Maximal microhypoellipticity or subellipticity for systems and applications to Witten Laplacians (After Helffer-Nourrigat, Maire, Nourrigat, Helffer-Nier and Derridj)

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Main goals

Revisit some aspects of the theory of hypoelliptic systems which could be useful in semi-classical analysis.

The approach which is developed here is based on the theory on nilpotent groups (old collaboration with J. Nourrigat) but the criteria which appear for specific first order systems are much more explicit and permit to go much further in the analysis.

Here we were mainly inspired (in the book with F. Nier) by the presentation, given by J. Nourrigat in a course in Recife [No1] of results which appear in a less explicit form in the book [HeNo3].
This is a particular aspect of the large program developed by J. Nourrigat at the end of the 80's for understanding the maximal hypoellipticity of differential systems of order 1 in connection with the characterization of subelliptic systems No1-No6.

As we shall see, an interest of this analysis of the maximal hypoellipticity by an approach based on the nilpotent Lie group techniques is that it provides

global, local or microlocal estimates,

leading to sufficient conditions for the compactness of the resolvent or to semiclassical local lower bounds.

More precisely, our aim is to analyze the maximal hypoellipticity of the system of n first order complex vector fields

$$L_j = (X_j - iY_j),\ \text{where } X_j = \partial_{x_j} \text{ and } Y_j = (\partial_{x_j}\Phi(x))\,\partial_t\ , \tag{1}$$

in a neighborhood $\mathcal{V}(0) \times \mathbb{R}_t$ of $0 \in \mathbb{R}^{n+1}$, where $\Phi \in C^\infty(\mathcal{V}(0))$.
We will show at the same time how the techniques used for this analysis will lead to some information on the question concerning the Witten Laplacian associated to Φ.

We assume that the real function Φ is such that the rank r Hörmander condition is satisfied for the vector fields $(X_j), (Y_j)$ at $(0,0)$.
This is an immediate consequence of the condition :

$$\sum_{1\leq|\alpha|\leq r} |\partial_x^\alpha \Phi(0)| > 0\ . \tag{2}$$

By maximal hypoellipticity for the system (1), we mean the existence of the inequality :

$$\sum_j ||X_j u||^2 + \sum_j ||Y_j u||^2 \leq C \left(\sum_j ||L_j u||^2 + ||u||^2 \right) . \tag{3}$$

The symbol of the system is the map :

$$\begin{aligned} & T^*(\mathcal{V}(0) \times \mathbb{R}) \setminus \{0\} \ni (x, t, \xi, \tau) \\ & \quad \mapsto \sigma(L)(x, t, \xi, \tau) := \big(i\xi_j + \tau(\partial_{x_j}\Phi)(x) \big)_{j=1,\dots,n} \in \mathbb{C}^n . \end{aligned} \tag{4}$$

The characteristic set is then by definition the set of zeroes of (the principal symbol of) $\sigma(L)$:

$$\sigma(L)^{-1}(0) = \{\xi = 0 \ , \ \nabla\Phi(x) = 0\} . \tag{5}$$

Outside this set the system is microlocally elliptic (its (principal) symbol does not vanish) and hence (maximally) microlocally hypoelliptic. So the local (maximal) hypoellipticity will result of the microlocal analysis in the neighborhood of the characteristic set, which has actually two connected components defined by $\{\pm\tau > 0\}$.

So we are more precisely interested in the microlocal hypoellipticity in a conic neighborhood $V_{\pm}$ of $(x, t; \xi, \tau) = (0; 0, \pm 1)$, that is with the microlocalized version of the inequality (3).

Due to the invariance of the problem with respect to the t variable, we look for an inequality which is local in x but global in the t variable and take the partial Fourier transform with respect to t in order to analyze the problem.

Observe that :

$$\sum_j ||L_j u||^2 = \sum_j \langle L_j^* L_j u \mid u \rangle \ , \tag{6}$$

and that

$$\sum_j L_j^* L_j = -\left(\sum_j X_j^2 + \sum_j Y_j^2 - i \sum_j [X_j, Y_j] \right) \ . \tag{7}$$

Microlocal hypoellipticity and semi-classical analysis

Global estimates for operators with t-independent coefficients lead after partial Fourier transform with respect to the t-variable to semiclassical results.

We look for $C > 0$, such that, for any $\tau \in \mathbb{R}$, :

$$\begin{aligned}\sum_j ||\pi_\tau(X_j)v||^2 + \sum_j ||\pi_\tau(Y_j)v||^2 \\ \leq C\left(\sum_j ||\pi_\tau(L_j)v||^2 + ||v||^2\right) \ ,\end{aligned} \tag{8}$$

in a ngbd $\mathcal{V}(0)$ of 0 in $\mathbb{R}^n$ awith

$$\pi_\tau(L_j) = \pi_\tau(X_j) - i\pi_\tau(Y_j) = \partial_{x_j} + \tau(\partial_j \Phi)(x) \ . \tag{9}$$

Two remarks :

1. The estimate (8) is trivial for τ in a bounded set.

2. Depending on which connected component of the characteristic set is concerned, we have to consider the inequality for $\pm\tau \geq 0$ (τ large).

From now on, we choose the $+$ component and assume

$$\tau > 0 \tag{10}$$

for simplicity. In any case, changing Φ into $-\Phi$ exchanges the roles of $\tau > 0$ and $\tau < 0$, so there is no loss of generality in this choice.

If we introduce the semi-classical parameter by :

$$h = \frac{1}{\tau}\ , \tag{11}$$

the inequality (8) becomes, after division by τ^2 :

$$\begin{array}{c}\sum_j ||(h\partial_{x_j})v||^2 + \sum_j ||(\partial_{x_j}\Phi)\, v||^2 \\ \leq C \left(\langle \Delta^{(0)}_{\Phi,h} v \mid v\rangle + h^2 ||v||^2 \right)\ ,\end{array} \tag{12}$$

for all $v \in C_0^\infty(\mathcal{V}(0))$, where

$$\Delta^{(0)}_{\Phi,h} = -h^2\Delta + |\nabla\Phi|^2 - h\Delta\Phi\ , \tag{13}$$

is the semi-classical Witten Laplacian on functions.

Hörmander's condition gives as a consequence of the microlocal subelliptic estimate (cf also [BoCaNo]) the existence of $\mathcal{V}(0)$, $h_0 > 0$ and $C > 0$ such that :

$$h^{2-\frac{2}{r}} ||v||^2 \leq C \left(\sum_j ||(h\partial_{x_j})v||^2 + \sum_j ||(\partial_{x_j}\Phi)\, v||^2 \right) \tag{14}$$

for $h \in]0, h_0]$ and $v \in C_0^\infty(\mathcal{V}(0))$.

So we finally obtain the existence of $\mathcal{V}(0)$, $h_0 > 0$ and $C > 0$ such that :

$$h^{2-\frac{2}{r}} ||v||^2 \leq C \, \langle \Delta^{(0)}_{\Phi,h} v \mid v \rangle \; , \; \forall v \in C_0^\infty(\mathcal{V}(0)) \; , \tag{15}$$

for $h \in]0, h_0]$.
So the maximal microhypoellipticity (actually the subellipticity would have been enough) in the "$+$" component implies some semi-classical localized lower bound for the semi-classical Witten Laplacian of order 0.

Of course, many semi-classical results can be obtained by other techniques, particularly in the case when Φ is a Morse function.

For the discussion of the different approaches, it is convenient to say that the semiclassical Witten Laplacian $\Delta^{(0)}_{\Phi,h}$ is said δ-subelliptic, $0 \leq \delta < 1$, in an open set Ω, if there exist $C > 0$ and $h_0 > 0$ such that the estimate,

$$h^{2\delta} \left\|v\right\|^2 \leq C \; ||d^{(0)}_{\Phi,h} v||^2 \; , \tag{16}$$

holds uniformly for all $h \in (0, h_0]$ and $v \in \mathcal{C}_0^\infty(\Omega)$.

The estimate (15) says that $\Delta^{(0)}_{\Phi,h}$ is $(1-\frac{1}{r})$-subelliptic in a neighborhood of $x = 0$. If one goes back to the system with $\tau = \frac{1}{h}$, the δ-subellipticity of $\Delta^{(0)}_{\Phi,h}$ gives for the system L_j :

$$\tau^{2-2\delta} \left\|v\right\|^2 \leq C \left(\sum_j \left\|\pi_\tau(L_j) v\right\|^2 \right) , \quad \forall \tau > 0 \; ,$$

which means that the system (L_j) is microlocally hypoelliptic near $(0; 0, +1)$ with loss of δ derivatives.

Analysis of the microhypoellipticity for systems

Let us now express what the group theoretical criteria of Helffer-Nourrigat will give for the semi-classical Witten Laplacian.

Definition
Given Φ satisfying Hörmander condition of rank r at 0, we denote by $\mathcal{L}_0$ the set of all polynomials P of degree less or equal to r vanishing at 0 ($P \in E_r$) such that there exists a sequence $x_n \to 0$, $\tau_n \to +\infty$ and $d_n \to 0$ such that :

$$d_n^{|\alpha|} \tau_n (\partial_x^\alpha \Phi)(x_n) \to \partial_x^\alpha P(0) \ . \tag{17}$$

Now, the translation of Helffer-Nourrigat's Theorem [HelNo3] provides the semiclassical estimate :

Theorem

We assume that (2) is satisfied at rank r. Then, if the condition :

No polynomial in $\mathcal{L}_0$ except 0 has a local minimum at the origin,

is satisfied then the operator $\Delta^{(0)}_{\Phi,h}$ is $(1-\frac{1}{r})$-subelliptic in a neighborhood of $x=0$ (inequality (15) holds for h small enough). Moreover, the condition is necessary for getting the maximal estimate (12).

Remarks

Equivalently the condition in Theorem is :

There exists a neighborhood V of 0 and two constants d_0 and c_0, such that :

$$\inf_{|x-x_1|\leq d}(\Phi(x)-\Phi(x_1)) \leq -c_0 \sup_{|x-x_1|\leq d}|\Phi(x)-\Phi(x_1)|\,,$$

for all $x_1 \in V$ and for all $d \in [0, d_0[$.

Ref : Tr, No1-No6,HeNo3 and Mai1).

In the case, when Φ is a Morse function with critical point at 0, the set $\mathcal{L}_0$ is simply the quadratic approximation of Φ at 0 up to translation and dilation.

When Φ is a Morse function, and if Φ has a local minimum at a point x_{min}, the implementation in (15) of the trial function $\chi \exp -\frac{\Phi}{h}$, where χ is a cut-off function localizing in the neighborhood of x_{min}, shows that there are no hope to have a subelliptic estimate. The right hand side in (15) becomes indeed exponentially small $\mathcal{O}(\exp -\frac{\alpha}{h})$, for some $\alpha > 0$.
This argument works more generally under the weaker assumption that Φ has isolated critical points, without assuming the Morse property.

In connection with previous work by F. Trèves [Tr2], Maire's results [Mai1] suggest the

Conjecture
Under the assumption that Φ is analytic and that Φ has no local minimum at the origin, then $\Delta^{(0)}_{\Phi,h}$ is δ-subelliptic in a neighborhood of 0 for some $\delta \in [0,1)$.

Maire's result on hypoellipticity relies crucially on the Lojaciewicz's inequality.

Maire [Mai1] is concerned with more general systems for which τ lies in a multidimensional space. The situation met here is one dimensional.

The proof of Maire leads only to a weak form of subellipticity, implying effectively microlocal hypoellipticity but giving only an L^∞ version of (16).

The system

$$\left\{ \begin{array}{ll} L_1 & = \partial_{x_1} - i\left((2\ell+1)x_1^{2\ell} - x_2^2\right)\partial_t \;, \\ L_2 & = \partial_{x_2} + 2ix_1x_2\partial_t \;, \end{array} \right. \qquad (18)$$

is L^2-microlocally subelliptic ([Der1]) with minimal loss of $\frac{2\ell}{2\ell+1}$ derivatives, but not maximally hypoelliptic if $\ell > 1$. This implies that the Witten Laplacian $\Delta^{(0)}_{\Phi,h}$, with

$$\Phi = x_1^{2\ell+1} - x_1x_2^2\,,$$

is $\frac{2\ell}{2\ell+1}$-subelliptic in a neighborhood of $x = 0$.

Around the proof of Theorem

The proof is based on a priori estimates obtained by a recursion argument strongly related to Kirillov's theory. All this section is strongly inspired by the presentation of J. Nourrigat [No1]. We first observe that the subset $\mathcal{L}_0$ (in the space E_r of the polynomials of degree r vanishing at 0, has some stability[1] properties :

1. If $P \in \mathcal{L}_0$ and $y \in \mathbb{R}^n$, then the polynomial defined by

   $$Q(x) = P(x+y) - P(y), \forall x \in \mathbb{R}^n \ ,$$

   is also in $\mathcal{L}_0$.

2. If $P \in \mathcal{L}_0$ and $\lambda > 0$, then $Q(x) = P(\lambda x)$ is also in $\mathcal{L}_0$.

[1] which are actually the translation of the group theoretical properties appearing in the definition of the set Γ_{x_0,ξ_0} in Helffer-Nourrigat's conjecture about maximal microhypoellipticity,

3. $\mathcal{L}_0$ is a closed subset of E_r .

A set in E_r satisfying the three previous stability conditions is called canonical set .

To each polynomial $P \in E_r$, we can associate a system of differential operators in $\mathbb{R}^n$ by

$$\begin{array}{c} \pi_P(X_j) = D_{x_j} \ , \ \pi_P(Y_j) = \partial_{x_j} P \ , \\ \pi_P(L_j) = \pi_P(X_j) - i\pi_P(Y_j) \ . \end{array} \tag{19}$$

The core of the proof is in the **Proposition** :
Let $\mathcal{L}$ be a canonical subset of E_r. We assume that for any $P \in \mathcal{L} \setminus \{0\}$, P has no local minimum in $\mathbb{R}^n$. Then there exists a constant $c_0 > 0$ such that :

$$\begin{array}{l} \sum_j ||\pi_P(X_j)u||^2 + \sum_j ||\pi_P(Y_j)u||^2 \\ \leq c_0 \sum_j ||\pi_P(L_j)u||^2 \ , \end{array} \tag{20}$$

for all $P \in \mathcal{L}$.

The proof involves a recursion on the rank :

Proposition

Let $\mathcal{L}$ be a canonical subset of E_r. We assume that for any $P \in (\mathcal{L} \cap E_{r-1}) \setminus \{0\}$, P has no local minimum in $\mathbb{R}^n$. Then there exists $c_1 > 0$ s. t. :

$$\begin{aligned} &\sum_j ||\pi_P(X_j)u||^2 + \sum_j ||\pi_P(Y_j)u||^2 \\ &\quad \leq c_1 \left(\sum_j ||\pi_P(L_j)u||^2 + \left[\sum_{|\alpha|=r} |P^{(\alpha)}(0)| \right]^{\frac{2}{r}} ||u||^2 \right) , \end{aligned} \tag{21}$$

for all $P \in \mathcal{L}$.

This will have interesting byproducts.

Spectral by-products for the Witten Laplacians

For a polynomial $\Phi \in E_r$, we denote by $\mathcal{L}_\Phi$ the smallest canonical closed set containing Φ.

Theorem
Let $\Phi \in E_r$ and let us assume that :

1. The representation π_Φ is irreducible[2].

2. The canonical set $\mathcal{L}_\Phi \cap E_{r-1}$ does not contain any non zero polynomial having a local minimum.

Then the Witten Laplacian $\Delta_\Phi^{(0)}$ has compact resolvent. Moreover we have maximal estimates for the system $d_\Phi^{(0)}$ and for the corresponding Laplacian $\Delta_\Phi^{(0)}$.

[2]We recall that this condition is equivalent to $k(\Phi) = n$ or to the property that $\sum_{|\alpha|>0} |D_x^\alpha \Phi(x)| \to +\infty$ as $|x| \to +\infty$.

Applications for homogeneous examples

Let Φ be an homogeneous polynomial of degree r without translational invariance. The set $\mathcal{L}_\Phi \cap E_{r-1}$ is obtained by determining the polynomials P_∞ of order $r-1$ appearing as limits :

$$P_\infty = \lim_{n\to+\infty} \left(\lambda_n^r \Phi(\cdot + h_n) - \lambda_n^r \Phi(h_n)\right) \ ,$$

for some sequence (λ_n, h_n) with

$$\lambda_n \to 0 \ .$$

The coefficients of this limiting polynomial P_∞ should satisfy :

$$\lim_{n\to+\infty} \lambda_n^r (\partial_x^\alpha \Phi)(h_n) = (\partial_x^\alpha P_\infty)(0) \ .$$

1. Elliptic case.

This corresponds to

$$\nabla\Phi(x) \neq 0 \ , \ \forall x \neq 0 \ . \tag{22}$$

One can show that the limit polynomial is necessarily of degree 1, which clearly cannot have any local minimum, except the 0 case.

This case can also be treated more directly by observing that, under assumption (22), $|\nabla\Phi(x)| \to +\infty$ as $|x| \to +\infty$ and by observing that $\Delta\Phi$ is of lower order. We indeed immediately obtain that :

$$\lim_{|x|\to+\infty} \left(|\nabla\Phi(x)|^2 - \Delta\Phi(x)\right) = +\infty \ .$$

2. Generic non-elliptic case

We assume now that

$$|\nabla\Phi|^{-1}(0) \cap (\mathbb{R}^n \setminus \{0\}) \neq \emptyset \,, \tag{23}$$

and introduce the non degeneracy condition :

$$\sum_{1\leq|\alpha|\leq 2} |\Phi^{(\alpha)}(x)| \neq 0 \;,\; \forall x \neq 0 \;. \tag{24}$$

Under this condition, all the limiting polynomials in $\mathcal{L}_\Phi \cap E_{r-1}$ should be of order less than 2. Because Φ is homogeneous :

$$\forall \omega \in (\nabla\Phi)^{-1}(\{0\}) \cap \mathbb{S}^{n-1} \;,\; \Phi''(\omega)\cdot\omega = 0 \;. \tag{25}$$

Then

Proposition

Under assumptions (24) and if, for all $\omega \in (\nabla\Phi)^{-1}(\{0\}) \cap \mathbb{S}^{n-1}$, the Hessian $\Phi''(\omega)$ restricted to $(\mathbb{R}\omega)^\perp$ is non degenerate and not of index 0, then the corresponding Witten Laplacian $\Delta_\Phi^{(0)}$ has a compact resolvent.

Let us consider as an example the case :

$$\Phi_\varepsilon(x_1, x_2) = \varepsilon x_1^2 x_2^2 \;, \quad \text{with } \epsilon = \pm 1 \;, \tag{26}$$

and let us determine more explicitly all the limiting polynomials. A computation gives the two types of limiting polynomials

$$\begin{array}{l} P_\infty(x) = \frac{\gamma}{2} x_1^2 + \ell_1 x_1 \;, \quad \text{(with } \varepsilon\gamma > 0) \;, \\ P_\infty(x) = \frac{\gamma}{2} x_2^2 + \ell_2 x_2 \;, \quad \text{(with } \varepsilon\gamma > 0) \;. \end{array} \tag{27}$$

The conclusion is : For $\Phi(x_1, x_2) = -x_1^2 x_2^2$, the corresponding Witten Laplacian has a compact resolvent.

When $\varepsilon = 1$, one can find a non trivial polynomial having a local minimum. The criterion does not apply. We will show later that the operator actually does not have a compact resolvent.

Another result of Helffer-Nier

Theorem (Helffer-Nier)
Let $\Phi \in \mathbb{R}[X_1, \ldots, X_d]$ be a polynomial potential.

i) If Φ is a sum of non negative monomials, then we have :

$$(Poincaré) \Leftrightarrow \left(\lim_{|x|\to\infty} \Phi(x) = +\infty\right) \Leftrightarrow \left((1+\Delta_\Phi^{(0)})^{-1} \text{ compact}\right) .$$

ii) If Φ is a sum of non positive monomials, then $\Delta_\Phi^{(0)}$ has a compact resolvent if and only if $q(x) := \sum_{|\alpha|>0} |D_x^\alpha \Phi(x)| \to +\infty$.

Examples :

(a) : $\Phi = x_1^2 x_2^2$ in $\mathbb{R}^2$.
0 belongs to the essential spectrum of $\Delta_\Phi^{(0)}$. $e^{-\Phi}$ is not in $L^2(\mathbb{R}^d)$ and 0 is not an eigenvalue of $\Delta_\Phi^{(0)}$.

(b) : $\Phi = x_1^2 x_2^2 (x_1^2 + x_2^2)$ in $\mathbb{R}^2$.
0 is an eigenvalue contained in the essential spectrum.

(c) : $\Phi = (x_1^2 + x_2^2)(x_2^2 + x_3^2) + (x_1^2 + x_3^2)$ in $\mathbb{R}^3$.
Then $\Delta_\Phi^{(0)}$ has a compact resolvent.

(d) : $\Phi = \left(1 + |x^2|\right)^{1/2}$ in $\mathbb{R}^d$.
The function $e^{-\Phi}$ belongs to $L^2(\mathbb{R}^d)$. The Poincaré inequality is satisfied, but since $V = |\nabla\Phi|^2 - \Delta\Phi$ belongs to $L^\infty(\mathbb{R}^d)$, the resolvent of $\Delta_\Phi^{(0)}$ is not compact.

(e) : $\Phi_\varepsilon = x_1^2 x_2^2 + \varepsilon(x_1^2 + x_2^2)$ in $\mathbb{R}^2$.
$\Delta_{\Phi_\varepsilon}^{(0)}$ has a compact resolvent for any $\varepsilon > 0$.

About the Poincaré inequality for an homogeneous potential.

Here we make the connection between the different approachs. We consider the simple case of an homogeneous potential near ∞ without sign condition :

$$\forall x \in \mathbb{R}^n, |x| \geq 1, \quad \Phi(x) = |x|^m \varphi(\frac{x}{|x|}) \ . \tag{28}$$

With the homogeneity degree m, we associate the integer

$$\widehat{m} = \max\{\mu \in \mathbb{N}, \mu < m\} \ . \tag{29}$$

We shall provide here various necessary and sufficient conditions for the compactness of the resolvent of $\Delta_{\Phi}^{(0)}$. The sufficient conditions will rely on maximal or non-maximal microhypoellipticity of associated complex differential systems and the comparison of the two cases will be done. When φ is a Morse function, we have necessary and sufficient conditions which depend on m.

Necessary conditions

Assume that the $\Phi \in C^\infty(\mathbb{R}^n)$ satisfies (28). If the Witten Laplacian $\Delta_\Phi^{(0)}$ has a compact resolvent, then

i) $m > 1$;

ii) φ does not vanish at order $\widehat{m} + 1$, $\sum_{|\alpha| \leq \widehat{m}} |\partial_\theta^\alpha \varphi(\theta)| > 0$;

iii) There is no pair $K \subset U$ with $K \subset \varphi^{-1}(\{0\})$ compact, $K \neq \emptyset$ and $U \subset \mathbb{S}^{n-1}$ open, such that

$$\forall \theta \in U \setminus K, \quad \varphi(\theta) > 0 \, .$$

The condition $m > 1$ is easy. [3] The condition (ii) is obtained by contradiction and construction of a Weyl's sequence.

[3] If not, $\Delta_\Phi^{(0)}$ is a bounded perturbation of $-\Delta$.

Sufficient conditions

The compactness of the resolvent of $\Delta_\Phi^{(0)}$, with $\Phi(r\theta) = |x|^m \varphi(\frac{x}{|x|})$ is related to the microhypoellipticity of the system $\partial_{\theta_j} + (\partial_{\theta_j}\varphi(\theta))D_t$ near the point $(\theta_0, t_0, \hat{\theta}, \tau = +1)$ in $(\mathbb{S}^{n-1} \times \mathbb{R}_t) \times (\mathbb{R}^n \setminus \{0\})$, where θ_0 is a zero of φ. A condition related to Remark permits maximal hypoellipticity arguments and leads to the compactness of the resolvent for $\Delta_\Phi^{(0)}$. Meanwhile the condition that φ is analytic with no 0-valued minimum, which leads to microhypoellipticity properties, will not be sufficient in general.

We start with a proposition. First we introduce for $c > 1$, the shell :

$$\mathcal{S}_c = \left\{x \in \mathbb{R}^d, \quad c^{-1} < |x| < c\right\} \ .$$

.

Proposition

Let $\Phi \in C^\infty(\mathbb{R}^n)$ be of the form (28), $m > 1$, and such that φ does not vanish at order $\widehat{m} + 1$. If for some $c > 1$ and $\mu \geq 1$, the function Φ is homogeneous in $\left\{|x| \geq c^{-1}\right\}$ and the semiclassical Witten Laplacian $\Delta^{(0)}_{\Phi,h}$ is $(1 - \frac{1}{\mu})$-subelliptic in $\mathcal{S}_c$, then the Witten Laplacian $\Delta^{(0)}_{\Phi}$ $(h = 1)$ is bounded from below by $C^{-1}\langle x\rangle^{2(\frac{m}{\mu}-1)} - C$ and has a compact resolvent if $\frac{m}{\mu} > 1$.

The idea behind the proof is the introduction of a dyadic decomposition in shells and an associated partition of unity.

Here is a sufficient condition for the compactness of the resolvent of $\Delta_\Phi^{(0)}$ relying on maximal hypoellipticity.

Proposition

Assume that $\Phi \in C^\infty(\mathbb{R}^n)$ has the form (28), $m > 1$, and satisfies

(1) φ does not vanish at order $\widehat{m} + 1$

and

(2) For all $\theta_0 \in \varphi^{-1}(\{0\})$, there exist a neighborhood $\mathcal{V}_{\theta_0}$ of θ_0 and two constants $d_{\theta_0} > 0$ and $c_{\theta_0} > 0$, such that, for all $d \in]0, d_{\theta_0}]$, for all $\theta_1 \in \mathcal{V}_{\theta_0}$,

$$\inf_{|\theta-\theta_1|\leq d}(\varphi(\theta) - \varphi(\theta_1)) \leq -c_{\theta_0} \sup_{|\theta-\theta_1|\leq d} |\varphi(\theta) - \varphi(\theta_1)| \,. \tag{30}$$

Then the Witten Laplacian $\Delta_\Phi^{(0)}$ has a compact resolvent.

In dimension $n = 2$, the condition (30) is equivalent to the absence of 0-valued local minimum.

This is

an application of the technical Proposition once we have checked that the semiclassical Witten Laplacian $\Delta^{(0)}_{\Phi,h}$ is $(1 - \frac{1}{\widehat{m}})$-subelliptic in any shell $\mathcal{S}_c$, $c > 1$.

On recent results of M. Derridj

As we have already mentioned the characterization of subellipticity for systems seems open. We will discuss here recent results obtained by M. Derridj.
Our aim is to give some results on subellipticity for some systems of complex vector fields defined on an open set in $\mathbb{R}^q$. When one has a system of p smooth real vector fields $(X_1, \dots, X_p)$, the famous result of L. Hörmander [Ho1] (with a precise measure of the subellipticity by L. Rothschild and E. Stein [RoSt]) gave a sufficient condition in terms of the Lie algebra generated by the vector fields X_j.
We associate to the C^∞ function $\theta \mapsto \varphi(\theta)$ such that $\varphi(0) = 0$, and for $j = 1, \cdots, n$, the following vector fields

$$L_j = \frac{\partial}{\partial \theta_j} - i \frac{\partial \varphi}{\partial \theta_j} \frac{\partial}{\partial t}, \quad , \theta \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{31}$$

We search sufficient conditions on φ, for the existence of $\epsilon > 0$, $0 \in \omega \subset \Omega$, $0 \in I \subset \mathbb{R}$ such that

$$||u||_{1-\delta} \leq C\left(\sum_{j=1}^{n} ||L_j u|| + ||u||\right), \quad \forall u \in \mathcal{D}(\omega \times I) \tag{32}$$

In fact, we make a finer study, obtaining micro-local subellipticity : if (η, τ) denotes the dual variables of (θ, t), we see from (31) that the system (L_j) is elliptic in the directions (η, τ) with $\eta \neq 0$. So one has to study subellipticity in conic neighborhoods of $(\tau > 0, \eta = 0)$ and $(\tau < 0, \eta = 0)$.

We mention that H. Maire studied these problems in [Mai1, Mai4] particularly in norms uniform in the θ-variables.
To simplify, we write :

$$L_j u = f_j,\ \ j = 1, ..., n \ \text{ or } \ Lu = f\,, u \in \mathcal{D}(\Omega \times \mathbb{R}) \tag{33}$$

Then using partial Fourier transform in t, one has :

$$\frac{\partial \widehat{u}}{\partial \theta_j}\ (\theta,\tau) + \tau \frac{\partial \varphi}{\partial \theta_j}(\theta)\ \widehat{u}\ (\theta,\tau) = \widehat{f}_j(\theta,\tau). \qquad (34)$$

Now, consider a neighborhood of 0 denoted ω with $\bar{\omega} \subset \Omega$ and let γ_θ be a piecewise smooth curve such that

$$\gamma_\theta(0) = \theta \in \omega,\ \gamma_\theta(1) \notin \omega,\ \gamma_\theta : [0,1] \to \Omega\,. \qquad (35)$$

Then we can integrate the system (34) along the curve $s \mapsto \gamma_\theta(s) := \gamma(\theta, s)$.

$$\widehat{u}(\theta,\tau) = -\textstyle\int_0^1 \exp - \Big[\tau \cdot \Big(\varphi(\theta) - \varphi\Big(\gamma_\theta(s)\Big)\Big)\Big] \\ \widehat{f}\Big(\gamma_\theta(s),\tau\Big)\ \cdot \gamma_\theta{}'(s)\, ds \qquad (36)$$

Sufficient condition for microlocal subellipticity in the positive direction $\tau > 0$: **Condition** (H_+)

1) There exist a neighborhood ω of 0 with $\bar{\omega} \subset \Omega$ and a finite number of subsets of ω denoted $\omega_1, ..., \omega_k$ such that $\omega \setminus \cup_{j=1}^{k} \omega_j$ has measure 0 and :

2) $\forall j \in \{1, ., k\}$, there exists $\gamma_j \colon \omega_j \times [0,1] \to \Omega$ with the following properties : $\forall \theta \in \omega_j$, the curve $\gamma_j(t, .)$ has finite C^1- pieces and :

i) $\gamma_j(\theta, 0) = \theta, \ \gamma_j(\theta, 1) \notin \omega, \ \forall \theta \in \omega_j$;

ii) γ_j is C^1, outside a negligible set E and satisfies :

$$\left\{ \begin{array}{l} |\gamma_j'| = |\frac{\partial \gamma_j}{\partial s}| \leq c_2 \ ; \ 0 < c_1 \leq |\det(D_t \gamma_j)| \leq c_2 \, , \\ \varphi\Big(\gamma_j(\theta, s)\Big) - \varphi(\theta) \leq -c_1 s^\alpha \, , \, (\theta, s) \in \omega_j \times [0,1] \\ \text{ where } c_1, c_2 \text{ and } s \text{ are positive constants} \end{array} \right. \qquad (37)$$

The second inequality in (37), will give the gain of subellipticity equal to $\frac{1}{\alpha}$.

Theorem

Assume the hypothesis (H_+) satisfied. Then there exists $C > 0$ such that

$$\int_{\omega\times\mathbb{R}^+} \tau^{2/\alpha} |\widehat{u}(\theta,\tau)|^2 d\theta d\tau \leq C \int_{\omega\times\mathbb{R}^+} |\widehat{f}(\theta,\tau)|^2 d\theta\, d\tau\,.$$

The proof is rather elementary and based on a tricky use of Cauchy-Schwarz inequality.

Remarks

One can similarly introduce a condition (H_-) for treating the case $(\tau < 0, \eta = 0)$.

The conditions are invariant by diffeomorphism.

Derridj gives various examples. For some of them one can use actually the result of L. Hörmander [Ho1] appliaed to a combination of the L_j's to deduce subellipticity but other examples can not be obtained through this trick.

Homogeneous functions in case $n = 2$

Consider a real function φ, C^1 and homogeneous in $\mathbb{R}^2$: $\varphi(\lambda\theta) = \lambda^m \varphi(\theta), \lambda > 0, m \in \mathbb{N}^*, \theta \in \mathbb{R}^2$. Denote by S the unit circle ; $\omega \in [-\pi, \pi[$ the variable on S. Let $\psi(\omega) = \varphi(\cos\omega, \sin\omega)$. So we assume Condition $(H)_\varphi$:

a) The function ψ vanishes at $\omega_1, ., \omega_k$ of S, where it changes sign
On the intervals when positive, ψadmits only one local maximum on the intervals where negative it admits only one local minimum.
b) There exist $c > 0$ and $\epsilon > 0$ such that
$|\psi(\omega) - \psi(\omega')| \geq c\, |\omega - \omega'|^m$,
for $\omega, \omega' \in [\omega_j - \epsilon, \omega_j + \epsilon]$, $j = 1, ., k$.

Proposition
Let φ as above, satisfying $(H)_\varphi$. Then φ satisfies the hypotheses (H_+) and (H_-), with $\alpha = m$.

Example

$\varphi(\theta) = \theta_1(\theta_1^{2\ell} - \theta_2^{2\ell}) + \widetilde{\varphi}(\theta_3, \cdots, \theta_n)$ with $\ell \in \mathbb{N}^*$ $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi} \in C^1(V, \mathbb{R})$, $0 \in V \subset \mathbb{R}^{n-2}$. In that case, one has just to study the function $\Psi(\theta) = \cos\omega\,(\cos^{2\ell}\omega - \sin^{2\ell}\omega)$ on $[\,0, \pi\,]$.

In fact, the example of H. Maire is a special case of quasihomogeneous functions for which Derridj [Der2] has a result analogous to the one given for the homogeneous functions.

In the case φ is real analytic it is believed that the non existence of a local minimum of φ in a neighborhood of 0 implies the microlocal subellipticity in the positive direction, and the non existence of a local maximum implies microlocal subellipticity in the negative direction.

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