Strong diamagnetism for general domains and applications

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Main goals

We consider the Neumann Laplacian with constant magnetic field on a regular domain. Let B be the strength of the magnetic field, and let $\lambda_1(B)$ be the first eigenvalue of the magnetic Neumann Laplacian on the domain. It is proved that $B \mapsto \lambda_1(B)$ is monotone increasing for large B.

We discuss applications of this monotonicity for the critical fields in superconductivity.

The Schrödinger operator with magnetic field

Let, for $B \in \mathbb{R}_+$, the magnetic Neumann Laplacian $\mathcal{H}(B)$ be the self-adj. operator (with Neumann boundary conditions) associated to the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto Q_{B,\vec{F}}(u) := \int_\Omega |(-i\nabla - B\vec{F})u|^2 \, dx \, ,$$

where Ω is a fixed, regular, bounded set in $\mathbb{R}^2$ and $\vec{F}$ is

$$\left.\begin{array}{c} \text{div } \vec{F} = 0 \\ \text{curl } \vec{F} = 1 \end{array}\right\} \quad \text{in } \Omega \, ,$$

and the boundary condition

$$\vec{F} \cdot \nu = 0 \ \text{ on } \partial\Omega \, .$$

We define $\lambda_1(B)$ as the lowest eigenvalue of $\mathcal{H}(B)$.

Basic domains

Here we can take

$$\vec{F}_0 = \left(-\frac{x_2}{2}\,,\,\frac{x_1}{2}\right)\,.$$

Whole space $\mathbb{R}^2$

The spectrum is discrete with eigenvalues of infinite multiplicity, which are called the Landau levels.
The bottom of the spectrum is $|B|$.

Half space $\mathbb{R}^2_+$

Continuous spectrum. The bottom of the spectrum is $\Theta_0|B|$, with $\Theta_0 \in]0,1[$. The spectral analysis is based on the analysis of the family

$$H(\xi) = D_t^2 + (t+\xi)^2\,,$$

on the half-line (Neumann at 0) whose lowest eigenvalue $\mu(\xi)$ admits a unique minimum at $\xi_0 < 0$.e

Two universal constants.

We have to keep in mind two universal constants attached to the problem on $\mathbb{R}^+$.

The first one is

$$\Theta_0 = \mu(\xi_0) \; . \tag{1}$$

It corresponds to the bottom of the spectrum of the Neumann realization in $\mathbb{R}^2_+$ (with $B = 1$).
Note that

$$\Theta_0 \in]0,1[\; .$$

The second one is defined as follows. If u_ξ denotes the L^2-normalized groundstate of $H(\xi)$, we will also meet later the universal constant

$$\mathcal{C}_1 = \frac{u_{\xi_0}(0)^2}{3} \tag{2}$$

The case of the disk

All the previous models are dilation invariants, so it was enough to treat $B = 1$.

The disk is an important model for understanding curvature effects. The first results are due to Giorgi-Phillips, but we give below a useful improvment for the control of the third term.

Theorem [Eigenvalue asymptotics for the disc]
Suppose that Ω is the unit disc. Define $\delta(m, B)$, for $m \in \mathbb{Z}$, $B > 0$, by

$$\delta(m, B) = m - \frac{B}{2} - \xi_0 \sqrt{B}. \tag{3}$$

Then there exist (computable) constants $\mathcal{C}_0, \delta_0 \in \mathbb{R}$ such that, with $\Delta_B = \inf_{m \in \mathbb{Z}} |\delta(m, B) - \delta_0|$,

$$\lambda_1(B) = \Theta_0 B - \mathcal{C}_1 \sqrt{B} + 3\mathcal{C}_1 \sqrt{\Theta_0} \big(\Delta_B^2 + \mathcal{C}_0\big) + \mathcal{O}(B^{-\frac{1}{2}}) \, , \tag{4}$$

as $B \to +\infty$.

Note that we can recover the result for the disk of radius R by dilation. So the second term in the expansion becomes

$$-\mathcal{C}_1 \frac{1}{R} \sqrt{B}\ ,$$

and will show the role of the curvature.

This has also the following important consequence.

Proposition[Case of the disk]
Let Ω be the disc. Then the left- and right-hand derivatives $\lambda'_{1,\pm}(B)$ exist and satisfy

$$\begin{aligned} &\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)\ , \\ &\liminf_{B\to+\infty} \lambda'_{1,+}(B) \geq \Theta_0 - \tfrac{3}{2}\mathcal{C}_1|\xi_0| > 0\ . \end{aligned} \tag{5}$$

In particular, $B \mapsto \lambda_1(B)$ is strictly increasing for large B.

Asymptotic expansions for $\lambda_1(B)$ in a bounded regular domain

All the asymptotics below are for B large.

Version 1 (Lu-Pan or DelPino-Fellmer-Sternberg (2000)).

$$\lambda_1(B) = \Theta_0 B + o(B)\,, \tag{6}$$

$+$ better upper bound.

Version 2 (Helffer-Morame (2001)).

$$\lambda_1(B) = \Theta_0 B - \mathcal{C}_1 k_{max}\sqrt{B} + o(\sqrt{B})\,, \tag{7}$$

where $\mathcal{C}_1 > 0$ is a spectral quantity attached to the half space problem, and k_{max} is the maximal curvature along the boundary.

Version 3 (Bernoff-Sternberg (1998) formal construction, Fournais-Helffer (2005)).

If $\partial\Omega$ has only a finite number of points of maximal curvature and that in addition these points are non degenerate, we have a complete expansion in fractional powers of $B^{-\frac{1}{8}}$:

$$\begin{aligned}\lambda_1(B)/B \\ &= \Theta_0 - C_1 k_{max} B^{-\frac{1}{2}} \\ &\quad + C_1 \Theta_0^{\frac{1}{4}} \sqrt{\frac{3k_2}{2}} B^{-\frac{1}{4}} \\ &\qquad + \sum_{j\geq 7} c_j B^{-\frac{j}{8}}\,,\end{aligned} \tag{8}$$

where

$$k_2 = \inf_{x\in\partial\Omega\ k(x)=k_{max}} (-k''(x))\,,$$

$k(x)$ being the curvature.

For corners, see Bonnaillie-Noël, Bonnaillie-Noël–Dauge.

Localization at the boundary

From the work of Helffer-Morame [HeMo2]

——(improving Del Pino-Fellmer-Sternberg and Lu-Pan)——

we know that, as $B \to +\infty$, the groundstate is localized in a neighborhood of the boundary.

The proof is based on semi-classical Agmon estimates, but the "Agmon distance" has to be replaced by the distance to the boundary.

Note that the Agmon estimates give first, for some $\alpha > 0$,

$$|| \exp \alpha \, B^{\frac{1}{2}} \, d(x, \partial\Omega) \psi ||_2^2 \leq C \, ||\psi||_2^2 \, ,$$

From semi-classical Agmon estimates to weak localization

When speaking of semi-classical analysis, we mean that the semi-classical parameter is $\frac{1}{B}$.

The previous inequality implies

$$||\psi||_2^2 \leq M \int_{d(x,\partial\Omega)\leq MB^{-\frac{1}{2}}} |\psi(x)|^2 dx \; ,$$

We will need the following weak form of this localization :

$$||\psi||_{L^2(\Omega)} \leq C\, B^{-\frac{1}{8}}\, ||\psi||_{L^4(\Omega)} \; , \tag{9}$$

which is true for B large enough.

Localization inside the boundary

The statement in dimension 2 is that the groundstate ψ is also localized in the tangential variable to a small zone around the points of maximal curvature.
Here we speak about an exponential tangential decay which is of lower rate $B^{\frac{1}{4}}$ (instead of $B^{\frac{1}{2}}$) and which is measured through an Agmon distance associated to the function $(k_{max} - k(s))$ playing the role of the potential.
The proof is inspired by what was done in the case of the Schrödinger operator with electric potential (mini-well case) in Helffer-Sjöstrand, with the following dictionary :

- The boundary plays the role of a degenerate well.
- The minima of the curvature inside the boundary play the role of the miniwells.

New results on Diamagnetism

We know, by Kato's inequality that

$$\lambda_1(B) \geq \lambda(0) \ .$$

But the monotonicity is unknown in full generality.

Here we refer to some recent results of [FoHe5].

Main Theorem
If Ω is bounded and has a regular boundary then $B \mapsto \lambda_1(B)$ is monotonically increasing for B large.

The case of the disk was treated previously.

We now assume that Ω is NOT a disk. We will play with the gauge invariance in the following way. Let $\widehat{\mathbf{A}}$ be any magnetic potential such that curl $\widehat{\mathbf{A}} = 1$. Then for a suitable choice of a ground state eigenfunction $\widehat{\psi}_{1,+}(B)$ of the Hamiltonian $\widehat{\mathcal{H}}(B)$ associated to the quadratic form $Q_{B,\widehat{A}}$), we can first calculate,

$$\lambda'_{1,+}(B) = \langle \widehat{\psi}_{1,+}(B) ; \left(\widehat{\mathbf{A}} \cdot p_{B\widehat{\mathbf{A}}} + p_{B\widehat{\mathbf{A}}} \cdot \widehat{\mathbf{A}}\right) \widehat{\psi}_{1,+}(B) \rangle \; .$$

Then, using the quadratic character of the operator with respect to B, we get for any $\beta > 0$, the following lower bound

$$\begin{aligned} &\lambda'_{1,+}(B) \\ &= \frac{Q_{B+\beta,\widehat{A}}(\widehat{\psi}_{1,+}(B)) - Q_{B,\widehat{A}}(\widehat{\psi}_{1,+}(B))}{\beta} - \beta \int_\Omega (\widehat{\mathbf{A}})^2 |\widehat{\psi}_{1,+}(B)|^2 \, dx \\ &\geq \frac{\lambda_1(B+\beta) - \lambda_1(B)}{\beta} - \beta \int_\Omega (\widehat{\mathbf{A}})^2 |\widehat{\psi}_{1,+}(B)|^2 \, dx . \end{aligned}$$

The last inequality is no more gauge invariant BUT .....

BUT we can now look look for a suitable choice of β and $\widehat{A}$.

The trick is that for a suitable gauge, we have

$$\int_\Omega (\widehat{\mathbf{A}})^2 |\psi_{1,+}(B)|^2\, dx \leq C \int_\Omega \text{dist }(x, \partial\Omega)^2 |\psi_{1,+}(B)|^2\, dx + \|\widehat{\mathbf{A}}\|_\infty^2 \int_{\Omega\setminus\Omega'} |\psi_{1,+}(B)|^2\, dx. \tag{10}$$

Here Ω' is some tubular simply connected region touching the boundary, whose complementary in Ω is a region where $|\psi_{1,+}(B)|$ is exponentially small as $B \to +\infty$.

That such a choice is possible is a consequence of

- the accurate normal Agmon estimates
- together with the (weak) tangential Agmon estimates,
- together with the assumption that the curvature is not constant.

Now in $\overline{\Omega}'$ we can choose a gauge for which $\widehat{A}$ vanishes at $\partial\Omega$.
Using these Agmon estimates, we therefore find

$$\int_\Omega (\widehat{\mathbf{A}})^2 |\psi_{1,+}(B)|^2 \, dx \leq C B^{-1}. \tag{11}$$

Now choose $\beta = \eta B$, where $\eta > 0$ is arbitrary, in the previous lower bound of $\lambda'_{1,+}(B)$. Using a weak asymptotics for $\lambda_1(B)$, we therefore find

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \Theta_0 - \eta C. \tag{12}$$

Since η was arbitrary this implies

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \Theta_0. \tag{13}$$

Applying the same argument to the left side derivative, $\lambda'_{1,-}(B)$, we get (the inequality gets turned since $\beta < 0$)

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \Theta_0. \tag{14}$$

Since, by perturbation theory, $\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)$ for all B, we get

$$\lim_{B\to\infty} \lambda'_{1,-}(B) = \Theta_0 = \lim_{B\to\infty} \lambda'_{1,+}(B) \ , \tag{15}$$

hence the monotonicity of $\lambda_1(B)$.

In the first proof we gave, we were obliged to have an asymptotic of $\lambda_1(B)$ modulo $o(1)$. This was leading to stronger assumptions on the boundary (isolated points of maximal curvature + non degeneracy assumption).

One could think that we now only use the knowledge of $\lambda_1(B)$ modulo $o(B)$.

This is not true because the tangential localization suppose a knowledge (or is proved simultaneously with the determination) of $\lambda_1(B)$ modulo $o(B^{\frac{1}{2}})$. This was obtained (by Helffer-Morame) in full generality when the boundary is compact and regular.

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \left\{ |\nabla_{\kappa H\vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl}\vec{A} - 1|^2 \right\} dx\,,$$

with Ω simply connected, $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$ and where $\nabla_{\vec{A}} = (\nabla - i\vec{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{div}\vec{A} = 0 \quad \text{in } \Omega\,,\ \vec{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Minimizers $(\psi, \vec{A})$ of the functional satisfy the Ginzburg-Landau equations,

$$\left.\begin{array}{r} -\nabla^2_{\kappa H \vec{A}}\psi = \kappa^2(1-|\psi|^2)\psi \\ \text{curl}\,^2\vec{A} = -\frac{i}{2\kappa H}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A} \end{array}\right\} \quad \text{in} \quad \Omega\,; \tag{16a}$$

$$\left.\begin{array}{r} (\nabla_{\kappa H \vec{A}}\psi)\cdot\nu = 0 \\ \text{curl}\,\vec{A} - 1 = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{16b}$$

Here $\text{curl}\,(A_1, A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$,

$$\text{curl}\,^2\vec{A} = (\partial_{x_2}(\,\text{curl}\,\vec{A}), -\partial_{x_1}(\,\text{curl}\,\vec{A}))\,.$$

Terminology for the minimizers

The pair $(0, \vec{F})$ is called the Normal State.

A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State.

In the other cases, one will speak about Mixed State.

The general question is to determine the topology of the subset in $\mathbb{R}^+ \times \mathbb{R}^+$ of the (κ, H) corresponding to minimizers belonging to each of these three situations.

Existence of the third critical field $\underline{H}_{C3}(\kappa)$

It is known that, for a given pair κ, H, the functional $\mathcal{E}$ has minimizers.

Moreover, after some analysis of the functional, one finds (see [GiPh]) that, for given κ, there exists $H(\kappa)$ such that if $H > H(\kappa)$ then $(0, \vec{F})$ is the unique minimizer of $\mathcal{E}_{\kappa,H}$ (up to change of gauge).

Following Lu and Pan [LuPa1], we define

$$\underline{H}_{C_3}(\kappa) = \inf\{H > 0 \; : \; (0, \vec{F}) \text{ minimizer of } \mathcal{E}_{\kappa,H}\} \, .$$

A central question in the mathematical treatment of Type II superconductors is to establish the asymptotic behavior of $\underline{H}_{C_3}(\kappa)$ for large κ.

Our first result [FoHe3] is the following strengthening of a result in [HePa].

Theorem A
Suppose Ω is a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let $k_{\max}$ be the maximal curvature of $\partial\Omega$. Then

$$\underline{H}_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{\mathcal{C}_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}}) \,. \tag{17}$$

Remark
The constants $\Theta_0, \mathcal{C}_1$ appear in the spectral analysis of our basic models.

Discussion of critical fields

Actually, we should define more than one critical field, instead of just $\underline{H}_{C_3}$.

We should also a priori define an upper third critical field, by

$$\begin{aligned}&\overline{H}_{C_3}(\kappa)\\ &\quad = \inf\{H > 0 \;:\; \forall H' > H\,, (0, \vec{F})\\ &\qquad\qquad\qquad \text{unique minimizer of } \mathcal{E}_{\kappa,H'}\}\,,\end{aligned}$$

Of course we have

$$\underline{H}_{C_3}(\kappa) \leq \overline{H}_{C_3}(\kappa)\,.$$

Note that one can prove that the asymptotics given for $\underline{H}_{C_3}(\kappa)$ is also valid for $\overline{H}_{C_3}(\kappa)$.

The local upper critical fields can now be defined by :

$$\overline{H}_{C_3}^{\mathrm{loc}}(\kappa) = \inf\{H > 0 \ : \ \forall H' > H, \lambda_1(\kappa H') \geq \kappa^2\} \ ,$$

and

$$\underline{H}_{C_3}^{\mathrm{loc}}(\kappa) = \inf\{H > 0 \ : \ \lambda_1(\kappa H) \geq \kappa^2\} \ .$$

The coincidence between $\overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ and $\underline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ immediately results if we prove the strict monotonicity of $B \mapsto \lambda_1(B)$.

Comparison Theorem B

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary, then, for κ large enough, all the critical fields coincide.

Questions, other results and Perspectives

This is far to be the end of the story. Here are some additional questions or remarks :

1. The case of corners was analyzed by Hadallah, Bonnaillie, and a numerical analysis of the tunneling in polygons was performed by Dauge-Bonnaillie. New results about critical fields have been obtained by Bonnaillie-Fournais.

2. Analyze the case when Ω is not simply connected !

3. Analyze the situation between $H_{C3}(\kappa)$ and $H_{C2}(\kappa)$ (Pan, Fournais-Helffer, Almog-Helffer, Sandier-Serfaty).

4. Analyze other conditions than Neumann (see the analysis of Lu-Pan and Kachmar for the De Gennes (Robin) conditions). Similar questions concern the Josephson junctions.

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