

Agmon estimates along 40 years (1979-2019).

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Abstract.

Since the proof of his estimates controlling the decay of eigenfunctions for N -body Schrödinger operators by S. Agmon¹ was diffused at the end of the seventies, a lot of applications and variants have been found, particularly in the context of semi-classical analysis.

We will discuss some of them at the light of more recent contributions.

¹S. Agmon 1922-

Agmon estimates' birthday 1979-2019

Agmon's birthday is 1922 !

- ▶ Decay at $+\infty$. Combes-Thomas, Agmon, Carmona-Simon
- ▶ Agmon estimates in semi-classical analysis Simon and Helffer-Sjöstrand.
- ▶ Comparison eigenfunction–quasi-mode after Helffer-Sjöstrand
- ▶ Application to tunneling
- ▶ Resonant wells
- ▶ Agmon estimates and magnetic Schrödinger operators. Helffer-Sjöstrand (Pisa). Martinez–Nakamura.

Agmon estimates continued

- ▶ Agmon estimates in tight-binding (Daumer,..)
- ▶ Agmon estimates and pseudo-differential operators: Klein-Gordon, Dirac, Kac, Harper,...
- ▶ Magnetic bottles.
- ▶ Decay estimates in superconductivity
- ▶ Decay estimates for the Robin problem.
- ▶ Agmon estimates and Landscape function.

Introduction of Carmona-Simon

We start from the introduction of the paper of Carmona-Simon [CaSi] (1981). The results of Agmon are known at this time [Ag1] (1979) but some are unpublished [Ag2]. Hence we learn Agmon's result through the paper [CaSi].

This paper is a contribution to the large literature on the decay at infinity of eigenvectors of Schrödinger operators $-\Delta + V$, associated to discrete spectrum [Many references given](#). For the leading behavior of the ground state, ϕ , our results are definitive in the sense that we will show that:

$$\lim_{|x| \rightarrow +\infty} -\frac{\log \phi(x)}{\rho(x)} = 1,$$

for an explicit function ρ and for a large class of potentials, V , including general N -body systems.

The upper bounds implicit above are not new: for multiparticle systems, they were found in successively more general cases by Mercuriev (1974), Deift et al., and Hoffmann-Ostenhof et al. (atoms with infinitely heavy nucleus)(1977) in the seventies and Agmon [Ag1] (1979) in the general case; for potentials going to infinity at infinity they were found by Lithner [Li] (1964) and rediscovered by Agmon [Ag1]. The Lithner-Agmon upper bounds are only proven to hold in some average sense, but it is easy to get pointwise bounds with minor extra restrictions on V .

Our primary goal here will be to find lower bound complementary to these various upper bounds which show that the upper bounds are "best possible". A major source of motivation for the approach we use is the part of Agmon's work [Ag1] which identifies the function ρ .

Let us initially describe the situation for the case $V \rightarrow +\infty$, $V \geq 1$ and continuous, a case treated by Lithner, with a related intuition. Agmon finds a sufficient condition for:

$$|\phi(x)| \leq c_\epsilon \exp(1 - \epsilon)\rho(x), \forall \epsilon > 0,$$

is the condition

$$|\nabla\rho(x)|^2 \leq V(x).$$

Related conditions were found using the Combes-Thomas method [CT] (1973).

Agmon estimates in the semi-classical context 1983-84

The application of Agmon estimates in semi-classical analysis appears chronologically along 1983 in three papers: one announcement by B. Simon [Sim1] exhibiting for the first time the role of the Agmon estimate in the semi-classical context for the double well symmetric problem, one preprint by Helffer-Sjöstrand [HS1] and a detailed version of his announcement by B. Simon [Sim2] (the two last ones were diffused as preprints in Fall 1983 and appear in 1984).

The techniques of [HS1] are more general but one can find in [Sim1] a probabilistic approach which has an independent interest at least for the two lowest eigenvalues.

Note that V. Maslov [Mas84] was also claiming some priority on the result on the double well problem but we do not see a published complete proof (see also [Pank84]) and the basic role of Agmon estimates is completely overlooked.

Energy inequalities

The main but basic tool is a very simple identity attached to the Dirichlet realization of the Schrödinger operator

$$P_{h,V} : -h^2 \Delta + V ,$$

where V is assumed to be non negative.

Proposition: Energy identity

Let Ω be a bounded open domain in \mathbb{R}^m with C^2 boundary. Let $V \in C^0(\bar{\Omega}; \mathbb{R})$, and ϕ a real valued lipschitzian function on $\bar{\Omega}$. Then, for any $u \in C^2(\bar{\Omega}; \mathbb{R})$ with $u|_{\partial\Omega} = 0$, we have

$$\int_{\Omega} |h \nabla(\exp \frac{\phi}{h} u)|^2 dx + \int_{\Omega} (V - |\nabla \phi|^2) \exp \frac{2\phi}{h} |u|^2 dx = \int_{\Omega} \exp \frac{2\phi}{h} (P_{h,V} u)(x) \cdot u(x) dx . \quad (1)$$

Under some conditions on V , we can also take $\Omega = \mathbb{R}^n$. Note that the identity is universal and this is only later that we have to play with the semi-classical parameter.

Immediate applications

If u_h is an eigenfunction of $P_{h,V}$ with eigenvalue λ_h , we get:

$$\int_{\Omega} |h \nabla(\exp \frac{\phi}{h} u_h)|^2 dx + \int_{\Omega} (V - |\nabla \phi|^2 - \lambda_h) \exp \frac{2\phi}{h} |u_h|^2 dx = 0, \quad (2)$$

which implies the simple estimate

$$\int_{\Omega} (V - |\nabla \phi|^2 - \lambda_h) \exp \frac{2\phi}{h} |u_h|^2 dx \leq 0, \quad (3)$$

or the weaker, assuming that $\lambda_h \leq E$,

$$\int_{\Omega} (V - |\nabla \phi|^2 - E) \exp \frac{2\phi}{h} |u_h|^2 dx \leq 0. \quad (4)$$

This is true for any ϕ . Hence the question is to determine if there is a clever choice for ϕ .

The Agmon distance

The Agmon metric attached to an energy E and a potential V is defined as $(V - E)_+ dx^2$ where dx^2 is the standard metric on \mathbb{R}^n . This metric is degenerate and is identically 0 at points living in the "classical" region: $\{x \mid V(x) \leq E\}$. Associated to the Agmon metric, we define a natural distance

$$(x, y) \mapsto d_{(V-E)_+}(x, y)$$

by taking the infimum :

$$d_{(V-E)_+}(x, y) = \inf_{\gamma \in \mathcal{C}^{1,pw}([0,1]; x, y)} \int_0^1 [(V(\gamma(t)) - E)_+]^{\frac{1}{2}} |\gamma'(t)| dt, \quad (5)$$

where $\mathcal{C}^{1,pw}([0, 1]; x, y)$ is the set of the piecewise (pw) C^1 paths in \mathbb{R}^n connecting x and y . When there is no ambiguity, we shall write more simply $d_{(V-E)_+} = d$.

Similarly to the Euclidean case, we obtain the following properties

- ▶ Triangular inequality

$$|d(x', y) - d(x, y)| \leq d(x', x), \quad \forall x, x', y \in \mathbb{R}^m. \quad (6)$$

- ▶

$$|\nabla_x d(x, y)|^2 \leq (V - E)_+(x), \quad (7)$$

almost everywhere.

We observe that the second inequality is satisfied for any derived distance like

$$d(x, U) = \inf_{y \in U} d(x, y).$$

If $U = \{x \mid V(x) \leq E\}$, $d(x, U)$ measures the distance to the classical region.

All these notions being expressed in terms of metrics, they can be easily extended on manifolds.

Decay of eigenfunctions.

When u_h is a normalized eigenfunction of the Dirichlet realization in Ω satisfying $P_{h,V} u_h = \lambda_h u_h$ then the energy identity gives roughly that $\exp \frac{\phi}{h} u_h$ is well controlled (in L^2) in a region

$$\Omega_1(\epsilon_1, h) = \{x \mid V(x) - |\nabla \phi(x)|^2 - \lambda_h > \epsilon_1 > 0\},$$

by $\exp \left(\sup_{\Omega \setminus \Omega_1} \frac{\phi(x)}{h} \right)$. The choice of a suitable ϕ (possibly depending on h) is related to the Agmon metric $(V - E)_+ dx^2$, when $\lambda_h \rightarrow E$ as $h \rightarrow 0$.

The typical choice is $\phi(x) = (1 - \epsilon)d(x)$ where $d(x)$ is the Agmon distance to the "classical" region $\{x \mid V(x) \leq E\}$. In this case we get that the eigenfunction is localized inside a small neighborhood of the classical region and we can measure the decay of the eigenfunction outside the classical region by

$$\exp(1 - \epsilon) \frac{d(x)}{h} u_h = \mathcal{O}\left(\exp \frac{\epsilon}{h}\right), \quad (8)$$

for any $\epsilon > 0$.

More precisely we get for example the following theorem

Theorem: localization of eigenfunctions

Let us assume that V is C^∞ , semibounded and satisfies

$$\liminf_{|x| \rightarrow \infty} V > \inf V = 0 \quad (9)$$

and

$$V(x) > 0 \text{ for } |x| \neq 0. \quad (10)$$

Let u_h be a (family of L^2 -) normalized eigenfunctions such that

$$P_{h,V} u_h = \lambda_h u_h, \quad (11)$$

with $\lambda_h \rightarrow 0$ as $h \rightarrow 0$. Then for all ϵ and all compact $K \subset \mathbb{R}^m$, there exists a constant $C_{\epsilon,K}$ such that for h small enough

$$\|h \nabla(\exp \frac{d}{h} \cdot u_h)\|_{L^2(K)} + \|\exp \frac{d}{h} \cdot u_h\|_{L^2(K)} \leq C_{\epsilon,K} \exp \frac{\epsilon}{h}. \quad (12)$$

First application

We can compare different Dirichlet problems corresponding to different open sets Ω_1 and Ω_2 containing a unique well U attached to an energy E . If for example $\Omega_1 \subset \Omega_2$, one can prove the existence of a bijection b between the spectrum of $P_{(h,\Omega_1)}$ in an interval $I(h)$ tending (as $h \rightarrow 0$) to E and the corresponding spectrum of $P_{(h,\Omega_2)}$ such that $|b(\lambda) - \lambda| = \mathcal{O}(\exp -S/h)$ (under a weak assumption on the spectrum at $\partial I(h)$).

Here S is chosen such that

$$0 < S < d_{(V-E)_+}(\partial\Omega_1, U).$$

This can actually be improved (using more sophisticated perturbation theory) as $\mathcal{O}(\exp -2S/h)$.

Second application: the symmetric double well problem

As discussed above, the double well problem in dimension ≥ 2 was first discussed in the note [Sim1] immediately followed by the two detailed papers [HS1] and [Sim2].

There is a huge litterature in (1D) including an exercise in Landau-Lifschitz, the book by Fröman-Fröman (1960), the french group in Marseille around J.M. Combes and the detailed mathematical proof for the tunneling by E. Harrell (1978) [Ha]. Agmon estimates are not needed because one can work directly with WKB solutions and the theory of ordinary differential equations.

Once the harmonic approximation is done, it is possible to construct an orthonormal basis of the spectral space attached to some interval $I(h) := [\inf V, \inf V + Ch]$ (C avoiding the eigenvalues of the approximating harmonic oscillators at each minimum), each of the elements of the basis being exponentially localized in one of the wells.

The computation of the matrix of the operator in this basis using WKB approximation leads to the so-called “interaction matrix” (See the books of Dimassi-Sjöstrand [DiSj] or Helffer [He0] for a pedagogical presentation).

We consider the case with two wells, say U_1 and U_2 . We assume that there is a symmetry² g in \mathbb{R}^m , such that $g^2 = Id$, $gU_1 = U_2$, and such that the corresponding action on $L^2(\mathbb{R}^m)$ defined by $gu(x) = u(g^{-1}x)$ commutes with the Laplacian.

In addition

$$gV = V.$$

We now define reference one well problems by introducing :

$$M_1 = \mathbb{R}^m \setminus B(U_2, \eta), \quad M_2 = \mathbb{R}^m \setminus B(U_1, \eta).$$

With this choice, we have $gM_1 = M_2$.

²Typically, in 2D the symmetry with respect to $\{x_2 = 0\}$.

The parameter $\eta > 0$ is free but can always be chosen arbitrarily small.

We denote by ϕ_j the corresponding ground state of the Dirichlet realization of $-h^2\Delta + V$ in M_j and corresponding to the ground state energy $\lambda_{M_1} = \lambda_{M_2}$. According to our result on the decay, these eigenfunctions decay like $\tilde{O}(\exp -\frac{d(x, U_j)}{h})$, where $\tilde{O}(f)$ roughly³ means $\exp \frac{\epsilon}{h} \cdot \mathcal{O}_\epsilon(f)$ for all $\epsilon > 0$ as $h \rightarrow 0$.

We can of course keep the relation

$$g\phi_1 = \phi_2 .$$

³More precisely, for any $\epsilon > 0$, one can choose above $\eta > 0$ such that...

Let us now introduce θ_j , which is equal to 1 on $B(U_j, \frac{3}{2}\eta)$ and with support in $B(U_j, 2\eta)$. We introduce

$$\chi_1 = 1 - \theta_2, \quad \chi_2 = 1 - \theta_1,$$

and we can also keep the symmetry condition :

$$g\chi_1 = \chi_2.$$

Our approximate eigenspace will be generated by

$$\psi_j = \chi_j \phi_j, \quad (j = 1, 2),$$

which satisfies

$$S_h \psi_j = \lambda_M \psi_j + r_j,$$

with

$$r_j = h^2(\Delta \chi_j) \phi_j + 2h^2(\nabla \chi_j) \cdot (\nabla \phi_j).$$

We note that the “smallness” of r_j can be immediately controlled using the decay estimates in $B(U_j, 2\eta) \setminus B(U_j, \frac{3}{2}\eta)$.

In order to construct an orthonormal basis of the eigenspace F corresponding to the two lowest eigenvalues near λ_M , we first project our basis ψ_j which was not far to be orthogonal and introduce:

$$v_j = \Pi_F \psi_j .$$

The resolvent formula shows that $v_j - \psi_j$ can be made very small (at least $\exp -\frac{S}{h}$ with $S < d(U_1, U_2)$ by choosing $\eta > 0$ small enough). More precisely, we have the following comparison.

Lemma

$$(v_j - \psi_j)(x) = \tilde{O}\left(\exp -\frac{\delta_j(x)}{h}\right), \quad (13)$$

in $\mathbb{R}^m \setminus B(U_{\hat{j}}, 4\eta)$, where $\hat{1} = 2$, $\hat{2} = 1$ and

$$\delta_j(x) = d(x, U_{\hat{j}}) + d(U_1, U_2) .$$

This is indeed an improvement of the control in L^2 .

We notice that :

$$\delta_j(x) \geq d(x, U_j) ,$$

What we see here is that the improved estimate does not lead to improvements near U_j , where we have modified ϕ_j into ψ_j by introducing a cut-off function but that the improvement is quite significant when keeping a large distance (in comparison with η) with U_j .

We then orthonormalize by the Gram-Schmidt procedure.

$$e_j = \sum_k (V^{-\frac{1}{2}})_{jk} v_k ,$$

with

$$V_{ij} = \langle v_i | v_j \rangle .$$

We note that

$$V_{ij} - \delta_{ij} = \mathcal{O}(\exp -\frac{S}{h}) .$$

At each step, we control the difference $e_j - \psi_j$, which satisfies also (13).

The matrix we would like to analyze is then simply the two by two matrix

$$M_{ij} = \langle (P_h - \lambda_M)e_i | e_j \rangle .$$

The eigenvalues of this matrix measure the dispersion of the two eigenvalues around λ_M .

We observe that symmetry considerations lead to :

$$M_{12} = M_{21} \text{ and } M_{11} = M_{22} .$$

So the eigenvalues are easy to compute and corresponding eigenvectors are $\frac{1}{\sqrt{2}}(1, 1)$ and $\frac{1}{\sqrt{2}}(-1, +1)$. As soon as we have the main behavior of M_{12} , we can deduce that the eigenvalues are simple and that the splitting between the two eigenvalues is given by $2|M_{12}|$.

It remains to explain how one can compute M_{12} . The analysis of the decay permits to show that

$$M_{12} = \frac{1}{2} (\langle r_2, \psi_1 \rangle + \langle r_1, \psi_2 \rangle) + \mathcal{R}_{12}, \quad (14)$$

with

$$\mathcal{R}_{12} = \mathcal{O}\left(\exp - \frac{2S}{h}\right), \quad (15)$$

for a suitable choice of $\eta > 0$ small enough.

An integration by parts leads (observing that $\nabla\chi_1 \cdot \nabla\chi_2 \equiv 0$ for our choice of η) to the formula

$$M_{12} = h^2 \int \chi_1(\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2) \nabla \chi_2 + \mathcal{R}_{12}. \quad (16)$$

A priori informations on the decay permit to restrict the integration in the right hand side of (16) to the set $\{d(x, U_1) + d(x, U_2) \leq d(U_1, U_2) + a\}$ for some $a > 0$. A computation based on the Stokes Lemma gives then the existence of $\epsilon_0 > 0$ such that:

$$M_{12} = h^2 \int_{\Gamma} [\phi_2 \partial_n \phi_1 - \phi_1 \partial_n \phi_2] d\nu_{\Gamma} + \mathcal{O}\left(\exp - \frac{S_{12} + \epsilon_0}{h}\right). \quad (17)$$

Here $S_{12} = d(U_1, U_2)$ and Γ is an open piece of hypersurface defined in the neighborhood of the minimal geodesic $\text{geod}(U_1, U_2)$ between the two points U_1 and U_2 , that we assume for simplification to be unique and ∂_n denotes the normal derivative to Γ , positively oriented from U_1 to U_2 .

The last step is to observe that in a neighborhood of the intersection γ_{12} of Γ with $\text{geod}(U_1, U_2)$, one can replace the function ϕ_j (or ψ_j) modulo $\mathcal{O}(h^\infty) \exp -\frac{d(x, U_j)}{h}$ by its WKB approximation $h^{-\frac{m}{4}} a_j(x, h) \exp -\frac{d(x, U_j)}{h}$.

This leads finally to

$$\begin{aligned}
 M_{12} = & h^{1-\frac{m}{2}} \exp -\frac{d(U_1, U_2)}{h} \times \\
 & \times \int_{\Gamma} \exp -\frac{(d(x, U_1) + d(x, U_2) - d(U_1, U_2))}{h} \times \\
 & \times (a_1(x, 0)a_2(x, 0)(\partial_n d(x, U_1) - (\partial_n d(x, U_2)) + \mathcal{O}(h)) d\nu_{\Gamma} ,
 \end{aligned}
 \tag{18}$$

where $d\nu_{\Gamma}$ is the induced measure on Γ .

With natural generic additional assumptions saying that the map

$$\Gamma \ni x \mapsto (d(x, U_1) + d(x, U_2) - d(U_1, U_2))$$

vanishes exactly at order 2 at γ_{12} , this finally leads to the formula giving the splitting after use of the Laplace integral method.

To look at very excited states far from the bottom (for example to another critical point for V could be extremely difficult (see A. Martinez in the 90-ties and the recent Astérisque (2018) of J.F. Bony, T. Fujie, T. Ramond and M. Zerzeri.

Non resonant wells

As soon as we leave the symmetric case many effects can be considered. This is analyzed by B. Simon [Sim3-4] under the name: "Flea of the elephant".

This is then analyzed more systematically in the second and third papers of the series [HS2-6] where the decay of an eigenfunction is analyzed when crossing a non resonant well.

Other cases are considered like the case of degenerate wells, miniwells (fifth and sixth of the series).

This question of localization appears also for Witten Laplacians (forth of the series) and then along the years since 1990 Bovier-Gaynard-Klein, Helffer-Nier, D. Le Peutrec, F. Nier, T. Lelièvre, L. Michel, G. Di Gesu, B. Nectoux...

The magnetic case– Helffer-Sjöstrand 1987

The previous analysis can be extended to operators

$$P_{h,A,V} := -\Delta_{h,A} + V,$$

where

$$\Delta_{h,A} := \sum_j (h\partial_{x_j} - iA_j)^2,$$

is the magnetic Laplacian associated with the magnetic potential \vec{A} .

The Agmon energy identity now reads

$$\int_{\Omega} |\nabla_{h,A}(\exp \frac{\phi}{h} u)|^2 dx + \int_{\Omega} (V - |\nabla\phi|^2) \exp \frac{2\phi}{h} |u|^2 dx = \Re \left(\int_{\Omega} \exp \frac{2\phi}{h} (P_{h,A,V} u)(x) \cdot \bar{u}(x) dx \right). \quad (19)$$

This leads to decay estimates (the same as for $A = 0$), which are not necessarily optimal ! Counter-examples are known (Erdős). The simplest example is to consider the case with linear magnetic potential and quadratic potential in (2D), where everything can be computed explicitly.

Hence except rather strong assumptions on the size of A and analyticity condition (see [HS2]), we do not get optimal information on the tunneling, a subject which remains rather open, particularly when $V = 0$.

Some decay can be measured without any electric potential due to the confining through the magnetic field. This will be discussed later.

Infinite number of wells (1990)

Motivated by discussions with J. Bellissard at the end of the eighties, the question is now to understand the effect of the magnetic field in the case when V is periodic with respect to a lattice Γ . It is usually assume that the magnetic field is constant but the analysis is also doable if the magnetic field is periodic on the same lattice.

In Carlsson (1990) [Carl] (see also Helffer-Sjöstrand [HS3]), there is a general approach (in the case of an infinite number of wells) for constructing at the bottom a basis for the eigenspace corresponding to a suitable band consisting of approximate eigenfunctions suitably localized (in the semi-classical limit) in each of the wells.

This strategy was also followed by F. Klopp [Kl1993] in the nineties for treating the case of Random electric potentials.

In this basis of localized functions, we get for the restriction of the Schrödinger to this eigenspace an infinite matrix acting on $\ell^2(\Gamma)$ where Γ is the set of the wells.

In the case when Γ is a lattice, V is periodic and the magnetic field is zero this matrix corresponds to a convolution on $\ell^2(\Gamma)$ and by duality we recover the Floquet theory and an eigenvalue $\lambda(\theta)$ as in [Out87, Sim3-4] which describes the first band which is close to the ground state energy of a "one well" reference Schrödinger operator.

From now on, I only describe (2D) results. Under suitable assumptions of symmetries, we get in the case of the square lattice an asymptotic of the type

$$\lambda(\theta, h) = \lambda_0(h) + \tau(h)(\cos \theta_1 + \cos \theta_2) + r(h),$$

with a good equivalent for the tunneling effect $\tau(h)$ (which behaves like $\exp -\frac{S}{h}$ for some $S > 0$) and a remainder which is exponentially small in comparison with $\tau(h)$.

Application to the tight-binding

The flexibility of Agmon's approach for estimating decay estimates can also be seen in the context of the tight-binding model. Here there is no semiclassical parameter but a large parameter $R > 0$ measuring the distance between the wells. The potential has for example the form

$$V_R(x) = v(x - Re_1) + v(x + Re_1),$$

where v is a potential having some decay at ∞ (the most simple case is when v has compact support but one can also consider Coulomb potential), e_1 is a non zero vector in \mathbb{R}^n .

Here for estimating the tunneling effect, this is the decay of the eigenfunctions at ∞ for the one-well model $-\Delta + v(x)$ which plays the important role. Hence we are coming back to the original Agmon estimates.

We refer to the work of F. Daumer [Da93, Da96] and to quite recent contributions (2018) by C. Fefferman, M. Weinstein in connection with graphene models.

Localization for other operators

If one wants to extend to other operators one naturally looks at the symbol of the operator

$$\exp -\frac{\phi}{h} P_h \exp \frac{\phi}{h}$$

In the case of the Schrödinger operator, this leads to the (semi-classical) symbol

$$(x, \xi) \mapsto (\xi + i\phi')^2 + V(x),$$

whose real part is $\xi^2 - \phi'(x)^2 + V(x)$.

Here ξ_j is associated to $\frac{1}{i}h\partial_{x_j} = hD_{x_j}$ (usually noted p_j in the physics literature).

An interesting case is the case of Klein-Gordon, whose initial symbol is

$$(x, \xi) \mapsto \sqrt{(1 + \xi^2) + V(x)},$$

What is important here is the analyticity in ξ of the symbol. What is done with the Agmon's proof can be considered as a particular case of an argument of microhyperbolicity (see the book of Martinez (2002)).

We note that in the case of Klein-Gordon there is a singularity in the complex for $\xi^2 = -1$ which can be seen when analyzing the tunneling (see Carmona-Masters-Simon [CaMaSi] and Helffer-Parisse [HelPa94]).

The condition on ϕ for Klein-Gordon reads, with $\epsilon > 0$,

$$|\nabla\phi(x)|^2 \leq \inf [(1 - (1 - V)^2), 1 - \epsilon] .$$

A comparison between decay estimates for eigenfunctions of the Dirac operator

$$\sum_j \alpha_j (hD_{x_j}) + V$$

and Klein-Gordon appears in [HelPa94] in connexion with the semi-classical analysis by X.P. Wang (1985) [Wa85].

Another interesting operator where decay estimates are important is the Harper model in the papers by Helffer-Sjöstrand starting in 1995 with [HS3] is the operator on $L^2(\mathbb{R})$

$$L^2(\mathbb{R}) \ni u \mapsto \frac{1}{2}(u(\cdot + h) + u(\cdot - h)) + V(x)u(x),$$

where $V(x)$ can be in particular the function $\cos x$.

Another one is the Kac operator $\exp -\frac{V}{2} \exp + h^2 \Delta \exp \frac{V}{2}$, which appears in statistical mechanics [He1992].

Magnetic bottles in semi-classical analysis.

Here I refer to papers with A. Morame, Y. Kordyukov, N. Raymond and S. Vu-Ngoc from 1996 to 2017. A good reference is the recent book by N. Raymond (2017) [Ray2017]. A different (non semi-classical) result is the very recent contribution of Mayboroda-Poggi [MP2018] (in continuation of earlier works of Fefferman-Phong (1983), Helffer-Nourrigat (1985), Helffer-Mohamed [HM1988] (1988), Nourrigat and students, Z. Shen (1995-99), Auscher-Ben Ali (2007)....).

When no electric potential $V(x)$ is present we can still have localization through the magnetic field.

To simplify, we describe the case of dimension 2. We assume that the magnetic field $B(x) = \text{curl}A$ is positive and that

$$0 \leq \inf B < \liminf_{|x| \rightarrow +\infty} B(x).$$

Considering $P_{hA,0}$ one can show that the spectrum below $h \liminf_{|x| \rightarrow +\infty} B$ is discrete and we would like to localize the groundstate (as $h \rightarrow 0$).

The question is then:

Does it exist a substitute for the Agmon strategy ?

So we should look for an effective electric potential. Here we simply observe that

$$\langle P_{h,A,0}u, u \rangle \geq h \int B(x)|u(x)|^2 dx.$$

This suggest to take $hB(x)$ as an effective electric potential. More precisely, we use the previous inequality in the form

$$\langle P_{hA,0}u, u \rangle \geq (1 - \epsilon)h \int B(x)|u(x)|^2 dx + \epsilon \langle P_{h,A,0}u, u \rangle,$$

and look for an optimal $\epsilon \in (0, 1)$.

A magnetic Agmon distance d_B is associated with $B - \inf B$ and one expects, assuming a unique minimum at 0 , a decay in $\exp -\alpha \frac{d_B(x,0)}{\sqrt{h}}$, for some $\alpha > 0$.

One can also analyze the decay at ∞ of the eigenfunctions under the conditions that for some $r \in \mathbb{N}$

$$\sum_{|\alpha| \leq r} |D_x^\alpha B(x)| \rightarrow +\infty$$

as $|x| \rightarrow +\infty$ and a technical (and important) condition controlling the derivatives of order $r + 1$ (see Brummelhuis [Bru91] and Helffer-Nourrigat [HelNo92]).

Similar ideas can be used for getting decay estimates for the eigenfunctions of

$$-\Delta + x^2 y^2.$$

We can for example observe that

$$\begin{aligned} -\Delta + x^2 y^2 &= \left(-\frac{d^2}{dx^2} + \frac{1}{2}x^2 y^2\right) + \left(-\frac{d^2}{dy^2} + \frac{1}{2}x^2 y^2\right) \\ &\geq -\Delta + \frac{1}{\sqrt{2}}(|y| + |x|), \end{aligned}$$

and use $\frac{1}{\sqrt{2}}(|x| + |y|)$ as new effective potential.

Decay estimates in superconductivity

Here we refer to works with A. Morame, X. Pan, S. Fournais and more recently with A. Kachmar and N. Raymond between 2002 till 2018. We refer to our book with S. Fournais for the state of the art in 2009.

The so-called Surface Superconductivity is strongly related with the Neumann realization of the Schrödinger operator with magnetic field (which in the initial papers is assumed to be constant and then variable). Here we observe a phenomenon of localization at the boundary and the role of the Agmon distance is played by the distance to the boundary.

More accurate localization is given in dimension 2 by the curvature of the boundary.

The decay along the boundary is measured by a tangential Agmon distance associated with the curvature. Again the Agmon's strategy plays an important role.

Agmon estimates and the Robin problem

Let $\Omega \subset \mathbb{R}^2$ be an open domain with $\Gamma = \partial\Omega$. First, we consider the following two assumptions.

- ▶ Ω is smooth with a bounded, non concave boundary.
- ▶ The curvature κ on the boundary Γ attains its maximum κ_{\max} at a finite number N of points on Γ and these maxima are non degenerate.

In the case when $N = 2$ one can carry out a refined analysis valid under the following additional (geometric) assumption:

- i) Ω is symmetric with respect to the y -axis.
- ii) The curvature κ on the boundary Γ attains its maximum at exactly two points a_1 and a_2 which are not on the symmetry axis.
- iii) The second derivative of the curvature (w.r.t. arc-length) at a_1 and a_2 is negative.

A simple example of a domain satisfying all the assumptions is the full ellipse. One want to do the semiclassical analysis of the operator

$$\mathcal{L}_h = -h^2 \Delta, \quad (20)$$

with domain

$$\text{Dom}(\mathcal{L}_h) = \{u \in H^2(\Omega) : \nu \cdot h^{\frac{1}{2}} \nabla u - u = 0 \text{ on } \Gamma\}, \quad (21)$$

where ν is the *outward* pointing normal and $h > 0$ is the semiclassical parameter.

The associated quadratic form is given by

$$\forall u \in H^1(\Omega), \quad \mathcal{Q}_h(u) = \int_{\Omega} |h \nabla u|^2 dx - h^{\frac{3}{2}} \int_{\Gamma} |u|^2 ds(x),$$

where ds is the standard surface measure on the boundary.

Let $(\mu_n(h))$ be the sequence of Rayleigh quotients of \mathcal{L}_h . It is known that the non concavity of the curvature implies that, for all $n \in \mathbb{N}$, $\mu_n(h)$ belongs to the discrete spectrum as soon as h is small enough and that it is precisely the n -th eigenvalue of \mathcal{L}_h counting multiplicities.

As $h \rightarrow 0$, the first eigenfunctions are exponentially localized near the boundary. Moreover, there is an Agmon distance related to the curvature which permits to localize these eigenfunctions near the point of maximal curvature. These results were obtained by Helffer-Kachmar-Raymond [HKR2015] (see also another example Helffer-Pankrashkin [HP] with an isosceles triangle). See also the talk of K. Pankrashkin in this conference.

Agmon estimates associated with the Landscape function

References: Filoche–Mayboroda (2012) [FM], Steinerberger (2015) [St], Arnold–David–Filoche–Jerison–Mayboroda [ADFJM2018]. For a short presentation, we follow the introduction of [St]. See also the talk of S. Mayboroda in this conference.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and

$$(-\Delta + V)\phi = \lambda\phi \text{ in } \Omega,$$

with Dirichlet boundary conditions, where V is a real-valued, nonnegative potential.

Anderson [An] noticed that for some potentials the low-lying eigenfunctions tend to strongly localize in a subregion of space in a very complicated manner (see many talks in this conference).

Filoche–Mayboroda (2012) [FM] have given an effective method to predict the behavior of low-energy eigenfunctions.

Their approach is based on the following inequality:
if we associate to the problem a landscape function u which is defined as the solution of

$$(-\Delta + V)u = 1 \text{ in } \Omega$$

with Dirichlet boundary conditions, then an eigenfunction ϕ associated with an eigenvalue λ , we have

$$|\phi(x)| \leq \lambda u(x) \|\phi\|_\infty$$

Note that at the level of decay estimate, this is not a very accurate estimate but should be considered as a first step localization.

The regions where u is small will be of particular interest because an eigenfunction can only localize in $\{x : \lambda u(x) \geq 1\}$ to compare with the so-called classical region $\{x : V(x) \leq \lambda\}$.

It is instructive to regard the graph of u as a landscape comprised of peaks and valleys. The valleys may then be understood as inducing a partition of the domain.

Numerical experiments suggest that low-lying eigenfunctions respect that partition and favor localization in one or at most a few elements in that partition.

Moreover, these localized eigenfunctions are almost compactly supported in the sense that in crossing from one element of the partition to another eigenfunctions seem to experience exponential decay when crossing the valley (see [FM]).

Concerning this exponential drop in size of an eigenfunction when crossing valley, this was recently made precise by Arnold, David, Filoche, Jerison, and Mayboroda [ADFJM2016, ADFJM2018] who point out that the inverse of the landscape function acts as an effective potential responsible for the exponential decay of the localized states.

The approach is based on writing an eigenfunction as

$$\phi = u\psi.$$

The spectral equation

$$(-\Delta + V)\phi = \lambda\phi,$$

then transforms into

$$\left[\frac{1}{u^2} \operatorname{div}(u^2 \nabla \psi) \right] + \frac{(-\Delta + V)u}{u} \psi = \lambda \psi.$$

We then get a new Schrödinger operator with a new potential

$$W = \frac{(-\Delta + V)u}{u} = \frac{1}{u}$$

for which we can use Agmon's approach and give explicit constants.

There are cases where a numerical evidence is given that this new decay estimate predicts a more accurate decay than the standard Agmon estimate.

Remarks

Note that we know about three choices of u for the quotient $W = \frac{(-\Delta+V)u}{u}$.

- ▶ $u = 1$ (so $W = V$)
- ▶ $u = u_0$ the groundstate of the Schrödinger operator (choice which can be used for the analysis of the splitting).
- ▶ The Landscape function (so $W = \frac{1}{u}$).

In the semi-classical context, the Landscape function is close to $\frac{1}{V}$ and there is probably no gain in working with it.

The remarkable efficiency of the Landscape analysis should be related to the existence of some "new" semiclassical parameter. This parameter does not appears clearly but one can use the fact that Agmon strategy permits to work with explicit constants [ADFJM2018].



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