A review on large k minimal spectral k-partitions and Pleijel's Theorem.

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Orsay (24 October 2013)

We review the properties of minimal spectral k-partitions in the two-dimensional case and revisit the connexions with Pleijel's theorem. The hexagonal conjecture corresponds to the idea that, when k is large, the minimal k-partitions will behave like the restriction of an hexagonal tiling (when far from the boundary). We focus on this large k problem (and the hexagonal conjecture) in connexion with two recent papers by J. Bourgain and S. Steinerberger on the Pleijel theorem and with I. Polterovich's conjecture, which could be the consequence of a "square" conjecture for minimal spectral bipartite partitions.

We consider mainly the Dirichlet Laplacian in a bounded domain $\Omega \subset \mathbb{R}^2$. We assume that Ω is sufficiently regular say with C^∞ boundary.

In [14] we have started to analyze the relations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy (= lowest eigenvalue) of the Dirichlet realization of the Laplacian $H(D_i)$ in D_i is minimal.

We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of real-valued eigenfunctions. The groundstate u_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions u_k must have zerosets. For any real-valued $u \in C_0^0(\overline{\Omega})$, we define the zero set as

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of u is called $\mu(u)$. These $\mu(u)$ nodal domains define a k-partition of Ω, with $k = \mu(u)$.

We recall that the Courant nodal theorem says that, for $k \geq 1$, and if λ_k denotes the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with λ_k, then, for all real-valued $u \in E(\lambda_k) \setminus \{0\}\,,\ \mu(u) \leq k\,.$

In dimension 1 the Sturm-Liouville theory says that we have always equality (for Dirichlet in a bounded interval) in the previous theorem (this is what we will call later a Courant-sharp situation).

A theorem due to Pleijel [16] in 1956 says that this cannot be true when the dimension (here we consider the $2D$-case) is larger than one.

Minimal spectral partitions

We now introduce for $k \in \mathbb{N}$ ($k \geq 1$), the notion of k-partition. We will call k-**partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint sets in Ω. We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected. We denote by $\mathfrak{O}_k(\Omega)$ the set of open connected partitions of Ω. A spectral minimal partition sequence is defined by

Definition

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k(\Omega)$, we set

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{2}$$

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{3}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ a minimal k-partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

More generally we can define, for $p \in [1, +\infty)$, $\Lambda^p(\mathcal{D})$ and $\mathfrak{L}_{k,p}$ by replacing in (2) $\max_i \lambda(D_i)$ by $\left(\frac{\sum \lambda(D_i)^p}{k}\right)^{\frac{1}{p}}$:

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda^p(\mathcal{D}). \tag{4}$$

Note we can minimize over non necessarily connected partitions and get the connectedness of the minimal partitions as a property [14].

If $k = 2$, it is rather well known that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a **nodal partition**, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to λ_2.

A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called **strong** if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{5}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$, which is called the **boundary set** of the partition :

$$N(\mathcal{D}) = \overline{\cup_i (\partial D_i \cap \Omega)} \,. \tag{6}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This suggests the following definition:

Definition

We call a partition $\mathcal{D}$ regular if its associated boundary set $N(\mathcal{D})$, has the following properties :
(i) Except for finitely many distinct $x_i \in \Omega \cap N$ in the neighborhood of which N is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geq 3$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the **equal angle meeting property**

The x_i are called the critical points and define the set $X(N)$. Similarly we denote by $Y(N)$ the set of the boundary points z_i. By **equal angle meeting property**, we mean that the half curves meet with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

We say that D_i, D_j are **neighbors** or $D_i \sim D_j$, if $D_{ij} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected. We associate with each $\mathcal{D}$ a **graph** $G(\mathcal{D})$ by associating with each D_i a vertex and to each pair $D_i \sim D_j$ an edge. We will say that the graph is **bipartite** if it can be colored by two colors (two neighbors having two different colors). We recall that the graph associated with a collection of nodal domains of an eigenfunction is always bipartite.

Pleijel's theorem revisited

Pleijel's theorem as stated in the introduction is the consequence of a more precise theorem and this is the aim of this section to present a formalized proof of the historical statement permitting to understand recent improvements and formulated conjectures. Generally, the classical proof is going through the proposition

Proposition 1

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}}\,, \tag{7}$$

where $\mu(\phi_n)$ is the cardinal of the nodal components of $\Omega \setminus N(\phi_n)$ and then to establishing a lower bound for $A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$.

Behind this statement, we have actually the proposition:

Proposition 2

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty} \frac{L_k(\Omega)}{k}}\,. \tag{8}$$

Here $L_k(\Omega)$ is the smallest eigenvalue (if any) such that there exists in the corresponding eigenspace an eigenfunction with k nodal domains. Otherwise, we take $L_k(\Omega) = +\infty$.

The proof of Proposition 2 is immediate observing first that for any subsequence n_ℓ, we have

$$\frac{\lambda_{n_\ell}}{n_\ell} \geq \frac{L_{\mu(\phi_{n_\ell})}}{n_\ell} = \frac{L_{\mu(\phi_{n_\ell})}}{\mu(\phi_{n_\ell})} \cdot \frac{\mu(\phi_{n_\ell})}{n_\ell}\,.$$

If we choose the subsequence n_ℓ such that

$$\lim_{\ell\to+\infty} \frac{\mu(\phi_{n_\ell})}{n_\ell} = \limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n}\,,$$

we observe that by Weyl's formula that

$$\lim_{\ell\to+\infty} \frac{\lambda_{n_\ell}}{n_\ell} = 4\pi/A(\Omega)\,,$$

and

$$\liminf_{\ell\to+\infty} \frac{L_{\mu(\phi_{n_\ell})}}{\mu(\phi_{n_\ell})} \geq \liminf_{k\to+\infty} \frac{L_k}{k}\,.$$

□

Proposition 1 is deduced from Proposition 2 by observing that it was established in [14] that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega)\,. \tag{9}$$

Moreover, and this is a deep theorem, the equality $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ implies $\mathfrak{L}_k(\Omega) = L_k(\Omega) = \lambda_k(\Omega)$.
We say in this case that we are in a Courant-sharp situation.

Towards improvements ??

If we think that only nodal partitions are involved in Pleijel's Theorem, it could be natural to introduce $\mathfrak{L}_k^\sharp(\Omega)$ where we take the infimum over a smaller non-empty class of k-partitions $\mathcal{D} = (D_1, \cdots, D_k)$. We call $\mathcal{O}_k^\sharp$ this undefined class, which should contain all the nodal k-partitions, if any.

Definition

$$\mathfrak{L}_k^\sharp(\Omega) := \inf_{\mathcal{D}\in\mathcal{O}_k^\sharp} \max \lambda(D_i)\,. \tag{10}$$

Of course we have always

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq \mathfrak{L}_k^\sharp(\Omega) \leq L_k(\Omega)\,. \tag{11}$$

Hence we have:

Proposition

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{A(\Omega)\liminf_{k\to+\infty} \frac{\mathfrak{L}_k^{\sharp}(\Omega)}{k}}, \tag{12}$$

Hence this is the right hand side of (12) which seems to be interesting to analyze.
It is clear from (11) that all these upper bounds are less than one, which corresponds to a weak asymptotic version of Courant's theorem.

Classical Pleijel's Theorem (based on the Faber-Krahn inequality) is the immediate consequence of the first proposition and of the lower bound

$$A(\Omega) \liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq \lambda(Disk_1)\,. \tag{13}$$

(Note that the proof of Pleijel uses only a weak form of this inequality, where $\mathfrak{L}_k$ is replaced by L_k.)
This leads to

Pleijel's Theorem

$$A(\Omega) \limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \nu_{Pl}\,, \tag{14}$$

with

$$\nu_{Pl} = \frac{4\pi}{\lambda(Disk_1)} \sim 0.691\,.$$

Remark

Note that the same result is true in the Neumann case (Polterovich [17]) under some analyticity assumption on the boundary.

Remark

Note that we have the better:

$$A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_{k,1}(\Omega)}{k}\geq\lambda(Disk_1)\,.$$

But this improvement has no incidence on Pleijel's Theorem. In particular, note that we do not have necessarily $\lambda_k \leq \mathfrak{L}_{k,1}$ (take $k = 2$ and use the criterion of Helffer-Hoffmann-Ostenhof [13]).

It is rather easy to prove that:

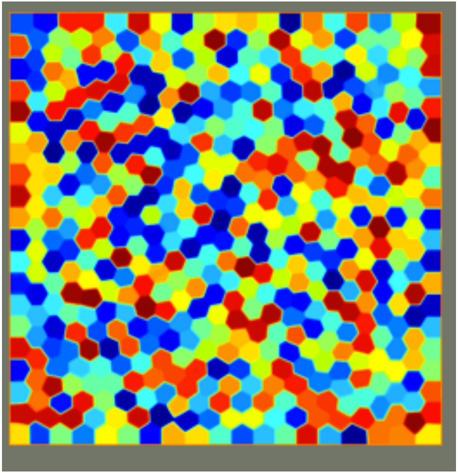
$$A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_{k}(\Omega)}{k}\leq A(\Omega)\limsup_{k\to+\infty}\frac{\mathfrak{L}_{k}(\Omega)}{k}\leq\lambda(Hexa_1)\,. \tag{15}$$

A now well known conjecture (hexagonal conjecture) (Van den Berg, Caffarelli-Lin [7]) was discussed in Helffer-Hoffmann-Ostenhof-Terracini [14], Bonnaillie-Helffer-Vial [6], Bourdin-Bucur-Oudet [4] and reads as follows:

Hexagonal conjecture

$$A(\Omega) \liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = A(\Omega) \limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(Hexa_1) \qquad (16)$$

This was computed for the torus by Bourdin-Bucur-Oudet [4] (for the sum), as the picture below shows.



This would lead to the conjecture that in Pleijel's estimate we have actually:

Hexagonal conjecture for Pleijel)

$$A(\Omega) \limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \nu_{Hex} \,, \tag{17}$$

with

$$\nu_{Hex} = \frac{4\pi}{\lambda(Hexa_1)} \sim 0.677 \,.$$

We note indeed that

$$\frac{\nu_{Hex}}{\nu_{Pl}} = \frac{\lambda(Disk_1)}{\lambda(Hexa_1)} \sim 0.977 \,.$$

If we think that we have lost some information by not using the nodal character of the partition, one can think that the hexagonal tiling should be replaced by a square tiling, the proof going through the research of a suitable $\mathcal{O}_k^\sharp$ as mentioned before. This leads to the stronger conjecture that

Square conjecture for Pleijel (Polterovich)

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \frac{4\pi}{\lambda(Sq_1)} = \frac{2}{\pi}\,. \tag{18}$$

This conjecture is due to Iosif Polterovich [17] on the basis of computations of Blum-Gutzman-Smilansky [3]. Due to the computations on the square ([16],[18]), this would be the optimal result.

Improving the use of Faber-Krahn by J. Bourgain and S. Steinerberger

The goal of Bourgain and Steinerberger was to improve the lower bound of $\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$. Bourgain gives an estimate of his improvement on the size of 10^{-9} and Steinerberger does not give any estimate.
In any case, it is clear that

$$\nu_{Hex} \leq \nu_{Bo} < \nu_{Pl} \,,$$

and

$$\nu_{Hex} \leq \nu_{St} < \nu_{Pl} \,,$$

where ν_{Bo} and ν_{St} are the constants of Bourgain [5] and Steinerberger [19].

Bourgain's improvement

One ingredient is a refinement of the Faber-Krahn inequality:

Lemma (Hansen-Nadirashvili)

For a nonempty simply connected bounded domain $\Omega \subset \mathbb{R}^2$, we have

$$A(\Omega)\,\lambda(\Omega) \geq \left(1 + \frac{1}{250}(1 - \frac{r_i(\Omega)}{r_0(\Omega)})^2\right)\lambda(\mathrm{Disk}_1)\,,$$

with $r_0(\Omega)$ the radius of the disk of same area as Ω and $r_i(\Omega)$ the inradius of Ω.

Actually, one needs a modified version for treating non simply connected domains. This is effectively unknown if we are in a non simply connected situation.
The other very tricky idea is to use quantitatively that all the open sets of the partition cannot be very close to disks (packing density) (see Blind [2]).

The inequality obtained by Bourgain is the following (see (26) in his note, first version) as $k \to +\infty$, is that for any $\delta \in (0, \delta_0)$

$$\frac{\mathfrak{L}_k(\Omega)}{k} \geq (1 + o(1))\lambda(\mathrm{Disk}_1)A(\Omega)^{-1} \times b(\delta) \qquad (19)$$

where

$$b(\delta) := (1 + 250\delta^{-3})(\frac{\pi}{\sqrt{12}}(1-\delta)^{-2} + 250\delta^{-3})^{-1}\,.$$

and $\delta_0 \in (0,1)$ is computed with the help of the packing condition. This condition reads

$$\frac{\delta_0^3}{250} = (\frac{1-\delta_0}{p})^2 - 1\,,$$

where p is a packing constant determined by Blind ($p \sim 0.743$).

But for $\delta > 0$ small enough, we get $b(\delta) > 1$ (as a consequence of $\frac{\pi}{\sqrt{12}} < 1$), hence Bourgain has improved what was obtained via Faber-Krahn.

As also observed by Steinerberger, one gets

$$\frac{\lambda(\mathrm{Hexa}_1)}{\lambda(\mathrm{Disk}_1)} \geq \sup_{\delta \in (0,\delta_0)} b(\delta) > 1\,,$$

which gives a limit for any improvement of the estimate.

In any case, we have

$$\liminf_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq \lambda(\mathrm{Disk}_1) A(\Omega)^{-1} \times \sup_{\delta \in (0,\delta_0)} b(\delta) \tag{20}$$

The uncertainty principle by S. Steinerberger

To explain this principle, we associate to a partition Ω_i of Ω

$$D(\Omega_i) = 1 - \frac{\min_j A(\Omega_j)}{A(\Omega_i)}\,,$$

and, with the notation $\Omega \triangle B = (\Omega \setminus B) \cup (B \setminus \Omega)$,

$$\mathcal{A}(\Omega) = \inf_B \frac{A(\Omega \triangle B)}{A(\Omega)}\,,$$

where the infimum is over the balls of same area.

Steinerberger's uncertainty principle reads:

Steinerberger's principle

There exists a universal constant $c > 0$, and $N_0(\Omega)$ such that, if the cardinal N of the partition $\geq N_0(\Omega)$, then

$$\sum_i (D(\Omega_i) + \mathcal{A}(\Omega_i)) \frac{A(\Omega_i)}{A(\Omega)} \geq c\,. \tag{21}$$

Application to equipartitions of energy λ

Let us show how we recover a lower bound for $\liminf_{k\to+\infty}(\mathfrak{L}_k(\Omega)/k)$. We consider a k-equipartition of energy λ. The uncertainty principle says that its is enough to consider two cases.
We first assume that

$$\sum_i D(\Omega_i)\frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2}\,.$$

We can rewrite this inequality in the form:

$$k \inf_j A(\Omega_j) \leq (1-\frac{c}{2})A(\Omega)\,.$$

After implementation of Faber-Krahn, we obtain

$$\frac{k}{\lambda}\lambda(Disk_1) \leq (1-\frac{c}{2})A(\Omega)\,. \tag{22}$$

We now assume that

$$\sum_i \mathcal{A}(\Omega_i)\frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2}\,.$$

This assumption implies

$$A\left(\cup_{\{\mathcal{A}(\Omega_i)\geq \frac{c}{6}\}}\Omega_i\right) \geq \frac{c}{6}A(\Omega)\,. \tag{23}$$

The role of $\mathcal{A}$ can be understood in the following inequality due to Brasco-De Philippis-Velichkov:
$\exists C > 0$ such that $\forall \omega$

$$A(\omega)\lambda(\omega) - \lambda(Disk_1) \geq C\mathcal{A}(\omega)^2\lambda(Disk_1)\,. \tag{24}$$

We apply this inequality with $\omega = \Omega_i$.
This reads

$$A(\Omega_i)\lambda - \lambda(Disk_1) \geq C\mathcal{A}(\Omega_i)^2\lambda(Disk_1)\,.$$

Hence we get for any i such that $\mathcal{A}(\Omega_i) \geq \frac{c}{6}$, to

$$\lambda(Disk_1)(1 + \frac{Cc^2}{36}) \leq A(\Omega_i)\lambda\,. \tag{25}$$

which is an improvement of Faber-Krahn for these Ω_i.
Summing over i and using the information (23) leads to

$$\frac{k}{\lambda}\lambda(Disk_1) \leq (1 + \frac{Cc^2}{36})^{-1} A(\Omega) \left(1 + (1 - \frac{c}{6})\frac{Cc^2}{36}\right)$$

and finally to

$$\frac{k}{\lambda}\lambda(Disk_1) \leq \left(1 - \frac{Cc^3}{216 + 6Cc^2}\right) A(\Omega) \tag{26}$$

Putting (22) and (25) together, we obtain that for k large enough the k-partition satisfies

$$\frac{k}{\lambda}\lambda(Disk_1) \leq \max\left((1-\frac{c}{2}),(1-\frac{Cc^3}{216+6Cc^2})\right)A(\Omega)\,. \tag{27}$$

If we apply this to minimal partitions ($\lambda = \mathfrak{L}_k(\Omega)$), this reads

$$\lambda(Disk_1) \leq \max\left((1-\frac{c}{2}),(1-\frac{Cc^3}{216+6Cc^2})\right)A(\Omega)\liminf_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}\,. \tag{28}$$

One recovers Bourgain's improvement (20) with a different constant.

Remark

Steinerberger obtains also a similar lower bound for $\liminf_{k\to+\infty}\frac{\mathfrak{L}_{k,1}(\Omega)}{k}$ using a convexity argument.

Considerations around rectangles

Take the square $Q = (0,1)^2$. The eigenvalues are given by $\lambda_{m,n} = \pi^2(m^2 + n^2)$. The following conjecture seems to be natural: *Let $\lambda_{m,n} = \pi^2(m^2 + n^2)$ and suppose that $\lambda_{m,n}$ has multiplicity $\mathfrak{m}(m,n)$. Let $\mu_{\max}(u)$ be the maximum of the number of nodal domains of the eigenfunctions in the eigenspace associated with $\lambda_{m,n}$.*

$$\mu_{\max} = \sup_j (m_j n_j)\,,$$

where the sup is computed over the pairs (m_j, n_j) such that

$$\pi^2(m_j^2 + n_j^2) = \lambda_{m,n}\,.$$

The problem is difficult because one has to consider, in the case of degenerate eigenvalues, linear combinations of the canonical eigenfunctions associated with the $\lambda_{m,n}$.

Actually, as stated above, this conjecture is wrong. According to Pleijel, this is wrong for the fifth eigenvalue on the square.

The eigenfunction $(x,y) \mapsto \sin x \, \sin y \, \sin(x+y) \, \sin(x-y) = \frac{1}{4}(\sin x \sin 3y - \sin y \sin 3x) = \psi_{1,3}(x,y)$ has four nodal domains, and the above quantity equals 3. More generally one can consider $\psi_{1,3}(2^k x, 2^k y)$ to get an eigenfunction associated with the eigenvalue $10 \cdot 4^k$ with 4^k nodal domains. The corresponding μ_{max} for this subsequence is asymptotic to $\frac{2}{5\pi}$. This does not infirm the Polterovich's conjecture.

The counterexample of Pleijel

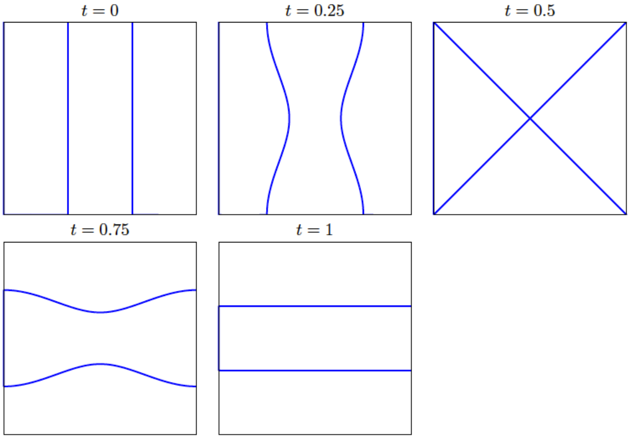


Figure: Nodal sets of $\cos\theta \sin x \sin 3y + \sin\theta \sin y \sin 3x$ for various θ's.

Thanks to C. Lena for transmitting the pictures.

We continue by collecting some observations.

Proposition

Let $\mathcal{R}(a,b) = (0, a\pi) \times (0, b\pi)$. For all $a, b > 0$ with $\frac{a^2}{b^2} \in \mathbb{R} \setminus \mathbb{Q}$ the Pleijel constant

$$Pl(\mathcal{R}(a,b)) = \frac{2}{\pi}. \tag{29}$$

Note that this has been already mentioned in [3] and in [17]. The case when $b^2/a^2 \in \mathbb{Q}$ depends on an alternative to the conjecture discussed before.

Proof

Since the Pleijel constant is scale invariant, it suffices to consider $\mathcal{R}(\pi, b\pi)$ for irrational b^2. The eigenvalues are given by

$$\lambda_{m,n} = m^2 + n^2/b^2 , \tag{30}$$

and the eigenfunctions by $u_{m,n}(x, y) = \sin mx \sin(ny/b)$. Since b is irrational the eigenvalues are simple and

$$\mu(u_{m,n}) = mn. \tag{31}$$

Weyl asymptotics tells us that with $\lambda = \lambda_{m,n}$:

$$k(m,n) := \#\{(\tilde{m},\tilde{n}) : \lambda_{\tilde{m},\tilde{n}}(b) < \lambda\} = \frac{b\pi}{4}(m^2 + n^2/b^2) + o(\lambda)\,. \tag{32}$$

We consider

$$P(m,n;b) = \frac{mn}{k(m,n)} = \frac{4mn}{\pi(m^2 b + n^2/b)} \leq \frac{2}{\pi} \tag{33}$$

Next we take a sequence $b = \lim_{k\to\infty} \frac{p_k}{q_k}$ with $\frac{p_k}{q_k} + \epsilon_k = b, \epsilon_k > 0$. We pick $n = mp_k/q_q$ and estimate with $a_j = p_j/q_j$

$$\frac{4mn}{\pi(m^2 b + n^2/b)} = \frac{2}{\pi}\,\frac{2a_j}{a_j + \epsilon_j + a_j^2(a_j + \epsilon_j)^{-1}}. \tag{34}$$

A simple calculation leads to

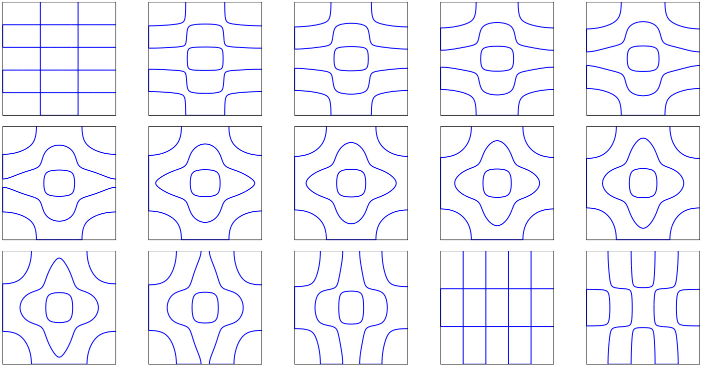
$$PI(j) := \frac{2}{\pi} \frac{2}{1 + \epsilon_j/a_j + 1/(1 + \epsilon_j/a_j)}. \tag{35}$$

We have $c_j := \epsilon_j/a_j \to 0$ and we can write for some finite constant C

$$|PI(j) - 2/\pi| < Cc_j^2. \tag{36}$$

This proves the proposition.

For varying θ, the nodal sets on $]0,\pi[^2$ of
$u(x,y) = \cos(\theta)\,\sin(3x)\,\sin(5y) + \sin(\theta)\,\sin(5x)\,\sin(3y)$.



Thanks to Corentin Lena for transmitting these pictures.

Bipartite partitions

We now start the discussion on possible choices of the class $\mathcal{O}_k^\sharp$. If we think that only nodal partitions are involved in Pleijel's theorem, it could be natural to consider as class $\mathcal{O}_k^\sharp$ the class $\mathcal{O}_k^{bp}$ of the bipartite strong regular connected k-partitions $\mathcal{D} = (D_1, \cdots, D_k)$ (we call this class). Note that there is some arbitrariness in the definition. One would have perhaps preferred to relax the assumptions on the partition like for $\mathfrak{L}_k$ but "strong" is necessary to define a bipartite partition. Note that ν_i should be even in the definition of "regular" and hence

$$\nu_i \geq 4\,.$$

Definition

$$\mathfrak{L}_k^{bp}(\Omega) := \inf_{\mathcal{D}\in\mathcal{O}_k^{bp}} \max \lambda(D_i)\,. \tag{37}$$

By definition, we know that $\mathfrak{L}_k^{bp}(\Omega) \leq L_k(\Omega)$, if the inequality is strict then it cannot by definition come from an eigenfunction. If we want this notion to be helpful for improving Pleijel's constant it is necessary that

$$\liminf_{k\to+\infty} \frac{\mathfrak{L}_k^{bp}(\Omega)}{k} > \liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k}\,.$$

Unfortunately one can show that

$$\mathfrak{L}_k^{bp}(\Omega) = \mathfrak{L}_k(\Omega)$$

Let us explain why on a simple example. We consider the Mercedes Star (MS). One easily sees that we have a bipartite 3-partition whose energy can be arbitrarily close to the energy of MS.

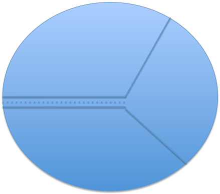


Figure: Scheme of the construction for the Mercedes Star

One can extend the previous construction in order to have the result in full generality. Hence this class do not lead to any improvement of the hexagonal conjecture for Pleijel's Theorem.

Almost nodal partitions

Here is a new try for a definition of $\mathcal{O}^\sharp$ in order to have a flexible notion of partition which is close to a nodal partition. We will say that a k-partition $\mathcal{D}$ of Ω of energy $\Lambda(\mathcal{D})$ is almost nodal with defect j, if there is a connected open set $\Omega' \subset \Omega$ and a $(k-j)$-subpartition $\mathcal{D}'$ of $\mathcal{D}$ such that $\mathcal{D}'$ is a nodal partition of Ω' of energy $\Lambda(\mathcal{D})$. Of course a nodal partition is almost nodal with defect j for any $j \leq k-1$.
The first useful observation is
If Ω is simply connected, there exists always an almost nodal k-partition with defect 1.
The proof is obtained using a sufficiently thin "square" $(k-1)$-partition in Ω and completing by the complementary in Ω of the closure of the union of the preceding squares.

We also note that an almost nodal partition in Ω is almost nodal in $\tilde{\Omega}$ if $\Omega \subset \tilde{\Omega}$. We can now introduce

Definition

Denoting by $\mathcal{O}_k^{anod,j}$ the set of the almost nodal partition with defect j, we introduce

$$\mathfrak{L}_k^{anod,j}(\Omega) = \inf_{\mathcal{D} \in \mathcal{O}_k^{anod,j}} \Lambda(\mathcal{D}) \,. \tag{38}$$

Assuming Ω simply connected it is enough to take $j = 1$ and we write simply $\mathfrak{L}_k^{anod}(\Omega)$.

Of course, we have

$$\mathfrak{L}_k(\Omega) \leq \mathfrak{L}_k^{anod}(\Omega) \leq L_k(\Omega)\,. \tag{39}$$

The next point is to observe, by playing with square tilings, that

$$A(\Omega) \limsup_{k\to+\infty} \frac{\mathfrak{L}_k^{anod}(\Omega)}{k} \leq \lambda(Sq_1)\,. \tag{40}$$

We can then continue in the same way as for $\mathfrak{L}_k^{bp}(\Omega)$, hoping this time that

$$A(\Omega) \liminf_{k\to+\infty} \frac{\mathfrak{L}_k^{anod}(\Omega)}{k} = \lambda(Sq_1)\,.$$

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