

Harmonic quasi-isometric maps between rank one symmetric spaces

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Abstract

We prove that a quasi-isometric map between rank one symmetric spaces is within bounded distance from a unique harmonic map. In particular, this completes the proof of the Schoen-Li-Wang conjecture.

1 Introduction

1.1 Main result

We first explain the title.

A *symmetric space* is a Riemannian manifold X such that, for every point x in X , there exists a symmetry centered at x , i.e. an isometry s_x of X fixing the point x and whose differential at x is minus the identity.

In this article we will call *rank one symmetric space* a symmetric space X whose sectional curvature is everywhere negative : $K_X < 0$. The list of rank one symmetric spaces is well known. They are the real hyperbolic spaces $\mathbb{H}_{\mathbb{R}}^p$, the complex hyperbolic spaces $\mathbb{H}_{\mathbb{C}}^p$, the quaternion hyperbolic spaces $\mathbb{H}_{\mathbb{Q}}^p$, with $p \geq 2$, and the Cayley hyperbolic plane $\mathbb{H}_{\mathbb{C}a}^2$.

A map $f : X \rightarrow Y$ between two metric spaces X and Y is said to be *quasi-isometric* if there exists a constant $c \geq 1$ such that f is *c-quasi-isometric* i.e. such that, for all x, x' in X , one has

$$c^{-1} d(x, x') - c \leq d(f(x), f(x')) \leq c d(x, x') + c. \quad (1.1)$$

Such a map f is called a *quasi-isometry* if one has $\sup_{y \in Y} d(y, f(X)) < \infty$.

A map $h : X \rightarrow Y$ between two Riemannian manifolds X and Y is said to be *harmonic* if its tension field is zero i.e. if it satisfies the elliptic nonlinear partial differential equation $\text{tr}(D^2h) = 0$ where D^2h is the second covariant derivative of h . For instance, an isometric map with totally geodesic image is always harmonic. The problem of the existence, regularity and uniqueness of harmonic maps under various boundary conditions is a very classical

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topic (see [9], [16], [8], [35], [34] or [22]). In particular, when Y is simply connected and has non positive curvature, a harmonic map is always C^∞ i.e. it is indefinitely differentiable, and is a minimum of the energy functional -see Formula (3.4)- among maps that agree with h outside a compact subset of X .

The aim of this article is to prove the following

Theorem 1.1. *Let $f : X \rightarrow Y$ be a quasi-isometric map between rank one symmetric spaces X and Y . Then there exists a unique harmonic map $h : X \rightarrow Y$ which stays within bounded distance from f i.e. such that*

$$\sup_{x \in X} d(h(x), f(x)) < \infty .$$

The uniqueness of h is due to Li and Wang in [21, Theorem 2.3]. In this article we prove the existence of h .

1.2 Previous results and conjectures

When X is equal to Y , Theorem 1.1 was conjectured by Li and Wang in [21, Introduction] extending a conjecture of Schoen in [31] for the case $X = Y = \mathbb{H}_{\mathbb{R}}^2$. When these conjectures were formulated, the case where $X = Y$ is either equal to $\mathbb{H}_{\mathbb{Q}}^p$ or $\mathbb{H}_{\mathbb{C}a}^2$ was already known by a previous result of Pansu in [27]. In that case the harmonic map h is an onto isometry.

The uniqueness part of the Schoen conjecture was quickly settled by Li and Tam in [20], and the proof was extended by Li and Wang to rank one symmetric spaces in their paper [21]. Partial results towards the existence statement in the Schoen conjecture were obtained in [36], [15], [29], [23], [5]. A major breakthrough was then achieved by Markovic who proved successively the Li-Wang conjecture for the case $X = Y = \mathbb{H}_{\mathbb{R}}^3$ in [25], for the case $X = Y = \mathbb{H}_{\mathbb{R}}^2$ in [24] thus solving the initial Schoen conjecture, and very recently with Lemm for the case $X = Y = \mathbb{H}_{\mathbb{R}}^p$ with $p \geq 3$ in [19].

As a corollary of Theorem 1.1, we complete the proof of the Li-Wang conjecture. In particular, we obtain the following.

Corollary 1.2. *For $p \geq 1$, any quasi-isometric map $f : \mathbb{H}_{\mathbb{C}}^p \rightarrow \mathbb{H}_{\mathbb{C}}^p$ is within bounded distance from a unique harmonic map $h : \mathbb{H}_{\mathbb{C}}^p \rightarrow \mathbb{H}_{\mathbb{C}}^p$.*

Another new feature in Theorem 1.1 is that one does not assume that X and Y have the same dimension. Even the following special case is new.

Corollary 1.3. *Any quasi-isometric map $f : \mathbb{H}_{\mathbb{R}}^2 \rightarrow \mathbb{H}_{\mathbb{R}}^3$ is within bounded distance from a unique harmonic map $h : \mathbb{H}_{\mathbb{R}}^2 \rightarrow \mathbb{H}_{\mathbb{R}}^3$.*

We finally recall that, according to a well-known result of Kleiner and Leeb in [18], every quasi-isometry $f : X \rightarrow Y$ between irreducible higher rank symmetric spaces stays within bounded distance of an isometric map,

after a suitable scalar rescaling of the metrics. Another proof of this result has also been given by Eskin and Farb in [10].

However one can not extend Theorem 1.1 to irreducible higher rank symmetric spaces. For instance, let X and Y be symmetric spaces such that $X \times \mathbb{R}$ embeds isometrically into Y and let $x_0 \in X$. Then the quasi-isometric map $f : X \rightarrow X \times [0, \infty[\subset Y$ given by $f(x) = (x, d(x_0, x))$ does not stay within bounded distance of a harmonic map. Indeed the second component of such an harmonic map would be a non-constant harmonic function that reaches its minimum value, a contradiction to the maximum principle.

1.3 Motivation

We now briefly recall a few definitions and facts that are useful to understand the context and the motivation of Theorem 1.1. None of them will be used in the other sections of this article.

Let X and Y be rank one symmetric spaces. Recall first that X is diffeomorphic to \mathbb{R}^k and has a visual compactification $X \cup \partial X$. The visual boundary ∂X is homeomorphic to a topological sphere \mathbb{S}^{k-1} . Choosing a base point O in X , this boundary is endowed with the Gromov quasidistance d' defined by $d'(\xi, \eta) := e^{-(\xi|\eta)_O}$ for ξ, η in ∂X where $(\xi|\eta)_O$ denotes the Gromov product (see [13, Sec. 7.3]). A non-constant continuous map $F : \partial X \rightarrow \partial Y$ between the boundaries is called *quasisymmetric* if there exists $K \geq 1$ such that for all ξ, η, ζ in ∂X with $d'(\xi, \eta) \leq d'(\xi, \zeta)$, one has $d'(F(\xi), F(\eta)) \leq K d'(F(\xi), F(\zeta))$.

The following nice fact, which is also true for a wider class of geodesic Gromov hyperbolic spaces, gives another point of view on quasi-isometric maps :

Fact 1.4. *Let X, Y be rank one symmetric spaces.*

- a) *Any quasi-isometric map $f : X \rightarrow Y$ induces a boundary map $\partial f : \partial X \rightarrow \partial Y$ which is quasisymmetric.*
- b) *Two quasi-isometric maps $f, g : X \rightarrow Y$ have the same boundary map $\partial f = \partial g$ if and only if f and g are within bounded distance from one another.*
- c) *Any quasisymmetric map $F : \partial X \rightarrow \partial Y$ is the boundary map $F = \partial f$ of a quasi-isometric map $f : X \rightarrow Y$.*

This fact has a long history. Point *a* is in the paper of Mostow [26] and was extended later by Gromov in [14, Sec. 7] (see also [13, Sec. 7]). Point *b* is in the paper of Pansu [27, Sec. 9]. Point *c* is in the paper of Bonk and Schramm [4, Thms 7.4 and 8.2] extending previous results of Tukia in [37], of Paulin in [28] (see also [3] and [6] for related questions).

Recall that a diffeomorphism f of X is said to be *quasiconformal* if the function $x \mapsto \|Df(x)\| \|Df(x)^{-1}\|$ is uniformly bounded on X . The original formulation of the Schoen conjecture involved quasiconformal diffeomorphisms instead of quasi-isometries : *Every quasisymmetric homeomorphism*

F of \mathbb{S}^1 is the boundary map of a unique quasiconformal harmonic diffeomorphism of $\mathbb{H}_{\mathbb{R}}^2$.

Relying on a previous result of Wan in [38, Theorem 13], Li and Wang pointed out in [21, Theorem 1.8] that a harmonic map between the hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$ and itself is a quasiconformal diffeomorphism if and only if it is a quasi-isometric map. This is why Li and Wang formulated in [21] the higher dimensional generalisation of the Schoen conjecture using quasi-isometries instead of quasiconformal diffeomorphisms.

Note that in dimension $k \geq 6$ there exist harmonic maps between compact manifolds of negative curvature which are homotopic to a diffeomorphism but which are not diffeomorphisms (see [11]).

1.4 Strategy

To prove our Theorem 1.1, we start with a c -quasi-isometric map $f : X \rightarrow Y$ between rank one symmetric spaces. We want to exhibit a harmonic map $h : X \rightarrow Y$ within bounded distance from f .

We will first gather in Chapter 2 a few properties of Hadamard manifolds : images of triangles under quasi-isometric maps, Hessian of the distance function, gradient estimate for functions with bounded Laplacian.

The first key point in our proof is the simple remark that, thanks to a smoothing process, we may assume without loss of generality that the c -quasi-isometric map f is \mathcal{C}^∞ and that its first and second covariant derivatives Df and D^2f are uniformly bounded on X (Proposition 3.4). We fix a point O in X . For $R > 0$, we denote by $B_R := B(O, R)$ the closed ball in X with center O and radius R and by ∂B_R the sphere that bounds B_R . We introduce the unique harmonic map $h_R : B_R \rightarrow Y$ whose restriction to the sphere ∂B_R is equal to f . This map h_R is \mathcal{C}^∞ on the closed ball B_R . The harmonic map h will be constructed as the limit of the maps h_R when R goes to infinity. In order to prove the existence of this limit h , using a classical compactness argument that we will recall in Section 3.3, we just have to check that on the balls B_R the distances

$$\rho_R := d(h_R, f)$$

are uniformly bounded in R . We will argue by contradiction and assume that we can find radii R with ρ_R arbitrarily large.

The second key point in our proof is what we call the boundary estimate (Proposition 3.8). It tells us that the ratio $\frac{d(h_R(x), f(x))}{d(x, \partial B_R)}$ is uniformly bounded for $R \geq 1$ and x in B_R . In particular, when ρ_R is large, the ball $B(O, R-1)$ contains a ball $B(x_R, r_R)$ whose center x_R satisfies $d(h_R(x_R), f(x_R)) = \rho_R$ and whose radius $r_R \geq 1$ is quite large. A good

choice for the radius r_R will be $r_R = \rho_R^{1/3}$. We will focus on the restriction of the maps f and h_R to this ball $B(x_R, r_R)$. Let $y_R := f(x_R)$. For z in $B(x_R, r_R)$, we will write

$$f(z) = \exp_{y_R}(\rho_f(z)v_f(z)) \quad \text{and} \quad h_R(z) = \exp_{y_R}(\rho_h(z)v_h(z)),$$

where $\rho_f(z), \rho_h(z)$ are non negative and $v_f(z), v_h(z)$ lie in the unit sphere $T_{y_R}^1 Y$ of the tangent space $T_{y_R} Y$. We write $v_R := v_h(x_R)$ and we denote by $\theta(v_1, v_2)$ the angle between two vectors v_1, v_2 of the sphere $T_{y_R}^1 Y$.

The third key point in our proof is to write for each point z on the sphere $S(x_R, r_R)$ the triangle inequality

$$\theta(v_f(z), v_R) \leq \theta(v_f(z), v_h(z)) + \theta(v_h(z), v_R)$$

and, adapting an idea of Markovic in [24], to focus on the set

$$W_R := \{z \in S(x_R, r_R) \mid \rho_h(z) \geq \rho_R - \frac{r_R}{2c} \text{ and } \rho_h(z_t) \geq \frac{\rho_R}{2} \text{ for } 0 \leq t \leq r_R\}$$

where $(z_t)_{0 \leq t \leq r_R}$ is the geodesic segment between x_R and z .

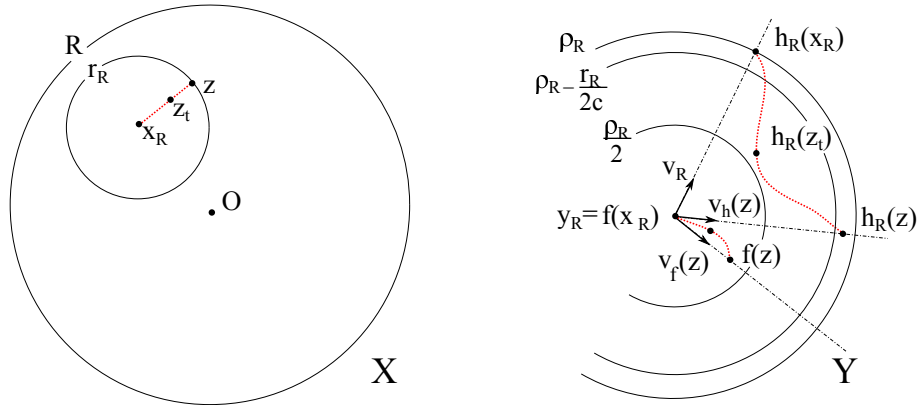


Figure 1: The vectors $v_f(z), v_h(z)$ and v_R when z belongs to W_R .

The contradiction will come from the fact that when both R and ρ_R go to infinity, the two angles $\theta_1 := \theta(v_f(z), v_h(z))$ and $\theta_2 := \theta(v_h(z), v_R)$ converge to 0 uniformly for z in W_R , while one can find $z = z_R$ in W_R such that the other angle $\theta_0 = \theta(v_f(z), v_R)$ stays away from 0. Here is a rough sketch of the arguments used to estimate these three angles.

To get the upper bound for the angle θ_1 (Lemma 4.5), we use the relation between angles and Gromov products (Lemma 2.1) and we notice that the set W_R has been chosen so that the Gromov product $(f(z)|h_R(z))_{y_R}$ is large.

To get the upper bound for the angle θ_2 (Lemma 4.6), we check that the gradient Dv_h is uniformly small on the geodesic segment between x_R and z .

This follows from the comparison inequality $2 \sinh(\rho_h/2) \|Dv_h\| \leq \|Dh_R\|$, from the bound for $\|Dh_R\|$ that is due to Cheng (Lemma 3.3), and from the definition of W_R that ensures that the factor $\sinh(\rho_h/2)$ stays very large on this geodesic.

To find a point $z = z_R$ in W_R such that the angle θ_0 is not small (Lemma 4.7), we use the almost invariance of the Gromov products -and hence of the angles- under a quasi-isometric map (Lemma 2.2). We also use a uniform lower bound on the measure of W_R (Lemma 4.4). This lower bound is a consequence of the subharmonicity of the function ρ_h (Lemma 3.2) and of Cheng's estimate.

We would now like to point out the difference between our approach and those of the previous papers. The starting point of Markovic's method in [24] is the fact that any K -quasisymmetric homeomorphism of the circle is a uniform limit of K -quasisymmetric diffeomorphisms. This fact has no known analog in high dimension. The starting point of the methods in both [25] and [19] is the fact that a quasisymmetric homeomorphism of the sphere \mathbb{S}^{k-1} is almost surely differentiable. This fact is not true on \mathbb{S}^1 . Since our strategy avoids the use of quasisymmetric maps, it gives a unified approach for all $\mathbb{H}_{\mathbb{R}}^p$ with $p \geq 2$, and also works when X and Y have different dimensions.

The assumption that X and Y have negative curvature is used in several places, for instance in the boundary estimates in Proposition 3.8 or in the angle estimates in Lemma 2.1. The assumption that X and Y are symmetric spaces is also used in several places, for instance in the Green formula in Lemma 4.4 or in the smoothing process in Proposition 3.4.

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2 Hadamard manifolds

In this preliminary chapter, we recall various estimates on a Hadamard manifold : for the angles of a geodesic triangle, for the Hessian of the distance function, and also for functions with bounded Laplacian.

2.1 Triangles and quasi-isometric maps

We first recall basic estimates for triangles in Hadamard manifolds and explain how one controls the angles of the image of a triangle under a quasi-isometric map.

All the Riemannian manifolds will be assumed to be connected and to have dimension at least two. We will denote by d their distance function.

A Hadamard manifold is a complete simply connected Riemannian manifold X of non positive curvature $K_X \leq 0$. For instance, the Euclidean space \mathbb{R}^k is a Hadamard manifold with zero curvature $K_X = 0$, while the rank one symmetric spaces are Hadamard manifolds with negative curvature $K_X < 0$. We will always assume without loss of generality that the metric on a rank one symmetric space X is normalized so that $-1 \leq K_X \leq -1/4$.

Let x_0, x_1, x_2 be three points on a Hadamard manifold X . The *Gromov product* of the points x_1 and x_2 seen from x_0 is defined as

$$(x_1|x_2)_{x_0} := (d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2))/2. \quad (2.1)$$

We recall the basic comparison lemma which is one of the motivations for introducing the Gromov product.

Lemma 2.1. *Let X be a Hadamard manifold with $-1 \leq K_X \leq -a^2 < 0$, let T be a geodesic triangle in X with vertices x_0, x_1, x_2 , and let θ_0 be the angle of T at the vertex x_0 .*

a) *One has $(x_0|x_2)_{x_1} \geq d(x_0, x_1) \sin^2(\theta_0/2)$.*

b) *One has $\theta_0 \leq 4e^{-a(x_1|x_2)_{x_0}}$.*

c) *Moreover, when $\min((x_0|x_1)_{x_2}, (x_0|x_2)_{x_1}) \geq 1$, one has $\theta_0 \geq e^{-(x_1|x_2)_{x_0}}$.*

Proof. Assume first that X is the hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$ with curvature -1 . Let $\ell_0 := d(x_1, x_2)$, $\ell_1 := d(x_0, x_1)$, $\ell_2 := d(x_0, x_2)$ be the side lengths of T and $m := (\ell_1 + \ell_2 - \ell_0)/2$ so that

$$(x_1|x_2)_{x_0} = m, \quad (x_0|x_2)_{x_1} = \ell_1 - m, \quad (x_0|x_1)_{x_2} = \ell_2 - m.$$

The hyperbolic law of cosines

$$\cosh(\ell_0) = \cosh(\ell_1)\cosh(\ell_2) - \cos(\theta_0)\sinh(\ell_1)\sinh(\ell_2)$$

can be rewritten as

$$\sin^2(\theta_0/2) = \frac{\sinh(\ell_1 - m)}{\sinh(\ell_1)} \frac{\sinh(\ell_2 - m)}{\sinh(\ell_2)}. \quad (2.2)$$

Hence, one has $\sin^2(\theta_0/2) \leq \frac{\ell_1 - m}{\ell_1}$. This proves a) when $K_X = -1$.

We still assume that $K_X = -1$. Equation (2.2) and the basic inequality

$$\frac{\sinh(\ell - m)}{\sinh(\ell)} \leq e^{-m} \quad \text{for } 0 \leq m \leq \ell$$

yield $\theta_0 \leq 4e^{-(x_1|x_2)_{x_0}}$.

When $K_X = -a^2$, one deduces the bounds a) and b) by a rescaling.

Assuming again $K_X = -1$, the bound c) follows from (2.2) and

$$\frac{\sinh(\ell - m)}{\sinh(\ell)} \geq e^{-m}/2 \quad \text{for } 0 \leq m \leq \ell - 1.$$

Finally, when the sectional curvature of X is pinched between -1 and $-a^2$, the triangle comparison theorems of Alexandrov and Toponogov (see [17, Theorems 4.1 and 4.2]) ensure that these results also hold in X . \square

We now recall the effect of a quasi-isometric map on the Gromov product.

Lemma 2.2. *Let X, Y be Hadamard manifolds with $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$, and let $f : X \rightarrow Y$ be a c -quasi-isometric map. There exists $A = A(a, b, c) > 0$ such that, for all x_0, x_1, x_2 in X , one has*

$$c^{-1}(x_1|x_2)_{x_0} - A \leq (f(x_1)|f(x_2))_{f(x_0)} \leq c(x_1|x_2)_{x_0} + A. \quad (2.3)$$

Proof. This is a general property of quasi-isometric maps between Gromov δ -hyperbolic spaces which is due to M. Burger. See [13, Prop. 5.15]. \square

2.2 Hessian of the distance function

We now recall basic estimates for the Hessian of the distance function and of its square on a Hadamard manifold.

When x_0 is a point in a Riemannian manifold X , we denote by d_{x_0} the distance function defined by $d_{x_0}(x) = d(x_0, x)$ for x in X . We denote by $d_{x_0}^2$ the square of this function. When F is a \mathcal{C}^2 function on X , we denote by DF the differential of F and by D^2F the Hessian of F , which is by definition the second covariant derivative of F .

Lemma 2.3. *Let X be a Hadamard manifold and $x_0 \in X$.*

a) *The Hessian of the square $d_{x_0}^2$ satisfies on X*

$$D^2d_{x_0}^2 \geq 2g_X, \quad (2.4)$$

where g_X is the Riemannian metric on X .

b) *Assume that $-b^2 \leq K_X \leq -a^2 < 0$. The Hessian of the distance function d_{x_0} satisfies on $X \setminus \{x_0\}$*

$$a \coth(ad_{x_0})g_0 \leq D^2d_{x_0} \leq b \coth(bd_{x_0})g_0, \quad (2.5)$$

where $g_0 := g_X - Dd_{x_0} \otimes Dd_{x_0}$.

Proof. a) When $X = \mathbb{R}^k$ is the k -dimensional Euclidean space, one has $D^2d_{x_0}^2 = 2g_X$. The general statement follows from this model case and the Alexandrov triangle comparison theorem.

b) Assume first that $X = \mathbb{H}_{\mathbb{R}}^2$ is the real hyperbolic plane with curvature -1 . Using the expression $\cosh(\ell_t) = \cosh(\ell_0) \cosh(t)$ for the length ℓ_t of the hypotenuse of a right triangle with side lengths ℓ_0 and t , one infers that

$$D^2d_{x_0} = \coth(d_{x_0})g_0.$$

The general statement follows by the same argument combined again with the Alexandrov and Toponogov triangle comparison theorems. \square

2.3 Functions with bounded Laplacian

We give a bound for functions defined on balls of a Hadamard manifold, when their Laplacian is bounded and their boundary value is equal to 0.

The Laplace-Beltrami operator Δ on a Riemannian manifold X is defined as the trace of the Hessian. In local coordinates, the Laplacian of a function F reads as

$$\Delta F = \operatorname{tr}(D^2 F) = \frac{1}{V} \sum_{i,j=1}^k \frac{\partial}{\partial x_i} (V g_X^{ij} \frac{\partial}{\partial x_j} F)$$

where $V = \sqrt{\det(g_X^{ij})}$ is the volume density. The function F is said to be harmonic if $\Delta F = 0$, subharmonic if $\Delta F \geq 0$ and superharmonic if $\Delta F \leq 0$. The study of harmonic functions on Hadamard manifolds has been initiated by Anderson and Schoen in [1].

Proposition 2.4. *Let X be a Hadamard manifold with $K_X \leq -a^2 < 0$. Let O be a point of X and $B_R = B(O, R)$ be the closed ball with center O and radius $R > 0$. Let G be a C^2 function on B_R and let $M > 0$. Assume that*

$$\begin{aligned} |\Delta G| &\leq M \quad \text{on } B_R \\ G &= 0 \quad \text{on } \partial B_R. \end{aligned} \tag{2.6}$$

Then, for all x in B_R , one has the upper bound

$$|G(x)| \leq \frac{M}{a} d(x, \partial B_R). \tag{2.7}$$

Remark. The assumption in Proposition 2.4 that the curvature is negative is essential. Indeed, the function $G := R^2 - d_O^2$ on the ball B_R of the Euclidean space $X = \mathbb{R}^k$ satisfies (2.6) with $M = 2k$ while the ratio $|G(x)|/d(x, \partial B_R)$ cannot be bounded independently of R .

The proof of Proposition 2.4 relies on the following.

Lemma 2.5. *Let X be a Hadamard manifold with $K_X \leq -a^2$ and x_0 be a point of X . Then, the function d_{x_0} is subharmonic. More precisely, the distribution $\Delta d_{x_0} - a$ is non-negative.*

Proof of Lemma 2.5. Since X is a Hadamard manifold, Lemma 2.3 ensures that the function d_{x_0} is C^∞ on $X \setminus \{x_0\}$ and satisfies $\Delta d_{x_0}(x) \geq a$ for $x \neq x_0$. It remains to check that the distribution $\Delta d_{x_0} - a$ is non-negative on X . The function d_{x_0} is the uniform limit when ε converges to 0 of the C^∞ functions $d_{x_0, \varepsilon} := (\varepsilon^2 + d_{x_0}^2)^{1/2}$. One computes their Laplacian on $X \setminus \{x_0\}$:

$$\Delta d_{x_0, \varepsilon} = \frac{d_{x_0}}{(\varepsilon^2 + d_{x_0}^2)^{1/2}} \Delta d_{x_0} + \frac{\varepsilon^2}{(\varepsilon^2 + d_{x_0}^2)^{3/2}}.$$

Hence, one has on $X \setminus \{x_0\}$:

$$\Delta d_{x_0, \varepsilon} \geq \frac{a d_{x_0}}{(\varepsilon^2 + d_{x_0}^2)^{1/2}} .$$

Since both sides are continuous functions on X , this inequality also holds on X . Since a limit of non negative distributions is non negative, one gets the inequality $\Delta d_{x_0} \geq a$ on X by letting ε go to 0. \square

Proof of Proposition 2.4. According to Lemma 2.5, both functions

$$G_{\pm} := \frac{M}{a}(R - d_O) \pm G$$

are superharmonic on B_R , i.e. one has $\Delta G_{\pm} \leq 0$. Since they vanish on the boundary ∂B_R , the maximum principle ensures that these functions G_{\pm} are non-negative on the ball B_R . \square

3 Harmonic maps

In this chapter we begin the proof of Theorem 1.1. We first recall basic facts satisfied by harmonic maps. We then explain why we can assume our c -quasi-isometric map f to be C^∞ with bounded covariant derivatives. We also explain why an upper bound on $d(h_R, f)$ implies the existence of the harmonic map h . Finally we provide this upper bound near the boundary ∂B_R .

3.1 Harmonic maps and the distance function

In this section, we recall two useful facts satisfied by a harmonic map h : the subharmonicity of the functions $d_{y_0} \circ h$, and Cheng's estimate for the differential Dh .

Definition 3.1. *Let $h : X \rightarrow Y$ be a C^∞ map between two Riemannian manifolds. The tension field of h is the trace of the second covariant derivative $\tau(h) := \text{tr} D^2 h$. The map h is said to be harmonic if $\tau(h) = 0$.*

The tension $\tau(h)$ is a Y -valued vector field on X , i.e. it is a section of the pulled-back of the tangent bundle $TY \rightarrow Y$ under the map $h : X \rightarrow Y$.

Lemma 3.2. *Let $h : X \rightarrow Y$ be a harmonic C^∞ map between Hadamard manifolds. Let $y_0 \in Y$ and let $\rho_h : X \rightarrow \mathbb{R}$ be the function $\rho_h := d_{y_0} \circ h$. Then the continuous function ρ_h is subharmonic on X .*

Proof. The proof is similar to the proof of Lemma 2.5. We first recall the formula for the Laplacian of a composed function. Let $f : X \rightarrow Y$ be a C^∞ map and $F \in C^\infty(Y)$ be a C^∞ function on Y . Then one has

$$\Delta(F \circ f) = \sum_{1 \leq i \leq k} D^2 F(D_{e_i} f, D_{e_i} f) + \langle DF, \tau(f) \rangle, \quad (3.1)$$

where $(e_i)_{1 \leq i \leq k}$ is an orthonormal basis of the tangent space to X .

Since Y is a Hadamard manifold, the continuous function $\rho_h = d_{y_0} \circ h$ is \mathcal{C}^∞ outside $h^{-1}(y_0)$. Using Formula (3.1), the harmonicity of h and Lemma 2.3, we compute the Laplacian on $X \setminus h^{-1}(y_0)$:

$$\Delta \rho_h = \sum_{1 \leq i \leq k} D^2 d_{y_0}(D_{e_i} h, D_{e_i} h) \geq 0.$$

The function ρ_h is the uniform limit when ε go to 0 of the \mathcal{C}^∞ functions $\rho_{h,\varepsilon} := (\varepsilon^2 + \rho_h^2)^{1/2}$. We compute their Laplacian on $X \setminus h^{-1}(y_0)$:

$$\Delta \rho_{h,\varepsilon} = \frac{\rho_h}{(\varepsilon^2 + \rho_h^2)^{1/2}} \Delta \rho_h + \frac{\varepsilon^2}{(\varepsilon^2 + \rho_h^2)^{3/2}} \geq 0.$$

It follows that the inequality $\Delta \rho_{h,\varepsilon} \geq 0$ also holds on the whole X .

One finally gets $\Delta \rho_h \geq 0$ as a distribution on X by letting ε go to 0. \square

Another crucial property of harmonic maps is the following bound for their differential due to Cheng.

Lemma 3.3. *Let X, Y be two Hadamard manifolds with $-b^2 \leq K_X \leq 0$. Let $k = \dim X$, x_0 be a point of X , $r_0 > 0$ and let $h : B(x_0, r_0) \rightarrow Y$ be a harmonic \mathcal{C}^∞ map such that the image $h(B(x_0, r_0))$ lies in a ball of radius R_0 . Then one has the bound*

$$\|Dh(x_0)\| \leq 2^5 k \frac{1+br_0}{r_0} R_0. \quad (3.2)$$

In the applications, we will use this inequality with $b = 1$ and $r_0 = 1$.

Proof. This is an explicit version of [7, Formula 2.9] in which we keep track of the constant. We use Cheng's formula 2.9 where the point called y_0 by Cheng is on the sphere $S(h(x_0), 2R_0)$ and the radii called a and b by Cheng are respectively equal to r_0 and $4R_0$. Correcting a misprint, this formula is

$$\frac{r_0^4 \|Dh(x_0)\|^2}{(12 R_0^2)^2} \leq C_k \max \left(\frac{K r_0^4}{7 R_0^2}, \frac{(1 + \sqrt{K} r_0) r_0^2}{7 R_0^2}, \frac{16 R_0^2 r_0^2}{49 R_0^4} \right),$$

where $-K$ is a lower bound for the Ricci curvature on X , and where C_k is a constant depending only on k . Choosing $K = kb^2$, one gets

$$\|Dh(x_0)\|^2 \leq 144 k C_k \max \left(\frac{b^2 R_0^2}{7}, \frac{(1 + br_0) R_0^2}{7 r_0^2}, \frac{16 R_0^2}{49 r_0^2} \right)$$

and hence, one has $\|Dh(x_0)\| \leq c_k \frac{1+br_0}{r_0} R_0$ for some constant c_k . Since the explicit value $c_k = 2^5 k$ for this constant is not crucial we omit the details of this calculation. \square

3.2 Smoothing quasi-isometric maps

The following proposition will allow us to assume in Theorem 1.1 that the quasi-isometric map f is C^∞ with bounded covariant derivatives.

Proposition 3.4. *Let X, Y be two symmetric spaces of non positive curvature and $f : X \rightarrow Y$ be a quasi-isometric map. Then there exists a C^∞ quasi-isometric map $\tilde{f} : X \rightarrow Y$ within bounded distance from f and whose covariant derivatives $D^p \tilde{f}$ are bounded on X for all $p \geq 1$.*

This regularized map \tilde{f} will be constructed as follows. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative C^∞ function with support $[-1, 1]$. For x in X , we introduce the positive finite measure on X

$$\alpha_x := (\alpha \circ d_x^2) \, \text{dvol}_X.$$

Let $\mu_x := f_* \alpha_x$ denote the image measure on Y . It is defined, for any positive function φ on Y , by

$$\mu_x(\varphi) = \int_X \varphi(f(z)) \alpha(d^2(x, z)) \, \text{dvol}_X(z).$$

We choose α so that each μ_x is a probability measure. By homogeneity of X , this fact does not depend on the point x . We will define $\tilde{f}(x) \in Y$ to be the center of mass of the measure μ_x . To be more precise, we will need the following Lemma 3.5 which is an immediate consequence of Lemma 2.3.

Lemma 3.5. *For x in X , let Q_x be the function on Y defined for y in Y by*

$$Q_x(y) = \int_X d^2(y, f(z)) \alpha(d^2(x, z)) \, \text{dvol}_X(z). \quad (3.3)$$

For x in X , the functions Q_x are proper and uniformly strictly convex. More precisely, for all x in X and y in Y , the Hessian admits the lower bound

$$D_y^2 Q_x \geq 2g_Y$$

where g_Y is the Riemannian metric on Y .

Proof of Proposition 3.4. For x in X , we define the point $\tilde{f}(x) \in Y$ to be the center of mass of μ_x , i.e. to be the unique point where the function Q_x reaches its infimum. Equivalently, the point $y = \tilde{f}(x) \in Y$ is the unique critical point of the function Q_x , i.e. it is defined by the implicit equation

$$D_y Q_x = 0.$$

Since the map f is c -quasi-isometric, the support of the measure μ_x lies in the ball $B(f(x), 2c)$. Since Y is a Hadamard manifold, the balls of Y are

convex (see [2]) so that the center of mass $\tilde{f}(x)$ also belongs to the ball $B(f(x), 2c)$. In particular, one has

$$d(f, \tilde{f}) \leq 2c.$$

We now check that the map $\tilde{f} : X \rightarrow Y$ is \mathcal{C}^∞ . Since the Hessians $D_y^2 Q_x$ are nondegenerate, this follows from the implicit function theorem applied to the \mathcal{C}^∞ map

$$\Psi : (x, y) \in X \times Y \longrightarrow D_y Q_x \in T^*Y.$$

To prove that the first derivative of \tilde{f} is bounded on X , we first notice that Lemma 3.5 ensures that the covariant derivative

$$D_y \Psi(x, y) = D_y^2 Q_x \in \mathcal{L}(T_y Y, T_y Y^*)$$

is an invertible linear map with

$$\|(D_y \Psi(x, y))^{-1}\| \leq 1/2.$$

We also notice that, since the point $y = \tilde{f}(x)$ is at distance at most $4c$ of all the points $f(z)$ with z in the ball $B(x, 1)$, the norm

$$\|D_x \Psi(x, \tilde{f}(x))\|$$

is also uniformly bounded on X . Hence the norm $\|D\tilde{f}\|$ of the derivative of \tilde{f} is uniformly bounded on X .

For the same reason, since Y is homogeneous, the norm of each covariant derivative

$$\|D_x^p D_y^q \Psi(x, \tilde{f}(x))\|$$

is uniformly bounded on X for $p, q \geq 0$. Hence the norm of each covariant derivative $\|D^p \tilde{f}\|$ is uniformly bounded on X for $p \geq 1$. \square

3.3 Existence of harmonic maps

In this section we prove Theorem 1.1, taking for granted Proposition 3.6 below.

Let X, Y be rank one symmetric spaces and $f : X \rightarrow Y$ be a c -quasi-isometric \mathcal{C}^∞ map whose first two covariant derivatives are bounded.

We fix a point O in X . For $R > 0$, we denote by $B_R := B(O, R)$ the closed ball in X with center O and radius R and by ∂B_R the sphere that bounds B_R . Since the manifold Y is a Hadamard manifold, there exists a unique harmonic map $h_R : B_R \rightarrow Y$ satisfying the Dirichlet condition $h_R = f$

on the sphere ∂B_R . Moreover, this harmonic map h_R is *energy minimizing*. This means that the map h_R achieves the minimum of the energy functional

$$E_R(h) := \int_{B_R} \|Dh(x)\|^2 \, \text{dvol}_X(x) \quad (3.4)$$

among all \mathcal{C}^1 maps g on the ball which agree with f on the sphere ∂B_R , i.e. one has

$$E_R(h_R) = \inf_g E_R(g).$$

These facts are due to Schoen (see [30] or [9, Thm 12.11]). Thanks to Schoen and Uhlenbeck in [32] and [33], the harmonic map h_R is known to be \mathcal{C}^∞ on the closed ball B_R . We denote by

$$d(h_R, f) = \sup_{x \in B(O, R)} d(h_R(x), f(x))$$

the distance between these two maps.

The main point of this article is to prove the following uniform estimate.

Proposition 3.6. *There exists a constant $M \geq 1$ such that, for any $R \geq 1$, one has $d(h_R, f) \leq M$.*

Even though the argument is very classical, we first explain how to deduce our main theorem from this proposition.

Proof of Theorem 1.1. As explained in Proposition 3.4, we may also assume that the c -quasi-isometric map f is \mathcal{C}^∞ with bounded covariant derivatives. Pick an increasing sequence of radii R_n converging to ∞ and let $h_{R_n} : B_{R_n} \rightarrow Y$ be the harmonic \mathcal{C}^∞ map which agrees with f on the sphere ∂B_{R_n} . Proposition 3.6 ensures that the sequence of maps h_{R_n} is locally uniformly bounded. More precisely there exists $M \geq 1$ such that, for all $S \geq 1$, for n large enough, one has

$$h_{R_n}(B_{2S}) \subset B(f(O), 2cS + M).$$

Using the Cheng Lemma 3.3 with $b = 1$ and $r_0 = 1$, it follows that the derivatives are also uniformly bounded on each ball B_S . More precisely one has, for all $S \geq 1$, for n large enough,

$$\sup_{x \in B_S} \|Dh_{R_n}(x)\| \leq 2^6 k (2cS + M).$$

The Ascoli-Arzelà theorem implies that, after extraction, the sequence h_{R_n} converges uniformly on every ball B_S towards a continuous map $h : X \rightarrow Y$.

By construction this limit map h stays within bounded distance from the quasi-isometric map f . We claim that the limit map h is harmonic. Indeed,

the harmonic maps h_{R_n} are energy minimizing and, on each ball B_S , the energies of h_{R_n} are uniformly bounded :

$$\limsup_{n \rightarrow \infty} E_S(h_{R_n}) < \infty .$$

Hence the Luckhaus compactness theorem for energy minimizing harmonic maps (see [35, Section 2.9]) tells us that the limit map h is also harmonic and energy minimizing. \square

Remark 3.7. By Li-Wang uniqueness theorem in [22], the harmonic map h which stays within bounded distance from f is unique. Hence the above argument also proves that the whole family of harmonic maps h_R converges to h uniformly on the compact subsets of X when R goes to infinity.

3.4 Boundary estimate

In this section we begin the proof of Proposition 3.6 : we bound the distance between h_R and f near the sphere ∂B_R .

Proposition 3.8. *Let X, Y be Hadamard manifolds and $k = \dim X$. Assume that $-1 \leq K_X \leq -a^2 < 0$. Let $c \geq 1$ and $f : X \rightarrow Y$ be a \mathcal{C}^∞ map with $\|Df(x)\| \leq c$ and $\|D^2f(x)\| \leq c$. Let $O \in X$, $R > 0$, $B_R := B(O, R)$.*

Let $h_R : B_R \rightarrow Y$ be the harmonic \mathcal{C}^∞ map whose restriction to the sphere ∂B_R is equal to f . Then, for all x in B_R , one has

$$d(h_R(x), f(x)) \leq \frac{4kc^2}{a} d(x, \partial B_R) . \quad (3.5)$$

An important feature of this upper bound is that it does not depend on the radius R , provided the distance $d(x, \partial B_R)$ remains bounded. This is why we call (3.5) the *boundary estimate*. The proof relies on an idea of Jost in [16, Section 4].

Proof. Let x be a point in B_R and w in ∂B_R such that $d(x, w) = d(x, \partial B_R)$. Since $h_R(w) = f(w)$, the triangle inequality reads as

$$d(f(x), h_R(x)) \leq d(f(x), f(w)) + d(h_R(w), h_R(x)) . \quad (3.6)$$

The assumption on f ensures that

$$d(f(x), f(w)) \leq c d(x, \partial B_R) . \quad (3.7)$$

To estimate the other term, we choose a point y_0 on the geodesic ray starting from $h_R(x)$ and passing by $h_R(w)$. This choice of y_0 ensures that one has the equality

$$d(h_R(w), h_R(x)) = d(h_R(x), y_0) - d(h_R(w), y_0) . \quad (3.8)$$

We also choose y_0 far enough so that

$$F(z) := d(f(z), y_0) \geq 1 \text{ for all } z \text{ in } B_R. \quad (3.9)$$

This function F is then \mathcal{C}^∞ on the ball B_R . Let $H : B_R \rightarrow \mathbb{R}$ be the harmonic \mathcal{C}^∞ function whose restriction to the sphere ∂B_R is equal to F . By Lemma 3.2, since h_R is a harmonic map, the function $z \mapsto d(h_R(z), y_0)$ is subharmonic on B_R . Since this function is equal to H on the sphere ∂B_R , the maximum principle ensures that

$$d(h_R(z), y_0) \leq H(z) \text{ for all } z \text{ in } B_R, \quad (3.10)$$

with equality for z in ∂B_R . Combining (3.8) and (3.10), one gets

$$d(h_R(w), h_R(x)) \leq H(x) - H(w). \quad (3.11)$$

To estimate the right-hand side of (3.11), we observe that the function $G := F - H$ vanishes on ∂B_R and has bounded Laplacian :

$$|\Delta G| \leq 3kc^2.$$

Indeed, using Formulas (3.1), (2.5) and (3.9), one computes

$$|\Delta G| = |\Delta(d_{y_0} \circ f)| \leq k\|D^2 d_{y_0}\| \|Df\|^2 + k\|Dd_{y_0}\| \|D^2 f\| \leq 3kc^2.$$

Using Proposition 2.4, one deduces that

$$|G(x)| \leq \frac{3kc^2}{a} d(x, \partial B_R)$$

and therefore, combining with (3.6), (3.7) and (3.11), one concludes that

$$\begin{aligned} d(f(x), h_R(x)) &\leq c d(x, \partial B_R) + |G(x)| + |F(x) - F(w)| \\ &\leq \left(\frac{3kc^2}{a} + 2c\right) d(x, \partial B_R). \end{aligned}$$

This proves (3.5). □

4 Interior estimate

In this chapter we complete the proof of Proposition 3.6. We follow the strategy explained in the introduction (Section 1.4).

4.1 Notation

We first explain more precisely the notation and the assumptions that we will use in the whole chapter.

Let X and Y be rank one symmetric spaces and $k = \dim X$. We start with a C^∞ quasi-isometric map $f : X \rightarrow Y$ all of whose covariant derivatives are bounded. We fix a constant $c \geq 1$ such that, for all x, x' in X , one has

$$\|Df(x)\| \leq c, \quad \|D^2f(x)\| \leq c \quad \text{and} \quad (4.1)$$

$$c^{-1}d(x, x') - c \leq d(f(x), f(x')) \leq cd(x, x'). \quad (4.2)$$

Note that the additive constant c on the right-hand side term of (1.1) has been removed since the derivative of f is bounded by c .

We fix a point O in X . For $R > 0$, we introduce the harmonic C^∞ map $h_R : B(O, R) \rightarrow Y$ whose restriction to the sphere $\partial B(O, R)$ is equal to f . We let

$$\rho_R := \sup_{x \in B(O, R)} d(h_R(x), f(x)).$$

We denote by x_R a point of $B(O, R)$ where the supremum is achieved :

$$d(h_R(x_R), f(x_R)) = \rho_R.$$

According to the boundary estimate in Proposition 3.8, one has,

$$d(x_R, \partial B(O, R)) \geq \frac{1}{8kc^2} \rho_R.$$

When ρ_R is large enough, we introduce a ball $B(x_R, r_R)$ with center x_R , and whose radius r_R is a function of R satisfying

$$1 \leq r_R \leq \frac{1}{16kc^2} \rho_R. \quad (4.3)$$

Note that this condition ensures the inclusion $B(x_R, r_R) \subset B(O, R-1)$. Later on, in section 4.5, we will assume that $r_R := \rho_R^{1/3}$.

We will focus on the restrictions of the maps f and h_R to this ball $B(x_R, r_R)$. We will express the maps f and h_R through the polar exponential coordinates (ρ, v) in Y centered at the point $y_R := f(x_R)$. For z in $B(x_R, r_R)$, we will thus write

$$\begin{aligned} f(z) &= \exp_{y_R}(\rho_f(z)v_f(z)), \\ h_R(z) &= \exp_{y_R}(\rho_h(z)v_h(z)), \\ h_R(x_R) &= \exp_{y_R}(\rho_R v_R), \end{aligned}$$

where $\rho_f(z) \geq 0$, $\rho_h(z) \geq 0$ and where $v_f(z)$, $v_h(z)$ and v_R belong to the unit sphere $T_{y_R}^1 Y$ of the tangent space $T_{y_R} Y$. Note that ρ_h and v_h are shorthands for ρ_{h_R} and v_{h_R} . For simplicity, we do not write the dependence on R .

We denote by $[x_R, z]$ the geodesic segment between x_R and z .

Definition 4.1. *We introduce the following subsets of the sphere $S(x_R, r_R)$:*

$$\begin{aligned} U_R &= \{z \in S(x_R, r_R) \mid \rho_h(z) \geq \rho_R - \frac{1}{2c} r_R\}, \\ V_R &= \{z \in S(x_R, r_R) \mid \rho_h(z_t) \geq \rho_R/2 \text{ for all } z_t \text{ in } [x_R, z]\}, \\ W_R &= U_R \cap V_R. \end{aligned}$$

4.2 Measure estimate

We first notice that one can control the size of $\rho_h(z)$ and of $Dh_R(z)$ on the ball $B(x_R, r_R)$. We will then give a lower bound for the measure of W_R .

Lemma 4.2. *Assume (4.3). For z in $B(x_R, r_R)$, one has*

$$\rho_h(z) \leq \rho_R + c r_R.$$

Proof. The triangle inequality and (4.2) give, for z in $B(x_R, r_R)$,

$$\rho_h(z) \leq d(h_R(z), f(z)) + d(f(z), y_R) \leq \rho_R + c r_R. \quad \square$$

Lemma 4.3. *Assume (4.3). For z in $B(x_R, r_R)$, one has*

$$\|Dh_R(z)\| \leq 2^8 k \rho_R.$$

Proof. For all z, z' in $B(O, R)$ with $d(z, z') \leq 1$, the triangle inequality and (4.2) yield

$$\begin{aligned} d(h_R(z), h_R(z')) &\leq d(h_R(z), f(z)) + d(f(z), f(z')) + d(f(z'), h_R(z')) \\ &\leq \rho_R + c + \rho_R \leq 4\rho_R. \end{aligned}$$

Applying Cheng's lemma 3.3 with $b = 1$ and $r_0 = 1$, one gets for all z in $B(O, R-1)$ the bound $\|Dh_R(z)\| \leq 2^8 k \rho_R$. \square

We now give a lower bound for the measure of W_R . We will denote by the same letter σ the probability measure on each sphere $S(x_R, t)$ that is invariant under all the isometries of X that fix the point x_R .

Lemma 4.4. *Assume (4.3). Then one has*

$$\sigma(W_R) \geq \frac{1}{3c^2} - 2^{12} k c \frac{r_R^2}{\rho_R}. \quad (4.4)$$

Proof. The proof relies on the subharmonicity of the function ρ_h on the ball $B(x_R, r_R)$ (see Lemma 3.2). We claim that, for any $0 < t \leq r_R$, one has

$$\int_{S(x_R, t)} (\rho_h(z) - \rho_R) d\sigma(z) \geq 0. \quad (4.5)$$

Let us give a short proof of this special case of the Green formula. Since X is a rank one symmetric space, the group Γ of isometries of X that fix the point x_R is a compact group which acts transitively on the spheres $S(x_R, t)$. Let $d\gamma$ be the Haar probability measure on Γ . The function $F := \int_{\Gamma} \rho_h \circ \gamma d\gamma$, defined as the average of the translates of the function ρ_h under Γ , is equal to a constant F_t on each sphere $S(x_R, t)$ of radius $t \leq r_R$. By the maximum

principle applied to this subharmonic function F , one gets $F_0 \leq F_t$ for all $t \leq r_R$. Since $F_0 = \rho_h(x_R) = \rho_R$, this proves (4.5).

First step. We prove

$$\sigma(U_R) \geq \frac{1}{3c^2}. \quad (4.6)$$

By Lemma 4.2, the function ρ_h is bounded by $\rho_R + cr_R$, hence Equation (4.5) implies

$$cr_R \sigma(U_R) - \frac{r_R}{2c} (1 - \sigma(U_R)) \geq 0$$

so that $\sigma(U_R) \geq (1 + 2c^2)^{-1} \geq c^{-2}/3$.

Second step. We prove

$$\sigma(V_R) \geq 1 - 2^{12} k c \frac{r_R^2}{\rho_R}. \quad (4.7)$$

For z in the complementary subset $V_R^c \subset S(x_R, r_R)$, we define

$$\begin{aligned} t_z &:= \inf\{t \in [0, r_R] \mid \rho_h(z_t) = \frac{1}{2}\rho_R\}, \\ s_z &:= \sup\{t \in [0, t_z] \mid \rho_h(z_t) = \frac{3}{4}\rho_R\}. \end{aligned}$$

We claim that, for each z in V_R^c , one has

$$t_z - s_z \geq 2^{-10} k^{-1}. \quad (4.8)$$

Indeed, the length of the curve $t \mapsto h_R(z_t)$ between $t = s_z$ and $t = t_z$ is at least $\frac{\rho_R}{4}$. Hence, using Lemma 4.3, one gets

$$\frac{\rho_R}{4} \leq (t_z - s_z) \sup_{B(x_R, r_R)} \|Dh_R\| \leq 2^8 k (t_z - s_z) \rho_R,$$

which prove (4.8).

The Green formula also gives the following variation of (4.5)

$$\int_{S(x_R, r_R)} \int_0^{r_R} (\rho_h(z_t) - \rho_R) dt d\sigma(z) \geq 0. \quad (4.9)$$

By Lemma 4.2, the function ρ_h is bounded by $\rho_R + cr_R$, hence Equation (4.9) implies

$$cr_R^2 + \int_{V_R^c} \int_{s_z}^{t_z} (\rho_h(z_t) - \rho_R) dt d\sigma(z) \geq 0. \quad (4.10)$$

Using the bound $\rho_h(z_t) \leq \frac{3}{4}\rho_R$, for all t in the interval $[s_z, t_z]$, one deduces from (4.8) and (4.10) that

$$cr_R^2 - 2^{-10} k^{-1} \frac{\rho_R}{4} \sigma(V_R^c) \geq 0.$$

This proves (4.7).

Since $W_R = U_R \cap V_R$, the bound (4.4) follows from (4.6) and (4.7). \square

4.3 Upper bound for $\theta(v_f(z), v_h(z))$

For all v in U_R , we give an upper bound for the angle between $v_f(z)$ and $v_h(z)$.

For two vectors v_1, v_2 of the unit sphere $T_{y_R}^1 Y$ of the tangent space $T_{y_R} Y$, we denote by $\theta(v_1, v_2)$ the angle between these two vectors.

Lemma 4.5. *Assume (4.3). Then, for z in U_R , one has*

$$\theta(v_f(z), v_h(z)) \leq 4e^{\frac{c}{4}} e^{-\frac{r_R}{8c}}. \quad (4.11)$$

Proof. For z in U_R , we consider the triangle with vertices $y_R, f(z)$ and $h_R(z)$. Its side lengths satisfy

$$\begin{aligned} \ell_0 &:= d(h_R(z), f(z)) \leq \rho_R && \text{by definition of } \rho_R, \\ \ell_1 &:= \rho_f(z) \geq \frac{1}{c} r_R - c && \text{since } f \text{ is } c\text{-quasi-isometric,} \\ \ell_2 &:= \rho_h(z) \geq \rho_R - \frac{1}{2c} r_R && \text{by definition of } U_R. \end{aligned}$$

Since $K_Y \leq -1/4$, applying Lemma 2.1 with $a = \frac{1}{2}$, one gets

$$\theta(v_f(z), v_h(z)) \leq 4e^{-\frac{1}{4}(\ell_1 + \ell_2 - \ell_0)} \leq 4e^{\frac{c}{4}} e^{-\frac{r_R}{8c}}. \quad \square$$

4.4 Upper bound for $\theta(v_h(z), v_R)$

For all v in V_R , we give an upper bound for the angle between $v_h(z)$ and v_R .

Lemma 4.6. *Assume (4.3). Then, for z in V_R , one has*

$$\theta(v_h(z), v_R) \leq \frac{8\rho_R^2}{\sinh(\rho_R/4)}. \quad (4.12)$$

Proof. Let us first sketch the proof. We recall that the curve $t \mapsto z_t$, for $0 \leq t \leq r_R$, is the geodesic segment between x_R and z . By definition, for each z in V_R , the curve $t \mapsto h_R(z_t)$ lies outside of the ball $B(y_R, \rho_R/2)$ and by Cheng's bound on $\|Dh_R(z_t)\|$ one controls the length of this curve.

We now detail the argument. For z in V_R , we have the inequality

$$\theta(v_h(z), v_R) \leq r_R \sup_{0 \leq t \leq r_R} \|Dv_h(z_t)\|.$$

Since $K_Y \leq -1/4$ the Alexandrov triangle comparison theorem and the Gauss lemma ([12, 2.93]) yield, for y in $Y \setminus \{y_R\}$,

$$2 \sinh(\rho(y)/2) \|Dv(y)\| \leq 1,$$

where $(\rho(y), v(y)) \in]0, \infty[\times T_{y_R}^1 Y$ are the polar exponential coordinates on Y centered at y_R . Since $\rho_h = \rho \circ h$ and $v_h = v \circ h$, one gets, for x in $B(O, R)$ with $h_R(x) \neq y_R$,

$$2 \sinh(\rho_h(x)/2) \|Dv_h(x)\| \leq \|Dh_R(x)\|.$$

Hence since the point z belongs to V_R one deduces

$$\theta(v_h(z), v_R) \leq \frac{r_R}{2 \sinh(\rho_R/4)} \sup_{0 \leq t \leq r_R} \|Dh_R(z_t)\|.$$

Hence using Lemma 4.3 one gets

$$\theta(v_h(z), v_R) \leq 2^8 k \rho_R \frac{r_R}{2 \sinh(\rho_R/4)}.$$

Using (4.3) this finally gives (4.12). \square

4.5 Lower bound for $\theta(v_f(z), v_R)$

When ρ_R is large enough, we find a point $z = z_R$ in W_R for which the angle between $v_f(z)$ and v_R is bounded below.

For a subset W of the unit sphere $T_{y_R}^1 Y$, we denote by

$$\text{diam}(W) := \sup(\{\theta(v, v') \mid v, v' \in W\})$$

the diameter of W .

Lemma 4.7. *Assume that there exists a sequence of radii R going to infinity such that ρ_R goes to infinity. Then, choosing $r_R := \rho_R^{1/3}$, the diameters*

$$\text{diam}(\{v_f(z) \mid z \in W_R\})$$

do not converge to 0 along this sequence.

Proof. Let $\sigma_0 := \frac{1}{4c^2}$. According to Lemma 4.4, one has

$$\liminf_{R \rightarrow \infty} \sigma(W_R) > \sigma_0 > 0.$$

There exists $\varepsilon_0 > 0$ such that every subset W of the Euclidean sphere \mathbb{S}^{k-1} whose normalized measure is at least σ_0 contains two points whose angle is at least ε_0 .

Hence if R and ρ_R are large enough one can find z_1, z_2 in W_R such that

$$\theta_{x_R}(z_1, z_2) \geq \varepsilon_0 \quad \text{and} \quad r_R \geq \frac{(A+1)c}{\sin^2(\varepsilon_0/2)}, \quad (4.13)$$

where $\theta_{x_R}(z_1, z_2)$ is the angle between z_1 and z_2 seen from x_R , and where A is the constant given by Lemma 2.2. According to Lemma 2.1.a, we infer that

$$\min((x_R|z_1)_{z_2}, (x_R|z_2)_{z_1}) \geq (A+1)c.$$

Using then Lemma 2.2, one gets

$$\min((y_R|f(z_1))_{f(z_2)}, (y_R|f(z_2))_{f(z_1)}) \geq 1. \quad (4.14)$$

We now have the inequalities

$$\begin{aligned} \theta(v_f(z_1), v_f(z_2)) &\geq e^{-(f(z_1)|f(z_2))_{y_R}} && \text{by Lemma 2.1.c and (4.14)} \\ &\geq e^{-A} e^{-c(z_1|z_2)_{x_R}} && \text{by Lemma 2.2} \\ &\geq e^{-A} (\varepsilon_0/4)^{2c} && \text{by Lemma 2.1.b and (4.13)}. \end{aligned}$$

This proves our claim. \square

End of the proof of Proposition 3.6. Assume that there exists a sequence of radii R going to infinity such that ρ_R goes also to infinity. We set $r_R = \rho_R^{1/3}$. Using Lemmas 4.5 and 4.6 and the triangle inequality, one gets

$$\lim_{R \rightarrow \infty} \sup_{z \in W_R} \theta(v_f(z), v_R) = 0. \quad (4.15)$$

This contradicts Lemma 4.7. \square

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