Semi-classical analysis and Harper's equation –An introduction– Crash course[1] at Tsinghua University

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[1]The author was kindly invited by YanHui Qu

Abstract

Since the description in 1976 of the beautiful butterfly by the physicist Hofstadter interpreted as the spectra of a family of operators (called almost Mathieu or Harper's operator) parametrized by some flux, a huge literature has been written for understanding the properties of these spectra. After a presentation of the subject, these lectures will be devoted to the description of the results of Helffer-Sjöstrand (at the end of the eighties) based on an illuminating strategy proposed by the physicist M. Wilkinson in 1985. This leads to the proof of the Cantor structure of the spectrum for the Harper model for a some specific family of irrational fluxes (characterized on its expansion in continuous fractions). This was a very particular case of the ten Martinis conjecture of M. Kac popularized by B. Simon and which was finally proved in (2009) by A. Avila, S. Jitomirskaya and coauthors for any irrational.

The goal is to explain how semi-classical analysis appears in the analysis of this problem. The analysis of the spectrum of the Harper's model can indeed be done for some fluxes by semi-classical analysis and in this case can give a more precise information on the spectrum than simply its Cantor structure. If it seems to be impossible in these lectures to give a complete proof of the results (the use of the FBI techniques mainly due to J. Sjöstrand is omitted here), we hope to give a flavor of the tools used in this context permitting an easier access to the research papers (which are sometimes written in french). Complementary material can be found in our Lecture Note in Sonderborg or in the CIME course of J. Sjöstrand in 1989.

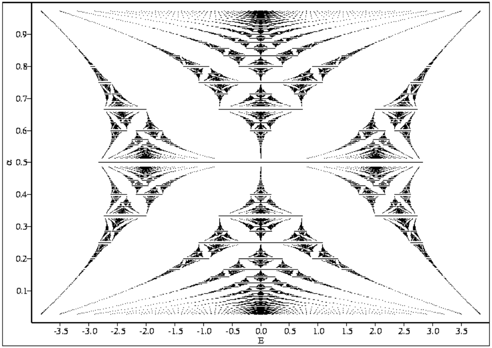
The lectures will present various connected points and the maximal program is the following:

- Recognize the Harper operator as a pseudo-differential operator.
- Give some introduction to h-pseudodifferential calculus and show how one gets the spectrum modulo $\mathcal{O}(h^{\infty})$.
- Describe the first step of a renormalization procedure leading to two new semi-classical models. This corresponds to a precise analysis of the tunneling effect and will actually be the main subject developed in these lectures.
- Give some hints for the complete renormalization procedure in the irrational case leading to the proof of the Cantor structure.
- Discuss other applications around the length of the spectrum in the rational case.

Introduction

The spectral properties of a charged particle in a two-dimensional system submitted to a periodic electric potential and a uniform magnetic field crucially depend on the arithmetic properties of the number α representing the magnetic flux quanta through the elementary cell of periods, see e.g. Bellissard [Be1991] for a description of various models.
Since the works by Azbel [Az1964] and Hofstadter [Hof] it is generally believed that for irrational α the spectrum is a Cantor set, that is a nowhere dense (the interior of the closure is empty) and perfect set (closed $+$ no isolated point), and the graphical presentation of the dependence of the spectrum on α shows a fractal behavior known as the Hofstadter butterfly.

The **Hofstadter's butterfly** is obtained in the following way. We put on the vertical axis the parameter proportional to the flux $\alpha = \frac{h}{2\pi} \in [0, 1]$ and on the horizontal line $y = \alpha$ the union over θ of the spectra of the family $H_\alpha(\theta)$. The picture results of computations for rational α's.



Let us consider more generally (introduction of $\lambda > 0$) the family of operators on $\ell^2(\mathbb{Z})$

$$(H_{\lambda,\alpha,\theta} u)_n = \frac{1}{2} (u_{n+1} + u_{n-1}) + \lambda \cos 2\pi(\theta + n\alpha) u_n \,.$$

Different names for this operator are given including Harper or Almost-Mathieu.
If $\alpha = \frac{p}{q}$ is rational the spectrum consists of the union of q intervals possibly touching at the end point. If α is irrational the spectrum is independent of θ and:

Ten Martini Theorem

The spectrum of the almost Mathieu operator $H_{\lambda,\alpha}$ is a Cantor set for all irrational α and for all $\lambda \neq 0$.

The Ten Martini conjectures was popularized by B. Simon in reference to some offer of M. Kac.
Computations for $\lambda \neq 1$ are proposed in a "numerical" paper of Guillement-Helffer-Treton [GHT].

After intensive efforts (we can mention Azbel (1964), Bellissard-Simon (1982), Van Mouche (1989), Helffer-Sjöstrand (1989), Puig (2004), Avila-Krikorian (2008)) this Cantor set structure was rigorously proved in 2009 by Avila-Jitomirskaya for all irrational values of α (see [AvJi] and references therein) for the models

$$u \mapsto (H_{\lambda,\alpha,\theta}\, u)_n = \frac{1}{2}\,(u_{n+1} + u_{n-1}) + \lambda \cos(2\pi(\alpha n + \theta))u_n \,.$$

with $\lambda > 0$.

Unfortunately Mark Kac died before to know that he has to buy these ten Martini. Fortunately M. Aizenman organized later some fest in Montreal (if I well remember) for all the contributors.

Coming back to mathematics, a more detailed analysis (Helffer and Sjöstrand – HSHarper1,HSHarper2,HSHarper3–in the years 1988-1990) shows that the study of some parts of the spectrum for the Schrödinger operator with a magnetic field and a periodic electric potentials reduces to the spectral problem for an operator pencil of one-dimensional quasiperiodic pseudodifferential operators.

Under some symmetry conditions for the electric potentials, the operator pencil reduces to the study of small perturbation of the continuous analog of the almost-Mathieu (=Harper) operator, which allowed one to carry out a rather detailed iterative analysis for special values of α.
In particular, in several asymptotic regimes a Cantor structure of the spectrum was proved.
This involved a semi-classical pseudo-differential calculus, whose relevance in this context was predicted by the physicist Wilkinson (from United Kingdom) in the middle of the eighties.
End of Introduction

Preliminary properties and first meeting with the pseudo-differential calculus

We are interested in $\cup_\theta \sigma(H(\theta))$. We observe that (with $h = 2\pi\alpha$)

$$H(\theta) = H(\theta+1) \text{ and } H(\theta+h) \text{ is unitary equivalent to } H(\theta)\,.$$

This implies that if $\alpha \notin \mathbb{Q}$, then the spectrum is independent of θ and secondly that

$$\cup_\theta\, \sigma(H(\theta)) = \sigma(\widetilde{H})\,,$$

where $\widetilde{H} : L^2(\mathbb{Z} \times [0,h)) \mapsto L^2(\mathbb{Z} \times [0,h))$ is defined by

$$\left(\widetilde{H}u\right)(\cdot,\theta) = H(\theta)u(\cdot,\theta)\,.$$

If we identify $L^2(\mathbb{Z} \times [0,h))$ with $L^2(\mathbb{R})$ by

$$u(k,\theta) = \widetilde{u}(\theta + hk)$$

the operator $\widetilde{H}$ becomes

$$\widetilde{H} = \frac{1}{2}(\tau_h + \tau_{-h}) + \lambda \cos x$$

where τ_h is the translation operator:

$$\tau_h v(x) = v(x - h)\,.$$

If we observe that $\tau_h = \exp ihD_x$, we can rewrite $\widetilde{H}$ as a h-pseudodifferential operator

$$\cos hD_x + \lambda \cos x$$

whose h-symbol is $\cos \xi + \lambda \cos x$.

In this last formalism, the Aubry duality is obtained by using a h-Fourier transform.

$$\mathcal{F}_h u(\xi) = (2\pi h)^{-\frac{1}{2}} \int e^{-ix\xi/h} u(x)\, dx\,.$$

By conjugation, the operator becomes

$$\lambda \cos(hD_\xi) + \cos\xi = \lambda(\cos(hD_\xi) + \frac{1}{\lambda}\cos\xi)\,.$$

The h-quantization of a symbol $p(x, \xi, h)$ with values in $M_n(\mathbb{C})$ is the pseudo-differential operator defined over $L^2(\mathbb{R}; \mathbb{C}^n)$ by

$$((\mathrm{Op}_h^w p)u)(x) = \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{i\frac{(x-y)\xi}{h}} p\left(\frac{x+y}{2}, \xi, h\right) u(y)\, dy\, d\xi\,. \tag{1}$$

We will come back to this definition later.

One remark on renormalization

If $\tau = \tau_{2\pi}$ and $\hat{\tau}$ is the multiplication operator by $e^{2\pi i x/h}$, then $\widetilde{H}$ commutes with τ and $\hat{\tau}$.
An important point is that τ and $\hat{\tau}$ do not necessarily commute.

$$\tau\,\hat{\tau} = \exp(-i(2\pi)^2/h)\,\hat{\tau}\,\tau = \exp -i\tilde{h}\,\hat{\tau}\,\tau\,,$$

with

$$(2\pi)/h = k + \tilde{h}/(2\pi)\,.$$

The map $h \to \tilde{h}$ plays a key role in the renormalization procedure.

The rational case

In order to compute the spectrum of $\widetilde{H}_\gamma$, we can start with the case when $\gamma/(2\pi)$ is a rational number. This is obtained by using the Floquet theory.
For $p, q \in \mathbb{N}^*$ we define the matrices $J_{p,q}, K_q \in \mathcal{M}_q(\mathbb{C})$ by

$$\begin{aligned} J_{p,q} &= \operatorname{diag}(\exp(2i\pi(j-1)p/q)) \\ (K_q)ij &= \begin{cases} 1 & \text{if } j = i+1 \pmod{q} \\ 0 & \text{if not} \end{cases} \end{aligned} \quad . \tag{2}$$

Note that

$$J_{p,q} = J_{1,q}^p \,.$$

Theorem

Let $\gamma = 2\pi p/q$ with $p, q \in \mathbb{N}^*$ relatively primes, $\lambda = 1$, and denote by σ_γ the spectrum of $\widetilde{H}_\gamma$. We have

$$\sigma_\gamma = \bigcup_{\theta_1,\theta_2 \in [0,1]} \sigma(M_{p,q,\theta_1,\theta_2}) \,, \tag{3}$$

where $M_{p,q,\theta_1,\theta_2}$ is given by

$$M_{p,q,\theta_1,\theta_2} = e^{i\theta_2} J^*_{p,q} + e^{-i\theta_2} J_{p,q} + e^{i\theta_1} K_q + e^{-i\theta_1} K^*_q \,. \tag{4}$$

Chambers formula

The Chambers formula gives a very elegant formula for this determinant:

$$\det(M_{p,q}(\theta_1, \theta_2) - \lambda) = f_{p,q}(\lambda) + (-1)^{q+1} 2 \left(\cos q\theta_1 + \cos q\theta_2\right) , \tag{5}$$

where $f_{p,q}$ is a polynomial of degree q. Each band I_ℓ is described by a solution $\lambda_\ell(\theta_1, \theta_2)$ of the Chambers equation which has the form

$$\lambda_\ell(\theta_1, \theta_2) = \varphi_{\ell,p,q}(2 \left(\cos q\theta_1 + \cos q\theta_2\right)) . \tag{6}$$

These bands do not overlap and do not touch except possibly at the center (Van Mouche).
Hence it remains to consider the irrational case.

Introduction to semi-classical analysis

The aim is to present the basic mathematical techniques in semi-classical analysis involving the theory of h-pseudodifferential operators and to illustrate how they permit to solve natural questions about spectral distribution and localization of eigenfunctions. More details are given in [He]. See also, the books of D. Robert (in french), Dimassi-Sjöstrand, A. Martinez, and the recent book of M. Zworski.

From classical mechanics to quantum mechanics

The initial goal of semi-classical mechanics is to explore the correspondence principle, due to Bohr in 1923 [Bo], which states that one should recover as the Planck constant h tends to zero the classical mechanics from the quantum mechanics. So we start with a very short presentation of these two theories.

Classical mechanics

We start (we present the Hamiltonian formalism) from a C^∞ function on $\mathbb{R}^{2n}$: $(x,\xi) \mapsto p(x,\xi)$ which will permit to describe the motion of the system in consideration and is called the Hamiltonian. The variable x corresponds in the simplest case to the position and ξ to the impulsion of one particle. The evolution is then described, starting of a given point (y,η), by the so called Hamiltonian equations

$$\begin{aligned} dx_j/dt &= (\partial p/\partial \xi_j)(x(t),\xi(t))\ ,\ \text{for}\ \ j=1,\cdots,n\ ; \\ d\xi_j/dt &= -(\partial p/\partial x_j)(x(t),\xi(t))\ ,\ \text{for}\ j=1,\cdots,n\ . \end{aligned} \tag{7}$$

The classical trajectories are then defined as the integral curves of a vector field defined on $\mathbb{R}^{2n}$ called the hamiltonian vector field associated with p and defined by $H_p = ((\partial p/\partial \xi), -(\partial p/\partial x))$. All these definitions are more generally relevant in the framework of symplectic geometry on a symplectic manifold M, but we choose for simplicity to explain the theory on $\mathbb{R}^{2n}$, which can be seen the cotangent vector bundle $T^*\mathbb{R}^n$, and is the "local" model of the general situation. This space is equipped naturally with a symplectic structure defined by giving at each point a non degenerate 2-form, which is here $\sigma \ := \sum_j d\xi_j \wedge dx_j$. This 2- form permits to associate canonically to a 1-form on $T^*\mathbb{R}^n_x$ a vector field on $T^*\mathbb{R}^n_x$. In this correspondence, if p is a function on $T^*\mathbb{R}^n_x$, H_p is associated with the differential dp.

We keep in mind as first guiding example the example of the Hamiltonian $p(x, \xi) = \xi^2 + V(x)$, also called the Schrödinger Hamiltonian and more specifically the case of the harmonic oscillator where $V(x) = \sum_{j=1}^{n} \mu_j x_j^2 \quad (\text{with } \mu_j > 0)$, which is the natural approximation of a potential near its minimum, when non degenerate.
But our main interest will be in the Hamiltonian ($n = 1$)

$$(x, \xi) \mapsto \cos x + \cos \xi.$$

In the framework of the classical mechanics the main questions could be :

- Are the trajectories bounded ?
- Are there periodic trajectories ?
- Is one trajectory dense in its energy level ?
- Is the energy level compact ? or a disjoint union of compacts ?

The solution of these questions could be very difficult. Let us just mention the trivial fact that, if $p^{-1}(\lambda)$ is compact for some λ, then the conservation of energy law

$$p(x(t), y(t)) = p(y, \eta) \,. \tag{8}$$

leads to the property that the trajectories starting of one point (y, η) remain in the set $\{p^{(-1)}(p(y, \eta))$ in $\mathbb{R}^{2n}$ and are hence bounded. This is in particular the case for the harmonic oscillator.

Energy levels

We recall that for a given energy E and an Hamiltonian p, the energy level is defined in $T^*\mathbb{R}^n$ by

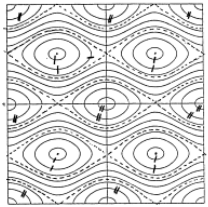
$$p^{-1}(E) = \{(x,\xi)\,,\, p(x,\xi) = E\}\,.$$

When $n = 1$ and if E is not a characteristic value of p, the level set is a ($1D$)-manifold (i.e. consisting of curves) .
We are mostly interested in the Harper Hamiltonian, which is defined for $\lambda \in (0,1]$ by

$$(x,\xi) \mapsto p_\lambda(x,\xi) = \cos\xi + \lambda\cos x\,.$$

and the analysis of the energy levels is easy.

Here are the pictures:

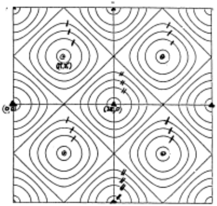


$-$: $-(1-\lambda)<E<(1-\lambda)$

$\nmid$: $-(1+\lambda)<E<-(1-\lambda)$

$\#$: $1-\lambda\leq E\leq 1+\lambda$

Pour $\lambda=1$ on trouve une structure plus simple :



$-$: $E=0$ $\quad$ $\circ$: $E=-2$ $\quad$ $\nmid$: $-2<E<0$ $\quad$ Δ : $E=2$ $\quad$ $\#$: $0<E<2$

The pictures for $\lambda \in (0,1)$ and $\lambda = 1$ are quite different.
When $\lambda = 1$, we note that the picture is stable by perturbation if we keep the periodicity and the symmetries $(x,\xi) \mapsto (x,-\xi)$ and $(x,\xi) \mapsto (\xi,-x)$. The critical points correspond indeed to a Morse function.

Quantum mechanics

The quantum theory is born around 1920. It is structurally related to the classical mechanics in a way that we shall describe very briefly. In quantum mechanics, our basic object will be a (possibly non-bounded) selfadjoint operator defined on a dense subspace of an Hilbert space $\mathcal{H}$. In order to simplify, we shall always take $\mathcal{H} = L^2(\mathbb{R}^n)$.

This operator can be associated with p by different techniques called quantizations. We choose here to present a procedure called the Weyl-quantization procedure, which under suitable assumptions on p and its derivatives will be defined for $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$\begin{aligned} & p^w(x, hD_x, h)u(x) = \\ & \quad (2\pi h)^{-n} \iint \exp(\tfrac{i}{h}(x-y)\cdot\xi)\; p(\tfrac{x+y}{2}, \xi, h)\; u(y)\; dy\, d\xi\;. \end{aligned} \tag{9}$$

The operator $p^w(x, hD_x, h)$ is called an h-pseudodifferential operator of Weyl-symbol p. One can also write $\mathrm{Op}_{\mathrm{h}}^{\mathrm{w}}(p)$ in order to emphasize that it is the operator associated to p by the Weyl quantization. Here h is a parameter which plays the role of the Planck constant.

Of course, one has to give a sense to these integrals and this is the object of the theory of the oscillatory integrals. If $p = 1$, we observe that the associated operator is nothing else, by Plancherel's formula, than the identity :

$$u(x) = (2\pi h)^{-n} \cdot \int\int \exp(\frac{i}{h}(x-y)\cdot\xi)\ u(y)\, dy\, d\xi\ .$$

A way to rewrite any h-differential operator $\sum_{|\alpha|\leq m} a_\alpha(x)(hD_x)^\alpha$ as an h-pseudodifferential operator is to apply it to the Plancherel identity. In particular, we observe that if $p(x,\xi)=\xi^2+V(x)$, then the h-Weyl quantization associated with p is the Schrödinger operator : $-h^2\Delta+V$. Other interesting examples appear naturally in solid state physics. We recall that for example the Harper's operator H has symbol $(x,\xi)\mapsto \cos\xi+\cos x$. and can also be written, for $u\in L^2(\mathbb{R}^n)$, by

$$(Hu)(x)=\frac{1}{2}(u(x+h)+u(x-h))+\cos x\ u(x)\,.$$

We shall later recall how to relate the properties of p and the properties of the associated operator.

We simply recall the case of the Schrödinger operator : $S_h = -h^2\Delta + V(x)$. If V is -say continuous- bounded from below, S_h, which is a priori defined on $\mathcal{S}(\mathbb{R}^n)$ as a differential operator, admits a unique selfadjoint extension on $L^2(\mathbb{R}^n)$.
We are first interested in the nature of the spectrum. If V tends to $+\infty$ as $|x| \to \infty$, one can show that S_h, more precisely its selfadjoint realization, has compact resolvent and its spectrum consists of a sequence of eigenvalues tending to ∞. We are next interested in the asymptotic behavior of these eigenvalues.

In the case of the harmonic operator, corresponding to

$$V(x) = \sum_{j=1}^{n} \mu_j x_j^2 \;(\text{with } \; \mu_j > 0)\,,$$

the criterion of compact resolvent is satisfied and the spectrum is described as the set of the

$$\lambda_\alpha(h) = \sum_{j=1}^{n} \sqrt{\mu_j}(2\alpha_j + 1)h\,,$$

for $\alpha \in \mathbb{N}^n$.

We have also in this case a complete description of the normalized associated eigenfunctions which are constructed recursively starting from the first eigenfunction corresponding to $\lambda_0(h) = \sum_j \sqrt{\mu_j}\, h$:

$$\phi_0(x;h) = (\prod_{j=1}^{n} \mu_j^{\frac{1}{8}})(\frac{2}{\pi})^{\frac{n}{2}} \cdot h^{-\frac{n}{4}} \cdot \exp(-\sum_j \sqrt{\mu_j} x_j^2 / h)\,. \qquad (10)$$

The eigenfunction ϕ_0 is strictly positive and decays exponentially. Moreover, (and here we enter in the semi-classical world), the local decay in a fixed closed set avoiding $\{0\}$ (which is measured by its L^2 norm) is exponentially small as $h \to 0$. In particular, this says that the eigenfunction lives asymptotically in the set $V(x) \leq \lambda(h)$ which has to be understood as the projection by the map $(x, \xi) \mapsto x$ of the energy level which is classically attached to the eigenvalue $\lambda(h)$, that is $\{(x, \xi), p(x, \xi) = \lambda(h)\}$.

From quantum mechanics to classical mechanics : semi-classical mechanics

Let us describe a few results which are typical in the semi-classical context. They concern **Weyl's asymptotics** and the localization of the eigenfunctions.
We start with the case of the Schrödinger operator S_h, but we emphasize however that the h-pseudodifferential techniques are not limited to this situation.
We assume that V is a C^∞ function on $\mathbb{R}^n$ which is semi-bounded and satisfies $\inf V < \underline{\lim}_{|x|\to\infty} V(x)$. The Weyl Theorem gives that the essential spectrum is contained in

$$[\, \underline{\lim}_{|x|\to\infty} V(x)\, ,\, +\infty\, [\, .$$

It is also clear that the spectrum is contained in $[\inf V, +\infty[$. In the interval $I = [\inf V, \underline{\lim}_{|x| \to \infty} V(x)[$, the spectrum is discrete, that is has only isolated eigenvalues with finite multiplicity. For any E in I, it is consequently interesting to look at the counting function of the eigenvalues contained in $[\inf V, E]$

$$N_h(E) = \sharp\{\lambda_j(h) \; ; \; \lambda_j(h) \leq E\} \, . \tag{11}$$

The main semi-classical result is then

Theorem : Weyl's asymptotics

With the previous assumptions, we have :

$$\lim_{h \to 0} h^n N_h(E) = L_n^c \int_{V(x) \leq E} (E - V(x))^{\frac{n}{2}} \, dx \, .$$

Another way is to write

$$\lim_{h\to 0} h^n N_h(E) = (2\pi)^{-n} \int_{\xi^2+V(x)\leq E} dx d\xi \,.$$

Under suitable assumptions, in particular

$$E < \liminf_{|x|+|\xi|\to+\infty} p(x,\xi)\,,$$

we get the following extension for the operator $p^W(x, hD_x)$

$$\lim_{h\to 0} h^n N_h(E) = (2\pi)^{-n} \int_{p(x,\xi)\leq E} dx d\xi \,.$$

In dimension 1, we will have much more precise results (see later).

Semi-classical localization

Let us start with very weak notion of localization. For a family $h \mapsto \psi_h$ of L^2-normalized functions defined in Ω, we will say that the family ψ_h lives (resp. fully lives) in a closed set U of $\overline{\Omega}$ if for any neighborhood $\mathcal{V}(U)$ of U,

$$\lim_{h\to 0} \int_{\mathcal{V}(U)\cap\Omega} |\psi_h|^2 dx > 0\ ,$$

respectively

$$\lim_{h\to 0} \int_{\mathcal{V}(U)\cap\Omega} |\psi_h|^2 dx = 1\ .$$

For example one expects that the groundstate of the Schrödinger operator $-h^2\Delta + V(x)$ fully lives in $V^{-1}(\inf V)$. Similarly, one expects that, if $\overline{\lim}_{h\to 0}\lambda(h) \leq E < \inf \sigma_{ess}(P_{h,V}) - \epsilon_0$ (for $\epsilon_0 > 0$ small enough) and ψ_h is an eigenvector associated to $\lambda(h)$, then ψ_h will fully live in $V^{-1}(]-\infty, E])$.
Of course the above is very heuristic but there are more accurate mathematical notions like the frequency set (see below or the book of D. Robert) permitting to give a mathematical formulation to the above vague statements.

Once we have determined a closed set U, where ψ_h fully lives (and hopefully the smallest), it is interesting to discuss the behavior of ψ_h outside U, and to measure how small ψ_h decays in this region.

To illustrate the discussion, one can start with the very explicit example of the harmonic oscillator. The ground state $x \mapsto ch^{-\frac{1}{4}} \exp -\frac{x^2}{h}$ of $-h^2\frac{d^2}{dx^2} + x^2$ lives at 0 and is exponentially decaying in any interval $[a, b]$ such that $0 \notin [a, b]$. This is this type of result that we want to recover but WITHOUT having an explicit expression for ψ_h.

Localization of the eigenfunctions

The localization property was already observed on the specific case of the harmonic oscillator. But this was a consequence of an explicit description of the eigenfunctions. This is quite important to have a good description of the decay of the eigenfunctions (as $h \to 0$) outside the classically permitted region without to have to know an explicit formula.

Various approachs can be used.

The first one fits very well in the case of the Schrödinger operator (more generally to h-pseudodifferential operators with symbols admitting holomorphic extensions in the ξ variable) and gives exponential decay. This is based on the so-called Agmon estimates (see Agmon [Ag], Helffer-Sjöstrand [**?**] or Simon [Si]). This is the starting point of the analysis of the tunneling (see [Hel], [DiSj] and [Mar]).

The second one is an elementary application of the h-pseudodifferential formalism which will be described later and leads for example to the following statement.

Proposition: localization of the eigenfunctions

Let E in I and let $(\lambda(h_j), \phi_{(h_j)}(x))$ a sequence in $I \times L^2(\mathbb{R}^n)$ where $\lambda(h_j) \to E$ and $h_j \to 0$ as $j \to \infty$, $x \mapsto \phi_{(h_j)}(x)$ is an L^2-normalized eigenfunction associated with $\lambda(h_j)$ with norm 1. Let Ω be a relatively compact set in $\mathbb{R}^n$ such that

$$V^{-1}(]-\infty, E]) \cap \bar{\Omega} = \emptyset \,.$$

Then,

$$||\phi_{(h_j)}||_{L^2(\Omega)} = \mathcal{O}(h_j^{+\infty}) \,.$$

Short introduction to the h-pseudodifferential calculus

Basic calculus : the class S^0 We shall mainly discuss the most simple called the S^0 calculus. Let us simply say here that the S^0 calculus is sufficient once we have suitably (micro)-localized the problem (for example by the functional calculus).
This class of symbols p is simply defined by

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta} \,,$$

for all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$.
The symbols can be h dependent.
More generally we can introduce, for $j \in \mathbb{R}$, the class S^j by the condition

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi,h)| \leq C_{\alpha,\beta}\, h^{-j} \,.$$

With a symbol, one can associate an h-pseudodifferential operator by (9). This operator is a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ but can also be defined by duality on $\mathcal{S}'(\mathbb{R}^n)$.
The first basic analytical result is the Calderon-Vaillancourt (see for example the book by Hörmander [Ho]) theorem establishing the L^2 continuity.

The second important property is the existence of a calculus. If a is in S^0 and b is in S^0 then the composition $a^w(x, hD_x) \circ b^w(x, hD_x)$ of the two operators is a pseudodifferential operator associated with an h-dependent symbol c in S^0:

$$a^w(x, hD_x) \circ b^w(x, hD_x) = c^w(x, hD_x; h) \ .$$

We immediately meet symbols admitting expansions in powers of h, called regular symbols, i.e. admitting expansions of the type

$$a(x, \xi; h) \sim \sum_j a_j(x, \xi) h^j \ , \ b(x, \xi; h) \sim \sum_j b_j(x, \xi) h^j \ .$$

In this case c has a similar expansion :

$$c(x,\xi;h) \sim \left[\exp\left(\frac{ih}{2}(D_x\cdot D_\eta - D_y\cdot D_\xi)\right)(a(x,\xi;h)\cdot b(y,\eta;h))\right]_{x=y\,;\,\xi=\eta}\,.$$

Note that, modulo $\mathcal{O}(h^\infty)$, the computation of c at (x,ξ) only depends on the germs of a and b at (x,ξ).
Hence if a and b (say independent of h) have disjoint support then $a^w(x,hD)\circ b^w(x,hD_x)$ has an $\mathcal{L}(L^2)$-norm which is $\mathcal{O}(h^\infty)$.

The symbol a_0 is called the principal symbol. At the level of principal symbols, the rule is that

$$c_0 = a_0 \cdot b_0 \ .$$

Another important property is the correspondence between commutator of two operators and Poisson brackets. The principal symbol of the commutator $\frac{1}{h}(a^w \circ b^w - b^w \circ a^w)$ is $\frac{1}{i}\{a_0, b_0\}$, where $\{f, g\}$ is the Poisson bracket of f and g :

$$\{f, g\}(x, \xi) = H_f g = \sum_j \left(\partial_{\xi_j} f \cdot \partial_{x_j} g \ - \ \partial_{x_j} f \cdot \partial_{\xi_j} g\right) \ .$$

Elliptic theory

Once one has a pseudo-differential calculus, the main point is to have a class of invertible operators, such that the inverse is also in the class. This is what we call an elliptic theory and the typical statement is:

Theorem: construction of the inverse

Let P be an h-pseudodifferential operator associated to a symbol p in S^0. (We write in this case $P \in \mathrm{Op}_h^w(S^0)$). We assume that it is elliptic in the sense that $p \neq 0$ and $\frac{1}{p}$ belongs to S^{reg}. Then there exists an h-pseudodifferential operator Q with symbol in S^{reg} such that

$$Q \cdot P = I + R \quad ; \quad P \cdot Q = I + S \,.$$

The remainders R and S are operators with symbols in $\mathcal{O}(h^\infty)$.

The proof is rather easy, once the formalism of composition and the notion of principal symbol have been understood. One can indeed start from the operator Q_0 of symbol $\frac{1}{p}$ and observe that

$$Q_0 P = I + R_1$$

with

$$R_1 \in \mathcal{O}(h)\,.$$

The operator

$$(I + R_1)^{-1} Q_0 \sim \left(\sum_{j \geq 0} (-1)^j R_1^j \right) Q_0$$

gives essentially the solution.
Note that for h small enough we get the inversibilty and using Beals's theorem we can show that the inverse belongs to $\mathrm{Op}_h^w(S^0)$.

Very often, p is not elliptic everywhere and we will be obliged to use "microlocal" inverses (see later).

The functional calculus

It is well known by the spectral theorem for a selfadjoint operator P that a functional calculus exists for Borel functions. What is important here is to find a class of functions (actually essentially C_0^∞) such that $f(P)$ is a nice pseudodifferential operator in the same class as P with simple rules of computation for the principal symbol.
We are starting from the general formula (see [DiSj])

$$f(P) = -\pi^{-1} \lim_{\epsilon \to 0^+} \iint_{|\Im z| \geq \epsilon} \frac{\partial \tilde{f}}{\partial \bar{z}}(x, y)\, (z - P)^{-1} dx\, dy$$

which is true for any selfadjoint operator and any f in $C_0^\infty(\mathbb{R})$.

Here $(x, y) \mapsto \tilde{f}(x, y)$ is a compactly supported, almost analytic extension of f in $\mathbb{C}$. This means that $\tilde{f} = f$ on $\mathbb{R}$ and that for any $N \in \mathbb{N}$ there exists a constant C_N such that $|\frac{\partial \tilde{f}(z)}{\partial \bar{z}}| \leq C_N |\Im z|^N$.
The main result due to Helffer-Robert [HelRob6] is that, for P an h-regular pseudodifferential operator satisfying suitable conditions and f in $C_0^\infty(\mathbb{R})$, then $f(P)$ is a pseudodifferential operator whose Weyl's symbol $p_f(x, \xi; h)$ admits a formal expansion in powers of h

$$p_f(x, \xi; h) \sim h^j p_{f,j}(x, \xi) ,$$

with

$$\begin{aligned} p_{f,0} &= f(p_0) \\ p_{f,1} &= p_1 \cdot f'(p_0) \\ p_{f,j} &= \sum_{k=1}^{2j-1} (-1)^k (k!)^{-1} d_{j,k} f^{(k)}(p_0) \;\; \forall j \geq 2 , \end{aligned}$$

where the $d_{j,k}$ are universal polynomial functions of the $\partial_x^\alpha \partial_\xi^\beta p_\ell$ with $|\alpha| + |\beta| + \ell \leq j$.

The main point in the proof is that we can construct a parametrix (= approximate inverse) for $(P - z)^{-1}$ for $\Im z \neq 0$ with a nice control as $\Im z \to 0$. The constants controling the estimates on the symbols are exploding as $\Im z \to 0$ but the choice of the almost analytic extension of f absorbs any negative power of $|\Im z|$.
As a consequence, we get that if in some interval I

$$(H) \quad p_0^{-1}(I + [-\epsilon_0, \epsilon_0]) \text{ is compact,}$$

for some $\epsilon_0 > 0$, then the spectrum is, for h small enough, discrete in I.

In particular, we get that, if $p(x,\xi) \to +\infty$ as $|x| + |\xi| \to +\infty$, then the spectrum of P_h is discrete (P_h has compact resolvent).

Another interest is that for suitable f (possibly h-dependent) the operator $f(P)$ could have better properties that the initial operator. It appears in particular very powerful in dimension 1 where we can in some interval of energy find a function $t \mapsto f(t;h)$ admitting an expansion in powers of h such that $f(P;h)$ has the spectrum of the harmonic oscillator. This is a way to get the Bohr-Sommerfeld conditions (See Helffer-Robert [HelRob7], in connexion with Maslov [Mas] or the work of Voros:

$$f(\lambda_n(h)\,;\, h) \sim (2n+1)h\ ,$$

modulo $\mathcal{O}(h^\infty)$.

Semiclassical microlocalization

We already speak of a notion of semi-classical localization for a family ψ_h. The notion of Frequency Set is more accurate (more "microlocal").

Definition

We say that (x_0, ξ_0) is NOT in the Frequency Set $FS(\psi_h)$ of the family ψ_h if there exists $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x_0) > 1$ and a neighborhood $\mathcal{V}(\xi_0)$ of ξ_0 such that

$$(\mathcal{F}_h \chi \psi_h)(\xi) = \mathcal{O}(h^\infty) \text{ in } \mathcal{V}(\xi_0)\,.$$

Another name (see in Zworski's book) is "semi-classical Wave front Set".

Examples

The Frequency Set of the family

$$\mathbb{R} \ni x \mapsto \psi_h(x) := h^{-\frac{1}{4}} \exp ibx/h \exp -\frac{(x-a)^2}{h}$$

is the point (a, b) in $\mathbb{R}^2$.
The Frequency Set of the family

$$\mathbb{R}^n \ni x \mapsto \psi_h(x) := a(x)h^{-\frac{n}{2}} \exp i\phi(x)/h$$

is the set

$$\Lambda_\phi := \{(x, \nabla\phi(x)), x \in \mathbb{R}^n\}.$$

Other remarks

- See the books by A. Martinez and M. Zworski for more information
- The parameter h can also be (instead of belonging to an interval $(0, h_0]$) a sequence h_j tending to 0.
- This is a refinement of the semi-classical localization.
- An eigenfunction ψ_h of $p^w(x, hD_x)$ associated with $\lambda(h)$ (with $\lambda(h)$ close to E), lives microlocally in its energy level $\{p(x,\xi) = E\}$.
- There is a h-pseudodifferential characterization of the Frequency Set.
- $$FS(p^w(x, hD_x)\psi_h) \subset FS(\psi_h)\,.$$
- There is a characterization of the Frequency set using the FBI transform (see the book of Martinez, p. 98).

More sophisticated properties based on Beals's theorem

The Beals Theorem is a characterization of the h-pseudodifferential operator by the properties of its commutators in $\mathcal{L}(L^2)$ with x_j and $\frac{\partial}{\partial x_j}$. We have already used the theorem when inverting $(I + R)$ in the elliptic theory. We omit the exact statement but we need some consequences of this theorem.

Localization property

If q, χ_1 and χ_2 belong to a bounded set in S^0 and if $\mathrm{dist}(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2) \geq \epsilon_0$ then for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$||\mathrm{Op}_h(\chi_1)\mathrm{Op}_h(q)\mathrm{Op}_h(\chi_2)||_{\mathcal{L}(L^2)} \leq C_N h^N \mathrm{dist}(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2)^{-N}\,.$$

This says that with h-pseudodifferential operators, we are not very far of the properties of differential operators.

We should also recognize in this theory the negligeable operators.

Negligeable operators

An operator K_h from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ can be written in the form $\mathrm{Op}_h(k_h)$ for some $k_h \in \cap_j S^j$ iff for any bounded set $B \subset S^0$ and for any $N \in \mathbb{N}$ there exists C such that, $\forall \chi_1, \chi_2$,

$$||\mathrm{Op}_h(\chi_1)\mathrm{Op}_h(k_h)\mathrm{Op}_h(\chi_2)||_{\mathcal{L}(L^2)} \\ \leq C_N h^N (1 + \mathrm{dist}(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2))^{-N} .$$

Semi-classical analysis of the one-well problem

We assume that $p(x,\xi)$ is elliptic at ∞. We introduce

$$U := p^{-1}(0)$$

(0 could be replaced by E, in this case we replace p by $p - E$).

Let I_h some compact interval tending to 0 with h and we want to analyze the spectrum of $P := p^w(x, hD_x)$ in h.
We assume some gap in the spectrum. More precisely, we assume that there exists N_0 and $h_0 > 0$ such that

$$a(h) \geq h^{N_0}\,, \forall h \in (0, h_0]\,,$$

and that P has no spectrum in $(I_h + [-2a(h), 2a(h)]) \setminus I_h$, for h small enough.
Note that we never try to control h_0. Hence it can change from line to line (but there is only a finite number of changes).

Properties of the projector

The spectral projector Π_I relative to I_h is defined by

$$\Pi_I = (2\pi i)^{-1} \int_{\partial\Omega_h} (z-P)^{-1}\,,$$

where $\Omega_h := \{z; \mathrm{dist}\,(z, I_h) \leq a(h)\}\,.$
It is an h-pseudodifferential operator with symbol in S^0 and has the property that there exists $\epsilon_0 > 0$ and for any N, C_N such that

$$\begin{aligned} &||\mathrm{Op}(\chi_1)\Pi \mathrm{Op}_h^w(\chi_2)|| \\ &\qquad \leq C_N h^N (\mathrm{dist}\,(\mathrm{supp}\chi_1, U) + \mathrm{dist}\,(\mathrm{supp}\chi_2, U))^{-N} \end{aligned}$$

if

$$\mathrm{dist}\,(\mathrm{supp}\chi_1, U) + \mathrm{dist}\,(\mathrm{supp}\chi_2, U) \geq \epsilon_0 > 0$$

Proof

We introduce χ with compact support in a neighborhood of U (arbitrarily small) and equal to 1 in a neighborhood of U. So $d((x,\xi),U)$ will be very close to $d((x,\xi),\mathrm{supp}\chi)$.
As before in the elliptic situation we first construct $Q_z^0 \in \mathrm{Op}(S^0)$ (holomorphic in $z \in \Omega_h$) s.t.

$$(P-z)Q_z^0 = I - \mathrm{Op}(\chi) - K_z \text{ with } K_z \in \mathrm{Op}(S^{-1})\,.$$

With $Q_z = Q_z^0(I-K_z^0)^{-1}$ we get

$$(P-z)Q_z = I - R_z \text{ with } R_z = \mathrm{Op}_h^w(\chi)(I-K_z^0)^{-1})\,.$$

Hence we have obtained a good approximate right inverse "outside of U".
Similarly, we construct an approximate left-inverse $\widetilde{Q}_z$:

$$\widetilde{Q_z}(P-z) = I - \widetilde{R_z} \text{ with } \widetilde{R_z} = (I-\widetilde{K_z^0})^{-1}\mathrm{Op}_h^w(\chi)\,.$$

We now assume that $z \in \partial\Omega_h$. In this case $(P - z)$ is invertible and we have

$$||(P-z)^{-1}||_{\mathcal{L}(L^2)} \leq \frac{1}{a(h)}\,.$$

By algebraic manipulations, we get

$$(P-z)^{-1} = \widetilde{Q}_z + \widetilde{R}_z Q_z + \widetilde{R}_z (P-z)^{-1} R_z\,.$$

The projector on the eigenspace of P attached to the interval I_h is given by the Cauchy integral

$$\Pi_I = (2\pi)^{-1} \int_{\partial\Omega_h} (P-z)^{-1} dz\,.$$

Using the above formula and the holomorphy, we obtain

$$\Pi_I = (2\pi)^{-1} \int_{\partial\Omega_h} \widetilde{R}_z (P-z)^{-1} R_z dz\,.$$

To prove the statement, we now simply write

$$\mathrm{Op}(\chi_1)\Pi_I \mathrm{Op}_h^w(\chi_2) = (2\pi)^{-1} \int_{\partial\Omega_h} \mathrm{Op}(\chi_1)\widetilde{R}_z(P-z)^{-1} R_z \mathrm{Op}(\chi_2) dz \,,$$

and get

$$||\mathrm{Op}(\chi_1)\Pi_I \mathrm{Op}_h^w(\chi_2)|| \leq \frac{C}{a(h)} \left(\sup_{z\in\Omega_h} ||\mathrm{Op}(\chi_1)\widetilde{R}_z|| \right) \left(\sup_{z\in\Omega_h} ||R_z \mathrm{Op}(\chi_2)|| \right)$$

For estimating the left hand side, we now observe that

$$\mathrm{Op}(\chi_1)\widetilde{R}_z = \mathrm{Op}(\chi_1)(I - \widetilde{K}_0)^{-1}\mathrm{Op}(\chi) \,,$$

and

$$R_z \mathrm{Op}(\chi_2) = \mathrm{Op}(\chi)(I - K_0)^{-1}\mathrm{Op}(\chi_2) \,.$$

We finally use our lower bound of $a(h)$ and a sufficiently good choice of the support of χ to achieve the proof.

Multiple wells

We assume that $p^{-1}(0)$ is a union (finite or indexed by $\mathbb{Z}^2$) of compact, disjoints sets.
We assume some uniformity for the family U_α:

- uniform bound on the diameter,
- $$\exists \epsilon_0 > 0 \text{ s.t. } U_\alpha + B(0,\epsilon_0) \cap U_\beta + B(0,\epsilon_0) = \emptyset\,,$$

  if $\alpha \neq \beta$,
- $\forall \alpha, \exists p_\alpha$ with $p_\alpha = p$ in $U_\alpha + B(0,\epsilon_0)\,,$
  uniformly bounded in $\mathcal{S}'$
  and uniformly elliptic outside any neighborhood of $\overline{U_\alpha}$.

With I_h and $a(h)$ as before we assume that the P_α admit the conditions given for P uniformly.

Multiple wells resolvent and projectors

Proposition: Properties of the resolvent

There exists h_0 such that for $h \in (0, h_0)$, $\partial\Omega_h$ belongs to the resolvent set of P. Moreover, there exists $\epsilon_0 > 0$ and for any N, C_N such that, $\forall z \in \partial\Omega_h$,

$$||\mathrm{Op}_h^w(\chi_1)(P-z)^{-1}\mathrm{Op}_h^w(\chi_2)|| \\ \leq C_N h^N (\widetilde{\mathrm{dist}}\,(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2))^{-N}$$

if

$$\widetilde{\mathrm{dist}}\,(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2) \geq \epsilon_0 > 0.$$

Here

$$\begin{aligned}&\widetilde{\mathrm{dist}}\left(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2\right)\\ &\quad := \inf\left(\mathrm{dist}\left(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2\right),\right.\\ &\qquad\qquad \left.\inf_\alpha\left(\mathrm{dist}\left(\mathrm{supp}\chi_1, U_\alpha\right) + \mathrm{dist}\left(\mathrm{supp}\chi_2, U_\alpha\right)\right)\right)\end{aligned}$$

The proof of this proposition is only sketched. We first construct an approximate resolvent by patching together microlocal resolvent in the neighborhood of each U_α and some holomorphic approximate resolvent outside of the wells. This will give the existence of the resolvent for $z \in \partial\Omega_h$ with the same structure as the approximate resolvent and we can then analyze the corresponding projector associated with the interval I_h. Anyway, I will give a few more details below.

Less sketchy presentation of the proof

We introduce a partition of unity χ_α associated with the wells U_α with uniformity properties and supported in small neighborhoods of U_α and we first reproduce the previous one well construction with χ replaced by $\sum_\alpha \chi_\alpha$.
We get

$$(P-z)Q_z = I - R_z \text{ with } R_z = \sum_\alpha \mathrm{Op}_h^w(\chi_\alpha)(I+L_z))\,,$$

with $L_z \in \mathrm{Op}(S^{-\infty})$.
We note that, at the difference of the one well case, we have also to show that $(P-z)$ is (like the P_α's) invertible for $z \in \partial\Omega_h$.

We construct an approximate right inverse by using the $(P_\alpha - z)^{-1}$:

$$R(z) = Q_z + \sum_\alpha (P_\alpha - z)^{-1} \mathrm{Op}_h^w(\chi_\alpha)(I + L_z)\,.$$

After some work, we obtain that

$$||R(z)|| \leq \frac{C}{a(h)}\,,$$

and like in the one well case

$$||\mathrm{Op}(\chi_1) R(z) \mathrm{Op}_h^w(\chi_2)|| \leq C_N h^N \,\mathrm{dist}\,(\mathrm{supp}\chi_1, \mathrm{supp}\chi_2)^{-N}\,,\ \forall N\,,$$

under the conditions of the proposition.
We have now to verify that this is indeed a good right inverse !

We start from

$$(P-z)R(z) = I + K_z\,,$$

with

$$K_z = \sum_\alpha (P-P_\alpha)(P_\alpha - z)^{-1}\mathrm{Op}(\chi_\alpha)(I+L_z)\,.$$

Here the support of $(p_\alpha - p)$ is disjoint of the support of χ_α by assumption and construction and we have (we should first treat the case when the support of the χ_j are bounded) , for each α,

$$\begin{array}{l} ||\mathrm{Op}_h^w(\chi_1)(P-P_\alpha)(P_\alpha - z)^{-1}\mathrm{Op}(\chi_\alpha)(I+L_z)\mathrm{Op}_h^w(\chi_2)|| \\ \quad \leq C_N h^N (1 + d(\mathrm{supp}\chi_1, U_\alpha) + d(U_\alpha, \mathrm{supp}\chi_2))^{-N} \end{array}$$

for any N with C_N independent of α.
By summation over α, we get immediately by the criterion on negligeable operators than $K_z \in \mathrm{Op}(S^{-\infty})$. In particular $(I+K_z)$ is invertible for h small enough and we obtain that $\partial\Omega_h$ is in the resolvent set of P and that

$$(P-z)^{-1} = R(z)(I+K_z)^{-1}\,.$$

We then obtain easily that

$$\|(P - z)^{-1}\| \leq \frac{C}{a(h)}\,,$$

This is what was mainly missing to extend the techniques of the one well problem to this more general situation.

Hence after some extrawork, we get as in the case of one well

Proposition: Properties of the projector

Let Π the projector associated with the spectum of P contained in Ω_h. Given a bounded set in S^0, there exists $\epsilon_0 > 0$, $h_0 > 0$ and for any N, C_N such that, for χ_1, χ_2 in this bounded set such that

$$\inf_\alpha \mathrm{dist}\,(\mathrm{supp}\chi_1, U_\alpha) + \mathrm{dist}\,(\mathrm{supp}\chi_2, U_\alpha) \geq \epsilon_0 > 0$$

and $h \in (0, h_0)$, we have

$$\begin{aligned} &||\mathrm{Op}_h^w(\chi_1)\Pi\mathrm{Op}_h^w(\chi_2)|| \\ &\quad \leq C_N h^N (\inf_\alpha \mathrm{dist}\,(\mathrm{supp}\chi_1, U_\alpha) + \mathrm{dist}\,(\mathrm{supp}\chi_2, U_\alpha))^{-N}\,. \end{aligned}$$

Construction of a wells-localized basis of the eigenspace attached to $\Omega_h \cap \mathbb{R}$

To simplify, we assume that

$$\sigma(P_\alpha) \cap I_h = \{\mu_\alpha\}$$

where μ_α has multiplicity 1 for $\alpha \in \mathbb{Z}^2$ and that

$$d(U_\alpha, U_\beta) \approx |\alpha - \beta| \, .$$

We consider the corresponding eigenfunction which satisfies

$$(P_\alpha - \mu_\alpha)\varphi_\alpha = 0\,,\; ||\varphi_\alpha|| = 1$$

and, observing that

$$\varphi_\alpha = \pi^{\alpha}_{I_h}\varphi_\alpha$$

and having in mind the one-well proposition we have

$$||\mathrm{Op}^w_h(\chi)\varphi_\alpha|| \leq C_N h^N d(\mathrm{supp}\chi, U_\alpha)^{-N}\,,\; \forall N\,,$$

if $d(\mathrm{supp}\chi, U_\alpha) \geq \epsilon_0 > 0$.

This implies that the φ_α are strongly (micro)-localized in U_α (in particular $FS(\varphi_\alpha) \subset U_\alpha$) and that we get an almost orthogonal basis

$$|\langle \varphi_\alpha \,|\, \varphi_\beta \rangle| \leq C_N h^N |\alpha - \beta|^{-N}, \ \text{ if } \alpha \neq \beta \,.$$

We can now introduce

$$v_\alpha = \Pi \varphi_\alpha \,,$$

and show that v_α also satisfy (use the properties of the projector)

$$||\mathrm{Op}_h^w(\chi) v_\alpha|| \leq C_N h^N d(\mathrm{supp}\chi, U_\alpha)^{-N} \,, \, \forall N \,,$$

In addition, v_α is very close to φ_α:

$$\|\mathrm{Op}_h^w(\chi)(v_\alpha - \varphi_\alpha)\| \leq C_N h^N (1 + d(\mathrm{supp}\chi, U_\alpha))^{-N} \,.$$

The proof is based on the previously established properties of the projector Π.
This implies that the distance between the space E generated by the φ_α to the spectral space $F(h)$ attached to Ω_h is $\mathcal{O}(h^\infty)$.
Note that the proof is in two steps. First we show that the v_α are an orthonormal basis of a closed subspace in $F(h)$ and then we have to show that $F'(h)$ is actually $F(h)$. For this, we observe that, using our approximate formula for the resolvent,

$$(P-z)^{-1} = Q_z + \sum_\alpha (P_\alpha - z)^{-1} \mathrm{Op}_h^w(\chi_\alpha) + \mathcal{O}(h^\infty)\,.$$

that if $u = \Pi u$ then

$$\begin{aligned} u &= \textstyle\sum_\alpha \pi_{I_h}^\alpha(\chi_\alpha u) + \mathcal{O}(h^\infty) \\ &= \textstyle\sum_\alpha c_\alpha \varphi_\alpha + \mathcal{O}(h^\infty)\,. \end{aligned}$$

Projecting again by Π, we get

$$u = \sum_\alpha c_\alpha v_\alpha + \mathcal{O}(h^\infty)\,.$$

Finally, we can replaced the quasi-orthogonal basis v_α, by the Schmidt orthogonalization procedure, to get an orthonormal basis of F with

$$e_\alpha = v_\alpha + \sum_\beta a_{\alpha,\beta}(h) v_\beta \, .$$

with

$$|a_{\alpha,\beta}(h)| \leq C_N \, h^N \, (1 + |\alpha - \beta|)^{-N} \, .$$

Exponential decay.

As in the case of the Schrödinger operator, where more information was needed about the decay of eigenfunctions outside the classical region, we have to improve the localization information (and actually do it microlocally).
We start the spectral analysis with an operator in a more general form that the Harper model:

$$P = (1 - \cos hD) + V(x)\,.$$

Hence, in comparison with Schrödinger, we have replaced $-h^2\Delta$ by $(1 - \cos hD)$.

In the case of Schrödinger, by playing with some easy energy estimate, we had:

Proposition-Schrödinger

If ϕ is Lipschitz, with ϕ' Lipschitz, $z \in \mathbb{C}$ and $F_\pm$ are non negative bounded functions satisfying

$$V - \Re z - \phi'^2 = F_+^2 - F_-^2 \,,$$

we have

$$\frac{1}{4}||(F_+ + F_-)u_\phi||^2 \leq ||\frac{1}{F_+ + F_-} e^{\frac{\phi}{h}}(P - z)u||^2 + ||F_- u_\phi||^2$$

for any $u \in C_0^\infty(\mathbb{R})$, where $u_\phi := e^{\frac{\phi}{h}} u$.

If we consider an eigenfunction ψ_h and $z = \lambda(h)$ we get

$$\frac{1}{4}||(F_+ + F_-)u_\phi||^2 \leq ||F_- u_\phi||^2$$

where

$$V - \lambda(h) - \phi'^2 = F_+^2 - F_-^2 \text{ and } u_\phi := e^{\frac{\phi}{h}} \psi_h \,.$$

This leads to a choice of an optimal ϕ close to the "Agmon distance" to the well

$$d_{(V-\lambda(h))_+}(x, \{V(x) \leq \lambda(h)\}) \,.$$

The Agmon distance $d_{(V-E)_+}$ is the distance associated (in the forbidden region) with the degenerate metric $(V-E)_+ g_0$ where g_0 is the standard metric in $\mathbb{R}^2$. In the case of one dimension this is immediate to compute.
By "close" we more specifically mean that we can take

$$\phi(x) = (1-\epsilon) d_{(V-\lambda(h))_+}(x, \{V(x) \leq \lambda(h)\})$$

where $\epsilon >$ can be chosen arbitrarily small.

Agmon distance in (1D)

In $(1D)$, one can actually be much more explicit. Hence the notion of Agmon distance in a geometric manner is not necessary.
For a given energy E, we assume that $U_E := \{V(x) \leq E\}$ is a a finite (or infinite) disjoint closed bounded intervals $U_j(E)$ (the wells). We then introduce $\Psi = \Psi_E$ as the nondecreasing function on the line such that

- $\Psi(U_{j_0}) = 0\,,$
- Ψ is constant on each $U_j(E)\,,$
- $\Psi'(x)^2 + V(x) = E$ in $\mathbb{R} \setminus \cup U_j$.

Hence, between two consecutive wells, $U_j = [x_j, y_j]$ and $U_{j+1} = [x_{j+1}, y_{j+1}]$ we have simply

$$\Psi(x) = \Psi(y_j) + \int_{y_j}^{x} \sqrt{V(t) - E}dt \text{ for } x \in (y_j, x_{j+1})\,.$$

The Agmon distance is then

$$d_{(V-E)_+}(x,y) := |\Psi_E(x) - \Psi_E(y)|\,.$$

The Agmon distance between U_j and U_{j+1} is

$$d_{(V-E)_+}(U_j, U_{j+1}) = d_{(V-E)_+}(y_j, x_{j+1}) = \int_{y_j}^{x_{j+1}} \sqrt{V(x) - E}\,dx\,.$$

We denote by S the minimal distance between U_j and U_ℓ for $j \neq \ell$.
We also observe that the distance is degenerate:

$$\forall x \in U_j, \forall y \in U_j, d_{(V-E)_+}(x,y) = 0\,.$$

From Schrödinger to Harper

Playing with some more sophisticated energy estimate, we get for our Harper like model the following result:

Proposition–Agmon estimates for the Harper like model

If ϕ is Lipschitz, with ϕ' Lipschitz and ϕ'' bounded, $z \in \mathbb{C}$ and $F_\pm$ are non negative bounded functions satisfying

$$V - \Re z - 2\sinh(\phi'/2)^2 = F_+^2 - F_-^2\,,$$

we have

$$\frac{1}{4}||(F_+ + F_-)u_\phi||^2 \leq ||\frac{1}{F_+ + F_-} e^{\frac{\phi}{h}}(P - z)u||^2 + ||F_- u_\phi||^2$$

for any $u \in C_0^\infty(\mathbb{R})$, where

$$u_\phi := e^{\frac{\phi}{h}} u\,.$$

Harper-Agmon distance

We only define it in the case when $V(x) = \cos x$. For a given energy $E \in (-2, 2) \setminus \{0\}$, we introduce Φ_E as the nondecreasing function such that

- $\Phi(0) = 0$,
- Φ is constant on each $U_j(E) = \pi_1 U_{jk}(E)$,
- $\cosh(\Phi') + \cos x = E$ in $\mathbb{R} \setminus \cup U_j$.

The associated distance is then

$$D_E(x, y) := |\Phi_E(x) - \Phi_E(y)| .$$

We denote by S the minimal distance between U_j and U_ℓ for $j \neq \ell$,

$$S := D_E(0, 2\pi) = \Phi_E(2\pi) - \Phi_E(0) .$$

A similar definition should be done in the ξ variable. Due to the symmetry of the symbol, we get a distance $\hat{D}_E$ associated with $\hat{\Phi}_E$ and we have

$$\hat{\Phi}_E(\xi) = \Phi_E(\xi)\,.$$

We will start by a rather simple result.

Application

We fix $\epsilon_0 > 0$ and would like to analyze the spectrum of $P = \cos(hD_x) + \cos x$ (we take $\lambda = 1$ but this is not necessary) in the interval $[\epsilon_0, 2 + \mathcal{O}(h)]$.
Hence we avoid the critical value of $p(x, \xi) = \cos x + \cos \xi$.

Semi-classical elementary exercise

We just want to prove that the distance of $\sigma(\cos(hD_x) + \cos x)$ is close to 2. More precisely

$$d(\sigma(\cos(hD_x) + \cos x), 2) = \mathcal{O}(h)\,.$$

One can be actually much more precise but to give an easy proof is probably enlightning.

We will construct an approximate eigenfunction whose frequency set is just the point $(0,0)$. If we think of the expansion of the symbol around this point, we get

$$\cos\xi + \cos x = 2 - \frac{1}{2}(\xi^2 + x^2) + \mathcal{O}((|x|^2 + |\xi|^2)^2)\,.$$

This suggests to consider the first eigenfunction of the Harmonic oscillator

$$x \mapsto u_h(x) := c_0 h^{-\frac{1}{4}} \exp -\frac{x^2}{h}\,,$$

where $c_0 \neq 0$ is such that $||u_h|| = 1$.

We have then to compute the L^2 norm of
$(\cos(hD_x) + \cos x - 2)u_h(x)$.
It is first clear that

$$||(1 - \cos x)u_h||_{L^2} = \mathcal{O}(h)\,.$$

We have also to estimate the L^2 norm of

$$\frac{1}{2}(\tau_h + \tau_h)u - u\,.$$

The easiest way is to compute the h-Fourier transform and we have to analyze the L^2-norm of $||(1 - \cos\xi)\mathcal{F}_h u||$.
Hence

$$(\cos(hD_x) + \cos x - 2)u_h(x) = \mathcal{O}(h) \text{ in } L^2(\mathbb{R})\,,$$

and by the spectral theorem for selfadjoint operators, we obtain as announced:

$$d(\sigma(\cos(hD_x) + \cos x), 2) = \mathcal{O}(h)\,.$$

Our choice for p_α

For $E = \mu$, we can define the wells $U_\alpha(\mu)$ ($\alpha \in \mathbb{Z}^2$) as the connected components of $p^{-1}(\mu)$.
We can now make explicit the choice of the p_α. We introduce $0 \leq \chi_{0,0} \in C_0^\infty(\cup_{0<\mu\leq 2} U_0(\mu))$ sufficiently large and then the χ_α by translation by $2\pi\alpha$. We then define

$$p_\alpha = p - \sum_{\beta \neq \alpha} \chi_\beta \,.$$

Hence p_α is deduced from $p_{0,0}$ by translation.
We choose $\chi_{0,0}$ such that

$$p_{0,0}^{-1}(\mu) = U_{(0,0)}(\mu) \,.$$

The quantum translations

We recall that we denote by $\tau = \tau_{2\pi}$ and $\hat{\tau}$ the multiplication operator by $e^{2\pi ix/h}$, then these two operators have the following property.

$$\tau\,\hat{\tau} = \exp(-i(2\pi)^2/h)\,\hat{\tau}\,\tau = \exp -i\tilde{h}\,\hat{\tau}\,\tau\,,$$

with

$$(2\pi)/h = k + \tilde{h}/(2\pi)\,,\ k \in \mathbb{Z}\,.$$

(in our renormalization procedure, we will later assume that $\tilde{h}$ is small enough and this can be seen on the continuous fraction expansion of $\tilde{h}/2\pi$.)

If p is a symbol, we have

$$\tau\,\mathrm{Op}_h^w(p) = \mathrm{Op}_h^w(\tau_* p)\,\tau\,,$$

and

$$\hat{\tau}\,\mathrm{Op}_h^w(p) = \mathrm{Op}_h^w(\hat{\tau}_* p)\,\hat{\tau}\,.$$

where, when acting on symbols,

$$\tau_* p(x,\xi) = p(x-2\pi,\xi)\,,\ \hat{\tau}_* p(x,\xi) = p(x,\xi-2\pi)\,.$$

In particular, in the case when p is doubly 2π-periodic, which is the case of Harper, then $\mathrm{Op}_h^w(p)$ commutes with τ and $\hat{\tau}$.
More generally, we can introduce for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$

$$T_\alpha = \tau^{\alpha_1} \, \hat{\tau}^{\alpha_2}$$

and observe that we have the two relations

$$T_\alpha \; T_\beta = \exp(i\alpha_2\beta_1\tilde{h}) \;\; T_{\alpha+\beta}$$

and

$$T_\alpha \, T_\beta = \exp(i\sigma(\alpha,\beta)\tilde{h}) \;\; T_\beta \, T_\alpha \, ,$$

where

$$\sigma(\alpha,\beta) = \alpha_2\beta_1 - \alpha_1\beta_2 \, .$$

Note also that

$$T_\alpha^{-1} = T_\alpha^* = \exp(i\alpha_1\alpha_2 \, \tilde{h}) \, T_{-\alpha} \, .$$

At the level of the operators, we have, with

$$P_\alpha = \mathrm{Op}_h^w(p_\alpha) = p_\alpha^w(x, hD_x)$$

the commutation relation

$$P_\alpha = T_\alpha P_{0,0} T_\alpha^{-1} \,.$$

Hence P_α is unitary equivalent with $P_{0,0}$ and it is enough to write the spectral theory for $P_{0,0}$ near the energy μ.

Commutation with Fourier

$$\mathcal{F}_h T_\alpha = \exp(-i\tilde{h}\alpha_1\alpha_2)\mathcal{F}_h T_{\kappa(\alpha)}$$

where

$$\kappa(x,\xi) = (-\xi, x)\,.$$

Spectral theory for $P_{0,0}$.

Although rather old in the case of Schrödinger, we can apply in this h-pseudodifferential situation the following result due to Helffer-Robert

Proposition

For $h \in (0, h_0]$ with $h_0 > 0$ small enough, we have

$$\sigma(P_{0,0}) \cap [\epsilon_0, 2 + Ch] = \cup_{j=0,\dots,N(h)} \{\mu_j(h)\} \,,$$

where the $\mu_j(h)$ are simple and ordered as a decreasing sequence. Moreover

$$\mu_j(h) - \mu_{j+1}(h) \approx h \,,\, 2 - \mu_0(h) \approx h \,.$$

There are various way to prove the statement: WKB solution, functional calculus, reduction to a model operator,...

Remarks

- $N(h) \approx \frac{C}{h}$. The limit of $hN(h)$ is indeed given by Weyl's formula:

$$\lim_{h\to 0} hN(h) = \int_{\epsilon_0 \leq p_0(x,\xi)} dx d\xi \,.$$

- For $\mu_0(h)$ (or a finite (independent of h) number of eigenvalues, we can apply the Harmonic approximation).
- A good approximation of $\mu_j(h)$ is given by the Bohr-Sommerfeld formula

$$\mu_j(h) \sim g((2j+1)h, h)$$

where $t \to g_0(t)$ is the inverse of $E \mapsto f_0(E)$ (so $f_0 \circ g_0 = Id$) defined

$$(\epsilon_0, 2] \ni E \mapsto f_0(E) := (2\pi)^{-1} \int_{p_{0,0}(x,\xi) \geq E} dx d\xi \,.$$

As E tends to 2, say for $E = 2 - \epsilon$ we have

$$\begin{aligned} f_0(2-\epsilon) &= (2\pi)^{-1} \int_{\cos x + \cos \xi \geq 2-\epsilon\,,\,(x,\xi)\in(-\pi,\pi)^2} dxd\xi \\ &\sim (2\pi)^{-1} \int_{\frac{x^2+\xi^2}{2}\leq \epsilon} dxd\xi \\ &\sim \epsilon\,. \end{aligned}$$

This gives

$$\mu_0(h) - 2 \sim -h\,,\ \mu_1(h) \sim -3h\,,\ \dots$$

as predicted by the harmonic approximation.
The reader can look at the Hofstadter butterfly near energy 2 to observe to what it corrresponds.

About WKB solutions

This has a long story for the $1D$-Schrödinger operator

$$-h^2\frac{d^2}{dx^2}+V(x)\,.$$

We assume that $V(x)\geq 0=V(0)$. If for some $E_0>0$, $V^{-1}(-\infty,E_0)$ is connected, bounded and if ∇V is not critical except at the minimum of V where V is assumed to be non degenerate. Then the whole spectrum in $(0,E_0)$ can be obtained modulo $\mathcal{O}(h^\infty)$ by the so-called generalized Bohr-Sommerfeld condition which reads

$$f(\lambda_n(h),h)=(n+\frac{1}{2})h\,.$$

The first step for getting this rule is to try to construct solution of the type $a(x,h)\exp \pm i\frac{\phi}{h}$ with energy E this is posssible except at $V^{-1}(\{E\})$. We have first to solve in $V^{-1}(-\infty, E)$ the so called equation

$$\phi'(x)^2 = E - V(x)\,.$$

This is when trying to match together these locally defined solution that we get that this is only possible for some h-dependent values of E.

In the case of the Harper model, if $E \in (-2,2)$, $E \neq 0$, we can perform a similar analysis whose first step is to solve

$$\cos\phi'(x) + \cos x = E\,.$$

One observes that there are many local solutions.

More detailed description in the well

If what preceeds is enough for the localization of the eigenvalue, we need more on the eigenfunction u_0 with energy $E = E(h)$.

We assume in our talk that $E(h)$ is far from the bottom (which implies that our labelling j depends on h as $h \to 0$.)
Note that this simplifies the presentation but in the real life we will also have to work in a transition region where we cannot be as explicit (this explains why we have this indirect construction at the beginning of Section 4, in HSHarper1).

Under this condition, the construction can be done by using the analytic WKB method which describes more precisely the eigenfunction u_0 near one microlocal well. The projection of the well is $[-x_0(E(h)), x_0(E(h))]$. We denote by $\phi_E(x)$ the solution of

$$\cos \phi_E'(x) + \cos x = E\,,$$
$$\phi_E(-x_0(E)) = 0, \phi_E'(-x_0(E)) = 0, \phi_E' > 0 \text{ in } (-x_0(E), x_0(E))\,.$$

We note that there were various choices of $\phi'(-x_0(E))$ which determine in which microlocal well (compute the frequency set of the WKB solution) we want to stay. We have chosen to do the construction in the microlocal well $U_{0,0}$.

Then we have in $(-x_0(E), x_0(E))$:

$$u_0(x,h) = a_E(x,h)\sin(\frac{\phi_E(x)}{h} + \frac{\pi}{4}),$$

modulo $\mathcal{O}(\exp(-\epsilon_K/h))$ uniformly on any compact of $(-x_0(E), x_0(E))$.
Here $a_E(x,h)$ is the realization of a formal analytic symbol

$$a_E(x,h) \sim \sum_j a_{E,j}(x) h^j .$$

Note that the first Bohr-Sommerfeld relation reads:

$$\phi_E(x_0(E)) = (2j + \frac{1}{2})h\,.$$

We refer to the Astérisque of J. Sjöstrand for the notion of analytic symbol. We say only here that it means for suitable complex neighborhoods ω_K of compact intervals K in $(-x_0(E), x_0(E))$ the a_j are holomorphic and satisfy

$$|a_j(x)| \leq C^{j+1} j!\,,\, \forall j, \forall x \in \omega_K\,.$$

This permits to define the sum of the series modulo $\mathcal{O}(\exp(-\epsilon/h))$ for some $\epsilon > 0$.

More on the eigenvalues

We also get from the "analytic" approach, that $E(h)$ admits the following expansion:

$$E(h) \sim \sum_{\ell} E_{\ell}((j+\frac{1}{2})h)\, h^{\ell} \;\text{ modulo } \mathcal{O}(\exp -\epsilon_0/h)\,,$$

where $j = j(h)$ is a suitable integer and

$$|E_{\ell}(t)| \leq C^{\ell+1}\ell!\,.$$

E_0 was obtained already. The new fact (in comparison with the previous C^∞-theory) are the "analytic" estimates.

More detailed description outside the well

To compute later the tunneling effect, we need to know the eigenfunction outside of the well. We already know that it is $\mathcal{O}(h^\infty)$ but, much more precisely, we have,

$$u_0(x,h) = b_E(x,h)\exp(-\varphi_E(x)/h)\,,\quad \text{in } (-2\pi, 2\pi)\setminus[-x_0(E(h)), x_0(E(h))]\,,$$

where the phase φ_E satisfies

- φ_E is the solution of the eikonal equation

$$\cosh\varphi_E'(x) + \cos x = E \text{ in } (x_0(E), 2\pi)\,,$$

- $\varphi_E(x_0(E)) = 0\,,$
- $\varphi_E(x) \geq 0\,,$

and the (analytic) symbol b_E satisfies

$$b_E(x,h) \sim \sum_j b_j(x,E)h^j \quad \text{in } (-x_0(E), x_0(E))\,.$$

Matching.

We have only described for simplicity the inside and the outside expressions of the WKB–eigenfunction. We actually know more on the solution and it implies

$$\frac{1}{4} \lim_{x \to x_0(E)^-} a_{E,0}(x)^2 \sin \phi'_E(x) = \lim_{x \to x_0(E)^+} b_{0,E}(x)^2 \sinh \varphi'_E(x) \, .$$

The proof involves Airy type integral.

Decay estimates for the resolvent

We are in the previously defined multiple-wells situation with $I_h = \{\mu_j(h)\}$ (I_h is moving but this is not important) and $a(h) \approx h$. With $\nu(h) = 2 - \mu(h)$ we get:

Theorem

For any $\epsilon > 0$ and $C_0 > 0$, there exists $C_\epsilon > 0$ such that, $\forall z \in \Omega_h$, $h \in \frac{1}{C_\epsilon}$, $\varphi \in C^{1,1}$ with $|\varphi''| \leq C_0$ and

$$((1 - \cos x) - \nu(h) - \epsilon)_+ - 2(\sinh(\varphi'/2)^2 \geq 0\,,$$

the operator $(P - z)^{-1}$ admits an extension from L^2_{comp} to

$$L^2_\varphi = \{u\,;\, e^{\varphi/h} u \in L^2\}$$

with norm $\leq C_\epsilon / h$.

Here L^2_{comp} is the space of the L^2 functions with compact support.

Decay estimates for the projector Π_F

Corollary

For any $\epsilon > 0$ and $C_0 > 0$, there exists $C_\epsilon > 0$ such that

$$||\Pi_F||_{\mathcal{L}(L^2_{\varphi_1}, L^2_{\varphi_2})} \leq C(\epsilon)\,,$$

for all φ_j as above such that

$$\varphi_1 = \varphi_2 \ \text{ on } \ \cup_\alpha \pi_1(U_\alpha) = \cup_{\alpha_1} \pi_1(U_{\alpha_1,0})\,.$$

Here $\pi_1(x,\xi) = x\,,\ \pi_2(x,\xi) = \xi$.
Note that the condition that $\varphi_1 = \varphi_2$ on each well means that we can not expect any gain in the classical region.
Using $\mathcal{F}_h$, one can get the same property for $\mathcal{F}_h \Pi_F \mathcal{F}_h$ where the weight are the same but in the ξ variable.

Structure of the interaction matrix

Once constructed an orthonormal basis u_α of $F(h)$ such that

$$u_\alpha = T_\alpha u_{0,0}$$

for a suitably carefully chosen $u_{0,0}$ (which is not exactly $e_{0,0}$, or the analytic WKB analytic solution, which in any case not defined everywhere and should be projected on $F(f)$) but indeed very close but with more information on his decay), we would like to write the (infinite) matrix $M_{\alpha,\beta}$ of $P_{/F(h)}$ which is given by

$$M_{\alpha,\beta} = \langle P u_\beta \mid u_\alpha \rangle = \mu(h) I + \langle (P-\mu) u_\beta \mid u_\alpha \rangle \,.$$

We now use the properties of the T_α.
We have

$$\begin{aligned}
\langle P u_\beta \mid u_\alpha \rangle &= \langle P T_\beta u_0 \mid T_\alpha u_0 \rangle \\
&= \langle u_0 \mid T_\beta^* P T_\alpha u_0 \rangle \\
&= \langle P u_0 \mid T_\beta^* T_\alpha u_0 \rangle \\
&= \exp(i\tilde{h}\beta_2(\alpha_1 - \beta_1))\, M_{\alpha-\beta,0} \,.
\end{aligned}$$

We also recall that the operator is selfadjoint:

$$M_{\alpha,\beta} = \overline{M_{\beta,\alpha}}\,.$$

At this stage, we have not used all the symmetries properties inherited of the Harper model but note already that

$$M_{\alpha,\alpha} = M_{0,0}\,.$$

Moreover $M_{0,0}$ equals the one well eigenvalue modulo $\mathcal{O}(h^\infty)$.

Fourier invariance

In the case of $\lambda = 1$ we can obtain the additional property that

$$\mathcal{F}_h u_{0,0} = \omega_0(h) u_{0,0} \,,$$

where $|\omega_0(h)| = 1$.
Using the commutation relation of $\mathcal{F}_h$ and T_α, and the commutation of $\mathcal{F}_h$ with P, one then obtains

$$M_{\alpha,\beta} = \exp(i\tilde{h}(\alpha_1\alpha_2 - \beta_1\beta_2)) M_{\kappa(\alpha),\kappa(\beta)} \,.$$

In particular, we get

$$M_{(1,0),(0,0)} = M_{(0,-1),(0,0)}$$

At the end we get the following formulas

Interaction matrix–first part

Let $M := \hat{\mu} I + W$, $W = W_{\alpha,\beta}$ $(\alpha \neq \beta)$, $W_{\alpha,\alpha} = 0$ the matrix of $P_{/F(h)}$ in the previously constructed basis u_α, then

$$W_{\alpha,0} = \begin{cases} \mathcal{O}(1)\exp(-(S+\epsilon_0)/h) & \text{if } ||\alpha||_\infty \geq 2 \\ \mathcal{O}(1)\exp(-S/h) & \text{if } ||\alpha||_\infty = 1, \alpha \neq (0,0) \end{cases}$$

Here $\epsilon_0 > 0$, $\hat{\mu}(h) = \mu(h) + \mathcal{O}(h^\infty)$.

Note that there is no need to localize $\mu(h)$ or $\hat{\mu}(h)$ more precisely (we just need to preserve the gap ! So this change by $\mathcal{O}(h^\infty)$ is more than enough).
We also recall that once we know $W_{\alpha,0}$ we know $W_{\alpha,\beta}$.

The treatment of $\alpha|_\infty \geq 2$ is done through the good knowledge of the decay (Agmon-Harper estimates) of $u_{0,0}$ and consequently of the u_α. We note that we have either $|\alpha_1| \geq 2$ or $|\alpha_2| \geq 2$. Using the Fourier inverse, we can without loss of generality assume that say $|\alpha_1| \geq 2$ and play with the Agmon estimates in the x-variable. We have to kind of estimates:

- A non optimal decay estimate with control with respect to $|\alpha_1|$, giving the existence of $\kappa > 0$ such that

$$|W_{\alpha,0}| \leq C \exp -\kappa \frac{|\alpha_1|}{h} \,.$$

- Then a more precise analysis for $|\alpha_1|$ smaller leading to the statement above.

Hence it remains to look at the estimates with $|\alpha|_\infty = 1$.

For this case, we prove

Interaction matrix-continued

If $||\alpha||_\infty = 1$

$$W_{\alpha,(0,0)} = -\langle u_{0,0} \,|\, [P,\chi] u_\alpha \rangle + \mathcal{O}(1) \exp(-(S+\epsilon)/h)\,.$$

(for some $\epsilon > 0$). Here $\epsilon > 0$, $\chi = 1_{(-\infty,\pi]}$ and S is the Harper-Agmon distance between two nearest wells with distinct space projection.

Note that if $||\alpha||_\infty = 1$, one can using the various invariances (in particular the Fourier invariance) reduce the computation to the computation of $W_{(1,0),(0,0)}$ and $W_{(1,1),(0,0)}$.

To show these new claims, we observe that

$$\begin{aligned} W_{\alpha,(0,0)} &= \langle P u_{0,0}, u_\alpha \rangle \\ &= \langle (P-\mu) u_{0,0}, u_\alpha \rangle \\ &= -\langle u_{0,0} \,|\, [P,\chi] u_\alpha \rangle \\ &\qquad + \langle (P-\mu) u_{0,0} \,|\, \chi u_\alpha) \rangle + \langle (1-\chi) u_{0,0}, (P-\mu) u_\alpha \rangle \,. \end{aligned}$$

We have then to show that for our choice of $u_{0,0}$ the two last terms are indeed $\mathcal{O}(1) \exp(-(S+\epsilon)/h)$, which is a non so easy proof in general (easier if we are far from the critical points) but is inspired by what was done for Schrödinger.

Main term

We have to show that $\langle u_{0,0} \,|\, [P, \chi] u_\alpha \rangle$ for $\alpha = (\pm 1, 0)$ has the right order and also that the case $\alpha = (\pm 1, \pm 1)$ is relatively small. Together with the previous results, this will imply that all the other terms for $\alpha \notin \{(\pm 1, 0), (0, \pm 1)\}$ are relatively small. At least for non critical energy, an explicit computation of the principal term (including prefactor) can be done.
This is what we sketch now in the case $\alpha = (1, 0)$.

We observe that

$$[\cos hD, \chi_\pi] = \frac{1}{2}\left([1_{[\pi,\pi+h]}\tau_h - 1_{[\pi,\pi-h]}\tau_{-h}\right) .$$

Hence we get

$$I_0(\pi, h) = -\int_\pi^{\pi+h} \Phi_0(x, h)dx ,$$

with

$$\Phi_0(x, h) = \frac{1}{2}\left(u_{0,0}(x)\overline{u_{1,0}(x-h)} - u_{0,0}(x-h)\overline{u_{1,0}(x)}\right) .$$

One can verify (a kind of Wronskian argument) that $\Phi_0(x, h)$ is essentially constant (modulo an error of size $\exp -\frac{S}{h} \times$ an exponentially small term), hence we get

$$I_0(\pi, h) = -h\Phi_0(\Pi, h) + \mathcal{O}(\exp -\frac{S+\epsilon}{h})) , \quad \text{for some } \epsilon > 0 .$$

In the case where we are far of all the critical points we can get the main term by using the analytic WKB approximation

$$I_0(\pi, h) \sim -h \exp -\frac{S}{h} b_{0,E}(\pi)^2 \sinh \varphi'(E) \,.$$

$b_{0,E}(\pi)$ can be explicitly computed by solving a transport equation. We findly get a nice explicit formula with a "classical" flavor

$$I_0(\pi, h) \sim -\frac{1}{T(E)}\, h \exp -\frac{S}{h} \,,$$

where $T(E)$ is the period of the motion on a connected component of $p(x, \xi) = E$.
Such an explicit computation is not in all the cases possible but we get in full generality the following weaker result.
For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$C_\epsilon^{-1} \exp -\frac{S+\epsilon}{h} \leq |W_{\pm 1,0}| \leq C_\epsilon exp - \frac{S-\epsilon}{h} \,.$$

Coming back to $\tilde{h}$ -pseudo-differential operators

We consider a bounded linear operator $C_\hbar$ acting on $\ell^2(\mathbb{Z}^2)$ given by an infinite matrix $\big(C(\alpha,\beta)\big)$, $\alpha,\beta\in\mathbb{Z}^2$, satisfying

$$C(\alpha+k,\beta+k,\hbar)=e^{-i\hbar k_2(\alpha_1-\beta_1)}C(\alpha,\beta,\hbar),\quad \alpha,\beta,k\in\mathbb{Z}^2, \quad (12)$$

for some $\hbar>0$.

Note that we meet these relations with $\hbar=\tilde{h}$.

Proposition A

Let $C_\hbar$ be a bounded self-adjoint operator in $\ell^2(\mathbb{Z}^2)$ with the property (12) and satisfying $|C(\alpha,\beta)| \leq ae^{-b|\alpha-\beta|}$ for some $a, b > 0$ and all $\alpha, \beta \in \mathbb{Z}^2$. Then the spectrum of $C_\hbar$ coincides with the spectrum of the Weyl $\hbar$-quantization of the symbol p given by

$$p(x,\xi) = \sum_{m,n\in\mathbb{Z}} c(m,n)e^{-imn\hbar/2}e^{i(mx+n\xi)}, \tag{13}$$

where $c(m,n) = C\big((0,0),(m,n)\big)$, $m, n \in \mathbb{Z}$.

Renormalization

We have reduced the analysis of the spectrum in Ω_h to the analysis of an infinite matrix satisfying a lot of properties. The first step of the renormalization will be achieved if we recognize that the spectrum of this matrix W (divided by $|W_{0,0}|$) is indeed the spectrum of an exponentially small perturbation of the initial Harper model but with a new semi-classical parameter $\tilde{h}$.

We actuall need more:

Defining suitably a neighborhood of the symbol $p(x, \xi) = \cos x + \cos \xi$ and $h_0 > 0$, we have to show that, if we start from a symbol in this neighborhood, with the same symmetries, then we can arrive for $h \in (0, h_0)$ to an operator relative to one eigenvalue $\mu(h)$ which, after division by an exponetilly small constant is in the SAME neighborhood. Assuming that $\tilde{h} \in (0, h_0)$, we can then iterate the procedure.

This leads to more technical efforts. For example, we have used the interpretation of $\cos hD$ as $\tau_h + \tau_h^*$ and this is no more possible.

The statement in Harper 1

This leads to the following theorem:

Theorem

For $\epsilon_0 > 0$, there exists $C_0 > 0$ such that if $h/(2\pi) \in (0,1) \setminus \mathbb{Q}$ and

$$h/(2\pi) = 1/(q_1 + 1/(q_2 + 1/q_3 + \cdots))))$$

with $q_j \in \mathbb{Z}$ and $|q_j| \geq C_0$, we have:

- The smallest closed interval J containing the spectrum $\sigma(H)$ has the form $[-2 + \mathcal{O}(1/|q_1|), 2 + \mathcal{O}(1/|q_1|)]$,
- $$\sigma(H) \subset \cup_{N_- \leq j \leq N_+} J_j$$
  where the J_j are closed intervals of positive length with $\partial J_j \subset \sigma(H)$,
- J_{j+1} is on the right of J_j at a distance of order $1/|q_1|$,

Theorem continued

- J_0 has length $2\epsilon_0 + \mathcal{O}(1/|q_1|)$ and contains 0 at a distance $\mathcal{O}(1/|q_1|)$ of its center
- The other bands have width $e^{-C(j)|q_1|}$ with $C(j)$ of order 1
- For $j \neq 0$, if κ_j denotes the affine function sending J_j onto $[-2, +2]$, then

$$\kappa_j(J_j \cap \sigma(P)) \subset \cup_k J_{j,k} \,,$$

where the $J_{j,k}$ have the same properties as the J_j with q_1 replaced by q_2 and so on.

Remarks

- This theorem is used [2] by Bourgain in order to give cases for which the integrated density of states is not Hölder.
- For the bands, which are far from the energy 2, we can give an asymptotic for $C(j)$.

[2] Thanks to Q. Zhou for this remark.

For having the complete structure of the spectrum, it remains at each step to treat the spectrum which is close to the critical value 0. In Harper I, we have avoided at each step a small zone.
The treatment of this critical zone was only obtained in [HS3] about one year later and is much more difficult.

But first we speak of Harper II.

About Harper II

Harper II is devoted to the semi-classical analysis near a rational. Only the two first steps are different.

After these two steps, we only are facing the same problems devoted to the perturbation of the Harper's model. Below, we denote for some irrational α by $[a_1, a_2, a_3, \cdots]$ its expansion in continuous fraction.

As continuation of Harper I, the following theorem is proved in Harper II. It is based on the semi-classical analysis of $M^w_{p,q}(x, hD_x)$.

Harper II- Step 1

Theorem Harper II

Let $\hat{m} \in \mathbb{N}$ ($\hat{m} \geq 2$) and $M \geq 2$. There exists $\epsilon_1 > 0$ and, for $\epsilon_0 \in (0, \epsilon_1)$, a constant $C = C(\hat{m}, M, \epsilon_0) > 0$ such that if $\alpha = [a_1, a_2, \ldots,]$ is irrational and satisfies for some $m \leq \hat{m}$

$$\begin{array}{ll} 1 \leq |a_j| \leq M & \text{for } 0 < j \leq m \\ |a_j| \geq C & \text{for } j \geq m+1\,, \end{array} \tag{14}$$

then the spectrum Σ_α of the Harper model is contained in the union of q_m intervals $I_\ell(h)$ ($\ell = 1, \cdots, q_m$) in the form $[\gamma_\ell(h), \delta_\ell(h)]$ with

$$\begin{array}{l} \gamma_\ell(h)\,,\ \delta_\ell(h) \in \Sigma_\alpha\,, \\ \gamma_\ell < \delta_\ell \leq \gamma_{\ell+1} < \delta_{\ell+1}\,, \\ \gamma_\ell(h) \geq \gamma_\ell - C|h| \ \text{ and } \delta_\ell(h) < \delta_\ell + C|h|\,, \\ \gamma_\ell(h) \geq \gamma_\ell + \frac{1}{C}\sqrt{h} \ \text{ if } \delta_{\ell-1} = \gamma_\ell\,, \end{array} \tag{15}$$

Theorem continued

where above

$$\alpha^{(m)} = [a_1, \ldots, a_m] = \frac{p_m}{q_m}\,, \tag{16}$$

$$h = 2\pi(\alpha - \alpha^{(m)})\,, \tag{17}$$

$$\cup_\ell\, [\gamma_\ell, \delta_\ell] = \Sigma_{\alpha^{(m)}}\,, \tag{18}$$

$$d(I_\ell(h), I_{\ell+1}(h)) \geq \frac{1}{C} \text{ if } \delta_\ell \neq \gamma_{\ell+1} \text{ and } \geq \frac{1}{C}\sqrt{|h|} \text{ if } \delta_\ell = \gamma_{\ell+1}\,. \tag{19}$$

Theorem (continued)

For each interval $I_\ell(h)$, $\Sigma_\alpha \cap I_\ell(h)$ can be described as living in a union of $N_{\ell,j}$ closed intervals $J_j^{(\ell)}$ (indexed by $j \in (-m_{\ell,j}, n_{\ell,j})$) of length $\neq 0$ with $\partial J_j^{(\ell)} \subset \Sigma_\alpha$, $J_{j+1}^{(\ell)}$ on the right of $J_j^{(\ell)}$ and

$$m_{\ell,j} \approx |a_{m+1}| \text{ and } n_{\ell,j} \approx |a_{m+1}| , \tag{20}$$

$$\frac{1}{|a_{m+1}|} \lesssim d(J_j^{(\ell)}, J_{j+1}^{(\ell)}) \lesssim \frac{1}{\sqrt{|a_{m+1}|}} , \tag{21}$$

$$J_0^{(\ell)} \text{ has length } 2\epsilon_0 + \mathcal{O}(\frac{1}{|a_{m+1}|}) . \tag{22}$$

Theorem (end)

The other bands have size

$$\exp\left(-C(j)|a_{m+1}|\right) \text{ with } C(j) \approx 1 . \qquad (23)$$

For $j \neq 0$, if $\kappa_j^{(\ell)}$ is the affine function sending $J_j^{(\ell)}$ in $[-2,+2]$, then

$$\kappa_j^{(\ell)}(J_j^{(\ell)}) \cap \Sigma_\alpha \subset \cup_k J_{j,k}^{(\ell)} ,$$

where the $J_{j,k}^{(\ell)}$ have analogous properties to the $J_j^{(\ell)}$ with a_{m+1} replaced by a_{m+2} and (21) can be improved in the form

$$d(J_{j,k}^{(\ell)}, J_{j,k+1}^{(\ell)}) \approx \frac{1}{|a_{m+2}|} . \qquad (24)$$

One can then iterate indefinitely using for the second step a generalization of Harper II and then starting from the third step Harper I (or Harper III).

Here in the statements $a \lesssim b$ means that $a/b \leq C$ where C depends only on C_0 and ϵ_0. The same is true when we use the notation $\mathcal{O}$ or $\approx$.

- ϵ_0 corresponds with the exclusion in each interval and at each step of the renormalization of a small interval of size $\approx 2\epsilon_0$ for which another analysis has to be done and which was the object of Harper III (see also Helffer-Kerdelhué) This corresponds to the energy 0 for the map $(x, \xi) \mapsto 2(\cos x + \cos \xi)$.
- This also gives an analysis for $\alpha = [a_1, \dots, a_m, a_{m+1}]$ when a_{m+1} is large. We can stop the analysis after two steps.
- The possibility of having $\delta_\ell = \gamma_{\ell+1}$ is due to the occurence of touching bands. Van Mouche has proven that it occurs only when q_m is even and for $\ell = \frac{q_m}{2}$. These two touching bands lead to the lower bound (19) and the weaker estimate in (21).

A typical example is for $q = 2$, where we get the matrix

$$M_{1,2}(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \cos\theta_2 & -\cos\theta_1 \end{pmatrix} \tag{25}$$

The eigenvalues are

$$\lambda_\pm(\theta_1, \theta_2) = \pm\sqrt{\cos^2\theta_1 + \cos^2\theta_2}$$

A semi-classical analysis of $M_{1,2}(hD_x, x)$ is possible including at the touching point. The harmonic approximation is replaced by a Dirac approximation

$$\begin{pmatrix} hD_x & x \\ x & -hD_x \end{pmatrix}$$

Main theorem on the nested structure

Here is now the result of the infinitely many steps procedure (this formulation is given in [HLQZ]):

Theorem

Fix $\widehat{m} \in \mathbb{N}$, and $M \geq 2$. Then there exist $\epsilon_1 > 0$ and, for $0 < \epsilon_0 \leq \epsilon_1$, some constants $C_1 > 0, b_2 > b_1 > 0\,, c_1 > 0, d_2 > d_1 > 0$ such that if $\alpha = [a_1, a_2, a_3, \cdots]$ and for some $0 \leq m \leq \widehat{m}$

$$\left\{ \begin{array}{ll} 1 \leq a_\ell \leq M\,, & \ell \leq m \\ a_\ell \geq C_1\,, & \ell \geq m+1 \end{array} \right. , \tag{26}$$

then there exists a sequence $\{(m_\theta, n_\theta) : \theta \in \Theta\}$ with

$$\begin{array}{ll} & b_1\, a_{k+m} \leq m_\theta \leq b_2\, a_{k+m} \\ \text{and} & b_1\, a_{k+m} \leq n_\theta \leq b_2\, a_{k+m}\,,\ \forall k \geq 1,\ \forall \theta \in \Theta_{k-1}\,, \end{array} \tag{27}$$

Theorem (continued)

and a family of bands

$$\{J_\theta : \theta \in \Omega \cup \Theta\}$$

such that:

(i) For each $k \geq 0$, $\{J_\theta : \theta \in \Omega_k \cup \Theta_k\}$ is a covering of Σ_α:

$$\Sigma_\alpha \subset \bigcup_{\theta \in \Omega_k \cup \Theta_k} J_\theta \,.$$

(ii) For each $k \geq 1$ and $\theta \in \Theta_{k-1}$,

$$\partial J_\theta \subset \Sigma_\alpha .$$

Theorem (end)

For each $i \in \mathcal{A}_\theta \cup \{0\}$,

$$J_{\theta \cdot i} \subset J_\theta ,$$

$J_{\theta\cdot(i+1)}$ is on the right of $J_{\theta\cdot i}$. Moreover,

$$\frac{c_1}{a_{k+m}} \leq \frac{d(J_{\theta\cdot(i+1)}, J_{\theta\cdot i})}{|J_\theta|} . \tag{28}$$

(iii) For each $k \geq 1$ and $\theta \in \Theta_{k-1}$,

$$\frac{|J_{\theta\cdot 0}|}{|J_\theta|} \leq \epsilon_0 \, ; \quad e^{-d_2 a_{k+m}} \leq \frac{|J_{\theta\cdot i}|}{|J_\theta|} \leq e^{-d_1 a_{k+m}}, \ (i \in \mathcal{A}_\theta) \, .$$

This result permits to give an interesting result on the Hausdorff measure of the spectrum (see Helffer-Liu-Qi-Zhou [HLQZ]).

Harper III

In [HSHarper3], in order to treat the Harper operator and perturbations of it occuring in a renormalization procedure, the following notion was introduced.

Definition

A symbol $L(x, \xi; \mu, h)$ will be called of strong type I if the following conditions are satisfied for all $h \in (0, h_0)$ with some $h_0 > 0$:

(a) L depends analytically on $\mu \in [-4, 4]$.

(b) There exists $\varepsilon > 0$ such that

(b1) $L(x, \xi; \mu, h)$ is holomorphic in

$$D_\varepsilon = \left\{ (\mu, x, \xi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} : |\mu| \leq 4, |\Im x| < \frac{1}{\varepsilon}, |\Im \xi| < \frac{1}{\varepsilon}, \right\},$$

(b2) for $(\mu, x, \xi) \in D_\varepsilon$, there holds

$$\left| L(x, \xi; \mu, h) - (\cos x + \cos \xi - \mu) \right| \leq \varepsilon.$$

Continuation of the definition

(c) The following symmetry conditions hold:

$$L(x,\xi;\mu,h) = L(\xi,x;\mu,h) = L(x,-\xi;\mu,h)$$
$$L(x,\xi;\mu,h) = L(x+2\pi,\xi;\mu,h) = L(x,\xi+2\pi;\mu,h).$$

By $\varepsilon(L)$ we will denote the minimal value of ε for which the above conditions hold.

In Harper I, this has the simpler form which permits to define the "neighborhhod of the symbol $\cos x + \cos \xi$ " stable in the renormalization procedure.

Definition

A symbol $L(x, \xi; h)$ will be called of strong type I if the following conditions are satisfied for all $h \in (0, h_0)$ with some $h_0 > 0$: There exists $\varepsilon > 0$ such that

(b1) $L(x, \xi; h)$ is holomorphic in

$$D_\varepsilon = \Big\{(x, \xi) \in \mathbb{C} \times \mathbb{C} : |\Im x| < \frac{1}{\varepsilon}, |\Im \xi| < \frac{1}{\varepsilon}, \Big\},$$

(b2) for $(x, \xi) \in D_\varepsilon$, there holds

$$\big|L(x, \xi; h) - (\cos x + \cos \xi)\big| \leq \varepsilon.$$

Continuation of the definition

(c) The following symmetry conditions hold:

$$L(x,\xi;h) = L(\xi,x;h) = L(x,-\xi;h)$$
$$L(x,\xi;h) = L(x+2\pi,\xi;h) = L(x,\xi+2\pi;h).$$

If one denotes by $\varepsilon(L)$ the minimal value of ε for which the above conditions hold, the neighborhoods in Harper 1 were parametrized by $\eta > 0$ and defined by $\{L\,,\,\epsilon(L) \leq \eta\}$.

The final result reads

Theorem HS

Let $L(\mu, h)$ be a strong type I symbol. There exist ϵ_0 , C s. t. if $\varepsilon(L) \leq \epsilon_0$ and if

$$\frac{h}{2\pi} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \dots}}}$$

with $n_j \geq C$, then the μ-spectrum of the associated operators $\mathrm{Op}_h^w(L(\mu))$ is a zero measure Cantor set.

In particular, this applies to the spectrum of the Harper's model. But the theorem says also that this is stable by perturbations respecting all the symmetries.

Critical points

Having in mind what was done in Harper I, the analysis in the interval J_0 is more delicate. For $E = 0$, the wells are no more compact and the previous construction does not work at all. The renormalization is much more involved. We need in particular a microlocal analysis of the model $h^2 D_x^2 - x^2$ and the renormalized operator is no more an Harper's model but a 2×2 system of h^{new}-pseudodifferential operator whose principal symbol is

$$Q(x,\xi) = \begin{pmatrix} b + \bar{a}e^{-i\xi} & \bar{b} + ae^{ix} \\ b + \bar{a}e^{-ix} & b + \bar{a}e^{i\xi} \end{pmatrix}$$

Fortunately, one can show that there are at the end four models permitting to complete the analysis after the first normalization.

On the way of proving the Cantor structure, we can try to understand other questions where the same analysis is relevant. I will describe one case where the first step of the procedure is enough.

Around some Thouless formula.

Y.Last get in the nineties that, for $0 \leq \lambda \leq 1$, the Lebesgue measure of the spectrum of $H_{\alpha,\lambda}$ is for a.e α equal to $2|1-|\lambda||$. The case when $\lambda = 1$ appears as a very important case and in this case Y.Last gets that the spectrum is a zero measure Cantor set for a.e α. More precisely the theorem is the following:

Theorem

If α is an irrational, s.t. there is a sequence of rationals p_n/q_n obeying:

$$\lim_{n\to\infty} q_n^2 |\alpha - \frac{p_n}{q_n}| = 0 \, , \tag{29}$$

then for every λ, $\theta \in \mathbb{R}$:

$$|\sigma(\alpha,\lambda,\theta)| = 2|1-|\lambda|| \, , \tag{30}$$

where $| \, . \, |$ denotes Lebesgue measure.

We now concentrate our study on the case when $\lambda = 1$ where the measure of the spectrum is proved to be 0. The proof of the theorem in this case is based on a careful study of the rational case and in this case the basic lemma is:

Lemma

If p and q are mutually prime (we then write $p \wedge q = 1$)

$$\frac{(\sqrt{5}+1)}{q} < |\Sigma(p/q, 1)| < \frac{4e}{q}, \tag{31}$$

with $e = \exp 1$.

A similar but weaker estimate of the lower bound was already obtained in Last-Wilkinson (1992).

This lemma is strongly related to a conjecture due to Thouless which says:

Thouless Conjecture

$$\lim_{\begin{cases} q \to \infty \\ p \wedge q = 1 \end{cases}} q|\Sigma(p/q, 1)| = 16 C_{Cat}/\pi \qquad (32)$$

where C_{Cat} is the so–called Catalan's constant

$$C_{cat} = \sum_{n \in \mathbb{N}} (-1)^n (2n+1)^{-2}.$$

which is approximately equal to:

$$C_{Cat} \approx 0.9159\ldots \quad .$$

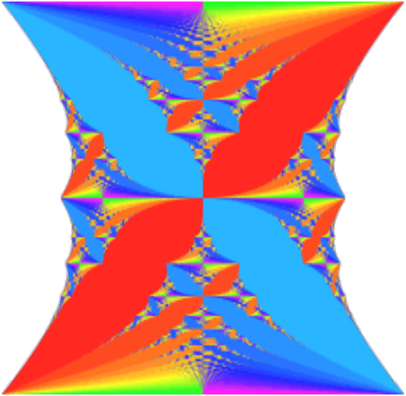
This conjecture has been studied numerically and theoretically in [Th1983], [Th1990b], [TaTh1991a], [TaTh1991b]. We have not followed if there is a more recent litterature.

Thouless (sometimes with collaborators) gives in particular semi-classical arguments justifying the conjecture in the case $p = 1$ and in the case $p = 2$; q odd. The proof is based on an analysis of the Green function but it is not completely clear to us if the proof is totally rigorous in the analysis of the remainders. Although complex WKB techniques are used in this approach, they are quite different of the approach we present here
(see however Buslaev-Fedotov).
The point of view of Y.Last and M.Wilkinson (1992) is more in the spirit of earlier works by M.Wilkinson (1984-1989) and uses semiclassical analysis in a microlocal spirit.

We shall see later that the whole spectrum is concentrated as $q \to \infty$ near 0. The reason is that outside a fixed interval $]-\epsilon, +\epsilon[$ the spectrum is a union of bands which are exponentially small. Consequently, for any $\epsilon_0 > 0$, the contribution in the total bandwidth which is outside $]-\epsilon_0, +\epsilon_0[$ is exponentially small. Moreover, the proof gives a renormalization procedure. For $j \neq 0$, the spectrum in each interval is given, after an affine transformation, sending J_j on essentially $[-2, 2]$, by the spectrum of a suitable perturbation of the Harper's operator to which the preceding theorem can be again applied if $|q_2| \geq C_0$. This perturbed Harper's operator is now an $\tilde{h}$-pseudodifferential operator with

$$\frac{\tilde{h}}{2\pi} = \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{q_4 + \dots}}}$$

Hofstadter Butterfly realized by Avron et al



In the rational case, the procedure stops after a finite number of steps. Note that if $\frac{h}{2\pi} = \frac{1}{q_1}$, we get $\tilde{h} = 0$, and the convention is that a $(\tilde{h} = 0)$-pseudodifferential operator of symbol p is then simply the operator of multiplication by p on $L^2_{x,\xi}(\mathbb{R}^2)$, whose spectrum is simply the set
$\{\lambda \in \mathbb{R} \text{ s.t. } \exists\, (x,\xi) \in \mathbb{R}^2$ with$\lambda = p(x,\xi)\}$.
In particular this says that the contribution in the total width as $q \to +\infty$ is exponentially small outside $]-\epsilon_0, \epsilon_0[$.

Hence we can not avoid the study of the spectrum of the Harper's operator near 0 which is much more difficult because 0 is a saddle point of the symbol of the operator:

$$(x,\xi) \to \cos x + \cos \xi \ .$$

It is consequently natural to think that one can also give a rigorous approach for the a priori easier problem consisting in measuring the total bandwidth. This problem is easier, in the sense that it appears as a one step problem and we shall not need the infinite sequence of approximate renormalizations used in Harper1-Harper3 in order to prove the Cantor structure.

The main goal will be consequently to see what gives this strategy and we shall prove that it works at least in the two cases where reasonable mathematical arguments were already given. We (i.e Helffer-Kerdelhué) consequently rigorously prove the:

Theorem 1 (Helffer-Kerdelhué CMP 1995)

$$\lim_{q\to\infty} q|\Sigma(1/q,1)| = (16/\pi)C_{Cat} \tag{33}$$

Theorem 2 (Helffer-Kerdelhué)

$$\lim_{q \to \infty} (2q+1)|\Sigma(\frac{2}{(2q+1)}, 1)| = (16/\pi) C_{Cat} \quad (34)$$

The second theorem will use also some techniques related to the semi-classical study near a rational [HeSj1990].

We now state what we know from the general theory in order to analyze our particular case and refer to [HeSj1989]. We only collect all the statements permitting to start a rigorous proof for the asymptotic behavior of the total bandwidth. This is also what we need to localize in the irrational case the intervals appearing in the first step near the critical value.

The main "extracted from Harper 3" result is the following (this gives a rigorous version of heuristic arguments due to Azbel Az1964):

Theorem (Helffer-Sjöstrand–Harper 3)

There exists $\epsilon_0 > 0$, $\epsilon_1 > 0$ and h_0 such that, in the interval $[-\epsilon_0, \epsilon_0]$ and for $0 < h < h_0$, μ is in the spectrum of the Harper's operator if and only if 0 is in the spectrum of a vector valued $\tilde{h}$-pseudodifferential operator $Q(x, \tilde{h}D_x, h, \mu')$ on $L^2(\mathbb{R};\mathbb{C}^2)$. Its symbol is a 2×2 matrix depending on a parameter μ' and given by

$$Q(x,\xi) = Q_0(x,\xi) + \mathcal{O}\left(\exp -\epsilon_1/h\right) \tag{35}$$

with

$$Q_0(x,\xi) = \begin{pmatrix} b + \bar{a}\exp -i\xi & \bar{b} + a\exp ix \\ \bar{b} + a\exp -ix & b + \bar{a}\exp i\xi \end{pmatrix} . \tag{36}$$

Theorem continued

The parameter $\tilde{h}$ is related to h by the relation

$$\frac{2\pi}{\tilde{h}} = \frac{h}{2\pi} \quad \mathrm{mod}\ \mathbb{Z}\ . \tag{37}$$

The parameter μ' is related to the spectral parameter μ by $\mu' = f(\mu, h)$ where f is the realization of a formal real valued symbol i.e. admits in $[-2\epsilon_0, 2\epsilon_0]$ the following expansion

$$f(\mu, h) \equiv f_0(\mu) + hf_1(\mu) + ... \tag{38}$$

$$f_0(0) = 0\ ;\ f_0'(0) = 1 \tag{39}$$

Theorem (end)

The parameters a and b are given by

$$b = (2\pi)^{-1/2}\Gamma(\frac{1}{2} - i\frac{\mu'}{h}) \exp\left[i\frac{\mu'}{h}\ln(\frac{1}{h}) + \pi\frac{\mu'}{2h} + i\frac{g(\mu', h)}{h}\right] \quad (40)$$

$$a = (2\pi)^{-1/2}\Gamma(\frac{1}{2} - i\frac{\mu'}{h}) \exp\left[i\frac{\mu'}{h}\ln(\frac{1}{h}) - \pi\frac{\mu'}{2h} + i\frac{g(\mu', h)}{h} - i\frac{\pi}{2}\right] \quad (41)$$

where Γ is the standard gamma function, g is a real classical analytic symbol of order ≤ 0.

Remark: Selfadjointization

The operator Q above is unfortunately not selfadjoint and it is quite useful in order to use perturbation theory to come back to a selfadjoint theory. That this is possible is of course not strange if we recall that our initial problem was selfadjoint. The proof given in Harper 3 keeps actually a "memory" of this property by giving an explicit way of selfadjointization.

More precisely, the proof gives also the existence of a family of operators $P_1(x, \tilde{h}D_x, \theta, h, \mu')$ of the same type of Q such that:

$$P_1^\star Q = Q^\star P_1 \tag{42}$$

and [3]

$$P_1(x,\xi,\theta) = P_{1,0}(x,\xi,\theta) + \mathcal{O}\left(\exp -\epsilon_1/h\right) \tag{43}$$

with

$$P_{1,0}(x,\xi,\theta) = \begin{pmatrix} b' + \bar{a}' \exp -i\xi & \bar{b}' + a' \exp ix \\ \bar{b}' + a' \exp -ix & b + \bar{a}' \exp i\xi \end{pmatrix}\;, \tag{44}$$

with

$$b' = b \exp i\theta \quad ; \quad a' = a \exp i\theta\;. \tag{45}$$

We have some freedom in the choice of θ which will be determined later.

[3] One can forget the green part at the first reading

Similarly, it is possible to define another family of operators P_2 of the same type such that:

$$QP_2^{\star} = P_2 Q^{\star} \tag{46}$$

Let us now establish useful relations. We get for $\mu' \in \mathbb{R}$ the following relations:

$$\begin{aligned} &|a|^2 + |b|^2 = 1 \quad , \\ &\arg(b) - \arg(a) = \pi/2 \, , \\ &a\bar{b} = -\bar{a}b = -i \; |a| \, . \, |b| \end{aligned} \tag{47}$$

$$|b| = \exp(\pi \frac{\mu'}{2h}) \; \left(\exp \pi \frac{\mu'}{h} + \exp -\pi \frac{\mu'}{h} \right)^{-1/2} \quad , \tag{48}$$

$$|a| = \exp(-\pi \frac{\mu'}{2h}) \; \left(\exp \pi \frac{\mu'}{h} + \exp -\pi \frac{\mu'}{h} \right)^{-1/2} \quad , \tag{49}$$

$$|a| \; |b| = 1/(2 \cosh(\pi \frac{\mu'}{h})) \; ; \; |a|^2 - |b|^2 = -\tanh(\pi \frac{\mu'}{h}) \, . \tag{50}$$

The determinant of the matrix Q_0 has the following form, as a function of a, b satisfying the conditions (40), (41):

$$\det Q_0(x,\xi) = 2i\left[\sin(2\arg b) + |a||b|(\cos\xi + \cos x)\right] . \qquad (51)$$

Similarly, the determinant of the matrix $P_{1,0}$ is given by

$$\det P_{1,0}(x,\xi,\theta) = 2i\left[\sin(2\arg b + 2\theta) + |a||b|(\cos\xi + \cos x)\right] . \qquad (52)$$

Let us now discuss from where comes the function f.

It is important to remark that the above theorem is not only proved for the Harper's equation, but also for small perturbations of this operator. This point is crucial in the renormalization analysis. But for our lecture it is better to forget the remark and to work on the unpertubed model.

The role of f will be clear if we recall the following theorem (Theorem b.1. in HeSj1989-Harper3).

Theorem- Normal form near the saddle point

Let $P(x, hD_x, h)$ be a formal classical analytic pseudodifferential operator, of order 0, formally selfadjoint, whose symbol is defined in a neighborhood of $(0,0)$. Let p be the principal symbol, and assume that p has a nondegenerate saddle point at $(0,0)$ with critical value 0. Then there is a real-valued analytic symbol: $\mu \to f(\mu, h)$ defined for μ in a neighborhood of 0, and a formal unitary analytic Fourier integral operator, whose associated canonical transformation (in the classical sense) is defined in a neighborhood of $(0,0)$, and maps this point onto itself, such that

$$U^\star f(P,h) U = \frac{1}{2}(x\, hD_x + hD_x\, x). \qquad (53)$$

A similar result exists in the case of a non-degenerate maxima. We then get the Harmonic oscillator.

One can concretely find the first term f_0 of f. The function $\mu \to f_0(\mu)$ is determined by the condition that the complex period $T(\mu)$ of the hamiltonian flow H_{p_0} on the energy level $p_0(x,\xi)=\mu$ becomes, by replacing p_0 by $f_0(p_0)$ and near the energy corresponding to the saddle point, independent of the energy and equal to $T_0 = 2i\pi$ which is the complex period of the hamiltonian flow attached to the model $(x,\xi) \to x \cdot \xi$. In the case when $p_0(x,\xi) = \cos x + \cos \xi$, we observe that the Taylor expansion at the order 2 is given for example the point $(0,\pi)$ is given by $(-x^2 + (\xi - \pi)^2)/2$ and this explains the conditions written for f_0 in the case of the Harper operator.

More precisely, we have in this case the following formula:

$$f_0(\mu) = \frac{1}{\pi}\mathrm{sign}(\mu) S(\mu) \ ,$$

where $S(\mu)$ is the tunneling parameter :

If $\mu \geq 0$, $S(\mu) = \int_a^{2\pi - a} \cosh^{-1}(\mu - \cos x) dx$,
with $\mu = 1 + \cos a$, $0 \leq a \leq \pi$;

If $\mu \leq 0$, $S(\mu) = S(-\mu) = \int_b^{4\pi - b} \cosh^{-1}(-\mu + \cos x) dx$,
with $\mu = -1 + \cos b$, $\pi \leq b \leq 2\pi$.

$f_0(\mu)$ can also be interpreted as a quantity attached to the Hamiltonian $\hat{p}_0(x, \xi) = \cosh \xi + \cos x$.

The asymptotic behavior of the total bandwidth depends actually only on f through $f_0'(0)$.

The function g contains a global information on the area of domains delimited by the energy surfaces near the critical one but this will not appear in the main term of our asymptotics.

We recall finally some properties of Q which are easily and directly verified for Q_0:

$$\begin{aligned} Q(-\xi,\ x) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q^{\star}(\ x,\ \xi) \qquad , \\ \bar{Q}(\ \xi,\ x) &= Q(\ x,\ \xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad , \\ Q(-x, -\xi) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q(\ x,\ \xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad . \end{aligned} \tag{54}$$

If

$$\alpha = \frac{h}{2\pi} = 1/q \ ,$$

then we get the special case

$$\tilde{h} = 0$$

and one is reduced to the study of a family of 2×2-matrices. In the irrational case, we will assume $\tilde{h}$ small for continuing the analysis but the band appearing in the analysis of $\tilde{h} = 0$ give the information on the localization of the spectrum at this first step.

The case $\alpha = 1/q$

Here $\tilde{h} = 0$ and, according to our conventions, the $\tilde{h}$-pseudodifferential operators have to be considered as multiplication operators defined on $L^2(\mathbb{R}^2_{x,\xi})$. So the theorem says in our particular case that there exists $\epsilon_0 > 0$ and $h_0 > 0$ such that, in the interval $[-\epsilon_0, \epsilon_0]$ and for $0 < h < h_0$, μ is in the spectrum of the Harper's operator if and only if there exists (x, ξ) s.t $Q(x, \xi, h, \mu')$ is not injective.

So we have the following

Proposition

There exists $\epsilon_0 > 0$ and h_0 such that, in the interval $[-\epsilon_0, \epsilon_0]$ and for $0 < h < h_0$, μ is in the spectrum of the Harper's operator if and only if

$$\text{there exists } (x, \xi) \ s.t \ \det Q(x, \xi, h, \mu') = 0 \ \text{ with } \mu' = f(\mu, h) \ . \tag{55}$$

Continued

Moreover we have

$$\det Q = \det Q_0 + \mathcal{O}(\exp -\epsilon_1/h) \tag{56}$$

with

$$\begin{aligned} \det Q_0(x,\xi) &= (i/\cosh(\pi\mu'/h)) \times \\ &\times [2\cosh(\pi\mu'/h)\sin(2\arg b) + \cos\xi + \cos x] \, . \end{aligned} \tag{57}$$

Let us denote by $\tilde{\sigma}_h$ the image by f of σ_h. This is not defined outside $[-\epsilon_0, \epsilon_0]$ but we have seen that $[-\epsilon_0, \epsilon_0]$ is the interesting region where the spectrum is concentrated as $h \to 0$.

We consider first

$$\frac{1}{h}|\tilde{\sigma}_h \cap ([-c, -\epsilon_2 h] \cup [\epsilon_2 h, c])|$$

where

- $c > 0$ is fixed sufficiently small s.t. $f^{-1}([-c, +c]) \subset]-\epsilon_0, \epsilon_0[$, $\pi c \leq \epsilon_1/2$
- $\epsilon_2 > 0$ will be chosen later arbitrarily small.

We only sketch an heuristic proof, forgetting the remainder term in (55) -(56) we get the condition

$$\cosh(\pi\mu'/h)\sin(2\arg b) \in [-1,+1] \tag{58}$$

In the interval $I(\epsilon_2, h)$ the variation of $\arg b$ is much larger in comparison with the variation of $1/\cosh\pi\frac{\mu'}{h}$.
We have indeed

$$|\partial_{\mu'}\arg b| = h^{-1}\ln(1/(h+|\mu'|)) + \mathcal{O}(1/h) \tag{59}$$

and

$$|\partial_{\mu'}\left(1/\cosh\pi\frac{\mu'}{h}\right)| \leq Ch^{-1}(\cosh\pi\frac{\mu'}{h})^{-1} .$$

In particular we observe that the quotient satisfies in the interval I

$$|\partial_{\mu'} \arg b| / |\partial_{\mu'} \left(1/ \cosh \pi \frac{\mu'}{h} \right) | \geq (1/C) \ln(1/h) \,.$$

We compute then approximately the length of the spectrum contained in $[\mu_1" h, \mu_2" h]$ by writing that

$$\frac{|\tilde{\sigma}_h \cap [\mu_1" h, \mu_2" h]|}{h(\mu_2" - \mu_1")} \approx \frac{\arcsin \left(1/ \cosh \pi \frac{\mu_0'}{h} \right)}{\frac{\pi}{2}} \,,$$

for some μ_0' in the interval.
After summation (using the Riemann approximation of the integral)

$$|\tilde{\sigma}_h|_{[\epsilon_2 h, c]} = \frac{4h}{\pi} \left(\int_{\epsilon_2}^{c/h} \arcsin(1/ \cosh \pi s) ds \right) \,.$$

It remains now to come back to the μ variable. Using (39), we get the existence of a constant C such that for any ϵ_3 s.t. $0 < \epsilon_3 < c$ and for any $\epsilon_2 > 0$:

$$\begin{aligned} |\tilde{\sigma}_h \cap [\epsilon_2 h, \epsilon_3]|(1 - C(h + \epsilon_3)| \\ \leq |f^{-1}(\tilde{\sigma}_h \cap [\epsilon_2 h, \epsilon_3])| \\ \leq |\tilde{\sigma}_h \cap [\epsilon_2 h, \epsilon_3]|(1 + C(h + \epsilon_3)| \,, \end{aligned}$$

$$|f^{-1}(\tilde{\sigma}_h \cap [-\epsilon_2 h, \epsilon_2 h])| \leq C \epsilon_2 h \,,$$

and

$$|f^{-1}(\tilde{\sigma}_h \cap [\epsilon_3, c])| \leq C(\epsilon_3) \exp -(C\,(\epsilon_3)/h) \,,$$

with $C(\epsilon_3) > 0$.

Let us also recall that:

$$f^{-1}(0) = \mathcal{O}(h)$$

and

$$\partial f / \partial \mu' = 1 + \mathcal{O}(h + c) \text{ on } [-c,\, c]\,.$$

We then get by combining the different estimates:

$$\frac{|\sigma_h|}{h} \to_{h\to 0} \frac{4}{\pi} \int_{-\infty}^{0} \arcsin(1/\cosh \pi s)ds = \frac{8}{\pi^2} C_{cat}\,. \tag{60}$$

We now take $h = 2\pi\alpha = 2\pi/q$ we finally get:

$$q|\sigma_{2\pi/q}| \to \frac{16}{\pi} C_{cat} \ , \tag{61}$$

which corresponds to the result of Thouless Th1990b.

On the size of the bands

Looking at p. 52-53 in Harper 3, one gets the following information on the length of the gaps and bands.

- If $|\mu'| \leq C_0 h$, then the set of solution μ' is a union of closed intervals of length $\sim h/\log(1/h)$. All these intervals are disjoint except possibly 2 (in the case of touching it is exactly for $\mu' = 0$. In a region where $|\mu'|/h \geq c_0 > 0$, the distance between two consecutive intervals is of the same order of magnitude as the length of these intervals. In the region where $|\mu'|/h$ is small, the gap between two consecutive intervals is of the order of magnitude $(2h/\log(1/h))|\sinh(\pi\mu'/h)|$. The point $\mu' = 0$ always belong to one of the intervals.
- If $|\mu'| \geq C_0 h$ (C_0 large), then the separation is of order $h/\log(1/|\mu'|)$ and the length of an interval of solution is $(2 + o(1))he^{-\pi|\mu'|/h}(\log(1/|\mu'|)^{-1}$ as $h \to 0$ and $|\mu'| \to 0$.

To compare with the formulas written outside $(-\epsilon_0, \epsilon_0)$ in Harper 1.

The end

Of course, this is only step 1. We should then analyze $Q^w(x, \tilde{h}D_x, \mu')$ semi-classically in each of these bands. This leads to two new models. The good point is that the further analysis in the next steps does not introduce new models and we can then show that the infinite renormalization procedure only involves these four models.

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