Spectral Theory for Schrödinger operator with magnetic field and analysis of the third critical field in superconductivity

Bernard Helffer
Mathématiques -
Université Paris Sud- UMR CNRS 8628
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Main goals

Using recent results by the authors on the spectral asymptotics of the Neumann Laplacian with magnetic field, we give precise estimates on the critical field, H_{C_3}, describing the appearance of superconductivity in superconductors of type II. Furthermore, we prove that the local and global definitions of this field coincide. Near H_{C_3} only a small part, near the boundary points where the curvature is maximal, of the sample carries superconductivity. We give precise estimates on the size of this zone and decay estimates in both the normal (to the boundary) and parallel variables.

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] =$$
$$\int_\Omega \Big\{ |p_{\kappa H\vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4$$
$$+\kappa^2 H^2 |\,\text{curl}\ \vec{A} - 1|^2 \Big\}\, dx\ ,$$

with $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$ and where $p_{\vec{A}} = (-i\nabla - \vec{A})$.

We fix the choice of gauge by imposing that

$$\text{Div}\ \vec{A} = 0 \quad \text{in } \Omega\ ,\ \vec{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\ .$$

Minimizers $(\psi, \vec{A})$ of the functional satisfy the Ginzburg-Landau equations,

$$\left.\begin{array}{r} p^2_{\kappa H \vec{A}}\psi = \kappa^2(1-|\psi|^2)\psi \\ \text{curl }^2\vec{A} = -\frac{i}{2\kappa H}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A} \end{array}\right\} \quad \text{in} \quad \Omega\,; \tag{1a}$$

$$\left.\begin{array}{r} (p_{\kappa H \vec{A}}\psi)\cdot\nu = 0 \\ \text{curl }\vec{A} - 1 = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{1b}$$

Here $\text{curl }(A_1, A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$,

$$\text{curl }^2\vec{A} = (\partial_{x_2}(\text{ curl }\vec{A}), -\partial_{x_1}(\text{ curl }\vec{A}))\,.$$

Let $\vec{F}$ denote the vector potential generating the constant exterior magnetic field

$$\left.\begin{array}{r} \text{Div }\vec{F} = 0 \\ \text{curl }\vec{F} = 1 \end{array}\right\} \quad \text{in } \Omega\,,\ \vec{F}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

The pair $(0, \vec{F})$ is called the Normal State.

A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.

In the other cases, one will speak about Mixed State.

The general question is to determine the topology of the sets of (κ, H) corresponding to minimizers belonging to each of these three situations.

Existence of the third critical field $\underline{H}_{C3}(\kappa)$

It is known that, for given values of the parameters κ, H, the functional $\mathcal{E}$ has minimizers.

However, after some analysis of the functional, one finds (see [GiPh]) that given κ there exists $H(\kappa)$ such that if $H > H(\kappa)$ then $(0, \vec{F})$ is the only minimizer of $\mathcal{E}_{\kappa,H}$ (up to change of gauge).

Following Lu and Pan [LuPa1], we define

$$\underline{H}_{C_3}(\kappa) = \inf\{H > 0 \; : \; (0, \vec{F}) \text{ minimizer of } \mathcal{E}_{\kappa,H}\} \, .$$

A central question in the mathematical treatment of Type II superconductors is to establish the asymptotic behavior of $\underline{H}_{C_3}(\kappa)$ for large κ.

We will also discuss the relevance of this definition and describe how $\underline{H}_{C_3}(\kappa)$ can be determined by the study of a linear problem.

Our first result is the following strengthening of a result in [HePa].

Theorem A

Suppose Ω is a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let $k_{\max}$ be the maximal curvature of $\partial\Omega$. Then

$$\underline{H}_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{C_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}}) \; , \qquad (2)$$

where C_1, Θ_0 are universal constants.

Remark

The constants Θ_0, C_1 are defined in terms of auxiliary spectral problems.

Localization at the boundary

From the work of Helffer-Morame [HeMo2] (improving Del Pino-Fellmer-Sternberg and Lu-Pan) (see also Helffer-Pan [HePa] for the non-linear case) we know that, when H is sufficiently close to $\underline{H}_{C3}(\kappa)$, minimizers of the Ginzburg-Landau functional are exponentially localized to a region near the boundary. This is called Surface Superconductivity.

Note that this localization leads to the proof of :

$$||\psi||_{L^2(\Omega)} \leq C\kappa^{-\frac{1}{4}}||\psi||_{L^4(\Omega)} \,, \tag{3}$$

which is true for κ large enough.

Localization at the points of maximal curvature

The statement is that, when H is rather close to the third critical field, the minimizers are also localized in the tangential variable to a small zone around the points of maximal curvature.

This leads in particular to the better

$$||\psi||_{L^2(\Omega)} \leq C\kappa^{-\frac{3}{8}}||\psi||_{L^4(\Omega)} \,, \tag{4}$$

Discussion of critical fields

Actually, we should define more than one critical field, instead of just $\underline{H}_{C_3}$. We define an upper third critical field, by

$$\begin{aligned}&\overline{H}_{C_3}(\kappa)\\&\quad = \inf\{H > 0 \ : \ \forall H' > H\,, (0, \vec{F})\\&\qquad\qquad \text{unique minimizer of } \mathcal{E}_{\kappa,H'}\}\,,\end{aligned}$$

Of course we have

$$\underline{H}_{C_3}(\kappa) \leq \overline{H}_{C_3}(\kappa)\,.$$

Note that one can prove that the asymptotics given before is valid for both fields.

The Schrödinger operator with magnetic field

Let, for $B \in \mathbb{R}_+$, the magnetic Neumann Laplacian $\mathcal{H}(B)$ be the self-adjoint operator (with Neumann boundary conditions) associated to the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto \int_\Omega |(-i\nabla - B\vec{F})u|^2 \, dx \; ,$$

We define $\lambda_1(B)$ as the lowest eigenvalue of $\mathcal{H}(B)$.

The local upper critical fields can now be defined :

$$\overline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \inf\{H > 0 \;:\; \forall H' > H, \lambda_1(\kappa H') \geq \kappa^2\}\ ,$$

and

$$\underline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \inf\{H > 0 \;:\; \lambda_1(\kappa H) \geq \kappa^2\}\ .$$

The coincidence between $\overline{H}^{\mathrm{loc}}_{C_3}(\kappa)$ and $\underline{H}^{\mathrm{loc}}_{C_3}(\kappa)$ is immediately related to lack of strict monotonicity of λ_1.

These critical fields appear when analyzing the (local) stability of the normal solution.

Comparison Theorem C

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary and let $\kappa > 0$, then the following general relations hold

$$\overline{H}_{C_3}(\kappa) \geq \overline{H}^{\mathrm{loc}}_{C_3}(\kappa)\ ,$$

and

$$\underline{H}_{C_3}(\kappa) \geq \underline{H}^{\mathrm{loc}}_{C_3}(\kappa)\ .$$

EASY and GENERAL.

Next theorem is new and more delicate !

Theorem D

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Then $\exists$ $\kappa_0 > 0$ such that, for $\kappa > \kappa_0$, we have

$$\overline{H}_{C_3}(\kappa) = \overline{H}_{C_3}^{\rm loc}(\kappa) \ ,$$

and

$$\underline{H}_{C_3}(\kappa) = \underline{H}_{C_3}^{\rm loc}(\kappa) \ ,$$

So the monotonicity of $\lambda_1(B)$ for B large immediately give the coincidence of the four fields !!

The second identity is a remark of R. Frank (but the proof is essentially analogous to the first one due to Fournais-Helffer)

This monotonicity has been shown in great generality under generic assumptions by Fournais-Helffer, who get in addition a complete asymptotic expansion.

Around the proof of Theorem D

The crucial point leads in the following argument.

If for some H there is a non trivial minimizer (ψ, A) so

$$\mathcal{E}(\psi, \vec{A}) \leq 0 \; .$$

then

$$0 < \Delta := \kappa^2 ||\psi||_2^2 - Q_{\kappa H \vec{A}}[\psi] = \kappa^2 ||\psi||_4^4 \; ,$$

where $Q_{\kappa H \vec{A}}[\psi]$ is the energy of ψ.

The last equality is a consequence of the first G-L equation.

Combining with (3), this gives

$$||\psi||_2 \leq C\kappa^{-\frac{3}{4}}\Delta^{\frac{1}{4}} \,.$$

By comparison of the quadratic forms Q respectively associated with $\vec{A}$ et $\vec{F}$, we get, with $\vec{a} = \vec{A} - \vec{F}$:

$$\Delta \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H\vec{F})\right] \|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \int_\Omega |\vec{a}\psi|^2 \, dx \,, \tag{5}$$

for all $0 < \rho < 1$.

Note that by the regularity of the system Curl-Div, combined with the Sobolev's injection theorem, we get

$$\|\vec{a}\|_4 \leq C_1\|\vec{a}\|_{W^{1,2}} \leq C_2\| \text{ curl } \vec{a}\|_2 \,.$$

Now Δ is also controlling $\|$ curl $\vec{a}\|_2^2$, so we get :

$$(\kappa H)^2 \|\vec{a}\|_4^2 \leq C\,\Delta\ .$$

Combining all these inequalities leads to :

$$\begin{array}{l} 0 < \Delta \leq \\ \quad \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H \vec{F})\right] \|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \|\vec{a}\|_4^2 \|\psi\|_4^2 \\ \quad \leq \left[\kappa^2 - \lambda_1(\kappa H \vec{F})\right] \|\psi\|_2^2 \\ \qquad\quad + C\rho\lambda_1(\kappa H)\Delta^{\frac{1}{2}}\kappa^{-\frac{3}{2}} + C\rho^{-1}\Delta^{\frac{3}{2}}\kappa^{-1}\ . \end{array}$$

Chosing $\rho = \sqrt{\Delta}\kappa^{-\frac{3}{4}}$, and using the rough upper bound $\lambda_1(\kappa H\,\vec{F}) < C\kappa^2$, we find

$$0 < \Delta \leq \left[\kappa^2 - \lambda_1(\kappa H)\right] \|\psi\|_2^2 + C\Delta\kappa^{-\frac{1}{4}}\ .$$

This shows finally, for κ large enough independently of H sufficiently close to "any" third critical field (they have the same asymptoics)

$$0 < \Delta \leq \tilde{C}\big[\kappa^2 - \lambda_1(\kappa H)\big]\,\|\psi\|_2^2\ ,$$

so in particular

$$\kappa^2 - \lambda_1(\kappa H) > 0\ .$$

Coming back to the definitions this leads to the statement.

Perspectives

This is far to be the end of the story. Here are some additional questions :

1. One can instead consider the more physical functional :

   $$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \Big\{ |p_{\kappa H\vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 \int_{\mathbb{R}^2} |\text{ curl } \vec{A} - 1|^2 \Big\} \, dx \ ,$$

   The difference is that the last integration is over $\mathbb{R}^2$! This is particularly important if Ω is not simply connected !

2. What is going on in Dimension 3 ?
   Results by Pan, Helffer-Morame, Fournais-Helffer.

3. Note also that other conditions than Neumann could be interesting.

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