Strong diamagnetism for general domains in $\mathbb{R}^3$ and applications to superconductivity

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Main goals

We consider the Neumann Laplacian with constant magnetic field on a regular domain. Let B be the strength of the magnetic field, and let $\lambda_1(B)$ be the first eigenvalue of the magnetic Neumann Laplacian on the domain. It is proved that $B \mapsto \lambda_1(B)$ is monotone increasing for large B.

This result was proved by Fournais-Helffer in the case of dimension 2 (first under a generic assumption, one year later in full generality). Our purpose (this is again a common work with S. Fournais) is to show here how one can prove the same result in dimension 3 (but under generic assumptions). The proof depends heavily on the two term asymptotics of $\lambda_1(B)$ obtained by Pan and Helffer-Morame in 2002.

If time permits,

we will discuss also applications of this monotonicity

for the identification of the critical fields in superconductivity

and

present similar questions in the context of the theory of liquid crystals.

Three models with parameters.

Model 1

The spectral analysis is based in particular on the analysis of the family

$$H(\xi) = D_t^2 + (t+\xi)^2 \,, \tag{1}$$

on the half-line (Neumann at 0) whose lowest eigenvalue $\mu(\xi)$ admits a unique minimum at $\xi_0 < 0$.

We have to keep in mind two universal constants attached to the problem on $\mathbb{R}^+$.

The first one is

$$\Theta_0 = \mu(\xi_0) \,. \tag{2}$$

It corresponds to the bottom of the spectrum of the Neumann realization in $\mathbb{R}^2_+$ (with $B = 1$).

Note that

$$\Theta_0 \in]0,1[\ .$$

The second constant is

$$\delta_0 = \frac{1}{2}\mu''(\xi_0) \ , \qquad (3)$$

Model 2

The second model is quite specific of the problem in dimension 3. We look in $\{x_1 > 0\}$ to

$$\mathfrak{L}(\vartheta, -i\partial_t) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (-i\partial_t + \cos\vartheta\, x_1 + \sin\vartheta\, x_2)^2 \,.$$

By Partial Fourier transform, we arrive to :

$$\mathfrak{L}(\vartheta, \tau) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (\tau + \cos\vartheta\, x_1 + \sin\vartheta\, x_2)^2 \,,$$

in $x_1 > 0$ and with Neumann condition on $x_1 = 0$.

It is enough to consider the variation with respect to $\vartheta \in [0, \frac{\pi}{2}]$.

The bottom the spectrum is given by :

$$\varsigma(\vartheta) := \inf \ \mathsf{Spec}\ (\mathfrak{L}(\vartheta, -i\partial_t)) = \inf_\tau (\inf \ \mathsf{Spec}\ (\mathfrak{L}(\vartheta, \tau))) \,.$$

We first observe the following lemma :

Lemma a.
If $\vartheta \in]0, \frac{\pi}{2}]$, then Spec $(\mathfrak{L}(\vartheta, \tau))$ is independent of τ.

This is trivial by translation in the x_2 variable.

One can then show that the function $\vartheta \mapsto \varsigma(\vartheta)$ is continuous on $]0, \frac{\pi}{2}[$.
This is based on the analysis of the essential spectrum of

$$\mathfrak{L}(\vartheta) := D_{x_1}^2 + D_{x_2}^2 + (x_1 \cos\vartheta + x_2 \sin\vartheta)^2 \,.$$

and that the bottom of the spectrum of this operator corresponds to an eigenvalue.
We then show easily that

$$\varsigma(0) = \Theta_0 < 1 \,.$$

and

$$\varsigma(\frac{\pi}{2}) = 1 \,.$$

Finally, one shows that $\vartheta \mapsto \varsigma(\vartheta)$ is monotonically increasing and that

$$\varsigma(\vartheta) = \Theta_0 + \alpha_1 |\vartheta| + \mathcal{O}(\vartheta^2) \;, \tag{4}$$

with

$$\alpha_1 = \sqrt{\frac{\mu''(\xi_0)}{2}} \;. \tag{5}$$

Model 3 : Montgomery's model.

When the assumptions are not satisfied, and that the magnetic field B vanishes. Other models should be considered. An interesting case is the case when B vanishes along a line. This model was proposed by Montgomery in connection with subriemannian geometry but this model appears also in the analysis of the dimension 3 case.

More precisely, we meet the following family (depending on ρ) of quartic oscillators :

$$D_t^2 + (t^2 - \rho)^2 \,. \tag{6}$$

Denoting by $\nu(\rho)$ the lowest eigenvalue, Kwek-Pan have shown that there exists a unique minimum of $\nu(\rho)$ leading to a new universal constant

$$\hat{\nu}_0 = \inf_{\rho \in \mathbb{R}} \nu(\rho) \,. \tag{7}$$

Main results

Here we will describe the results of Helffer-Morame, Lu-Pan, Pan and the recent paper of Fournais-Helffer.

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary, let $\beta \in \mathbb{R}^3$ be a unit vector, and define $\mathbf{F}$ to be a vector field such that

$$\text{curl}\,\mathbf{F} = \beta, \qquad \text{in } \Omega, \qquad \qquad \mathbf{F} \cdot N = 0 \qquad \text{on } \partial\Omega, \tag{8}$$

where $N(x)$ is the unit interior normal vector to $\partial\Omega$. Define Q_B to be the closed quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto Q_B(u) := \int_\Omega |(-i\nabla + B\mathbf{F})u(x)|^2\,dx. \tag{9}$$

Let $\mathcal{H}(B)$ be the self-adjoint operator associated to Q_B.

In other words, $\mathcal{H}(B)$ is the differential operator $(-i\nabla + B\mathbf{F})^2$ with domain

$$\{u \in W^{2,2}(\Omega)\,:\, N \cdot \nabla u|_{\partial\Omega} = 0\}\,.$$

The operator $\mathcal{H}(B)$ has compact resolvent and we introduce

$$\lambda_1(B) := \inf\ \mathsf{Spec}\,\mathcal{H}(B)\,. \tag{10}$$

We will prove (under a generic assumption on the domain Ω) that the mapping $B \mapsto \lambda_1(B)$ is monotonically increasing for sufficiently large values of B.

We will work under the following geometric assumption.

“Generic” Assumptions = G-Ass.
We assume that the set of boundary points where β is tangent to $\partial\Omega$, i.e.

$$\Gamma := \{x \in \partial\Omega \,\big|\, \beta \cdot N(x) = 0\}, \qquad (11)$$

is a regular submanifold of $\partial\Omega$:

$$\text{grad}\,'(\beta \cdot N)(x) \neq 0 \ , \ \forall x \in \Gamma \ . \qquad (12)$$

We finally assume that the set of points where β is tangent to Γ is finite.

These assumptions are rather generic and for instance satisfied for ellipsoids.

We will need the known two-term asymptotics of the groundstate energy of $\mathcal{H}(B)$. The following result was proved by Helffer-Morame (the corresponding upper bound was also given by Pan).

Theorem 1
If Ω and β satisfy **G-Assumptions**, then as $B \to +\infty$

$$\lambda_1(B) = \Theta_0 B + \widehat{\gamma}_0 B^{\frac{2}{3}} + \mathcal{O}(B^{\frac{2}{3}-\eta}), \tag{13}$$

for some $\eta > 0$.
Here $\widehat{\gamma}_0$ is defined by

$$\widehat{\gamma}_0 := \inf_{x\in\Gamma} \widetilde{\gamma}_0(x), \tag{14}$$

where

$$\widetilde{\gamma}_0(x) := 2^{-2/3}\widehat{\nu}_0\delta_0^{1/3}|k_n(x)|^{2/3}\Big(\delta_0+(1-\delta_0)|T(x)\cdot\beta|^2\Big)^{1/3}\ . \tag{15}$$

Here $T(x)$ is the oriented, unit tangent vector to Γ at the point x and

$$k_n(x) = |\ \text{grad}\ '(\beta\cdot N)(x)|\ .$$

The new result obtained in collaboration with S. Fournais is the

Theorem 2
Let $\Omega \subset \mathbb{R}^3$ and β satisfying G- Assumptions

Let $\{\Gamma_1, \dots, \Gamma_n\}$ be the collection of disjoint smooth curves making up Γ. We assume that, for all j there exists $x_j \in \Gamma_j$ such that $\widetilde{\gamma}_0(x_j) > \widehat{\gamma}_0$.

Then the directional derivatives
$\lambda'_{1,\pm} := \lim_{t \to 0\pm} \frac{\lambda_1(B+t) - \lambda(B)}{t}$,
exist.

Moreover

$$\lim_{B \to \infty} \lambda'_{1,+}(B) = \lim_{B \to \infty} \lambda'_{1,-}(B) = \Theta_0. \tag{16}$$

Localization estimates

We start by recalling the decay of a groundstate in the direction normal to the boundary. We will often use the notation

$$t(x) := \text{dist } (x, \partial\Omega). \tag{17}$$

Now, if $\phi \in C_0^\infty(\Omega)$, i.e. has support away from the boundary, a simple integration by parts implies that

$$Q_B(\phi) \geq B\|\phi\|_2^2. \tag{18}$$

It is a consequence of this elementary inequality (and the fact that $\Theta_0 < 1$) that groundstates are exponentially localized near the boundary.

Theorem 3
There exist constants $C, a_1 > 0, B_0 > 0$ such that

$$\begin{aligned}\int_\Omega e^{2a_1 B^{1/2} t(x)} \Big(|\psi_B(x)|^2 \\ + B^{-1} |(-i\nabla + B\mathbf{F})\psi_B(x)|^2 \Big)\, dx \\ \leq C\, \|\psi_B\|_2^2,\end{aligned} \tag{19}$$

for all $B \geq B_0$, and all groundstates ψ_B of the operator $\mathcal{H}(B)$.

We will mainly use this localization result in the following form.

Corollary 4
For all $n \in \mathbb{N}$, there exists $C_n > 0$ and $B_n \geq 0$ such that, $\forall B \geq B_n$,

$$\int t(x)^n |\psi_B(x)|^2\, dx \leq C_n\, B^{-n/2} \|\psi_B\|_2^2\, .$$

We work in tubular neighborhoods of the boundary as follows. For $\epsilon > 0$, define

$$B(\partial\Omega, \epsilon) = \{x \in \Omega \,:\, t(x) \leq \epsilon\}. \tag{20}$$

For sufficiently small ϵ_0 we have that, for all $x \in B(\partial\Omega, 2\epsilon_0)$, there exists a unique point $y(x) \in \partial\Omega$ such that $t(x) = \text{ dist } (x, y(x))$.

Define, for $y \in \partial\Omega$, the function $\vartheta(y) \in [-\pi/2, \pi/2]$ by

$$\sin\vartheta(y) := -\beta \cdot N(y). \tag{21}$$

We extend ϑ to the tubular neighborhood $B(\partial\Omega, 2\epsilon_0)$ by $\vartheta(x) := \vartheta(y(x))$.

In order to obtain localization estimates in the variable normal to Γ, we use the following operator inequality (due to Helffer-Morame).

Theorem 5
Let B_0 be chosen such that $B_0^{-3/8} = \epsilon_0$ and define, for $B \geq B_0, C > 0$ and $x \in \Omega$,

$$W_B(x) := \begin{cases} B - CB^{1/4}, & t(x) \geq 2B^{-3/8}, \\ B\varsigma(\vartheta(x)) - CB^{1/4}, & t(x) < 2B^{-3/8}. \end{cases} \tag{22}$$

Then, for C large enough

$$\mathcal{H}(B) \geq W_B, \tag{23}$$

(in the sense of quadratic forms) for all $B \geq B_0$.

We use this energy estimate to prove Agmon type estimates on the boundary.

Theorem 6
Suppose that $\Omega \subset \mathbb{R}^3$ and β satisfy G-Assumptions. Define for $x \in \partial\Omega$,

$$d_\Gamma(x) := \text{ dist } (x, \Gamma),$$

and extend d_Γ to a tubular neighborhood of the boundary by $d_\Gamma(x) := d_\Gamma(y(x))$.
Then there exist constants $C, a_2 > 0, B_0 \geq 0$, such that

$$\int_{B(\partial\Omega,\epsilon_0)} e^{2a_2 B^{1/2} d_\Gamma(x)^{3/2}} |\psi_B(x)|^2 \, dx \leq C\|\psi_B\|_2^2, \tag{24}$$

for all $B \geq B_0$ and all groundstates ψ_B of $\mathcal{H}(B)$.

We have the following easy consequence.

Corollary 7
Suppose that $\Omega \subset \mathbb{R}^3$ satisfies G-Assumptions relatively to β. Then for all $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\int_{B(\partial\Omega,\epsilon_0)} d_\Gamma(x)^n |\psi_B(x)|^2 \, dx \leq C_n B^{-n/3} \|\psi_B\|_2^2, \tag{25}$$

for all $B > 0$ and all groundstates ψ_B of $\mathcal{H}(B)$.

Consider now the set $\mathcal{M}_\Gamma \subset \Gamma$ where the function $\widetilde{\gamma}_0$ is minimal,

$$\mathcal{M}_\Gamma := \{x \in \Gamma \,:\, \widetilde{\gamma}_0 = \widehat{\gamma}_0\}. \tag{26}$$

For simplicity, we asume that Γ is connected.

Theorem 8

Suppose that $\Omega \subset \mathbb{R}^3$ satisfies G-Assumptions relatively to β and let $\delta > 0$. Then for all $N > 0$ there exists C_N such that if ψ_B is a groundstate of $\mathcal{H}(B)$, then

$$\int_{\{x\in\Omega :\ \text{dist }(x,\mathcal{M}_\Gamma)\geq\delta\}} |\psi_B(x)|^2\,dx \leq C_N B^{-N}, \tag{27}$$

for all $B > 0$.

Proposition 9

Let d_Γ be the function defined in Theorem 6 . Let $s_0 \in \Gamma$ and define, for $\epsilon > 0$,

$$\Omega(\epsilon, s_0) = \{x \in \Omega : \text{ dist } (x, \Gamma) < \epsilon \text{ and } \text{ dist } (x, s_0) > \epsilon\}.$$

Then, if ϵ is sufficiently small, there exists a function $\phi \in C^\infty(\overline{\Omega})$ such that

$$\widehat{\mathbf{A}} := \mathbf{F} + \nabla\phi \,,$$

and satisfies

$$|\widehat{\mathbf{A}}(x)| \leq C\Big(t(x) + d_\Gamma(x)^2\Big),$$

for all $x \in \Omega(\epsilon, s_0)$.

[Proof of Prop. 9]

We use adapted coordinates (r, s, t) near Γ ($N = 1$). Γ is parametrized by arc-length as

$$\frac{|\Gamma|}{2\pi}\mathbb{S}^1 \ni s \mapsto \Gamma(s) \in \partial\Omega.$$

Given $x \in \Omega$, close to Γ, there is a unique point $\Gamma(s(x)) \in \Gamma$ such that $\text{dist}_{\partial\Omega}(y(x), \Gamma) = \text{dist}_{\partial\Omega}(y(x), \Gamma(s(x)))$, where $\text{dist}_{\partial\Omega}$ denotes the geodesic distance on the boundary. The coordinates (r, s, t) associated to the point x now satisfy

$$|r| = \text{dist}_{\partial\Omega}(y(x), \Gamma), \quad s = s(x), \quad t = \text{dist}\,(x, \partial\Omega).$$

Notice that $d_\Gamma(x) \sim |r(x)|$, so we may replace d_Γ by r in the proposition.

Let $\widetilde{A}_1 dr + \widetilde{A}_2 ds + \widetilde{A}_3 dt$ be the magnetic one-form $\omega_{\mathbf{A}} = \mathbf{A} \cdot d\mathbf{x}$ pulled-back (or pushed forward) to the new coordinates (r, s, t). Also write

$$d\omega_A = \widetilde{B}_{12} dr \wedge ds + \widetilde{B}_{13} dr \wedge dt + \widetilde{B}_{23} ds \wedge dt.$$

Clearly, $\widetilde{B}_{ij} = \partial_i \widetilde{A}_j - \partial_j \widetilde{A}_i,$ for $i < j$. Here we identify $(1, 2, 3)$ with (r, s, t) for the derivatives.

The magnetic field β corresponds to the magnetic two-form via the Hodge-map. In particular, since β is tangent to $\partial\Omega$ at Γ we get that

$$\widetilde{B}_{12}(0, s, 0) = 0. \tag{28}$$

We now find a particular solution $\widetilde{\mathbf{A}}$ such that curl $\widetilde{\mathbf{A}} = \tilde{B}$.

We make the *Ansatz*

$$\widetilde{A}_1 = -\int_0^t \widetilde{B}_{13}(r,s,\tau)\,d\tau, \tag{29}$$

$$\widetilde{A}_2 = -\int_0^t \widetilde{B}_{23}(r,s,\tau)\,d\tau + \int_0^r \widetilde{B}_{12}(\rho,s,0)\,d\rho, \tag{30}$$

$$\widetilde{A}_3 = 0. \tag{31}$$

One verifies by inspection that with these choices

$$|\widetilde{\mathbf{A}}| \leq C(r^2 + t). \tag{32}$$

Transporting this $\widetilde{\mathbf{A}}$ back to the original coordinates gives an $\widehat{\mathbf{A}}$ with

$$\text{curl } \widehat{\mathbf{A}} = 1, \qquad |\widehat{\mathbf{A}}(x)| \leq C(t(x) + d_\Gamma(x)^2).$$

Since $\Omega(\epsilon, s_0)$ is simply connected (for sufficiently small ϵ) $\widehat{\mathbf{A}}$ is gauge equivalent to $\mathbf{F}$ and the proposition is proved.

Monotonicity

We now prove how one can derive the monotonicity result from the known asymptotics of the groundstate energy and localization estimates for the groundstate itself.

Based on these estimates the proof of Theorem is very similar to the two-dimensional case.

Proof of Theorem 2
For simplicity, we assume that Γ is connected. Applying analytic perturbation theory to $\mathcal{H}(B)$ we get the first part.

Let $s_0 \in \Gamma$ be a point with $\widetilde{\gamma}(s_0) > \widehat{\gamma}_0$. Let $\widehat{\mathbf{A}}$ be the vector potential defined in Proposition 9.

Let $\widehat{Q}_B$ the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto \widehat{Q}_B(u) = \int_\Omega |-i\nabla u + B\widehat{\mathbf{A}}u|^2 dx\,,$$

and $\widehat{\mathcal{H}}(B)$ be the associated operator.
Then $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$ are unitarily equivalent: $\widehat{\mathcal{H}}(B) = e^{iB\phi}\mathcal{H}(B)e^{-iB\phi}$, for some ϕ independent of B.
With $\psi_1^+(\,\cdot\,;\beta)$ being a suitable choice of normalized groundstate, we get (by analytic perturbation theory applied to $\mathcal{H}(B)$ and the explicit relation between $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$,

$$\begin{aligned} &\lambda_{1,+}'(B) \\ &= \langle \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\,,\, p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\rangle \\ &\qquad + \langle p_{B\widehat{\mathbf{A}}}\psi_1^+(\,\cdot\,;B)\,,\, \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\rangle\,. \end{aligned} \tag{33}$$

We now obtain for any $b > 0$,

$$\lambda'_{1,+}(B) = \frac{\widehat{Q}_{B+b}(\psi_1^+(\,\cdot\,;B)) - \widehat{Q}_B(\psi_1^+(\,\cdot\,;B))}{b} \tag{34}$$

$$- b\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx$$

$$\geq \frac{\lambda_1(B+b) - \lambda_1(B)}{b} - b\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx\,. \tag{35}$$

We choose $b := MB^{\frac{2}{3}-\eta}$, with η from (13) and $M > 0$ (to be taken arbitrarily large in the end). Then, using (13), (34) becomes

$$\begin{aligned} \lambda'_{1,+}(B) \quad &\geq \Theta_0 + \widehat{\gamma}_0 B^{-1/3}\tfrac{(1+b/B)^{2/3}-1}{b/B} \\ &-CM^{-1} - b\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx\,, \end{aligned} \tag{36}$$

for some constant C independent of M, B.

If we can prove that

$$B^{\frac{2}{3}} \int_{\Omega} |\widehat{\mathbf{A}}|^2 \, |\psi_1^+(x;B)|^2 \, dx \leq C, \tag{37}$$

for some constant C independent of B, then we can take the limit $B \to \infty$ in (36) and obtain

$$\liminf_{B\to\infty} \lambda_{1,+}'(B) \geq \Theta_0 - CM^{-1}. \tag{38}$$

Since M was arbitrary this implies the lower bound for $\lambda_{1,+}'(B)$. Applying the same argument to the derivative from the left, $\lambda_{1,-}'(B)$, we get (the inequality gets turned since $b < 0$)

$$\limsup_{B\to\infty} \lambda_{1,-}'(B) \leq \Theta_0. \tag{39}$$

Since, by perturbation theory, $\lambda_{1,+}'(B) \leq \lambda_{1,-}'(B)$ for all B, we get (16).

Thus it remains to prove (37).

By Proposition 9 we can estimate

$$\begin{aligned}
&\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx \\
&\leq C \int_{\Omega\left(\epsilon, s_0\right)} (t^2 + r^4)|\psi_1^+(x;B)|^2\, dx \\
&\quad + \|\widehat{\mathbf{A}}\|_\infty^2 \int_{\Omega\setminus\Omega\left(\epsilon, s_0\right)} |\psi_1^+(x;B)|^2\, dx\,.
\end{aligned}$$

Combining Corollaries 4 and 7 and Theorem 8, we therefore find the existence of a constant $C > 0$ such that :

$$\int_\Omega |\widehat{\mathbf{A}}|^2\, |\psi_1^+(x;B)|^2\, dx \leq C\, B^{-1}\,, \qquad (40)$$

which is stronger than the needed estimate (37).

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\mathbf{A}] = \int_\Omega \Big\{ |\nabla_{\kappa H \mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl} \mathbf{A} - \beta|^2 \Big\}\, dx\,,$$

with
Ω simply connected,
$(\psi,\mathbf{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^3)$,
$\beta = (0,0,1)$
and where
$\nabla_{\mathbf{A}} = (\nabla + i\mathbf{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{div} \mathbf{A} = 0 \quad \text{in } \Omega\,,\ \ \mathbf{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Terminology for the minimizers

The pair $(0, \mathbf{F})$ is called the Normal State.

A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.

In the other cases, one will speak about Mixed State.

The general question is to determine the topology of the subset in $\mathbb{R}^+ \times \mathbb{R}^+$ of the (κ, H) corresponding to minimizers belonging to each of these three situations.

Theorem 10 (Lu-Pan-Fournais-Helffer)

There exists κ_0 such that, $\forall \kappa \geq \kappa_0$, $(0, \mathbf{F})$ is a global minimizer of $\mathcal{E}_{\kappa,H}$ iff $\lambda_1(\kappa H) < \kappa^2$.

Remark.

This makes our analysis of the monotonicity of λ_1 (for B large), which implies, for κ large, the existence of a unique H such that

$$\lambda_1(\kappa H) = \kappa^2 \ .$$

particularly interesting.

Same questions in the theory of Liquid crystals

A similar arises in the theory of Liquid crystals. The simples question is to consider the case when the magnetic vector field is only of constant module ! Typically one is interested in the case when

$$\beta(x) = B\,(\cos\tau x_3, \sin\tau x_3, 0)\ ,$$

as $B \to +\infty$. There are partial results for this model obtained by Almog and Pan.

The energy for this model can be written as

$$\mathcal{E}[\psi,\mathbf{n}] = \int_\Omega \Big\{ |\nabla_{q\mathbf{n}}\psi|^2 - \mathbf{k^2}|\psi|^\mathbf{2} + \frac{\mathbf{k^2}}{\mathbf{2}}|\psi|^4 + K_1\,|\ \mathrm{div}\ \mathbf{n}|^\mathbf{2} + \mathbf{K_2}\,|\mathbf{n}\cdot\ \mathrm{curl}\ \mathbf{n} + \tau|^\mathbf{2} + K_3\,|\mathbf{n}\times\ \mathrm{curl}\ \mathbf{n}|^\mathbf{2} \Big\}\,\mathbf{dx}\,,$$

where :

- Ω is the region occupied by the liquid crystal,

- ψ is a complex-valued function called the *order parameter*,

- $\mathbf{n}$ is a real vector field of unit length called *director field*,

- q is a real number called *wave number*,

- τ is a real number measuring the chiral pitch in some liquid crystal materials,

- K_1, K_2 and K_3 are positive constants called the elastic coefficients,

and

- $\mathbf{k}$ is a positive constant which depends on the material and on the temperature.

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