Kähler-Einstein metrics and projective embeddings *

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1. In this paper, we will consider compact kählerian manifolds with negative or zero first Chern class. Since the work of Aubin-Calabi-Yau, it is known that such a manifold M carries a unique Kähler-Einstein metric in each Kähler class if $c_1(M) = 0$, and a unique Kähler-Einstein metric with Einstein constant -1 if $c_1(M) < 0$.

When $c_1(M) < 0$, or more generally when M is projective, one can ask whether one of the Kähler-Einstein metrics carried by M can be realized by a complex isometric embedding of M into a complex projective space equipped with its Fubini-Study metric g_{F-S}. The following asserts that this never happens.

THEOREM : *Let $(M^n, g) \hookrightarrow (\mathbb{P}^N, g_{\mathrm{F-S}})$ be an Einstein compact complex submanifold of the projective space. Then the Einstein constant of M is strictly positive.*

(In the sequel, we will normalize the Fubini metric g_{F-S} so that it has constant holomorphic sectional curvature 4, and will place no restriction on the value of the Einstein constant of M.)

This result can be seen as an extension of the well-known theorem by E. Calabi [5], which states that $(\mathbb{P}^N, g_{F-S})$ admits no complex submanifold with nonpositive constant holomorphic sectional curvature.

On the other hand, flag manifolds provide us with a series of Fano (homogeneous) examples of complex submanifolds of $(\mathbb{P}^N, g_{F-S})$ which are Einstein for the induced metric [6].

When $c_1(M) < 0$, that is when the canonical bundle K_M of M is ample, it is worth

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comparing our result with the following asymptotic theorem obtained by G. Tian [7] and T. Bouche [3] :

> Let g be the unique Kähler-Einstein metric on M with Einstein constant -1. One can build, from each (high) power K_M^m of the cano–nical bundle of M, a projective embedding $i_m \ : M \hookrightarrow (\mathbb{P}^{N(m)}, g_{\text{F-S}})$, so that if we denote by g_m the corresponding induced metric, the normalized sequence of metrics g_m/m converges C^2 to g.

Here, the metric g is obtained as a limit of metrics induced by complex embeddings into projective spaces (which dimensions and holomorphic sectional curvatures are unbounded).

Let us finally note, as a particular case of the above theorem, the fact that none of the Calabi-Yau metrics carried by an algebraic $K3$ surface can be realized by a projective embedding. The question was raised in [4] by J-P. Bourguignon.

The sequel of the paper is devoted to the proof of the theorem.

2. We work in the projective space $\mathbb{P}^N = (\mathbb{C}^{N+1}\setminus\{0\})/\mathbb{C}^*$ equipped with its Fubini metric (with constant holomorphic sectional curvature 4) and consider a (connected) complex submanifold $i : M^n \hookrightarrow \mathbb{P}^N$ endowed with the induced Kähler metric g.

Let us pick up a point m in M, and choose a unitary frame $(e_0, \ldots, e_N)$ for $\mathbb{C}^{N+1}$ with $m = [\mathbb{C}\cdot e_0]$, and in such a way that, if $\widehat{M} \subset \mathbb{C}^{N+1}\setminus\{0\}$ denotes the cone above M, the tangent space to $\widehat{M}$ at any point $\hat{m} \in \mathbb{C}\cdot e_0$ is spanned by the first $(n+1)$ vectors $(e_0, \ldots, e_n)$.

Let then $\mathbb{P}^{N-1}(m)$ be the hyperplane at infinity relative to the point m, that is the set of all complex lines in $\mathbb{C}^{N+1}$ which are perpendicular to e_0 ; the homogeneous coordinate system $[1; z_1, \ldots, z_n; z_{n+1}, \ldots, z_N]$ associated to our frame allows us to identify $\mathbb{P}^N\setminus\mathbb{P}^{N-1}(m)$ with $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^{N-n}$; the immersion i is then given around m by a graph

$$z = (z_1, \ldots, z_n) \in \mathbb{C}^n \longrightarrow [1, z_1, \ldots, z_n; f_1(z), \ldots, f_{N-n}(z)] \in \mathbb{C}^n \times \mathbb{C}^{N-n},$$

where the $(f_j)_{1\leq j\leq N-n}$ are holomorphic functions which vanish at the order 2 at the origin.

Let us denote by $M(m) = M \cap \mathbb{P}^{N-1}(m)$ the part of M which lies in the hyperplane at infinity relative to m. Assuming that (M, g) is Einstein, we will prove that the restriction to $M \setminus M(m) \subset \mathbb{C}^n \times \mathbb{C}^{N-n}$ of the n first homogeneous coordinates $(z_1, \ldots, z_n)$ of $\mathbb{P}^N$ –which are holomorphic on $M\setminus M(m)$– actually provides us with a local coordinate system in the neighbourhood of any point of $M\setminus M(m)$ (although this was *a priori* only true in the neighbourhood of m) ; moreover we will derive a simple identity linking the riemannian volume element of $M\backslash M(m)$, and the euclidean volume element of the chart $\mathbb{C}^n$.

The following proof was inspired by [5] and [1], who exhibit, in the neighbourhood of any point of a Kähler-analytic manifold, a preferred potential and local coordinate system.

3. The function $\log(1+\sum_{i=1}^N |z_i|^2)$ is a Kähler potential for the Fubini-Study metric on $\mathbb{P}^N\backslash\mathbb{P}^{N-1}(m)$. It induces by restriction to $M\backslash M(m)$ a Kähler potential for g, which reads in our chart around m as :

$$D(z) = \log(1 + \sum_{\alpha=1}^{n} |z_\alpha|^2 + \sum_{j=1}^{N-n} |f_j|^2) = \log(1 + |z|^2 + |f|^2)\,.$$

Let us denote by ω and ρ the Kähler and Ricci forms of g ; around m,

$$\begin{aligned} \omega &= \frac{i}{2}\sum g_{\alpha\bar\beta}\, dz_\alpha \wedge d\bar z_\beta = \frac{i}{2}\,\partial\bar\partial\, D \\ \rho &= -i\,\partial\bar\partial\, \log(\det g_{\alpha\bar\beta}) \end{aligned}$$

hold, where $\det g_{\alpha\bar\beta} = \det(\partial^2 D/\partial z_\alpha \partial\bar z_\beta)$ denotes the riemannian volume element for (M, g) expressed in our chart (z_α).

Let us assume from now on that (M, g) is Einstein with Einstein constant $2k$; then $\rho = 2k\omega$, and there exists around m an holomorphic function φ satisfying

$$\log\det(\frac{\partial^2 D}{\partial z_\alpha\,\partial\bar z_\beta}) = -k\,D + \varphi + \bar\varphi\ ;$$

now since

$$D = |z|^2 + \sum_{|a|\geq 2, |b|\geq 2} c_{a,b}\, z^a\,\bar z^b\ ,$$

only mixed terms (that is of the form $z^a\bar z^b$ with $a \neq 0$ and $b \neq 0$) will show up in the $(z, \bar z)$ series expansion of the left side of the above identity ; this will force $\varphi + \bar\varphi = 0$, hence around m :

$$\det(\frac{\partial^2 D}{\partial z_\alpha\,\partial\bar z_\beta}) = (1 + |z|^2 + |f|^2)^{-k} = \mathrm{e}^{-kD}\,,$$

or better, denoting by v_g the riemannian volume form for (M, g) :

$$v_g = i^n 2^{-n}\,(1 + \sum_{\alpha=1}^{n} |z_\alpha|^2 + \sum_{j=1}^{N-n} |z_j|^2)^{-k}\, dz_1 \wedge d\bar z_1 \wedge \ldots \wedge dz_n \wedge d\bar z_n\,. \qquad (*)$$

Both sides of this identity being real analytic on $M\backslash M(m)$, we infer that $(*)$ is actually satisfied on the whole (connected) $M\backslash M(m)$. In particular, the projection

$$\pi_m\ :\ [1; z_1, \ldots, z_n; z_{n+1}, \ldots, z_N] \in M\backslash M(m) \longrightarrow (z_1, \ldots, z_n) \in \mathbb{C}^n$$

provides us in the neighbourhood of any point of $M\backslash M(m)$ with a local coordinate system.

4. From now on, we will assume that M is compact hence algebraic by Chow's theorem, and that the Einstein constant $2k$ of (M,g) is nonpositive. Then, the identity $(*)$ implies that, at each point of $M\backslash M(m)$, the riemannian volume element of (M,g) is bounded below by the euclidean volume element of the chart π_m. This allows us to estimate from below the riemannian volume of M :

$$\mathrm{vol}\,(M,g) \geq \mathrm{vol}_{\mathrm{eucl}}\,(\pi_m(M\backslash M(m))\,. \qquad (**)$$

On the other hand, the algebraic map $\pi_m : M\backslash M(m) \longrightarrow \mathbb{C}^n$ is open, hence its image is Zariski dense in $\mathbb{C}^n$ (that is, π_m is a dominant morphism). Thus Chevalley's theorem ([2]) asserts that this image actually contains a Zariski open subset of $\mathbb{C}^n$, hence is of infinite euclidean volume, a contradiction with $(**)$: the theorem is proved.

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