{η_1} has semistable reduction⁹ over the normalization (S_1, s_1, η_1) of S in η_1 , i.e., if A_1/S_1 is the Néron model of A_{η_1} , the connected component $(A_1)_{s_1}^0$ of its special fiber is an extension of an abelian variety by a torus.*

The proof of 3.2 occupies over 300 pages in ([2], VII, VIII, IX). However, the idea is quite simple.

First of all, one rephrases 3.1 in terms of the *Tate module* of A_η ,

$$T_\ell(A_\eta) = \varprojlim A_{\bar{\eta}}[\ell^n],$$

where $[\ell^n]$ means the kernel of the multiplication by ℓ^n , a free \mathbf{Z}_ℓ -module of rank $2g$, where g is the dimension of A_η , equipped with a continuous action of $\text{Gal}(\bar{\eta}/\eta)$ (equivalently, a lisse \mathbf{Z}_ℓ -sheaf, free of rank $2g$, over η).

From now on, let us work with $\Lambda = \mathbf{Z}_\ell$.

By Serre-Lang,

$$H^1(A_{\bar{\eta}}) = T_\ell(A_\eta)^\vee (:= \text{Hom}(T_\ell(A_{\bar{\eta}}), \mathbf{Z}_\ell))$$

as Galois modules. Hence, in the notation of 3.1, for all $g \in I_1$, $(g-1)^2 = 0$ on $T_\ell(A_\eta)$. To prove 3.2 it therefore suffices to prove the following theorem (*cohomological criterion for semistable reduction*) ([2], IX, 3.5):

Theorem 3.3. *In the situation of 3.1, assume that for all $g \in I$, $(g-1)^2 = 0$ on $T_\ell(A_\eta)$. Then A_η has semistable reduction over S .*

3.4

The main ingredient in the proof of 3.3 is the so-called *orthogonality theorem*, which I will now recall. In the situation of 3.1, let A be the Néron model of A_η over S (so that A_η is the generic fiber of A). The Tate module $T_\ell(A_\eta)$ admits a $\text{Gal}(\bar{\eta}/\eta)$ -equivariant 2-step filtration

$$(3.4.1) \quad T_\ell(A_\eta)^t \subset T_\ell(A_\eta)^f \subset T_\ell(A_\eta),$$

where, as a Galois module, $T_\ell(A_\eta)^f = T_\ell(A_\eta)^I$ is the *fixed part* under I , which, by the universal property of the Néron model, is also canonically isomorphic to $T_\ell(A_s) = T_\ell(A_s^0)$, and $T_\ell(A_\eta)^t$ the *toric part*, i.e., $T_\ell(T)$, where T is the

⁹Today one often prefers to say “semi-abelian reduction”, to avoid confusion with semistable reduction as a scheme.

maximal subtorus of A_s^0 . In ([2], IX), Grothendieck likes to write (3.4.1) in the form

$$W \subset V \subset U.$$

This is a filtration by free, finitely generated \mathbf{Z}_ℓ -modules, and the quotients U/V , V/W are torsion-free¹⁰. Let $A'_\eta = \mathcal{E}xt^1(A_\eta, \mathbf{G}_m)$ be the dual abelian variety, A' its Néron model, and let $(W' \subset V' \subset U')$ be the corresponding filtration of $U' = T_\ell(A'_\eta)$. The Poincaré bi-extension of $A_\eta \times A'_\eta$ by \mathbf{G}_m defines a perfect pairing (cf. (*loc.cit.*, 1.0.3)) (the *Weil pairing*):

$$(3.4.2) \quad \langle , \rangle : U \otimes U' \rightarrow \mathbf{Z}_\ell(1).$$

The orthogonality theorem is the following formula (*loc. cit.*, 2.4):

$$(3.4.3) \quad W = V \cap V'^\perp,$$

where $(-)^{\perp}$ means the orthogonal for the pairing (3.4.2). Let g be the dimension of A_η , μ be that of the torus T , and α (resp. λ) be the abelian (resp. unipotent) rank¹¹ of A_s^0 . By (3.4.3) we have $\text{rk}(U/(V + V'^\perp)) = \text{rk}(W) = \mu$, and $\text{rk}((V + V'^\perp)/V'^\perp) = \text{rk}(V/W) = 2\alpha$, so, as $g = \alpha + \lambda + \mu$, we get

$$(3.4.4) \quad \text{rk}(V'^\perp/W) = 2\lambda.$$

By definition, A_η has semistable reduction over S if and only if $\lambda = 0$, which, by (3.4.4) is equivalent to $V'^\perp \subset V$.

From this it is immediate to prove 3.3. Indeed, as its action is unipotent, I acts on U (and U') through its tame quotient $I_t = \widehat{\mathbf{Z}}'(1)$ (and even through its ℓ -primary part $\mathbf{Z}_\ell(1)$). We have to show $V'^\perp \subset V$, i.e., if g is a topological generator of I_t , that $g - 1$ is zero on V'^\perp . But, by assumption, $(g - 1)^2$ is zero on U , hence on U' , hence $g - 1$ is zero on U'/V' , hence on $(U'/V')^\vee(1)$, but under (3.4.2), $(U'/V')^\vee(1) = V'^\perp$.

The first appearance of (3.4.3) is in a paper of Igusa [27]. Igusa considers the case where A_η is the Jacobian of X_η , for X/S a proper curve with geometrically connected fibers and semistable reduction, smooth outside a unique rational point x of X_s . He deduces (3.4.3) from what is called, in today's language, a *Picard-Lefschetz formula* at x for ℓ -adic vanishing cycles. This inspired to Grothendieck his theory of the monodromy pairing, that we discuss in the next section. However, Grothendieck's proof of (3.4.3) does not involve any vanishing cycles. These are, somehow, replaced by the Néron

¹⁰For U/V this is because $U^I = (U \otimes \mathbf{Q}_\ell)^I \cap U$, for V/W because over an algebraic closure of s , A_s^0/T becomes an extension of an abelian variety by a unipotent group.

¹¹i.e., the dimension of the quotient abelian variety (resp. unipotent part) of A_s^0/T over an algebraic closure of s .

models, and Grothendieck obtains (3.4.3) as a consequence of a vast *theory of bi-extensions*, developed in ([2] VII, VIII), considerably generalizing — and, should I say, simplifying — the notion initially introduced by Mumford for formal groups [45]. The inclusion $W \subset V \cap V'^\perp$ is more or less formal. The fact that it is an equality is proved in ([2], IX 2.4) as a corollary of an ampleness criterion of Raynaud ([50], XI 1.11).

Remark 3.5. (a) The converse of 3.3 holds: if A_η has semistable reduction over S , then, for all $g \in I$, $(g - 1)^2 = 0$ on $T_\ell(A_\eta)$. Indeed, as $V'^\perp = (U'/V')^\vee(1)$ is contained in $V = U^I$, $(g - 1)^2$ is zero on U' , hence on U .

(b) There is a variant of 3.3 (and its converse) for the case of *good reduction*, namely A_η has good reduction over S (i.e., A is an abelian scheme over S) if and only if I acts trivially on $T_\ell(A_\eta)$ ([2], IX 2.2.9). This is the so-called *Néron-Ogg-Shafarevich criterion* for good reduction. The proof is easy. It does not use the orthogonality relation (3.4.3).

(c) In ([2], IX 2.6 a)) Grothendieck observes that the inclusion $W \subset V \cap V'^\perp$ can be proved by an arithmetic argument, independent of the theory of bi-extensions, using Weil's theorem on the *weights* of Frobenius for abelian varieties over finite fields. Pushing the argument further, Deligne was able to prove the semistable reduction theorem 3.2, bypassing (3.4.3) ([2], I 6). However, this argument only yields that the inclusion $W \subset V \cap V'^\perp$ is an isogeny.

4 Grothendieck's monodromy pairing

4.1

Let (S, s, η) and A_η be as in 3.1. We denote by A'_η the dual abelian variety, by A (resp. A') the Néron model of A_η (resp. A'_η), and by T (resp. T') the maximal subtorus of A_s^0 (resp. $A'_s{}^0$). *From now on — unless otherwise stated — we assume that A_η has semistable reduction over S* , i.e., A_s^0 is extension of an abelian variety B by the torus T . It follows from the criterion 3.3 that A'_η also has semistable reduction, i.e., $A'_s{}^0$ is extension of an abelian variety B' by T' .

As in the previous section, we consider the Tate modules $U = T_\ell(A_\eta)$, $U' = T_\ell(A'_\eta)$, and their 2-step filtrations $(W \subset V \subset U)$, $(W' \subset V' \subset U')$, where $V = U^I = T_\ell(A_s^0)$, $W = T_\ell(T)$, and similarly $V' = U'^I = T_\ell(A'_s{}^0)$, $W' = T_\ell(T')$. Following Grothendieck's notation, we denote by M the character group of T' :

$$(4.1.1) \quad M = \mathcal{H}om(T', \mathbf{G}_m),$$

so that the co-character group is $M^\vee = \mathcal{H}om(\mathbf{G}_m, T') = \mathcal{H}om(M, \mathbf{Z})$ (these are free finitely generated \mathbf{Z} -modules with action of $\text{Gal}(\tilde{s}/s)$ (in the notation (1.1.1)), and similarly

$$(4.1.2) \quad M' = \mathcal{H}om(T, \mathbf{G}_m),$$

with $M'^\vee = \mathcal{H}om(\mathbf{G}_m, T) = \mathcal{H}om(M', \mathbf{Z})$. As $T'[\ell^n] = \mathcal{H}om(\mathbf{Z}/\ell^n, T') = \mathcal{H}om(\mu_{\ell^n}, T') \otimes \mu_{\ell^n} = (M^\vee/\ell^n M^\vee) \otimes \mu_{\ell^n}$, we have

$$(4.1.3) \quad W' = M^\vee \otimes \mathbf{Z}_\ell(1),$$

and similarly

$$(4.1.4) \quad W = M'^\vee \otimes \mathbf{Z}_\ell(1).$$

Let $M_\ell := M \otimes \mathbf{Z}_\ell$, $M'_\ell := M' \otimes \mathbf{Z}_\ell$. As A_η has semistable reduction, by (3.4.3) and (3.4.4), we have

$$(4.1.5) \quad W = V'^\perp = (U'/V')^\vee(1),$$

hence by (4.1.4),

$$(4.1.6) \quad U'/V' = M'_\ell,$$

and similarly

$$(4.1.7) \quad U/V = M_\ell,$$

which formulas are probably the reason for the *a priori* strange notation (4.1.1).

As I acts on U through its quotient $t_\ell : I \rightarrow \mathbf{Z}_\ell(1)$, and unipotently of exponent ≤ 1 , there exists a unique homomorphism (the *monodromy operator*)

$$(4.1.8) \quad N : U \rightarrow U(-1)$$

such that $gx = x + t_\ell(g)Nx$ for all $x \in U$ and $g \in I$. We have $N^2 = 0$, and N is $\text{Gal}(\bar{\eta}/\eta)$ -equivariant. We again denote by $N : U' \rightarrow U'(-1)$ the monodromy operator corresponding to A'_η . By definition, V and V' are the kernels of N . As the Weil pairing (3.4.2) is Galois equivariant, in particular, I -equivariant, it satisfies the formula

$$(4.1.9) \quad \langle Nx, y \rangle + \langle x, Ny \rangle = 0.$$

This implies that $N(U) \subset V'^{\perp}(-1)$ ($= W(-1)$ by (4.1.5)), and, as $N(V) = 0$, N induces a homomorphism

$$N : U/V \rightarrow W(-1),$$

which, by (4.1.5) and (4.1.6), can be rewritten

$$(4.1.10) \quad u_{\ell} : M_{\ell} \rightarrow M_{\ell}^{\vee},$$

or, equivalently, a pairing

$$(4.1.11) \quad \langle \cdot, \cdot \rangle_{\ell} : M_{\ell} \otimes M'_{\ell} \rightarrow \mathbf{Z}_{\ell}.$$

This is *Grothendieck's monodromy pairing*. By definition, u_{ℓ} is injective, hence is an isogeny. The pairing (4.1.11) is symmetric, i.e., the pairing deduced by exchanging A_{η} and A'_{η} (and identifying A_{η} with $(A'_{\eta})'$ by the biduality isomorphism) is obtained from (4.1.11) by $\langle x, y \rangle \mapsto \langle y, x \rangle$. Grothendieck's main result is the following ([2], IX 10.4) (discussed for the first time in a letter to Serre, dated October 3-5, 1964, see ([3], p. 207, 209)):

Theorem 4.2. (a) *There exists a unique homomorphism*

$$(4.2.1) \quad u : M \rightarrow M^{\vee}$$

such that $u \otimes \mathbf{Z}_{\ell} = u_{\ell}$ for all ℓ . Let

$$(4.2.2) \quad \langle \cdot, \cdot \rangle : M \otimes M' \rightarrow \mathbf{Z}$$

denote the pairing defined by (4.2.1).

(b) Let $\xi : A_{\eta} \rightarrow A'_{\eta}$ be a polarization, $\xi^* : M \rightarrow M'$ the homomorphism deduced by functoriality. Then the pairing

$$\langle \cdot, \cdot \rangle_{\xi} : M \otimes M \rightarrow \mathbf{Z}$$

defined by $\langle x, y \rangle_{\xi} = \langle x, \xi^* y \rangle$ is symmetric, and negative definite¹².

We will sketch Grothendieck's proof at the end of §4. An alternate construction of u and proof of (4.2 (b)) was given by Raynaud, using rigid methods ([52] and (SGA7, IX 14)).

¹²In *loc. cit.*, it is asserted to be positive definite. This discrepancy seems to be due to a sign in the Picard-Lefschetz formula.

Remark 4.3. The construction of u_ℓ (4.1.10) makes essential use of the hypothesis $\ell \neq p$. Let $M_p := M \otimes \mathbf{Z}_p$, $M'_p := M' \otimes \mathbf{Z}_p$. For S of *mixed characteristic*, using Tate's theorem on homomorphisms of Barsotti-Tate groups ([64], Th. 4), Grothendieck directly constructs in ([2], IX 9) a homomorphism $u_p : M_p \rightarrow M'_p$ in terms of the pro- p -groups $T_p(A^0)$, $T_p(A'^0)$, and, by an analytic argument, shows (in [2], IX 12) that (4.2 (a)) extends to $\ell = p$, i.e., $u_p = u \otimes \mathbf{Z}_p$. Now that thanks to de Jong [39] Tate's theorem has been established in equal characteristic, the restriction on S is superfluous¹³.

In the sequel, we assume again $\ell \neq p$. An immediate consequence of 4.2, actually just the fact that u_ℓ is an isogeny, is:

Corollary 4.4. (a) Consider the filtration $(W \subset V \subset U)$ (3.4.1) as an increasing filtration $(M_i)_{i \in \mathbf{Z}}$, with $M_i = U$ for $i \geq 1$, $M_0 = V$, $M_{-1} = W$, $M_i = 0$ for $i \leq -2$. Then $M_i \otimes \mathbf{Q}_\ell$ is the monodromy filtration of $U \otimes \mathbf{Q}_\ell$ with respect to the nilpotent operator N , i.e., $N(M_i \otimes \mathbf{Q}_\ell) \subset M_{i-2} \otimes \mathbf{Q}_\ell(-1)$, and $N^i : \mathrm{gr}_i^M(U \otimes \mathbf{Q}_\ell) \rightarrow \mathrm{gr}_{-i}^M(U \otimes \mathbf{Q}_\ell)(-i)$ is an isomorphism for all $i \geq 0$.

(b) Assume that k is a finite field \mathbf{F}_q . Then the filtration (M_i) is pure, i.e., equals, up to a shift, the weight filtration of $U \otimes \mathbf{Q}_\ell$ in the sense of Deligne ([21], 1.7.5): $\mathrm{gr}_i^M U$ is pure¹⁴ of weight $i - 1$.

As $H^1(A_{\bar{\eta}}) := H^1(A_{\bar{\eta}}, \mathbf{Z}_\ell)$ is dual to $U = T_\ell(A_\eta)$, the filtration, still denoted M_\bullet on $H^1(A_{\bar{\eta}})$ dual to the filtration M_\bullet on U , is again the monodromy filtration (for the monodromy operator N), and when $k = \mathbf{F}_q$, is the weight filtration up to shift : $\mathrm{gr}_i^M H^1(A_{\bar{\eta}})$ is pure of weight $i + 1$.

As we mentioned in (3.5 (c)), Deligne gave an alternate short proof of this. However, weight arguments don't work for the next corollary 4.6, which lies deeper. We need a little preliminary before stating it.

4.5

For the moment, we don't assume that A_η has semistable reduction over S . Consider the finite commutative étale group scheme over s of connected components of the special fiber of the Néron model A of A_η ,

$$(4.5.1) \quad \Phi_0 := \Phi_0(A_\eta) := A_s/A_s^0,$$

¹³A construction of u_p valid without assuming S of mixed characteristic, and using only Tate's theorem (but over a higher dimensional normal base, a formal moduli scheme as in ([2], IX 12.8)), has been made by B. Conrad (private communication).

¹⁴i.e., the eigenvalues of the geometric Frobenius F are q -Weil numbers of weight $i - 1$; moreover (by Weil), $\det(1 - Ft, \mathrm{gr}_i^M(U \otimes \mathbf{Q}_\ell))$ has coefficients in \mathbf{Z} and is independent of ℓ .

and similarly define $\Phi'_0 := \Phi_0(A'_\eta) := A'_s/A_s^0$. In ([2], IX), Grothendieck :

(i) defined a canonical pairing $\Phi_0 \times \Phi'_0 \rightarrow \mathbf{Q}/\mathbf{Z}$, which he conjectured to be perfect;

(ii) in the semistable reduction case, constructed a canonical isomorphism between Φ_0 and the cokernel of $u : M \rightarrow M^\vee$ (4.2.1).

Let me first discuss (ii). We now assume that A_η has semistable reduction. For simplicity, assume that S is strictly local (so that $S = \tilde{S}$ in the notation (1.1.1), and Φ_0, Φ'_0 are usual finite groups), and take $\ell \neq p$. Let $\Phi_0(\ell)$ be the ℓ -primary component of Φ_0 . The ℓ -primary component of $\text{Coker}(u)$ is

$$\text{Coker}(u)(\ell) = \text{Ker}(u_\ell \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell : M_\ell \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \rightarrow M_\ell^\vee \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

On the other hand, as A_s^0 is ℓ -divisible, we have

$$\Phi_0(\ell) = A_s(\ell)/A_s^0(\ell),$$

where $(-)(\ell) = \varinjlim(-)[\ell^n]$, hence

$$(4.5.2) \quad \Phi_0(\ell) = (U \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)^I / (V \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell),$$

with the above notation $U = T_\ell(A_\eta)$, $V = U^I = T_\ell(A_s^0)$ ¹⁵. We thus have an injection

$$(4.5.3) \quad \Phi_0(\ell) \hookrightarrow (U/V) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \stackrel{(4.1.7)}{=} M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell.$$

Now using that A_η has semistable reduction, Grothendieck extends the definition of (4.5.3) to $\ell = p$, and proves:

Corollary 4.6. *For A_η having semistable reduction, the homomorphism (4.5.3) induces a short exact sequence*

$$(4.6.1) \quad 0 \rightarrow \Phi_0(\ell) \rightarrow M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \xrightarrow{u \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell} M^\vee \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \rightarrow 0$$

for all primes ℓ , where u is the homomorphism (4.2.1). In other words, the maps (4.6.1) induce an isomorphism

$$(4.6.2) \quad \Phi_0 \xrightarrow{\sim} \text{Coker}(u : M \rightarrow M^\vee).$$

The proof for $\ell \neq p$ is easy: from (4.5.2) a simple calculation shows that $\Phi_0(\ell)$ is the torsion subgroup of $H^1(I, U) = U/NU(1)$ ([2], IX (11.3.8)), i.e., $W/NU(1)$, which, by (4.1.5) and the definition of u_ℓ is just $\text{Coker}(u_\ell)$. The proof for $\ell = p$ is more delicate.

¹⁵This formula does not use that A_η has semistable reduction.

Remark 4.7. When A_η is the Jacobian of the generic fiber of a proper, flat curve X/S , with geometrically connected fibers and semistable reduction (in other words, X is regular, X_η is smooth, and X_s is a (reduced) divisor with normal crossings), i.e., $A_\eta = \text{Pic}_{X_\eta/\eta}^0$, (4.6.2) leads (via the Picard-Lefschetz formula) to a purely *combinatorial description* of Φ_0 , in terms of the irreducible components and double points of the special fiber X_s . We will briefly discuss this in 6.1 and 6.3. Similar descriptions of Φ_0 under milder assumptions on X were given independently by Raynaud [51], see also ([13], 9.6), [28].

An interesting special case computed by Deligne, and discussed by Mazur-Rapoport in [42] is when $S = \text{Spec } \mathbf{Z}_p$, X is the modular curve $X_0(p)$ localized over S , with $p \geq 5$. Then, over an algebraic closure of \mathbf{F}_p , $\Phi_p := \Phi_0(J_p)$ (J_p the special fiber of the Néron model) is a cyclic group of order the numerator of $(p-1)/12$ ([42], A1), having as a generator the image of the \mathbf{Q} -rational divisor $(0) - (\infty)$. As an application, Mazur shows that, for $\ell \neq p$, the Hecke operator T_ℓ on Φ_p equals $1 + \ell$ (*loc. cit.*, 9.7). Ribet proved, more generally, that $T_\ell = 1 + \ell$ on $\Phi_0(J_p(X_0(pN)))$ (for $(\ell, pN) = 1$, $(p, N) = 1$) ([53], (3.12)), a result he used to prove that the Shimura-Taniyama-Weil conjecture implies Fermat.

4.8

Let us recall the definition of the pairing mentioned in (4.5 (i)). Let

$$(4.8.1) \quad w_\eta \in \text{Ext}^1(A_\eta \otimes^L A'_\eta, \mathbf{G}_m)$$

be the Poincaré bi-extension (tensor product taken over \mathbf{Z}), and consider the immersions $i : s \hookrightarrow S$, $j : \eta \hookrightarrow S$. As $R^1 j_* \mathbf{G}_m = 0$ (Hilbert 90), the exact sequence (of group schemes over S)

$$(4.8.2) \quad 0 \rightarrow \mathbf{G}_m \rightarrow j_* \mathbf{G}_m \rightarrow i_* \mathbf{Z} \rightarrow 0$$

(where the middle term is the *Néron-Raynaud model* of \mathbf{G}_m , a smooth commutative group scheme over S) yields an exact sequence

$$(4.8.3) \quad \text{Ext}^1(A \otimes^L A', \mathbf{G}_m) \rightarrow \text{Ext}^1(A_\eta \otimes^L A'_\eta, \mathbf{G}_m) \xrightarrow{c} \text{Ext}^1(A \otimes^L A', i_* \mathbf{Z}).$$

On the other hand, a boundary map gives a canonical isomorphism

$$(4.8.4) \quad \delta : \text{Hom}(\Phi_0 \otimes \Phi'_0, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \text{Ext}^1(A \otimes^L A', i_* \mathbf{Z}).$$

Grothendieck defines the pairing ([2], IX (1.2.1))

$$(4.8.5) \quad w_0 : \Phi_0 \otimes \Phi'_0 \rightarrow \mathbf{Q}/\mathbf{Z}$$

as the image of w_η (4.8.1) by $\delta^{-1}c$. This is the obstruction to extending w_η to a bi-extension $w \in \text{Ext}^1(A \otimes^L A', \mathbf{G}_m)$. Grothendieck conjectured that the pairing w_0 is perfect. Here is a brief history of the question:

- In the semistable reduction case, 4.6 gives the existence of a perfect pairing w'_0 of the form (4.8.5) and Grothendieck conjectures that, up to a sign that should be determined, it coincides with w_0 ([2], IX 11.4)¹⁶.

- Various cases were treated by Bégueri [6], McCallum [43], Bosch [14], Bosch-Lorenzini [15].

- Counter-examples for k not perfect were given by Bertapelle-Bosch [10], using Weil restrictions, and by Bosch-Lorenzini [15] for Jacobians.

- Proofs in the general case (k perfect) were recently given independently by Suzuki [63] (and, independently, Pépin (unpublished)).

While in the works of Bertapelle, Bosch, Bosch-Lorenzini, Werner, the main tools are those provided by the geometry of abelian varieties, Bégueri's approach exploits another ingredient, namely Serre's geometric local class field theory, the perfection of Grothendieck's pairing appearing as a by-product of the reciprocity isomorphism. It seems to me, however, that the relation between these various methods is not yet fully understood. For example, in the case of the Jacobian of a proper, smooth curve having semistable reduction, the pairing (4.1.11), as described below by (4.9.14) via Picard-Lefschetz, should be more directly related to Néron's height pairing: one would like to exhibit the vanishing cycles hidden in ([15], 4.4).

4.9

As in (4.2 (a)) the uniqueness of u is clear (one ℓ even suffices). The construction of u is easily reduced to the case where A_η is the Jacobian of a proper smooth and geometrically irreducible curve X_η/η ([2], IX 10.5.1-10.5.3). At this point, Grothendieck uses a result that was proved at about the same time by Deligne and Mumford¹⁷ [16] as a corollary to 3.2, the so-called *semistable reduction theorem for curves*, thanks to which, after a finite extension of η , X_η admits a proper, flat model X/S with semistable reduction. Thus, after a further reduction, we may assume that X_η is of this form, and also that the residue field k is separably closed. In this situation, we have combinatorial

¹⁶The definition of a perfect pairing of the form w_0 on the ℓ -primary components, $\ell \neq p$, using (4.5.2) is easy (*loc. cit.* 11.3) and doesn't need the semi-stability assumption; its coincidence with w_0 was checked by Bertapelle [11]. In the semistable case, the verification of the coincidence between w'_0 and w_0 was made by Werner [67], using the rigid geometry of Raynaud's extensions in [52].

¹⁷Other proofs, independent of 3.2, were found later: Artin-Winters [5], T. Saito [54], Temkin [65].

descriptions of M and u_ℓ in terms of the dual graph of X_s , and they suggest the definition of u .

(a) *Description of M .* By the canonical polarization of $A_\eta = \text{Pic}_{X_\eta/\eta}^0$, we identify A_η and A'_η , hence their Néron models A and A' , the maximal tori T and T' in their special fibers, and their character groups M and M' . Thus, by (4.1.1), M^\vee is the co-character group of T . On the other hand, by a result of Raynaud, $A_s^0 = \text{Pic}_{X_s/s}^0$ ([2], IX (12.1.12)). Put $Y := X_s$, denote by $\Gamma(Y)$ its dual graph. A simple calculation (([2], IX 12.3), ([28], 2.3)) shows that

$$(4.9.1) \quad M = H_1(\Gamma(Y), \mathbf{Z})$$

(and $M^\vee = H^1(\Gamma(Y), \mathbf{Z})$). More explicitly, if J is the set of irreducible components of Y , Σ the set of double points of Y , and, if for each $x \in \Sigma$ we choose an order on the set J_x of the two branches passing through x (points x_1, x_2 of the normalization of Y sitting over x), so that we have a basis $\delta'(x) = (1, -1)$ of the kernel $\mathbf{Z}'(x)$ of the sum map $\mathbf{Z}^{J_x} \rightarrow \mathbf{Z}$, then we have an exact sequence

$$(4.9.2) \quad 0 \rightarrow M \rightarrow \bigoplus_{x \in \Sigma} \mathbf{Z}'(x) \rightarrow \mathbf{Z}^J,$$

where $J_x = (x_1, x_2)$ and the second map sends $\delta'(x)$ to the difference $C_{x_1} - C_{x_2}$ of the components of Y corresponding to x_1 and x_2 . Dually¹⁸, we have an exact sequence

$$(4.9.3) \quad \mathbf{Z}^J \rightarrow \bigoplus_{x \in \Sigma} \mathbf{Z}(x) \rightarrow M^\vee \rightarrow 0,$$

where $\mathbf{Z}(x)$ is the cokernel of the diagonal map $\mathbf{Z} \rightarrow \mathbf{Z}^{J_x}$, with basis $\delta(x)$ dual to $\delta'(x)$, and the first map is dual to the second map in (4.9.2).

(b) *Description of N .* The *specialization sequence* (1.2.4) for $K = \mathbf{Z}_\ell(1)$ gives an exact sequence

$$(4.9.4) \quad 0 \rightarrow H^1(X_s)(1) \rightarrow H^1(X_{\bar{\eta}})(1) \rightarrow \Phi^1(1) \rightarrow H^2(X_s)(1) \rightarrow H^2(X_{\bar{\eta}})(1) \rightarrow 0,$$

where $H^i(-) = H^i(-, \mathbf{Z}_\ell)$, $\Phi^i := H^i(X_s, R\Phi(\mathbf{Z}_\ell))$. The vanishing cycle groups $R^q\Phi(\mathbf{Z}_\ell)$ were calculated by Deligne in ([2], XV 3) (as a special case of pencils with ordinary quadratic singularities). The complex $R\Phi(\mathbf{Z}_\ell)$ is concentrated in degree 1 and on the set Σ of double points of X_s , so that

$$(4.9.5) \quad \Phi^1(1) = \bigoplus_{x \in \Sigma} \Phi^1(1)_x,$$

¹⁸The cokernel of the last map is \mathbf{Z} , as Y is geometrically connected, hence $\Gamma(Y)$ connected.

and, with the choice of the ordering on J_x made above, $\Phi^1(1)_x$ has a distinguished basis δ'_x , called the *vanishing cycle* at x :

$$(4.9.6) \quad \Phi^1(1)_x = \mathbf{Z}_\ell \delta'_x.$$

By definition (4.1.7), $M_\ell = T_\ell(A_\eta)/T_\ell(A_s^0) = H^1(X_{\bar{\eta}})(1)/H^1(X_s)(1)$, hence, by (4.9.4),

$$(4.9.7) \quad M_\ell = \text{Ker}(\Phi^1(1) \rightarrow H^2(X_s)(1)).$$

Dually to (4.9.5) and (4.9.6), we have

$$(4.9.8) \quad H_\Sigma^1(X_s, R\Psi(\mathbf{Z}_\ell)) = \bigoplus_{x \in \Sigma} H_x^1(X_s, R\Psi(\mathbf{Z}_\ell)),$$

with $H_x^1(X_s, R\Psi(\mathbf{Z}_\ell))$ dual to $\Phi_x^1(1)$ (with values in \mathbf{Z}_ℓ), with dual basis δ_x :

$$(4.9.9) \quad H_x^1(X_s, R\Psi(\mathbf{Z}_\ell)) = \mathbf{Z}_\ell \delta_x$$

The monodromy operator N on $U = T_\ell(A_\eta) = H^1(X_{\bar{\eta}})$ factors (by definition of $R\Phi$) through a sum of local variation maps

$$(4.9.10) \quad N_x : \Phi_x^1(1) = \mathbf{Z}_\ell \cdot \delta'_x \rightarrow H_x^1(X_s, R\Psi(\mathbf{Z}_\ell)) = \mathbf{Z}_\ell \cdot \delta_x,$$

i.e., we have a commutative diagram

$$(4.9.11) \quad \begin{array}{ccc} H^1(X_{\bar{\eta}})(1) & \longrightarrow & \bigoplus_{x \in \Sigma} \Phi_x^1(1) \\ N \downarrow & & \bigoplus N_x \downarrow \\ H^1(X_{\bar{\eta}}) & \longleftarrow & \bigoplus_{x \in \Sigma} H_x^1(X_s, R\Psi(\mathbf{Z}_\ell)) \end{array} ,$$

where N_x is an isomorphism, the *Picard-Lefschetz isomorphism*, given by

$$(4.9.12) \quad N_x(\delta'_x) = -\delta_x.$$

Using duality and the cospecialization sequence, dual to (4.9.4),

$$(4.9.13) \quad H^1(X_{\bar{\eta}}) \leftarrow H_\Sigma^1(R\Psi) \leftarrow H^0(\tilde{X}_s) \leftarrow H^0(X_{\bar{\eta}}) \leftarrow 0,$$

where \tilde{X}_s is the normalization of X_s , one checks that the factorization (4.9.11) is refined into the following one:

$$(4.9.14) \quad \begin{array}{ccccc} H^1(X_{\bar{\eta}})(1) & \longrightarrow & M \otimes \mathbf{Z}_\ell & \longrightarrow & \bigoplus_{x \in \Sigma} \Phi_x^1(1) \\ N \downarrow & & u \otimes \mathbf{Z}_\ell \downarrow & & \bigoplus N_x \downarrow \\ H^1(X_{\bar{\eta}}) & \longleftarrow & M^\vee \otimes \mathbf{Z}_\ell & \longleftarrow & \bigoplus_{x \in \Sigma} H_x^1(X_s, R\Psi(\mathbf{Z}_\ell)) \end{array} ,$$

in which $u : M \rightarrow M^\vee$ is the map making the following square commute:

$$(4.9.15) \quad \begin{array}{ccc} M & \longrightarrow & \bigoplus_{x \in \Sigma} \mathbf{Z}\delta'(x), \\ u \downarrow & & \downarrow -\text{Id} \\ M^\vee & \longleftarrow & \bigoplus_{x \in \Sigma} \mathbf{Z}\delta(x) \end{array}$$

where the upper (resp. lower) row is the injection (4.9.2) (resp. surjection (4.9.3)). In other words, u is induced by the *negative definite* quadratic form $\sum -t_i^2$ on $\mathbf{Z}^r = \sum_{x \in \Sigma} \mathbf{Z}$.

In the case considered by Igusa [27] — which was for Grothendieck the starting point of the whole theory — the special fiber X_s is irreducible, and Σ consists of a single point x . The rows of (4.9.15) are isomorphisms (hence u also). If we identify δ_x with its image in $M^\vee \otimes \mathbf{Z}_\ell \subset H^1(X_s) \subset H^1(X_{\bar{\eta}})$ by the bottom arrow of (4.9.14), (4.9.4) yields a short exact sequence

$$(4.9.16) \quad 0 \rightarrow H^1(X_s)(1) \rightarrow H^1(X_{\bar{\eta}})(1) \xrightarrow{N} \mathbf{Z}_\ell \delta_x \rightarrow 0,$$

where $H^1(X_s)$ is the *fixed part*, and $\mathbf{Z}_\ell \delta_x$ the *toric part*

$$M^\vee \otimes \mathbf{Z}_\ell = W(-1) = T_\ell(T)(-1) = H^1(X_s)(1)^\perp,$$

where T is the maximal torus in A_s^0 , i.e., the 1-dimensional torus at x defined by $\pi_* \mathbf{G}_m / \mathbf{G}_m$, $\pi : \tilde{X}_s \rightarrow X_s$ being the normalization map. Elements σ of the inertia group I act on $H^1(X_{\bar{\eta}})(1)$ by the symplectic transvections $a \mapsto a - t_\ell(\sigma) \langle a, \delta_x \rangle \delta_x$ ([2], XV 3.4).

In general, such an exact sequence (4.9.16) exists only after tensoring by \mathbf{Q}_ℓ , as u_ℓ is only an isogeny.

5 Grothendieck's dreams

In ([2], IX, Introduction), Grothendieck writes: “... le présent exposé peut aussi être considéré comme une étude détaillée des *phénomènes de monodromie locale pour les H^1 ℓ -adiques* (ou mieux encore, pour les H^1 “motiviques”) *des variétés projectives et lisses sur K* . Dans cette optique, il semble clair que les principaux résultats du présent exposé sont destinés à être englobés dans une “théorie de Néron” pour des motifs de poids quelconque, i. e. pour des H^i (ℓ -adiques, ou de de Rham, ou de Hodge, etc.) avec i quelconque, qu’on ne commence qu’à entrevoir à l’heure actuelle. (Cf. à ce sujet [P. A. Griffiths, Report on variation of Hodge structures, à paraître]¹⁹,

¹⁹See [24]. See also [25], Deligne’s report [17], and Griffiths-Schmid’s survey [26], giving the state of the art in 1975, taking into account Deligne’s work on Hodge theory.

et plus particulièrement les conjectures de Deligne 9.8 à 9.13 du rapport cité.)”

These questions have been at the origin of several vast theories:

- mixed Hodge theory
- theory of weights in ℓ -adic cohomology
- p -adic Hodge theory
- mixed motives.

Hodge theory came first, with Deligne’s fundamental work, and subsequent developments on the analytic and algebraic theory of variations of Hodge structures by Griffiths, Schmid, and many others. Grothendieck’s conjectural “yoga” of weights over finite fields had inspired Deligne for his mixed Hodge theory. In turn, mixed Hodge theory gave some guidelines in his theory of mixed ℓ -adic sheaves in [21].

At the end of ([2], IX, Introduction), Grothendieck observes that, in the notation of 4.1, the Galois module $T_p(A_\eta)$ (p the residual characteristic) behaves quite differently from its ℓ -adic analogue, $\ell \neq p$ (as indeed Tate’s seminal article [64] had shown). He adds that Barsotti-Tate groups over η play the role of p -adic local systems over η , those which appear in the p -adic analysis of the H^1 of projective, smooth varieties over η . He suggests that in order to understand the higher H^i ’s from a p -adic viewpoint, the category of Barsotti-Tate groups should be suitably enlarged, using inputs from crystalline cohomology, this new theory he had just introduced. This can be seen as the origin of p -adic Hodge theory, which really started only a couple of years later with Fontaine’s foundational work on Grothendieck’s problem of the *mysterious functor*.

The degenerating abelian varieties studied in ([2] IX) are the prototype of mixed motives (over S , in cohomological degree 1). Grothendieck’s dream of generalizing this theory to higher degree is far from being fulfilled today, even in the case of a base field, despite extensive work during the past forty years (starting with Deligne’s theory of 1-motives in [19], up to the recent achievements accomplished by Voevodsky and many others, for which even a rough survey would by far exceed the scope of this report).

I will limit myself to a very brief update on ℓ -adic vanishing cycles and monodromy.

6 Update on the ℓ -adic side

6.1 Picard-Lefschetz

The Picard-Lefschetz formula in ℓ -adic cohomology proved by Deligne in ([2] XV) — and used by Grothendieck in the proof of 4.2 — was the key tool in the cohomological theory of Lefschetz pencils, developed in ([2] XVIII), which provided the basic framework for Deligne’s first proof of the Weil conjecture [18]. In the odd relative dimension n case (and already for $n = 1$), the proof given in ([2] XV) is of transcendental nature, using a lifting to characteristic zero, a comparison theorem ([2] XIV) between ℓ -adic and Betti nearby cycles, and an explicit topological calculation in the Betti case. A purely algebraic proof was found later [31], as a by-product of Rapoport-Zink’s description of the monodromy operator N in the semistable reduction case (see 6.3).

6.2 Structure of $R\Psi$

Let S and Λ be as in 1.1, and let X be of finite type over S . Deligne proved in ([20], Th. finitude) that for $K \in D_c^b(X_\eta)$, we have $R\Psi_X(K) \in D_c^b(X_{\bar{s}})$. Finer results were obtained later in relation with the theory of perverse sheaves. It was proved in [7] that the functor $R\Psi$ is right t-exact. Combined with a result of Gabber to the effect that $R\Psi$ commutes with duality, it implies that $R\Psi$ is t-exact, and in particular transforms perverse sheaves into perverse sheaves. Moreover, it was also proved by Gabber that, for K perverse on X , $R\Phi_X(K)[-1]$ is perverse, and that $R\Psi$ commutes with external products (for all this, see [8], [29]).

6.3 The semistable case

Assume that X/S has semistable reduction: X is regular, X_η is smooth, and X_s is a reduced divisor with normal crossings in X , i.e., étale locally at any geometric point \bar{x} of X_s , X is defined by an equation of the form $\prod_{1 \leq i \leq r} t_i = \pi$, where π is a uniformizing parameter of S and the t_i ’s are part of a system of regular parameters at the strict localization of X at \bar{x} . In [49], Rapoport and Zink proved that the absolute purity assumption needed to justify the calculation (2.7.1) was satisfied, and, moreover, that the action of the inertia I on the the nearby cycle groups $R^q\Psi\Lambda$ was tame, hence trivial (by (2.7.1)). It follows that I acts through its tame quotient I_t , and if T is a topological generator of I_t , the action of $T - 1$ on $R\Psi\Lambda$ is nilpotent. In fact, imitating a construction of Steenbrink, they gave an explicit description of this action, using a realization of $R\Psi\Lambda$ as the total

complex of a certain bicomplex (the *Rapoport-Zink bicomplex*), at least in the *strict* semistable reduction case, i.e., when the special fiber X_s is a sum of smooth components D_i , $1 \leq i \leq m$ (see [49], [29]).

If $d = \dim(X_s) = \dim(X_\eta)$ is the relative dimension of X/S , one has $(T-1)^{d+1} = 0$ on $R\Psi\Lambda$. As $\Lambda_{X_\eta}[d]$ is perverse, so is $R\Psi\Lambda[d]$ (6.2). When $d \leq 1$, or $m \leq 2$ (in which cases $(T-1)^2 = 0$), or $\Lambda = \mathbf{Q}_\ell$, the monodromy operator

$$(6.3.1) \quad N : R\Psi\Lambda(1) \rightarrow R\Psi\Lambda$$

such that $\sigma = \exp(Nt_\ell(\sigma))$ for all $\sigma \in I$ is defined, and it is more convenient to work with N , which does not depend of the choice of a generator of I_t , and is Galois equivariant. Let $\text{Per}(X_{\bar{s}})$ denote the category of perverse sheaves on $X_{\bar{s}}$. As N is a (twisted) nilpotent endomorphism of $R\Psi\Lambda$ in the abelian category $\text{Per}(X_{\bar{s}})[-d]$, it defines a *monodromy filtration*

$$(6.3.2) \quad \cdots \subset M_i \subset M_{i+1} \subset \cdots,$$

characterized by $NM_i(1) \subset M_{i-2}$, and $N^i : \text{gr}_i^M \xrightarrow{\sim} \text{gr}_{-i}^M(-i)$ for $i \geq 0$. As a by-product of [49], T. Saito [55] calculated the associated graded object:

$$(6.3.3) \quad \text{gr}_k^M R\Psi\Lambda = \bigoplus_{p-q=k, p \geq 0, q \geq 0} K_{p,q},$$

with

$$K_{p,q} := (a_{p+q*}\Lambda)[-p-q](-p),$$

where

$$a_n : X_s^{(n)} := \coprod_{J \subset \{1, \dots, m\}, |J|=n+1} \bigcap_{i \in J} D_i \rightarrow X_s$$

is the natural projection. The operator $N : \text{gr}_k^M \rightarrow \text{gr}_{k-2}^M(-1)$ sends $K_{p,q}$ to $K_{p-1, q+1}(-1)$ by the identity²⁰.

Example. If $m = 2$, so that X_s consists of a pair D_1, D_2 of smooth divisors (of dimension d) crossing transversally, we have the following picture, where $C = D_1 \cap D_2$:

$$(6.3.4) \quad \begin{array}{ccc} \text{gr}_1 & & \Lambda_C-1, \\ & & \downarrow N \\ \text{gr}_0 & \Lambda_{D_1} \oplus \Lambda_{D_2} & \\ & & \downarrow \\ \text{gr}_{-1} & & \Lambda_C[-1] \end{array}$$

²⁰See also [32] for an alternate proof of the tameness of $R\Psi\Lambda$ and an exposition of the above calculations.

where the isomorphism $N : \mathrm{gr}_1 \xrightarrow{\sim} \mathrm{gr}_{-1}(-1)$ is the identity of Λ_C-1. Note that $\Lambda_{D_1} \oplus \Lambda_{D_2} = \underline{IC}_{X_s}[-d]$, where \underline{IC}_{X_s} is the intersection complex of X_s , i.e., $j_{1*}(\Lambda[d])$ ($j : X_s - C \hookrightarrow X_s$ the inclusion). The object (6.3.4) appears in the Picard-Lefschetz formula in odd relative dimension (cf. [31]²¹). The simplest case is $X = S[t_1, t_2]/(t_1 t_2 - \pi)$. It is sometimes called the *Picard-Lefschetz oscillator* (cf. [58]), as the triple $(\mathrm{gr}_1, \mathrm{gr}_{-1}, N)$ uniquely extends to the standard representation of SL_2 over Λ .

6.4 The weight-monodromy conjecture

Let X/S be *proper*, with strict semistable reduction. The monodromy filtration 6.3.2 induces a spectral sequence, called the *weight spectral sequence*

$$(6.4.1) \quad E_1^{i,j} = H^{i+j}(X_{\bar{s}}, \mathrm{gr}_{-i}^M R\Psi\Lambda) \Rightarrow H^{i+j}(X_{\bar{\eta}}),$$

whose initial term can be re-written

$$(6.4.2) \quad E_1^{-r, n+r} = \bigoplus_{q \geq 0, r+q \geq 0} H^{n-r-2q}(X_{\bar{s}}^{(r+1+2q)})(-r-q)$$

thanks to (6.3.3). The differential d_1 is a sum of restriction and Gysin maps. Note that though (E_1, d_1) depends only on X_s , (6.4.1) does depend on X^{22} .

The weight spectral *degenerates at E_2* . This was first proved for k finite [49], as a consequence of Weil II.²³ The general case was proved by Nakayama [47] and Ito [37], independently.

Let \widetilde{M}_{\bullet} be the abutment filtration of (6.4.1). For $m \in \mathbf{Z}$, the monodromy operator N on $H^m(X_{\bar{\eta}})$ sends \widetilde{M}_i to $\widetilde{M}_{i-2}(-1)$. A central problem in the theory is the following conjecture, called the *weight monodromy conjecture*:

Conjecture 6.5. Assume that $\Lambda = \mathbf{Q}_{\ell}$. Then, for all $m \in \mathbf{Z}$, the filtration \widetilde{M}_{\bullet} on $H^m(X_{\bar{\eta}})$ is the monodromy filtration M_{\bullet} associated with the nilpotent operator N , i.e., for all $i \geq 0$, $N^i : \mathrm{gr}_i^{\widetilde{M}} H^m \xrightarrow{\sim} \mathrm{gr}_{-i}^{\widetilde{M}} H^m(-i)$.

By the description of N given in (6.3.3), N induces isomorphisms at the E_1 -level. As (6.4.1) degenerates at E_2 , 6.5 is equivalent to saying that N induces isomorphisms at the E_2 -level. When k is finite, it follows from Weil II that \widetilde{M}_{\bullet} on $H^n(X_{\bar{\eta}})$ is the weight filtration, up to shift: \widetilde{M}_r is the piece of weight $\leq i+n$. Therefore, in this case, 6.5 is equivalent to saying that the monodromy filtration M_{\bullet} is pure, i.e., the graded pieces of gr_{\bullet}^M are pure.

Here is the status of 6.5:

²¹There is a typo on p. 251, l. 18: $|i| > -1$ should be replaced by $|i| > 1$.

²²actually only via $X \otimes R/(\pi^2)$, where $S = \mathrm{Spec} R$, as shown by Nakayama [46].

²³The complex analogue had been proved by Steenbrink [61], using Hodge theory

- for k finite, X/S coming by localization from a proper, flat scheme over a smooth curve over k , with semistable reduction at a closed point, 6.5 was proved by Deligne ([21], 1.8.5);
- in the general equicharacteristic p case, by Ito [37];
- for k finite and $\dim(X/S) \leq 1$ (resp. $\dim(X/S) \leq 2$) by Grothendieck (4.4 (b)) (resp. by Rapoport-Zink ([49], 2.13, 2.23);
- for certain 3-folds X_η , and certain p -adically uniformized varieties X_η ([36], [38]);
- for X_η a set-theoretic complete intersection in a projective space (or in a smooth projective toric variety), by Scholze [59].

The general case is still open.

6.6 Euler-Poincaré characteristics of ℓ -adic sheaves

Deep relations between ℓ -adic nearby cycles and global Euler-Poincaré characteristics were discovered by Deligne in the mid 1970's, spurring a new line of research which has been active for the past forty years. See ([33], [35]) for (partial) surveys. Breakthroughs were made recently by Beilinson [9] and T. Saito [57] in their work on singular supports and characteristic cycles of ℓ -adic sheaves.

6.7 Vanishing cycles over higher dimensional bases

Though, in general, vanishing cycles don't behave well in families, in the early 1980's Deligne proposed a theory of functors $R\Psi$ and $R\Phi$ over general bases. It was summarized in [41], and revisited and completed in [48], [34]. This formalism is used in [23] and [57].

7 Appendix: proof of Theorem 2.3

The following lemma is well known. We give it for lack of a suitable reference.

Lemma 7.1. *Let $S = (S, s, \eta)$ be a strictly local trait, Λ be as in 1.1, (X, Z) a strict semistable pair over S in the sense of de Jong ([40], 6.3). Let $u : X - Z_f \hookrightarrow X$ be the open immersion, where Z_f is the horizontal part of Z , in de Jong's notation. Then, for all $q \in \mathbf{Z}$, the inertia group I acts trivially on $R^q\Psi_X(Ru_{\eta*}\Lambda)$ (resp. $R^q\Psi_X(u_{\eta!}\Lambda)$).*

Proof. The assertion relative to $Ru_{\eta*}\Lambda$ is a particular case of ([46], 3.5) (see also ([30], 8.4.4)). A direct proof can be given as follows. The conclusion has to be checked on the stalks at geometric points x of X_s . By the local

description of strict semistable pairs ([40], 6.4), étale locally at such a point x , X is isomorphic to $X_1 \times_S X_2$, where $X_1 = S[t_1, \dots, t_n]/(t_1 \cdots t_n - \pi)$, $X_2 = S[s_1, \dots, s_m]$, and the horizontal part Z_f is $X_1 \times_S D$, where $D = S[s_1, \dots, s_m]/(s_1 \cdots s_r)$, with $1 \leq r \leq m$, and π is a uniformizing parameter of S . We may therefore assume that $X = X_1 \times_S X_2$ and $Z_f = X_1 \times_S D$. Then $X_\eta - Z_\eta = X_1 \times_S (X_2 - D)$, and $u = \text{Id}_{X_1} \times_S v$, where $v : X_2 - D \hookrightarrow X_2$ is the inclusion. As $Ru_{\eta^*}\Lambda = \Lambda_{X_{1\eta}} \boxtimes Rv_{\eta^*}\Lambda$ (smooth base change), the commutation of $R\Psi$ with external tensor products ([29], 4.7) implies:

$$(7.1.1) \quad R\Psi_X(Ru_{\eta^*}\Lambda) = R\Psi_{X_1}(\Lambda) \boxtimes^L R\Psi_{X_2}(Rv_{\eta^*}\Lambda).$$

As X_2 is smooth over S and D is a relative divisor with normal crossings in X_2 , $R\Phi_{X_2}(Rv_{\eta^*}\Lambda) = 0^{24}$, so that

$$(7.1.2) \quad R\Psi_{X_2}(Rv_{\eta^*}\Lambda) = Rv_{s^*}\Lambda,$$

and in particular I acts trivially on $R\Psi_{X_2}(Rv_{\eta^*}\Lambda)$. On the other hand, I acts trivially on $R^q\Psi_{X_1}(\Lambda)$ for all q (cf. 6.3) for all q . Moreover, by (2.7.1), the stalks of $R^q\Psi_{X_1}(\Lambda)$ are finitely generated and free over Λ , and the same is true of the $R^q\Psi_{X_2}(Rv_{\eta^*}\Lambda)$ by (7.1.2). Therefore, by Künneth, (7.1.1) gives

$$(7.1.3) \quad R^q\Psi_X(Ru_{\eta^*}\Lambda) = \bigoplus_{i+j=q} R^i\Psi_{X_1}(\Lambda) \boxtimes R^j v_{s^*}(\Lambda).$$

As I acts trivially on both factors of each summand in the right hand side, the conclusion follows in this case. Similarly, we have

$$(7.1.4) \quad R\Psi_X(u_{\eta^!}\Lambda) = R\Psi_{X_1}(\Lambda) \boxtimes^L R\Psi_{X_2}(v_{\eta^!}\Lambda),$$

and

$$(7.1.5) \quad R\Psi_{X_2}(v_{\eta^!}\Lambda) = v_{s^!}\Lambda,$$

([2], vol. II, XIII 2.1.11, p.105)²⁵ So (7.1.4) and (7.1.5) yield

$$(7.1.6) \quad R\Psi_X(u_{\eta^!}\Lambda) = u_{s^!}R\Psi_{X-Z_f}(\Lambda).$$

In particular, I acts trivially on $R^q\Psi_X(u_{\eta^!}\Lambda)$ for all q , which finishes the proof. \square

²⁴As can be checked by induction on r , using relative purity, see ([56], Prop. 3.15) for a generalization.

²⁵It can also be deduced from (7.1.2) by duality, as $R\Psi$ commutes with duality. See again ([56], Prop. 3.15) for a generalization.

7.2

Let's prove 2.3. Fix $n \in \mathbf{Z}$. We may assume S strictly local, and, furthermore, complete (which doesn't change the inertia I nor $H^n(X_{\bar{\eta}})$ (resp. $H_c^n(X_{\bar{\eta}})$)). By de Jong's main theorem ([40], 6.5) and a standard step by step construction, up to replacing S by a finite extension, we may assume that we have a compactification $u_{\eta} : X_{\eta} \rightarrow \bar{X}_{\eta}$ (i.e., u_{η} dense open, \bar{X}_{η} proper over η), a simplicial scheme \bar{X}_{\bullet} over S , a closed simplicial scheme $Z_{\bullet} \hookrightarrow \bar{X}_{\bullet}$, and a cartesian diagram

$$(7.2.1) \quad \begin{array}{ccc} (X_{\bullet})_{\eta} & \xrightarrow{(u_{\bullet})_{\eta}} & (\bar{X}_{\bullet})_{\eta} \\ \downarrow & & \varepsilon \downarrow \\ X_{\eta} & \xrightarrow{u_{\eta}} & \bar{X}_{\eta} \end{array}$$

where ε is a proper hypercovering, (\bar{X}_i, Z_i) is a strict semistable pair over S for $i \leq n+1$, and $u_{\bullet} : X_{\bullet} = \bar{X}_{\bullet} - (Z_f)_{\bullet} \hookrightarrow \bar{X}_{\bullet}$ is the inclusion, $(Z_f)_{\bullet}$ denoting the horizontal part of Z_{\bullet} .

(a) *Calculation for H^n .* By proper cohomological descent, we have

$$\begin{aligned} H^n(X_{\bar{\eta}}) &= H^n((X_{\leq n+1})_{\bar{\eta}}) \\ &= H^n((\bar{X}_{\leq n+1})_{\bar{\eta}}, R(u_{\leq n+1})_{\bar{\eta}*}\Lambda) \\ &= H^n((\bar{X}_{\leq n+1})_s, R\Psi_{\bar{X}_{\leq n+1}} R(u_{\leq n+1})_{\eta*}\Lambda). \end{aligned}$$

By (7.1) (applied to the (\bar{X}_i, Z_i) 's) the inertia I acts trivially on the cohomology sheaves $R^q\Psi_{\bar{X}_{\leq n+1}} R(u_{\leq n+1})_{\eta*}\Lambda$. By the spectral sequence

$$\begin{aligned} E_2^{ij} &= H^i((\bar{X}_{\leq n+1})_s, R^j\Psi_{\bar{X}_{\leq n+1}}(R(u_{\leq n+1})_{\eta*}\Lambda)) \\ &\Rightarrow H^{i+j}((\bar{X}_{\leq n+1})_s, R\Psi_{\bar{X}_{\leq n+1}} R(u_{\leq n+1})_{\eta*}\Lambda), \end{aligned}$$

it follows that I acts unipotently with exponent at most n on $H^n(X_{\bar{\eta}})$.

(b) *Calculation for H_c^n .* The proof is similar. By proper cohomological descent, we have

$$H_c^n(X_{\eta}) = H^n((\bar{X}_{\leq n+1})_s, R\Psi_{\bar{X}_{\leq n+1}}(u_{\leq n+1})_{\eta!}\Lambda),$$

and again, by (7.1), I acts trivially on the cohomology sheaves $R^q\Psi_{\bar{X}_{\leq n+1}}(u_{\leq n+1})_{\eta!}\Lambda$, hence unipotently with exponent at most n on $H_c^n(X_{\bar{\eta}})$.

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References

- [1] SGA 5. *Cohomologie l -adique et fonctions L* . Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin-New York, 1977. Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie.
- [2] SGA 7. *Groupes de monodromie en géométrie algébrique*. Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969, I dirigé par A. Grothendieck, II par P. Deligne et N. Katz, Lecture Notes in Math. 288, 340, Springer-Verlag 1972, 1973.
- [3] P. Colmez and J.-P. Serre, editors. *Correspondance Grothendieck-Serre*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 2. Société Mathématique de France, Paris, 2001.
- [4] V. I. Arnol'd. Critical points of smooth functions. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974)*, Vol. 1, pages 19–39. Canad. Math. Congress, Montreal, Que., 1975.
- [5] M. Artin and G. Winters. Degenerate fibres and stable reduction of curves. *Topology*, 10:373–383, 1971.
- [6] L. Bégueri. Dualité sur un corps local à corps résiduel algébriquement clos. *Mém. Soc. Math. France (N.S.)*, (4):121, 1980/81.
- [7] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.
- [8] A. Beilinson and J. Bernstein. A proof of Jantzen conjectures. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 1–50. Amer. Math. Soc., Providence, RI, 1993.
- [9] A. Beilinson. Constructible sheaves are holonomic. *arXiv*, 2015.
- [10] A. Bertapelle and S. Bosch. Weil restriction and Grothendieck’s duality conjecture. *J. Algebraic Geom.*, 9(1):155–164, 2000.
- [11] A. Bertapelle. On perfectness of Grothendieck’s pairing for the l -parts of component groups. *J. Reine Angew. Math.*, 538:223–236, 2001.
- [12] P. Berthelot. Altérations de variétés algébriques (d’après A. J. de Jong). *Astérisque*, (241):Exp. No. 815, 5, 273–311, 1997. Séminaire Bourbaki, Vol. 1995/96.
- [13] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

- [14] S. Bosch. Component groups of abelian varieties and Grothendieck’s duality conjecture. *Ann. Inst. Fourier (Grenoble)*, 47(5):1257–1287, 1997.
- [15] S. Bosch and D. Lorenzini. Grothendieck’s pairing on component groups of Jacobians. *Invent. Math.*, 148(2):353–396, 2002.
- [16] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.
- [17] P. Deligne. Théorie de Hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. Gauthier-Villars, Paris, 1971.
- [18] P. Deligne. La conjecture de Weil. I. *Publ. Math. Inst. Hautes Études Sci.*, (43):273–307, 1974.
- [19] P. Deligne. Théorie de Hodge. III. *Publ. Math. Inst. Hautes Études Sci.*, (44):5–77, 1974.
- [20] P. Deligne. *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.
- [21] P. Deligne. La conjecture de Weil. II. *Publ. Math. Inst. Hautes Études Sci.*, (52):137–252, 1980.
- [22] K. Fujiwara. A proof of the absolute purity conjecture (after Gabber). In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 153–183. Math. Soc. Japan, Tokyo, 2002.
- [23] A. Genestier and V. Lafforgue. Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale. Preprint, 2015.
- [24] P. A. Griffiths. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.
- [25] P. A. Griffiths. A transcendental method in algebraic geometry. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 113–119. Gauthier-Villars, Paris, 1971.
- [26] P. Griffiths and W. Schmid. Recent developments in Hodge theory: a discussion of techniques and results. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 31–127. Oxford Univ. Press, Bombay, 1975.

- [27] J.-i. Igusa. Abstract vanishing cycle theory. *Proc. Japan Acad.*, 34:589–593, 1958.
- [28] L. Illusie. Réalisation ℓ -adique de l'accouplement de monodromie d'après A. Grothendieck. *Astérisque*, (196-197):7, 27–44 (1992), 1991. Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).
- [29] L. Illusie. Autour du théorème de monodromie locale. *Astérisque*, (223):9–57, 1994. Périodes p -adiques (Bures-sur-Yvette, 1988).
- [30] L. Illusie. An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. *Astérisque*, (279):271–322, 2002. Cohomologies p -adiques et applications arithmétiques, II.
- [31] L. Illusie. Sur la formule de Picard-Lefschetz. In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 249–268. Math. Soc. Japan, Tokyo, 2002.
- [32] L. Illusie. On semistable reduction and the calculation of nearby cycles. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 785–803. Walter de Gruyter, Berlin, 2004.
- [33] L. Illusie. Miscellany on traces in ℓ -adic cohomology: a survey. *Jpn. J. Math.*, 1(1):107–136, 2006.
- [34] L. Illusie. Exposé XI. Produits orientés. *Astérisque*, (363-364):213–234, 2014. Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents.
- [35] L. Illusie. From Pierre Deligne's secret garden: looking back at some of his letters. *Jpn. J. Math.*, 10(2):237–248, 2015.
- [36] T. Ito. Weight-monodromy conjecture for certain threefolds in mixed characteristic. *Int. Math. Res. Not.*, (2):69–87, 2004.
- [37] T. Ito. Weight-monodromy conjecture over equal characteristic local fields. *Amer. J. Math.*, 127(3):647–658, 2005.
- [38] T. Ito. Weight-monodromy conjecture for p -adically uniformized varieties. *Invent. Math.*, 159(3):607–656, 2005.
- [39] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995.
- [40] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.

- [41] G. Laumon. Vanishing cycles over a base of dimension ≥ 1 . In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1983.
- [42] B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.
- [43] W. G. McCallum. Duality theorems for Néron models. *Duke Math. J.*, 53(4):1093–1124, 1986.
- [44] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.
- [45] D. Mumford. Bi-extensions of formal groups. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pages 307–322. Oxford Univ. Press, London, 1969.
- [46] C. Nakayama. Nearby cycles for log smooth families. *Compositio Math.*, 112(1):45–75, 1998.
- [47] C. Nakayama. Degeneration of l -adic weight spectral sequences. *Amer. J. Math.*, 122(4):721–733, 2000.
- [48] F. Orgogozo. Modifications et cycles proches sur une base générale. *Int. Math. Res. Not.*, pages Art. ID 25315, 38, 2006.
- [49] M. Rapoport and T. Zink. Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. *Invent. Math.*, 68(1):21–101, 1982.
- [50] M. Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin-New York, 1970.
- [51] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, (38):27–76, 1970.
- [52] M. Raynaud. Variétés abéliennes et géométrie rigide. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 473–477. Gauthier-Villars, Paris, 1971.
- [53] K. A. Ribet. On modular representations of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.
- [54] T. Saito. Vanishing cycles and geometry of curves over a discrete valuation ring. *Amer. J. Math.*, 109(6):1043–1085, 1987.

- [55] T. Saito. Weight spectral sequences and independence of ℓ . *J. Inst. Math. Jussieu*, 2(4):583–634, 2003.
- [56] T. Saito. Wild ramification and the cotangent bundle. *arXiv*, 2013. To appear in *J. Alg. Geom.*
- [57] T. Saito. The characteristic cycle and the singular support of a constructible sheaf. *arXiv*, 2015.
- [58] S. Schieder. The Drinfeld-Lafforgue-Vinberg degeneration : Picard-Lefschetz oscillators. *arXiv*, 2014.
- [59] P. Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.
- [60] J.-P. Serre and J. Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.
- [61] J. Steenbrink. Limits of Hodge structures. *Invent. Math.*, 31(3):229–257, 1975/76.
- [62] S. Sun and W. Zheng. Parity and symmetry in intersection and ordinary cohomology. *arXiv*, 2014.
- [63] T. Suzuki. Grothendieck’s pairing on Néron component groups: Galois descent from the semistable case. *arXiv*, 2015.
- [64] J. T. Tate. p -divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.
- [65] M. Temkin. Stable modification of relative curves. *J. Algebraic Geom.*, 19(4):603–677, 2010.
- [66] N. Umezaki. A Uniform Bound for the Order of Monodromy. *arXiv*, 2014.
- [67] A. Werner. On Grothendieck’s pairing of component groups in the semistable reduction case. *J. Reine Angew. Math.*, 486:205–215, 1997.